"How To" for MA1101R

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This cute file is provided for the students in MA1101R. I hope it is helpful to "kill" the Final Examination. Any corrections and suggestions are always welcome. Please feel free to email me at `zhoufeng@nus.edu.sg`.

User guide: For each how-to question, I will list the most powerful methods, related examples (chosen from Examples in the textbook[1] and Exercises in the tutorial questions) and remarks if necessary.

1 Chapter 1

Question 1. *How to identify a row-echelon form (REF) and a reduced row-echelon form (RREF)?*

Answer. [Page 8] A matrix is said to be in row-echelon form (REF) if it has properties (1) and (2):

(1) If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

(2) In any two successive rows that do not consist entirely of zeros, the first nonzero number in the lower row occurs farther to the right than the first nonzero number in the higher row.

A matrix is said to be in reduced row-echelon form (RREF) if it is has properties (1), (2), (3), and (4):

(3) The leading entry of every nonzero row is 1.

(4) In each pivot column, except the pivot point, all other entries are zero.

Properties (3) and (4) are the differences between a REF and a RREF.

➢ Example: Exercises 1.28 and 1.29.

♣ Remark: In the textbook, the REF and RREF are defined for the augmented matrix, but they can be generalized for any arbitrary matrix. □

Question 2. *Given a matrix, how to obtain a row-echelon form or a reduced row-echelon form for it?*

Answer. [Page 11] If we want to get a REF, we should apply Gaussian Elimination. If we want to get a RREF, then we should apply Gauss-Jordan Elimination.

➢ Example: Example 1.4.4.

♣ Remark: In the textbook, Gaussian Elimination and Gauss-Jordan Elimination are defined for the augmented matrix, but they can be generalized for any arbitrary matrix, too. □

Question 3. *How to tell the number of solutions of a linear system from* REF*?*

Answer. [Page 15–16] There are the following three cases:

- A linear system has no solution if the last column of a REF of the augmented matrix is a pivot column, i.e. there is a row with nonzero last entry but zero elsewhere.

[1] Ma Siu Lun, Ng Kah Loon, Victor Tan: Linear Algebra – Concepts and Techniques on Euclidean Spaces (Revised Edition), McGraw-Hill, 2014

- A linear system has exactly one solution if except the last column, every column of a REF of the augmented matrix is a pivot column. That is, a linear system has exactly one solution if it is consistent and (♯ variables = ♯ nonzero rows).
- A linear system has infinitely many solutions if apart from the last column, a REF of the augmented matrix has at least one more non-pivot column. That is, a linear system has infinitely many solutions if it is consistent and (♯ variables > ♯ nonzero rows).

  In this case, its general solution has (♯ variables − ♯ nonzero rows) arbitrary parameter(s).

➢ Example: Example 1.4.10 and Exercise 1.22. □

Question 4. *If a linear system is consistent, how to find a general solution of it?*

Answer. We need follow this process:

1. Transfer the linear system to the corresponding augmented matrix;
2. Apply Gaussian Elimination (resp. Gauss-Jordan Elimination) to obtain a REF (resp. a RREF) of the augmented matrix;
3. Identify the pivot columns and non-pivot columns;
4. For any i, if the i-th column is a non-pivot column, then take the i-th variable to be a parameter.
5. Express each of the remaining variable(s) as an expression of the parameter(s).

➢ Example: Example 1.4.7. □

2 Chapter 2

Question 5. *How to determine whether an $m \times n$ matrix $\boldsymbol{A}$ is invertible?*

Answer. If $\boldsymbol{A}$ is not a square matrix, it cannot be invertible. If $\boldsymbol{A}$ is a square matrix, we have the following four methods:

- First method: [Page 65] $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$. So after computing the determinant of $\boldsymbol{A}$, it is easy to identify the invertibility of $\boldsymbol{A}$.
- Second method: [Page 181] $\boldsymbol{A}$ is invertible if and only if $\text{rank}(\boldsymbol{A}) = n$. So we may apply Gaussian Elimination to obtain a REF for $\boldsymbol{A}$, then find the $\text{rank}(\boldsymbol{A})$ and determine whether $\text{rank}(\boldsymbol{A}) = n$.
- Third method: [Page 45–46] If we can find a matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}$ or $\boldsymbol{BA} = \boldsymbol{I}$, then it follows from the definition that $\boldsymbol{A}$ is invertible.

  ➢ Example: Examples 2.3.3 and 2.3.8.
- Fourth method: [Page 53] $\boldsymbol{A}$ is invertible if and only if the RREF of $\boldsymbol{A}$ is an identity matrix.

  ➢ Example: Example 2.4.11 and Exercise 2.43.

♣ Remark: The second method and the fourth method are essentially the same. □

Question 6. *How to find the inverse of an invertible matrix $\boldsymbol{A}$?*

Answer. There are the following three methods:

- First method: [Page 54]
  1. Consider the $n \times 2n$ matrix $(\boldsymbol{A} \mid \boldsymbol{I})$.
  2. By Gauss-Jordan Elimination, there exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that $\boldsymbol{E}_k \cdots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{I}$, that is,

  $$\boldsymbol{A}^{-1} = \boldsymbol{E}_k \cdots \boldsymbol{E}_2\boldsymbol{E}_1.$$

  ➢ Example: Example 2.4.9.

- Second method: [Page 67] When $\boldsymbol{A}$ is sparse (nonzero entries are few), we may apply $\mathbf{adj}(\boldsymbol{A})$ to compute $\boldsymbol{A}^{-1}$:

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A}).$$

  ➢ Example: Example 2.5.26.

- Third method: [Page 45–46] If we can find a matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}$ or $\boldsymbol{BA} = \boldsymbol{I}$, then by definition $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1} = \boldsymbol{B}$.

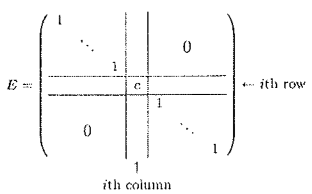
  ➢ Example: Exercises 2.26, 2.27, and 2.29.

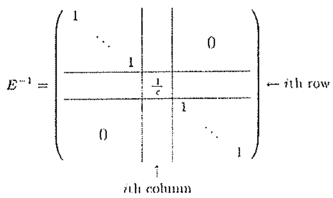
□

Question 7. *How to convert elementary row operations to elementary matrices, and vice versa?*

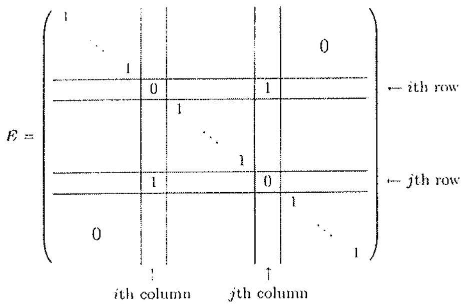
Answer. [Page 48–51] There are the following three cases:

- Multiply a row by a constant:

