MA1506 TUTORIAL 7

1. Find the Laplace transforms of the following functions [where u denotes the unit step function and the answers are given in brackets]:

   (a) t^2e^{-3t}. $\qquad [\frac{2}{(s+3)^3}]$

   (b) $tu(t-2)$. $\qquad [e^{-2s}\{\frac{1}{s^2}+\frac{2}{s}\}]$

2. Find the inverse Laplace transforms of the following functions:

   (a) $\frac{s}{s^2+10s+26}$. $\qquad [e^{-5t}(\cos t - 5\sin t)]$

   (b) $e^{-2s}\,\frac{1+2s}{s^3}$. $\qquad [(\frac{1}{2}t^2-2)u(t-2)]$

3. Solve the following initial value problems using Laplace transforms:

   (a) $y' = tu(t-2), \;\; y(0) = 4$. $\qquad [(\frac{1}{2}t^2-2)u(t-2)+4]$

   (b) $y'' - 2y' = 4, \;\; y(0) = 1, \;\; y'(0) = 0$. $\qquad [e^{2t} - 2t]$

4. (i) Show, from the definition of the Laplace transform, that for any function f(t),

$$L[tf(t)] = -F'(s),$$

   where F(s) is the Laplace transform of f(t). Hence find the Laplace transform of t sin(t).

   (ii) Use Laplace transforms to solve the resonance equation

$$\ddot{y} + y = cos(t),$$

   where $y(0) = \dot{y}(0) = 0$. You should recognise the solution!

5. The oil tanker in Tutorial 4 is at rest in an almost calm sea. Suddenly, at time t = T > 0, it is hit by a single rogue wave [http://en.wikipedia.org/wiki/Rogue_wave] which imparts to it a vertical [upward] momentum P, doing so almost instantaneously. Neglecting friction, solve for x(t), the downward displacement of the ship, and graph it. How far down

does the ship go [if it doesn't sink!]? [Hint: according to Newton's second law, momentum is the time integral of force. So to get the force as a function of time in this problem, you have to find a function which is zero except at t = T, and which has an integral equal to P. Note that the delta function has units of 1/time.] [Answer: the ship goes down either to P/ωM or to the bottom of the sea.]

6. In Question 5, suppose that you don't want to assume that the wave hits instantaneously: you want to model the situation by assuming that the momentum P is imparted to the ship over a short but non-zero period of time τ, starting at t = T. Explain how you would do this, using the step function. [Remember how to use the step function to model situations where something is turned on, then turned off.] Compute the Laplace transform of the displacement function. Show, using L'Hopital's rule, that, in the limit $\tau \to 0$, this Laplace transform tends to the one you found in Question 5. That is reasonable, right?

7. When music is recorded digitally, for example for a CD, it can only be *sampled*; that is, you record it only at discrete times, not continuously. [The theory of this is called signal processing in engineering http://en.wikipedia.org/wiki/Signal_processing .] This sampling process can be represented mathematically using an "impulse train", also known as *Dirac's comb*, defined by

$$\Delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

You can see that multiplying a continuous function by the comb essentially throws away information about the function except at t = 0, t = ±T, t = ±2T, etc; this is what is meant by "sampling". Note that the comb is periodic with period T. Find the Fourier series of the comb. If you graph the sum of the first few terms [say, the first 10] of this series you will see why $\Delta_T(t)$ is called a "comb".