NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

$\mathbb{MA}$ 1505 Mathematics I
Tutorial 1

1. Let $f(x) = \frac{6}{x}$ and $g(x) = \sqrt{|3-x|}$. Find an expression for $(g \circ f)(x) - (f \circ g)(x)$.
   Ans. $\sqrt{|3 - \frac{6}{x}|} - \frac{6}{\sqrt{|3-x|}}$.

2. Find the first derivatives of the following functions.

   (a) $y = \dfrac{ax+b}{cx+d}$ (b) $y = \sin^n x \, \cos(mx)$

   (c) $y = e^{x^2+x^3}$ (d) $y = x^3 - 4(x^2 + e^2 + \ln 2)$

   (e) $y = \left(\dfrac{\sin\theta}{\cos\theta - 1}\right)^2$ (f) $y = t\tan(2\sqrt{t}) + 7$

   (g) $r = \sin(\theta + \sqrt{\theta+1})$ (h) $s = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$

   Ans. (a) $y' = \dfrac{ad-bc}{(cx+d)^2}$ (b) $y' = n\sin^{n-1} x \cos x \cos mx - m \sin^n x \sin mx$

   (c) $y' = e^{x^2+x^3}(2x + 3x^2)$ (d) $y' = 3x^2 - 8x$

   (e) $y' = -2\sin\theta(\cos\theta - 1)^{-2}$ (f) $y' = \sqrt{t}\sec^2(2\sqrt{t}) + \tan(2\sqrt{t})$

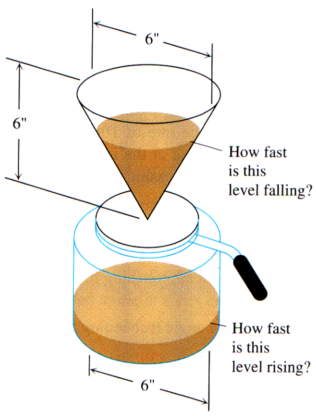
   (g) $r' = \frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}}\cos(\theta + \sqrt{\theta+1})$ (h) $s' = 4\tan x \sec x - \csc^2 x$

3. Coffee is drained from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.

   (a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

   (b) How fast is the level in the cone falling then?

   (Volume of cone: $\frac{1}{3}\times$ base area $\times$ height)



Ans. (a) $\frac{10}{9\pi}$ in/min; (b) $\frac{8}{5\pi}$ in/min.

4. For the following functions, find y' and y''.

   (a) $x^{2/3} + y^{2/3} = a^{2/3}, \ 0 < x < a, \, 0 < y$

   (b) $y = (\sin x)^{\sin x}, \ 0 < x < \frac{\pi}{2}$

   (c) $x = a\cos t, \, y = a\sin t$

   Ans. (a) $y' = -\sqrt{(\frac{a}{x})^{2/3} - 1}, \quad y'' = \dfrac{a^{2/3}}{3x^{4/3}\sqrt{a^{2/3} - x^{2/3}}}.$

   (b) $y' = (\sin x)^{\sin x}(1 + \ln\sin x)\cos x,$

   $y'' = (\sin x)^{\sin x}[(1 + \ln\sin x)^2\cos^2 x + \dfrac{\cos^2 x}{\sin x} - (1 + \ln\sin x)\sin x].$

   (c) $y' = -\cot t\,, \, y'' = -\dfrac{1}{a\sin^3 t}.$

5. For each of the following functions:

   (a) $y = \dfrac{x+1}{x^2+1}, \, x \in [-3, 3]$ (b) $y = (x-1)\sqrt[3]{x^2}, \, x \in (-\infty, \infty)$

   determine

   (i) the critical points;

   (ii) the intervals where it is increasing and decreasing;

   (iii) the local and absolute extreme values.

   Ans. (a) local min. $-\frac{1}{2(\sqrt{2}+1)}$ at $x = -1 - \sqrt{2}$ and $\frac{2}{5}$ at $x = 3$;
   local max. $\frac{1}{2(\sqrt{2}-1)}$ at $x = -1 + \sqrt{2}$ and $-\frac{1}{5}$ at $x = -3$.
   (b) local min. $-\frac{3}{5}\left(\frac{2}{5}\right)^{2/3}$ at $x = \frac{2}{5}$; local max. 0 at $x = 0$.

6. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart. If it takes 1.4 times as much energy to fly over water as land, find the distance between B and C.

   Ans. 5.1 km.

7. Let a and b be two constants with $0 < b < a$. Let E denote the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let C denote the circle $x^2 + y^2 = a^2$. Let O be the point $(-a, 0)$ and let L denote the line passing through O and making an angle θ with the x-axis, where $0 \leq \theta \leq \frac{\pi}{2}$. Suppose L intersects E and C at the points A and B respectively (see the diagram on the next page for illustration). Prove that the length AB attains a maximum value when $\theta = \cos^{-1}\left\{\sqrt{\frac{3-e^2-\sqrt{(9-e^2)(1-e^2)}}{2e^2}}\right\}$, where $e = \sqrt{\frac{a^2-b^2}{a^2}}$ is the eccentricity of the ellipse E.

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