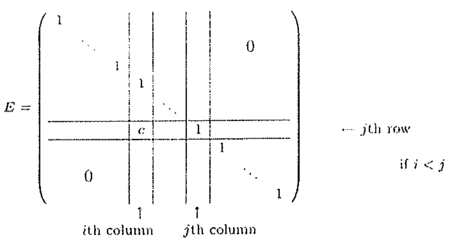


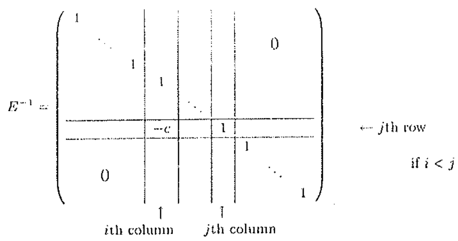


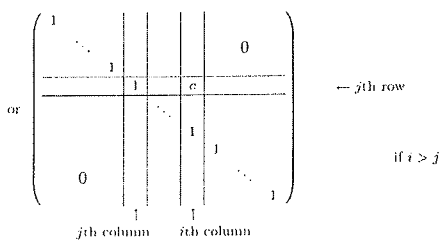
- Interchange two rows:

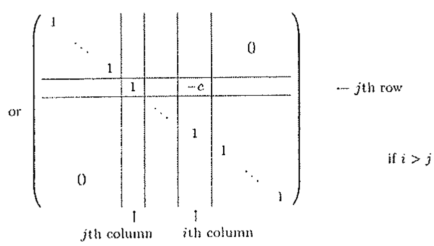


- Add a multiple of a row to another row:









□

Question 8. *Given two row equivalent matrices A and B, how to find an invertible matrix* $\boldsymbol{D}$ *such that* $\boldsymbol{DA} = \boldsymbol{B}$*?*

Answer. [Page 52] $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by elementary row operations. For each of these elementary row operations, we write down the corresponding elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$:

$$\boldsymbol{A} \xrightarrow{\boldsymbol{E}_1} \xrightarrow{\boldsymbol{E}_2} \quad \cdots \quad \xrightarrow{\boldsymbol{E}_k} \boldsymbol{B}.$$

It follows that $\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{B}$. So we can take $\boldsymbol{D} = \boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1$.

➢ Example: Example 2.4.5 and Exercise 2.33. □

Question 9. *How to compute the determinant of a matrix* $\boldsymbol{A}$ *using various methods (and not just cofactor expansion)?*

Answer. We have the following three methods:

- First method: [Page 58–60] Apply the definition or cofactor expansion.

  ➢ Example: Examples 2.5.4 and 2.5.7.

- Second method: [Page 59] For the 2×2 and 3×3 matrices, we have the following formulas:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi.$$

- Third method: [Page 62–65] Apply Gauss Elimination or Gauss-Jordan Elimination to get a REF or a RREF $\boldsymbol{R}$ for $\boldsymbol{A}$, that is, there exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that

$$\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{R}.$$

  Then we have

$$\det(\boldsymbol{A}) = \det(\boldsymbol{E}_1)^{-1} \det(\boldsymbol{E}_2)^{-1} \cdots \det(\boldsymbol{E}_k)^{-1} \det(\boldsymbol{R}),$$

  where $\boldsymbol{R}$ is an upper triangular matrix whose determinant is the product of the diagonal entries.

  ➢ Example: Example 2.5.17 and Exercise 2.52.

□

Question 10. *How does a row (column) operation change the determinant?*

Answer. [Page 62–64] There are the following three cases:

- If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by multiplying one row of $\boldsymbol{A}$ by a constant k, then $\det(\boldsymbol{B}) = k \det(\boldsymbol{A})$;
- If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by interchanging two rows of $\boldsymbol{A}$, then $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$;
- If $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by adding a multiple of one row of $\boldsymbol{A}$ to another row, then $\det(\boldsymbol{B}) = \det(\boldsymbol{A})$.

□

Question 11. *How are invertibility of a matrix and the homogeneous system related?*

Answer. [Page 53] A matrix $\boldsymbol{A}$ is invertible if and only if the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.

➢ Example: Exercise 2.41. □

Question 12. *How are invertibility and determinant of a matrix related?*

Answer. [Page 65] A matrix $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

➢ Example: Example 2.5.20. □

3 Chapter 3

Question 13. *How to write down implicit and explicit set notations for lines and planes in* $\mathbb{R}^3$*?*

Answer. [Page 87] An implicit form for a line in $\mathbb{R}^3$ is

$$\{\,(x,y,z) \mid a_1x+b_1y+c_1z=d_1,\ a_2x+b_2y+c_2z=d_2\,\},$$

where $a_1,b_1,c_1,d_1,a_2,b_2,c_2,d_2$ are real constants, a_1,b_1,c_1 are not all zero, and a_2,b_2,c_2 are not all zero.

An explicit form for a line in $\mathbb{R}^3$ is

$$\{\,(a_0,b_0,c_0)+t(a,b,c) \mid t\in\mathbb{R}\,\},$$

where a_0,b_0,c_0,a,b,c are real constants and a,b,c are not all zero.

An implicit form for a plane in $\mathbb{R}^3$ is

$$\{\,(x,y,z) \mid ax+by+cz=d\,\},$$

where a,b,c,d are real constants and a,b,c are not all zero.

An explicit form for a plane in $\mathbb{R}^3$ is

$$\begin{cases} \{\,(\frac{d-bs-ct}{a},s,t) \mid s,t\in\mathbb{R}\,\}, & \text{if } a\neq 0;\\ \{\,(s,\frac{d-as-ct}{b},t) \mid s,t\in\mathbb{R}\,\}, & \text{if } b\neq 0;\\ \{\,(s,t,\frac{d-as-bt}{c}) \mid s,t\in\mathbb{R}\,\}, & \text{if } c\neq 0.\end{cases}$$

□

Question 14. *How to find a line in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *when you are given the direction of the line and a point on the line?*

Answer. [Page 87] In $\mathbb{R}^2$, given a point (a_0,b_0) on the line and the direction (a,b), then the line is

$$\{\,(a_0,b_0)+t(a,b) \mid t\in\mathbb{R}\,\}.$$

In $\mathbb{R}^3$, given a point (a_0,b_0,c_0) on the line and the direction (a,b,c), then the line is

$$\{\,(a_0,b_0,c_0)+t(a,b,c) \mid t\in\mathbb{R}\,\}.$$

If we want to get an implicit form for the line $\{\,(a_0,b_0,c_0)+t(a,b,c) \mid t\in\mathbb{R}\,\}$, we should cancel the parameter t, and construct two equations in terms of x,y,z. Indeed, let $x=a_0+ta$, $y=b_0+tb$, and $z=c_0+tc$. Now we need to remove t. From $x=a_0+ta$ and $y=b_0+tb$, we will obtain

$$bx-ay=a_0b-b_0a. \tag{1}$$

Similarly, from $y=b_0+tb$ and $z=c_0+tc$, we will have

$$cy-bz=b_0c-c_0b. \tag{2}$$

Then the set

$$\{\,(x,y,z) \mid bx-ay=a_0b-b_0a,\ cy-bz=b_0c-c_0b\,\}$$

gives us an implicit form for the line. □

Question 15. *How to find the equation of a plane in* $\mathbb{R}^3$ *when you are given three points on the plane?*

Answer. [Page 87] Assume the plane is represented by the equation $ax+by+cz=d$, where a,b,c are not all zero. Since there are three points (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3) on the plane, they satisfy the equation $ax+by+cz=d$. Then consider the following linear system in terms of a,b,c,d:

$$\begin{cases} x_1a+y_1b+z_1c-d=0\\ x_2a+y_2b+z_2c-d=0\\ x_3a+y_3b+z_3c-d=0\end{cases}$$

We may have a general solution for a, b, c, d (should be determined by one parameter). By taking a particular one, we will get an equation for the plane.

