NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

$\mathbb{MA}$ 1505 Mathematics I
Tutorial 5

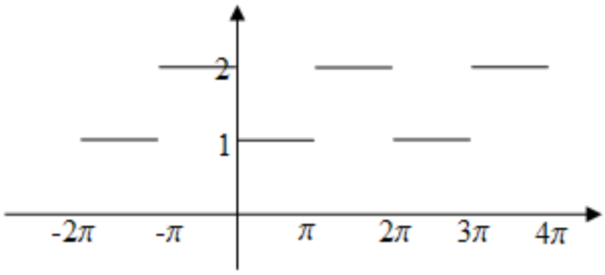
1. Show that the Fourier series for

$$f(x) = \frac{1}{2}(x + |x|), -\pi < x < \pi; \quad f(x + 2\pi) = f(x)$$

is given by

$$\frac{\pi}{4} + \sum_{1}^{\infty} \left\{ \frac{(-1)^n - 1}{\pi n^2} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right\}.$$

2. Find the Fourier series that represent the following graph:



Ans. $\displaystyle \frac{3}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$

3. Sketch the graph of the following periodic function. Determine whether it is an even or odd function, and find its Fourier series.

$$f(x) = \begin{cases} 1 & -\pi/2 < x < \pi/2 \\ 0 & \pi/2 < x < 3\pi/2 \end{cases}, \quad f(x + 2\pi) = f(x).$$

Ans. $\displaystyle \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(2n-1)x}{2n-1}$

4. Find the Fourier series of a half-wave rectifier

$$u(t) = \begin{cases} 0 & -\pi/w < t < 0 \\ \sin wt & 0 < t < \pi/w \end{cases}$$

and $u(t + \frac{2\pi}{w}) = u(t)$.

Ans. $\displaystyle \frac{1}{\pi} + \frac{1}{2} \sin wt - \frac{2}{\pi} \left(\frac{1}{1 \cdot 3} \cos 2wt + \frac{1}{3 \cdot 5} \cos 4wt + \cdots \right)$

5. Find the Fourier series for the function

$$f(x)=\begin{cases} -2-x & -2<x<0 \\ 2-x & 0<x<2 \end{cases}$$

and $f(x+4)=f(x)$.

Ans. $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$

6. Find the sine and cosine Fourier half range expansion for

$$f(x)=x, \quad 0<x<\pi.$$

Also, sketch the corresponding expansions of f.

Ans. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\sin nx}{n}$; $\qquad \dfrac{\pi}{2}+\displaystyle\sum_{n=1}^{\infty}\frac{2((-1)^n-1)}{\pi}\frac{\cos nx}{n^2}$

7. Let $f(x)$ be an odd function on the open interval $(-\pi,\pi)$ and assume that $f(x)$ has been extended to a 2π-periodic function for all x. Assume that $f(x)$ has the Fourier sine series representation $\sum\limits_{n=1}^{\infty} b_n \sin nx$ and that $(f(x))^2$ can be obtained by squaring this series formally and term by term integration of this squared series is valid. Prove the following form of Parseval's Identity:

$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx=\sum\limits_{n=1}^{\infty}(b_n)^2.$

From Example 6.2.12 of the lecture notes, we know that the Fourier sine series for the function $f(x)=x$ on $(-\pi,\pi)$ is $\sum\limits_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$. Use Parseval's Identity to prove that

$\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$