NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

$\mathbb{MA}$ 1505 Mathematics I
Tutorial 6

1. In an electric circuit, the voltage of V volts (V), current of I amperes (A), and resistance of R ohms (Ω) are governed by Ohm's Law $V = I \times R$.

   (i) If the resistance is fixed at 15 Ω, how fast is the current increasing with respect to voltage?

   (ii) If the voltage is fixed at 120 V, how fast is the current increasing with respect to resistance at the instant when resistance is 20 Ω?

   (iii) If the resistance is slowly increasing as the resistor heats up, how is the current changing at the moment when $R = 400\Omega$, $I = 0.08$A, $dV/dt = -0.01$ V/s and $dR/dt = 0.03$ Ω/s?

   Ans. (i) ≈ 0.0667 A/V; (ii) *decreasing* at 0.3 A/Ω; (iii) decreasing at 3.1×10^{-5} A/s

2. Find the directional derivative of $f(x, y) = xe^{2y-x}$ at $P(-2, -1)$ in the direction

   (i) $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; (ii) $3\mathbf{i} + 4\mathbf{j}$;

   Find the direction that gives the *largest possible* directional derivative of f at P.

   Ans. (i) $-\sqrt{2}/2$; (ii) $-7/5$; $f_x(-2, -1)\mathbf{i} + f_y(-2, -1)\mathbf{j} = 3\mathbf{i} - 4\mathbf{j}$

3. Let $f(x, y, z) = \sin(xyz)$ and $P = (\frac{1}{2}, \frac{1}{3}, \pi)$.

   (i) Find the rate of change of f at P in the direction $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$.

   (ii) Suppose P moves 0.1 unit along $\mathbf{u}$ in part (i). How much will the value of f have changed?

   Ans. (i) $\frac{1}{12}(1 - \pi)$; (ii) decreases by ≈ 0.01785.

4. Find the local maximum and minimum values and saddle points (if any) of each of the following functions.

   (i) $f(x, y) = \ln(x^2y) - xy - 2x$, where $x > 0,\ y > 0$

   (ii) $g(x, y) = xy(1 - x - y)$

   (iii) $h(x, y) = x^2 + y^2 + x^{-2}y^{-2}$, where $x \neq 0,\ y \neq 0$

   Ans. (i) $f(1/2, 2) = -\ln 2 - 2$ is a local maximum, (ii) $(0, 0)$, $(1, 0)$, $(0, 1)$ are saddle points, $g(1/3, 1/3) = 1/27$ is a local maximum, (iii) $h(\pm 1, \pm 1) = h(\pm 1, \mp 1) = 3$ are local minima.

5. Let $u = u(x, y)$ be a twice differentiable function of x and y. If u satisfies $u > 0$ and

$$u\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y},$$

   prove that $\frac{\partial(\ln u)}{\partial y}$ is a function of one variable y only.

6. The production output P (units) of a certain factory is estimated by

$$P(L, K) = 50L^{2/5}K^{3/5},$$

where L = amount of labour; and

K = capital investment.

The quantities L and K are measured in thousand dollars. If \$150,000 is available, how should this amount be allocated between labour and capital investment to generate the largest possible output?