♣ Remark: if the three points are on a line, the plane cannot be determined uniquely. □

Question 16. *How to determine whether a vector $\boldsymbol{v}$ is a linear combination of a given set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$?*

Answer. [Page 88–89] Assume $\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k$. Then we will have a linear system in terms of $c_1, c_2, \dots, c_k$:

$$\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{v},$$

where $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k, \boldsymbol{v}$ are column vectors.

Since we have that $\boldsymbol{v}$ can be expressed as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ if and only if the above linear system is consistent, it suffices to see whether the linear system is consistent.

➢ Example: Example 3.2.2. □

Question 17. *How to express a vector $\boldsymbol{v}$ as a linear combination of a given set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$?*

Answer. Following the process in the last question, if the linear system is consistent, and we can find a solution $(c_1, c_2, \dots, c_k)$, then we have

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k.$$

➢ Example: Example 3.2.2. □

Question 18. *How to show a linear span* $\mathrm{span}(S_1)$ *is contained in another one* $\mathrm{span}(S_2)$*, where* $S_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$, $S_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$?

Answer. [Page 92–93] Since $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$ if and only if each $\boldsymbol{u}_i$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m$, it suffices to show that each of the last k columns in the following REF is <u>not</u> a pivot column:

$$\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_m \mid \boldsymbol{u}_1 \mid \boldsymbol{u}_2 \mid \cdots \mid \boldsymbol{u}_k \end{pmatrix},$$

where $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m$ are column vectors.

➢ Example: Example 3.2.11 and Exercise 3.12. □

Question 19. *How to show that a set V is a subspace of $\mathbb{R}^n$?*

Answer. Firstly, we should show that V is a subset of $\mathbb{R}^n$. Then there are the following three methods:

- First method: [Page 95] If we can find a set S which spans V, then V is a subspace of $\mathbb{R}^n$ by definition directly.

  ➢ Example: Example 3.3.3, and Exercises 3.20, 5.7.

- Second method: [Page 99] V is a subspace of $\mathbb{R}^n$ if and only if $V \neq \emptyset$ and for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$, $c\boldsymbol{u} + d\boldsymbol{v} \in V$.

  ➢ Example: Exercises 3.20, 3.21, 3.22, 3.24, and 5.7.

- Third method: [Page 97] If V is the solution set of a homogeneous linear system, then V is a subspace of $\mathbb{R}^n$.

  ➢ Example: Exercises 3.21, 3.22, and 5.7.

□

Question 20. *How to show that a set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ is linearly (in)dependent?*

Answer. [Page 99–100] Apply the working definition. The equation

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \boldsymbol{0}$$

gives us a linear system in terms of $c_1, c_2, \ldots, c_k$:

$$\begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \cdots & \boldsymbol{u_k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{0},$$

where $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}, \boldsymbol{0}$ are column vectors.

By Gaussian Elimination, we can identify whether this linear system has only the trivial solution, and we have the fact that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ is linearly independent if and only if this linear system has only the trivial solution.

➢ Example: Example 3.4.3. □

Question 21. *How to show that a set S is a basis for a vector space V?*

Answer. [Page 103–111] We have the following four conditions:

- Condition (1): $S \subseteq V$;
- Condition (2-1): S is linearly independent;
- Condition (2-2): S spans V;
- Condition (2-3): $|S| = \dim(V)$.

If condition (1) and any <u>two</u> of conditions (2-1), (2-2) and (2-3) hold, then S is a basis for V.

♣ Remark: if we know $\dim(V)$, then we can choose conditions (1), (2-1) and (2-3) to check; if we do not know $\dim(V)$, we need to check conditions (1), (2-1), and (2-2).

➢ Example: Examples 3.5.5, 3.6.8, and Exercises 3.32, 3.38, 3.45. □

Question 22. *How to find a basis for a vector space V?*

Answer. [Page 104–110]

1. Find a set $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ which spans V, where $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are column vectors;
2. Apply Gaussian Elimination to the matrix $\begin{pmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_k^T \end{pmatrix}$, we obtain a REF $\boldsymbol{R}$;
3. Let S' be the set of nonzero rows in $\boldsymbol{R}$, then S' is a basis for V.

♣ Remark: S' is not necessarily unique.

➢ Example: Examples 3.4.6, 3.6.4, and Exercise 4.7. □

Question 23. *Given a basis $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ for a vector space V and a vector $\boldsymbol{v} \in V$, how to find coordinate vectors $[\boldsymbol{v}]_S$ and $(\boldsymbol{v})_S$?*

Answer. [Page 106–107] Let $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$. By solving the following linear system

$$\begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \cdots & \boldsymbol{u_k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{v},$$

where $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}, \boldsymbol{v}$ are column vectors, we will find a solution $(c_1, c_2, \ldots, c_k)$. So the coordinate vectors are

$$[\boldsymbol{v}]_S^T = (\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k).$$

➢ Example: Example 3.5.9. □

Question 24. *How to compute the dimension for a vector space?*

Answer. [Page 109–111] Following the process in **Question 22**, S' is a basis for V . Then we have

$$\dim(V) = |S'|.$$

➢ Example: Example 3.6.4.

♣ Remark: Except $\mathbb{R}^n$ and $\{\mathbf{0}\}$, we can identify $\dim(V)$ only when we have found a basis for it. □

Question 25. *How to compute the transition matrix from $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ to $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$, where S and T are two bases for a vector space V?*

Answer. There are the following two methods:

- First method: [Page 114–115] By solving linear systems, we can find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T, \ldots, [\boldsymbol{u}_k]_T$. Then the transition matrix from S to T is

$$\boldsymbol{P} = \begin{pmatrix} [\boldsymbol{u}_1]_T & [\boldsymbol{u}_2]_T & \cdots & [\boldsymbol{u}_k]_T \end{pmatrix}.$$

  ➢ Example: Example 3.7.4.

- Second method: [Page 116] If we know the transition matrix from T to S is $\boldsymbol{Q}$, then the transition matrix from S to T is $\boldsymbol{P} = \boldsymbol{Q}^{-1}$.

  ➢ Example: Example 3.7.6.

□

4 Chapter 4

Question 26. *How to find bases for the row space and the column space of $\boldsymbol{A}$?*

Answer. [Page 129–134] We may apply the similar method in **Question 22**:

Apply Gaussian Elimination to $\boldsymbol{A}$ to obtain a REF $\boldsymbol{R}$ of $\boldsymbol{A}$. Then the set of nonzero rows in $\boldsymbol{R}$ forms a basis for the row space of $\boldsymbol{A}$.

Besides, by choosing the pivot columns, we will find a set of columns of $\boldsymbol{R}$ which forms a basis for the column space of $\boldsymbol{R}$. Then the set of corresponding columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$.

➢ Example: Examples 4.1.12, 4.1.14, and Exercises 4.3, 4.4. □

Question 27. *How to extend a linearly independent set S to a basis for $\mathbb{R}^n$?*

Answer. [Page 134] We need follow this procedure:

1. Form a matrix $\boldsymbol{A}$ using the vectors in S as rows;
2. Reduce $\boldsymbol{A}$ to a REF $\boldsymbol{R}$;
3. Identify the non-pivot columns in $\boldsymbol{R}$;
4. For each non-pivot column identified in Step 3, get a vector such that the leading entry of the vector is at that column;
5. Now, $S \cup$ (the set of vectors obtained in Step 4) is a basis for $\mathbb{R}^n$.

➢ Example: Example 4.1.14. □

Question 28. *How to find a basis for the nullspace of $\boldsymbol{A}$?*

Answer. [Page 138–139] Solve the homogeneous system $\boldsymbol{Ax} = \mathbf{0}$. Then find a basis for the solution space.

➢ Example: Example 4.3.3. □

Question 29. *How to find the rank and the nullity of a matrix* $\boldsymbol{A}$*?*

Answer. [Page 136–140] Apply Gaussian Elimination to reduce $\boldsymbol{A}$ to a REF $\boldsymbol{R}$. Then

$$\operatorname{rank}(\boldsymbol{A}) = (\sharp \text{ pivot columns in } \boldsymbol{R}),$$

and

$$\begin{aligned} \operatorname{nullity}(\boldsymbol{A}) &= (\text{the dimension of the solution space of } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}) \\ &= (\sharp \text{ columns of } \boldsymbol{A}) - \operatorname{rank}(\boldsymbol{A}). \end{aligned}$$

➢ Example: Examples 4.2.2, 4.3.3, and Exercise 4.15. □

5 Chapter 6

Question 30. *How to find eigenvalues and eigenvectors of a matrix* $\boldsymbol{A}$*?*

Answer. There are the following three methods:

- First method: [Page 180] Solve the characteristic equation $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$.

  ➢ Example: Examples 6.1.7, 6.1.10, and Exercise 6.3.

- Second method: [Page 178] By definition, if we find a scalar λ and a nonzero column vector $\boldsymbol{u}$ in $\mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$, then λ is an eigenvalue of $\boldsymbol{A}$, and $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ.

  ➢ Example: Exercise 6.16.

- Third method: If we have such an equation

  $$\boldsymbol{A}^k + a_{k-1}\boldsymbol{A}^{k-1} + \cdots + a_1\boldsymbol{A} + a_0\boldsymbol{I} = \boldsymbol{0},$$

  then any solution for the equation $\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda + a_0 = 0$ is an eigenvalue of $\boldsymbol{A}$.

  ➢ Example: Exercise 6.4.

♣ Remark: If $\boldsymbol{A}$ is an "abstract" matrix (e.g. a stochastic matrix), we can only apply the second method. □

Question 31. *How to find a basis for the eigenspace of a matrix* $\boldsymbol{A}$*?*

Answer. [Page 182–184] Suppose we are given an eigenvalue λ of $\boldsymbol{A}$. By solving the linear system

$$(\lambda\boldsymbol{I} - \boldsymbol{A})x = \boldsymbol{0},$$

we will have a general solution, say $\boldsymbol{x} = s_1\boldsymbol{u_1} + s_2\boldsymbol{u_2} + \cdots + s_k\boldsymbol{u_k}$, where $s_1, s_2, \ldots, s_k$ are arbitrary parameters. Then the eigenspace of $\boldsymbol{A}$ associated with the eigenvalue λ is $E_\lambda = \operatorname{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$.

➢ Example: Example 6.1.12. □

Question 32. *How are eigenvalues and the invertibility of a matrix related?*

Answer. A matrix $\boldsymbol{A}$ is invertible if and only if 0 is <u>not</u> an eigenvalue of $\boldsymbol{A}$. □

Question 33. *How to determine whether a matrix* $\boldsymbol{A}$ *is diagonalizable?*

Answer. There are the following two methods:

- First method: [Page 188] If $\boldsymbol{A}$ has n distinct eigenvalues, then $\boldsymbol{A}$ is diagonalizable.

  ➢ Example: Examples 6.2.8 and 6.2.11.

- Second method: [Page 186–188] Apply Algorithm 6.2.4.

  ➢ Example: Example 6.2.6 and Exercise 6.13.

□

Question 34. *How to diagonalize a matrix?*

Answer. [Page 186–188] Apply Algorithm 6.2.4.

➢ Example: Example 6.2.6 and Exercise 6.11. □

Question 35. *How to compute powers of a matrix $\boldsymbol{A}$ using diagonalization?*

Answer. Assume there exists an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Then for any positive integer m, we have

$$\boldsymbol{A}^m = \underbrace{\boldsymbol{A}\cdots\boldsymbol{A}}_{\text{m times}} = \underbrace{(\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1})\cdots(\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1})}_{\text{m times}} = \boldsymbol{P}\boldsymbol{D}^m\boldsymbol{P}^{-1} = \boldsymbol{P}\begin{pmatrix} \lambda_1^m & & & \\ & \lambda_2^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{pmatrix}\boldsymbol{P}^{-1}.$$

➢ Example: Example 6.2.11. □

Question 36. *How to solve linear recurrence relations using diagonalization?*

Answer. [Page 190] We need follow this process:

1. Transfer the linear recurrence relation to a matrix equation. For example, given a linear recurrence relation $a_{n+1} = a_n + a_{n-1} + a_{n-2}$, then the related matrix equation is

$$\begin{pmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix},$$

where $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

2. Applying Algorithm 6.2.4, we will find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$, where $\boldsymbol{D} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$.

3. Thus

$$\begin{pmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} a_{n-2} \\ a_{n-1} \\ a_n \end{pmatrix} = \cdots = \boldsymbol{A}^{n-1}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} \lambda_1^{n-1} & & \\ & \lambda_2^{n-1} & \\ & & \lambda_3^{n-1} \end{pmatrix}\boldsymbol{P}^{-1}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

4. By computing the last expression, we will get an explicit form for a_n.

➢ Example: Example 6.2.11 and Exercise 6.21. □

6 Chapter 5

Question 37. *Given an orthogonal basis S or an orthonormal basis T for a vector space V and a vector $\boldsymbol{w} \in V$, how to find $[\boldsymbol{w}]_S$, $(\boldsymbol{w})_S$, $[\boldsymbol{w}]_T$, and $(\boldsymbol{w})_T$?*

Answer. [Page 154] We discuss by cases:

- If $S = \{u_1, u_2, \ldots, u_k\}$ is an orthogonal basis for V, then

$$w = \frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k} u_k,$$

  and hence

$$[w]_S^T = (w)_S = \left(\frac{w \cdot u_1}{u_1 \cdot u_1}, \frac{w \cdot u_2}{u_2 \cdot u_2}, \ldots, \frac{w \cdot u_k}{u_k \cdot u_k} \right).$$

- If $T = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis for V, then

$$w = (w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdot v_k)v_k,$$

  and hence

$$[w]_T^T = (w)_T = (w \cdot v_1, w \cdot v_2, \ldots, w \cdot v_k).$$

➢ Example: Example 5.2.9. □

Question 38. *How to find an orthogonal basis and an orthonormal basis for a vector space V?*

Answer. [Page 158] Following the process in **Question 22**, we have a basis $\{u_1, u_2, \ldots, u_k\}$ for V. Then applying Gram-Schmidt Process, we will get an orthogonal basis $\{v_1, v_2, \ldots, v_k\}$ for V. Then by normalizing every vector v_i, we will obtain an orthonormal basis $\{w_1, w_2, \ldots, w_k\}$ for V.

➢ Example: Example 5.2.20. □

Question 39. *How to find the projection of a vector w onto a subspace V of $\mathbb{R}^n$?*

Answer. [Page 156] Suppose we have an orthogonal basis $\{u_1, u_2, \ldots, u_k\}$ and an orthonormal basis $\{v_1, v_2, \ldots, v_k\}$ for V. Then

$$\frac{w \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k} u_k,$$

and

$$(w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdot v_k)v_k$$

are the projection of w onto V.

➢ Example: Examples 5.2.14 and 5.2.16. □

Question 40. *How to find the least squares solution to a system $Ax = b$?*

Answer. [Page 162–163] Solve the linear system $A^T A x = A^T b$.

➢ Example: Example 5.3.11. □

Question 41. *How to identify a matrix to be an orthogonal matrix?*

Answer. [Page 166] Given a square matrix A, then the following statements are equivalent:

- (Definition 5.4.3) A is orthogonal, i.e. $A^{-1} = A^T$;
- (Remark 5.4.4) $AA^T = I$;
- (Remark 5.4.4) $A^T A = I$;
- (Theorem 5.4.6) The rows of A form an orthonormal basis for $\mathbb{R}^n$;
- (Theorem 5.4.6) The columns of A form an orthonormal basis for $\mathbb{R}^n$;
- (Exercise 5.32) $\|Au\| = \|u\|$ for any vector $u \in \mathbb{R}^n$;
- (Exercise 5.32) $Au \cdot Av = u \cdot v$ for any vectors $u, v \in \mathbb{R}^n$.

➢ Example: Example 5.4.5. □

Question 42. *How is an orthogonal matrix related to an orthonormal basis?*

Answer. [Page 166] $\boldsymbol{A}$ is an orthogonal matrix if and only if the columns (rows) of $\boldsymbol{A}$ form an orthonormal basis for $\mathbb{R}^n$.

➢ Example: Exercise 5.34 (a), (b). □

Question 43. *How is a transition matrix related to an orthogonal matrix?*

Answer. [Page 167] Let S and T be two orthonormal bases for a vector space, and let $\boldsymbol{P}$ be the transition matrix from S to T. Then $\boldsymbol{P}$ is orthogonal. □

Question 44. *How to orthogonally diagonalize a symmetric matrix?*

Answer. [Page 192] Apply Algorithm 6.3.5.

➢ Example: Example 6.3.7, and Exercises 6.25, 6.29. □

7 Chapter 7

Question 45. *How are linear transformations related to matrices?*

Answer. [Page 209] Apply Definition 7.1.1.

➢ Example: Exercises 7.7 and 7.9. □

Question 46. *How to show that a mapping $T : V \to W$ is a linear transformation?*

Answer. There are the following two methods:

- First method: [Page 209] If we can find the standard matrix for T, then T is a linear transformation.
- Second method: [Page 210] If $T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$, then T is a linear transformation.

➢ Example: Example 7.1.2. □

Question 47. *How to find a basis for $R(T)$ (resp.* $\mathrm{Ker}(T)$*)?*

Answer. [Page 215–218] Let $\boldsymbol{A}$ be the standard matrix for T. Note that

$$\begin{aligned} R(T) &= \text{the column space of } \boldsymbol{A}, \\ \mathrm{Ker}(T) &= \text{the nullspace of } \boldsymbol{A}. \end{aligned}$$

Then follow the process in **Question 26** (resp. **Question 28**).

➢ Example: Examples 7.2.6 and 7.2.11. □

Question 48. *How to find* $\mathrm{rank}(T)$ *and* $\mathrm{nullity}(T)$*?*

Answer. [Page 218] Let $\boldsymbol{A}$ be the standard matrix for T. Note that

$$\begin{aligned} \mathrm{rank}(T) &= \mathrm{rank}(\boldsymbol{A}), \\ \mathrm{nullity}(T) &= \mathrm{nullity}(\boldsymbol{A}). \end{aligned}$$

Then follow the process in **Question 29**.

➢ Example: Exercise 7.12. □

Thank you for choosing to study in this tutorial group!

I hope your time here is an enjoyable and rewarding experience.

If you have any questions or comments, please feel free to contact me.

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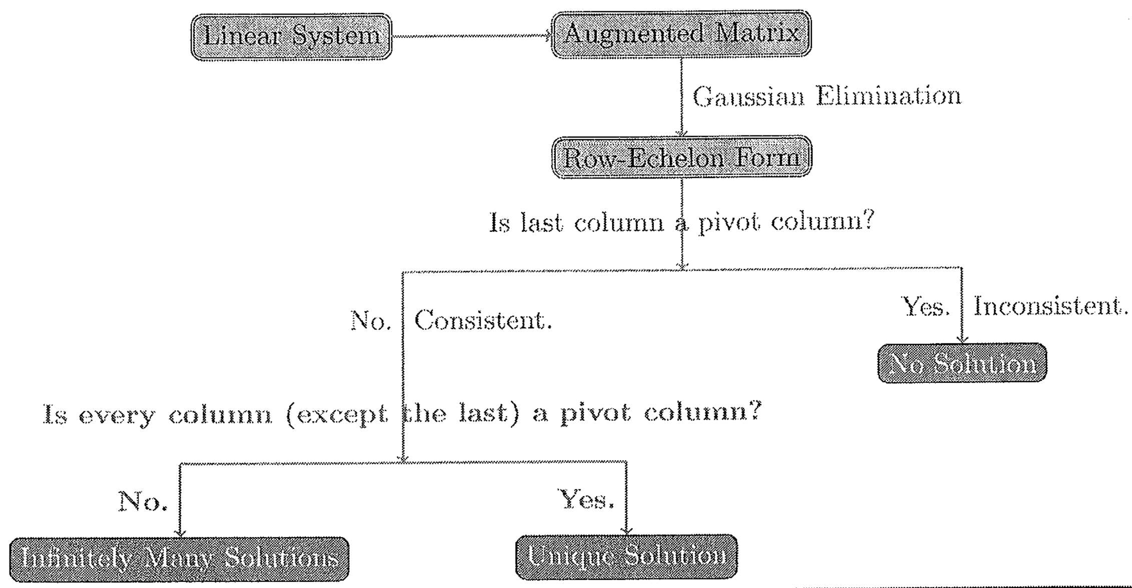
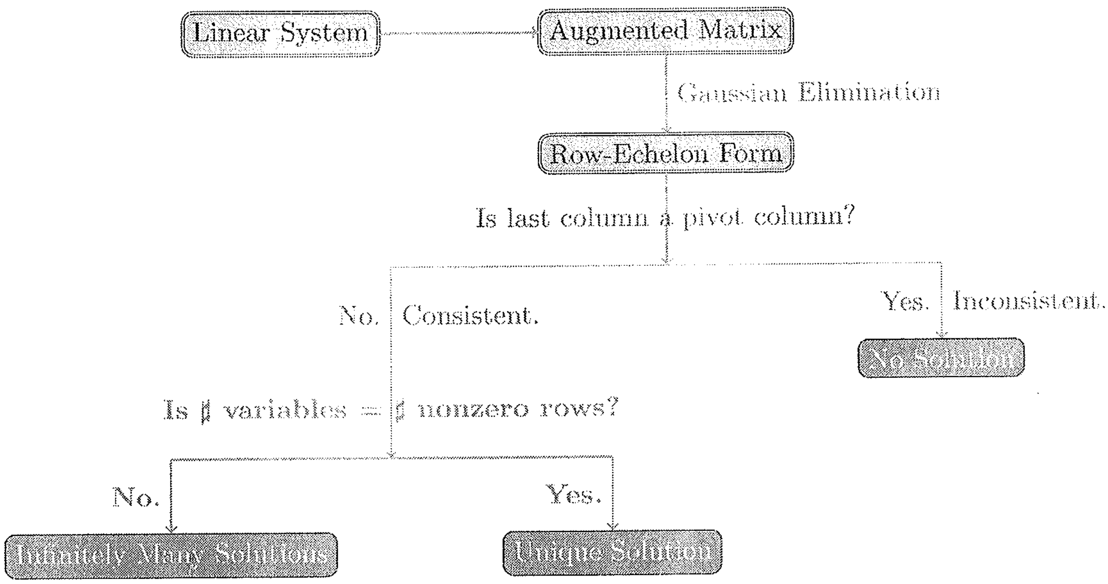
Tutorial Outline

- 11 tutorial classes: 4 before the recess week, and 7 after it.
- Before attending tutorial class, you are expected to attempt all the tutorial problems.
- When you are sitting in class,
  1. Take attendance: **print your signature**, rather than just a tick.
     If any mistakes occur on the attendance sheet, please let me know.
  2. **Collect handout** for each tutorial.
     Your homework scripts and test paper will be returned to you during tutorial class.
  3. **Presentation**: call for volunteers.
- After class, please check your NUS mailbox and **download the slides** for each tutorial.
  Tutorial slides style: quick review, tutorial problems with solutions, and additional material.
  Please **keep mailbox capacity below quota limits**.
  Thank you for your cooperation!

Summary of Remark 1.4.8





Additional Material

Question 23 (Exercise 1 of textbook). Determine the values of a and b so that the system

$$\left\{\begin{array}{rcrcrcl} ax & & & + & bz & = & 2 \\ ax & + & ay & + & 4z & = & 4 \\ & & ay & + & 2z & = & b \end{array}\right.$$

(a) has no solution;

(b) has exactly one solution;

(c) has infinitely many solutions and a general solution has one arbitrary parameter;

(d) has infinitely many solutions and a general solution has two arbitrary parameters.

Question 1 (a) in 2006/2007 Mid-Term Test.

Given

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 3 & 9 \\ 0 & 0 & 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 1 & 9 \end{array}\right)$$

is the row echelon form of the augmented matrix for a linear system in variables x_1, x_2, x_3, x_4, x_5.

Write down the general solution of the system.

Question 3 (a) in 2008/2009 Mid-Term Test.

Let the augmented matrix of a linear system be given by

$$\left(\begin{array}{ccc|c} 1-a & 0 & 0 & a \\ 0 & a & a & a \\ 0 & 0 & 1-a & 1-a \end{array}\right)$$

Find all the values of a so that the system

(i) has no solution;

(ii) has exactly one solution;

(iii) has infinitely many solutions.

A List of Examples

The following statements are stated in such a way that they are all true.

☞ 1. There exist two nonzero matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{0}$.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

☞ 2. There exists a nonzero matrix $\boldsymbol{A}$ such that $\boldsymbol{A}^2 = \boldsymbol{0}$.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

☞ 3. There exist two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{0}$ but $\boldsymbol{BA} \neq \boldsymbol{0}$.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

☞ 4. There exist matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ such that $\boldsymbol{A} \neq \boldsymbol{0}$, $\boldsymbol{AB} = \boldsymbol{AC}$ but $\boldsymbol{B} \neq \boldsymbol{C}$.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\boldsymbol{C} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

☞ 5. In general, $(\boldsymbol{AB})^k \neq \boldsymbol{A}^k\boldsymbol{B}^k$, where $k \in \mathbb{Z}^+$.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $k = 2$.

☞ 6. In general, $(\boldsymbol{AB})^{\mathrm{T}} \neq \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}$, where $\boldsymbol{A}^{\mathrm{T}}$ is the transpose of $\boldsymbol{A}$.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

☞ 7. In general, $(\boldsymbol{AB})^{-1} \neq \boldsymbol{A}^{-1}\boldsymbol{B}^{-1}$, where $\boldsymbol{A}^{-1}$ is the inverse of $\boldsymbol{A}$.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

☞ 8. There exist two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{A}^2 = \boldsymbol{B}^2 = \boldsymbol{I}$ but $(\boldsymbol{AB})^2 \neq \boldsymbol{I}$.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$.

☞ 9. There exist two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that both $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric, but $\boldsymbol{AB}$ is not symmetric.

✍ *Example.* $\boldsymbol{A} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Additional Material

Question 2 in 2007/2008 Mid-Term Test.

The following is a *row-echelon form* of the augmented matrix of a linear system with $a, b, c, d, e, f, g, h, j, k, l, m$ representing some real numbers.

$$M = \left(\begin{array}{cccc|c} a & b & c & d & e \\ 0 & f & g & h & j \\ 0 & 0 & k & l & m \end{array}\right)$$

Determine whether the following statements are true or false. Give brief justification for you answers.

(a) This linear system has 4 variables and 3 equations.

(b) If e, j, m are all 0, then the linear system must have non-trivial solutions.

(c) If the linear system is consistent, the general solution will have exactly one parameter.

(d) If $k \neq 0$, then the linear system will be consistent.

(e) If $f = 0$, then the second column of M cannot be a pivot column.

Question 1 in 2008/2009 Mid-Term Test (Make up).

Let $\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(a) Solve $\boldsymbol{Ax} = \boldsymbol{b}$ using *Gaussian Elimination*.

(b) Without performing Gaussian Elimination again, find a matrix $\boldsymbol{Y}$ such that

$$\boldsymbol{AY} = \begin{pmatrix} \pi & 2\pi \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) If $\boldsymbol{C}$ is a matrix row equivalent to $\boldsymbol{A}$, find the solution set of $\boldsymbol{Cx} = \boldsymbol{0}$. Justify your answer.

Question 3 (b) in 2008/2009 Mid-Term Test (Make up).

Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are square matrices of the same size such that $\boldsymbol{X}^2 = \boldsymbol{Y}^2$.

Is it necessary that $\boldsymbol{X} = \boldsymbol{Y}$ or $\boldsymbol{X} = -\boldsymbol{Y}$? Justify your answer.

Invertible Matrices

In Theorem 2.4.7 of the textbook, several equivalent conditions for a square matrix to be invertible were listed. Here is a list containing more equivalent conditions. For some of the conditions below, you may not understand at this moment, but you will learn them later in this module. Thus you are suggested to read statements 1-13 first and reread this part when you have finished learning the corresponding chapters.

Suppose that $\boldsymbol{A}$ is an $n \times n$ matrix, then the following statements are equivalent:

1. $\boldsymbol{A}$ is invertible.
2. There exists an $n \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}_n = \boldsymbol{BA}$.
3. There exists an $n \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}_n$.
4. There exists an $n \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{BA} = \boldsymbol{I}_n$.
5. The transpose $\boldsymbol{A}^{\mathrm{T}}$ is an invertible matrix.
6. $\boldsymbol{A}$ can be obtained from the identity matrix $\boldsymbol{I}_n$ by performing a series of elementary row operations.
7. $\boldsymbol{A}$ can be obtained from the identity matrix $\boldsymbol{I}_n$ by performing a series of elementary column operations (defined similarly as row operations).
8. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.
9. The reduced row-echelon form of $\boldsymbol{A}$ is the identity matrix $\boldsymbol{I}_n$.
10. A row-echelon form of $\boldsymbol{A}$ has n pivot points.
11. The determinant of $\boldsymbol{A}$ is not equal to 0.
12. The homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
13. The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has exactly one solution for every $\boldsymbol{b} \in \mathbb{R}^n$.
14. The linear span of the rows of $\boldsymbol{A}$ is $\mathbb{R}^n$. (See page 89 of the textbook for the definition of the linear span of a set of vectors.)
15. The linear span of the columns of $\boldsymbol{A}$ is $\mathbb{R}^n$.
16. The rows of $\boldsymbol{A}$ are linearly independent. (See page 99 of the textbook for the definition of linear independence.)
17. The columns of $\boldsymbol{A}$ are linearly independent.
18. The rows of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$. (See page 104 of the textbook for the definition of a basis.)
19. The columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.
20. $\mathrm{rank}(\boldsymbol{A}) = n$. (See page 136 of the textbook for the definition of the rank of a matrix.)
21. The number 0 is not an eigenvalue of $\boldsymbol{A}$. (See page 178 of the textbook for the definition of an eigenvalue of a square matrix.)
22. The linear transformation mapping $\boldsymbol{x}$ to $\boldsymbol{Ax}$ is a bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$. (See page 209 of the textbook for the definition of a linear transformation.)

Invertible Matrices

In Theorem 2.4.7 of the textbook, several equivalent conditions for a square matrix to be invertible were listed. Here is a list containing more equivalent conditions. For some of the conditions below, you may not understand at this moment, but you will learn them later in this module. Thus you are suggested to read statements 1-13 first and reread this part when you have finished learning the corresponding chapters.

Suppose that $\boldsymbol{A}$ is an $n \times n$ matrix, then the following statements are equivalent:

1. $\boldsymbol{A}$ is invertible.
2. There exists an $n \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}_n = \boldsymbol{BA}$.
3. There exists an $n \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}_n$.
4. There exists an $n \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{BA} = \boldsymbol{I}_n$.
5. The transpose $\boldsymbol{A}^{\mathrm{T}}$ is an invertible matrix.
6. $\boldsymbol{A}$ can be obtained from the identity matrix $\boldsymbol{I}_n$ by performing a series of elementary row operations.
7. $\boldsymbol{A}$ can be obtained from the identity matrix $\boldsymbol{I}_n$ by performing a series of elementary column operations (defined similarly as row operations).
8. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.
9. The reduced row-echelon form of $\boldsymbol{A}$ is the identity matrix $\boldsymbol{I}_n$.
10. A row-echelon form of $\boldsymbol{A}$ has n pivot points.
11. The determinant of $\boldsymbol{A}$ is not equal to 0.
12. The homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
13. The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has exactly one solution for every $\boldsymbol{b} \in \mathbb{R}^n$.
14. The linear span of the rows of $\boldsymbol{A}$ is $\mathbb{R}^n$. (See page 89 of the textbook for the definition of the linear span of a set of vectors.)
15. The linear span of the columns of $\boldsymbol{A}$ is $\mathbb{R}^n$.
16. The rows of $\boldsymbol{A}$ are linearly independent. (See page 99 of the textbook for the definition of linear independence.)
17. The columns of $\boldsymbol{A}$ are linearly independent.
18. The rows of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$. (See page 104 of the textbook for the definition of a basis.)
19. The columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.
20. $\operatorname{rank}(\boldsymbol{A}) = n$. (See page 136 of the textbook for the definition of the rank of a matrix.)
21. The number 0 is not an eigenvalue of $\boldsymbol{A}$. (See page 178 of the textbook for the definition of an eigenvalue of a square matrix.)
22. The linear transformation mapping $\boldsymbol{x}$ to $\boldsymbol{Ax}$ is a bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$. (See page 209 of the textbook for the definition of a linear transformation.)

Adjoint Matrices

Let A be a square matrix of order n. The *adjoint* of A is the $n \times n$ matrix

$$\mathbf{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} is the (i,j)-cofactor of A.

We have the fact that

$$A[\mathbf{adj}(A)] = [\mathbf{adj}(A)]A = \det(A)I,$$

no matter whether A is invertible.

Summary.

☞ **Theorem 2.5.25**

If A is invertible, then $A^{-1} = \frac{1}{\det(A)}\mathbf{adj}(A)$.

☞ **Exercise 2.60**

If A is invertible, then $\mathbf{adj}(A)$ is invertible and $\mathbf{adj}(A)^{-1} = \frac{1}{\det(A)}A$.

☞ **Question 3 (b) in 2009/2010 Mid-Term Test**

If $\mathbf{adj}(C)$ is invertible, then C is also invertible.

In linear algebra, the adjoint matrix of a square matrix is something similar to the inverse of a square matrix: When A is invertible, the difference between $\mathbf{adj}(A)$ and A^{-1} is just a scalar.

However, the adjoint matrix can be defined for any square matrix, not necessarily invertible.

Here is a list of more properties of adjoint matrices:

1. $\mathbf{adj}(I) = I$.
2. $\mathbf{adj}(A^{\mathrm{T}}) = \mathbf{adj}(A)^{\mathrm{T}}$.
3. If A is an $n \times n$ matrix, then $\mathbf{adj}(cA) = c^{n-1}\mathbf{adj}(A)$.
4. If A is an $n \times n$ invertible matrix , then $\det(\mathbf{adj}(A)) = [\det(A)]^{n-1}$.
5. If A is an $n \times n$ matrix with $n > 2$, then $\mathbf{adj}(\mathbf{adj}(A)) = [\det(A)]^{n-2}A$.
6. If A is invertible, then $\mathbf{adj}(A^{-1}) = [\mathbf{adj}(A)]^{-1} = \frac{1}{\det(A)}A$.

Additional Material

Question 2 (a) in 2006/2007 Mid-Term Test.

Let $\boldsymbol{A}$ be a 3×3 matrix.

By multiplying second row of $\boldsymbol{A}$ by scalar 3, we get a matrix $\boldsymbol{B}$.

By interchanging second and third row of $\boldsymbol{B}$, we get a matrix $\boldsymbol{C}$.

By adding 5 times of first row of $\boldsymbol{C}$ to its third row, we get the identity matrix $\boldsymbol{I}$.

Find the following matrices:

(i) $\boldsymbol{A}^{-1}$;

(ii) $\boldsymbol{B}\boldsymbol{C}^{-1}$;

(iii) $\boldsymbol{A}\boldsymbol{B}^{-1}\boldsymbol{C}$.

Question 3 in 2006/2007 Mid-Term Test.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $n \times n$ matrices that are row equivalent.

Determine whether the following statements are true. Justify your answers.

(a) $\det(\boldsymbol{A}) = \det(\boldsymbol{B})$.

(b) If $\boldsymbol{A}$ is invertible, then $\boldsymbol{B}$ is invertible.

(c) If the homogeneous system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a non-trivial solution, then $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions.

(d) $\boldsymbol{A}^2$ and $\boldsymbol{B}^2$ are row equivalent.

Question 3 (c) in 2008/2009 Mid-Term Test (Make Up).

Suppose $\boldsymbol{A}$ is a square matrix such that $(\boldsymbol{A} + 3\boldsymbol{I})(\boldsymbol{A} + 2\boldsymbol{I})(\boldsymbol{A} + \boldsymbol{I}) = \boldsymbol{0}$.

Is $\boldsymbol{A}$ invertible? Justify your answer.

Question 3 (b) in 2009/2010 Mid-Term Test.

Let $\boldsymbol{C}$ be an $n \times n$ matrix. Suppose $\mathbf{adj}(\boldsymbol{C})$ is invertible.

(i) Show that $\boldsymbol{C}$ is invertible.

(ii) Write down an expression for $[\mathbf{adj}(\boldsymbol{C})]^{-1}$ in terms of $\boldsymbol{C}$.

Question 2 (a) in 2011/2012 Mid-Term Test.

Let $\boldsymbol{A} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$ where a, b, c, d, e, f are some real numbers.

(i) Find $\det(\boldsymbol{A})$ and write down the condition on a, b, c, d, e, f for $\boldsymbol{A}$ to be invertible.

(ii) Find $\mathbf{adj}(\boldsymbol{A})$.

(iii) Find $\boldsymbol{A}^{-1}$ (without using any elementary row operation).

Summary: Lines in $\mathbb{R}^2, \mathbb{R}^3$ and Planes in $\mathbb{R}^3$.

- Lines in $\mathbb{R}^2$:

  Implicit form: $\{ (x, y) \mid ax + by = c \}$,

  where a, b, c are real constants and a, b are not both zero.

  Explicit form: $\{$ (general solution) $\mid$ 1 parameter $\}$, i.e.

  $$\{ (a_0, b_0) + t(a, b) \mid t \in \mathbb{R} \},$$

  where a_0, b_0, a, b are real constants and a, b are not both zero.

- Planes in $\mathbb{R}^3$:

  Implicit form: $\{ (x, y, z) \mid ax + by + cz = d \}$,

  where a, b, c, d are real constants and a, b, c are not all zero.

  Explicit form: $\{$ (general solution) $\mid$ 2 parameters $\}$, i.e.

  $$\left\{ \left(\frac{d - bs - ct}{a}, s, t \right) \mid s, t \in \mathbb{R} \right\}, \quad \text{if } a \neq 0;$$

  $$\left\{ \left(s, \frac{d - as - ct}{b}, t \right) \mid s, t \in \mathbb{R} \right\}, \quad \text{if } b \neq 0;$$

  $$\left\{ \left(s, t, \frac{d - as - bt}{c} \right) \mid s, t \in \mathbb{R} \right\}, \quad \text{if } c \neq 0.$$

- Lines in $\mathbb{R}^3$:

  Implicit form: $\{ (x, y, z) \mid a_1x + b_1y + c_1z = d_1,\ a_2x + b_2y + c_2z = d_2 \}$,

  where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are real constants, a_1, b_1, c_1 are not all zero, and a_2, b_2, c_2 are not all zero.

  Explicit form: $\{$ (general solution) $\mid$ 1 parameter $\}$, i.e.

  $$\{ (a_0, b_0, c_0) + t(a, b, c) \mid t \in \mathbb{R} \},$$

  where a_0, b_0, c_0, a, b, c are real constants and a, b, c are not all zero.

Linear Combinations and Linear Spans.

Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ be vectors in $\mathbb{R}^n$. For any real numbers $c_1, c_2, \ldots, c_k$, the vector

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

is called a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$.

Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ be a set of vectors in $\mathbb{R}^n$. Then the set of all linear combinations of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$,

$$\{ c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} \mid c_1, c_2, \ldots, c_k \in \mathbb{R} \},$$

is called the linear span of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ (or the linear span of S).

Notation: $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ or $\text{span}(S)$.

Question: *How to determine whether a vector $\boldsymbol{v}$ is a linear combination of a given set of vectors $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$?*

Answer. Assume $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$. Then we will have a linear system in terms of $c_1, c_2, \ldots, c_k$:

$$\begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \cdots & \boldsymbol{u_k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{v},$$

where $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}, \boldsymbol{v}$ are column vectors.

Since we have that $\boldsymbol{v}$ can be expressed as a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ if and only if the above linear system is consistent, it suffices to see whether the linear system is consistent. ∎

Question: *How to express a vector $\boldsymbol{v}$ as a linear combination of a given set of vectors $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$?*

Answer. Following the process in the last question, if the linear system is consistent, and we can find a solution $(c_1, c_2, \ldots, c_k)$, then we have $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$. ∎

Question: *How to show a linear span $\text{span}(S_1)$ is contained in another one $\text{span}(S_2)$, where $S_1 = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$, $S_2 = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m}\}$?*

Answer. Since $\text{span}(S_1) \subseteq \text{span}(S_2)$ if and only if each $\boldsymbol{u_i}$ is a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m}$, it suffices to show that each of the last k columns in the following REF is <u>not</u> a pivot column:

$$\left(\begin{array}{cccc|c|c|c|c} \boldsymbol{v_1} & \boldsymbol{v_2} & \cdots & \boldsymbol{v_m} & \boldsymbol{u_1} & \boldsymbol{u_2} & \cdots & \boldsymbol{u_k} \end{array}\right),$$

where $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}, \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m}$ are column vectors. ∎

Additional Material

Question 3 in 2007/2008 Mid-Term Test.

Consider the following subsets of $\mathbb{R}^3$.

(Note that vectors in $\mathbb{R}^3$ can be written in row or column form and regarded as the same.)

$$\begin{aligned} S &= \{(x,y,z) \mid 2x - 3y + z = 10 \text{ and } x - z = 5\}, \\ T &= \text{the solution set of the linear system } \begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & -1 \\ 3 & -3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 15 \end{pmatrix}, \\ U &= \{(t+2, t-3, t-3) \mid t \in \mathbb{R}\}. \end{aligned}$$

1. Determine whether the vector $(3, -2, -2)$ belongs to each of the three sets.
2. Describe the three sets geometrically (i.e. whether they represent points, lines, planes or others).
3. Which of the three sets are the same, if any? Justify your answers.

Question 4 in 2008/2009 Mid-Term Test (Make up).

(a) Let $V = \{\, (x,y,z) \mid x + y + z = 0 \text{ and } x + y - z = 0 \,\}$ and $W = \{\, t(1,1,1) \mid t \in \mathbb{R} \,\}$.

(i) What geometrical objects do V and W represent in $\mathbb{R}^3$?

(ii) Find $V \cap W$.

(iii) Write down the equation of a plane in $\mathbb{R}^3$ that contains $V \cup W$.

Justify your answers.

(b) Consider a linear system in 3 variables which has two solutions $(1,2,0)$ and $(0,1,2)$. Suppose the general solution of the system has only one parameter. Write down an explicit set notation for the solution set of this system. Justify your answer.

Question 1 (a) in Semester 1 Final Exam 2010-2011.

(a) Let $S_1 = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ be a set of vectors in $\mathbb{R}^n$, $n > 3$.

If $S_2 = \{\boldsymbol{u} - 2\boldsymbol{v}, \boldsymbol{v} - 2\boldsymbol{w}, \boldsymbol{w}\}$, show that $\operatorname{span}(S_1) = \operatorname{span}(S_2)$.

Methods for proving subspace

We have the following *four* methods for showing a subset V of $\mathbb{R}^n$ to be a subspace of $\mathbb{R}^n$.

1. Express V as a linear span.
2. (Exercise 3.31) show that V is non-empty and satisfies the closed condition.
3. Show that V is a solution set of some homogeneous linear system.
4. (For $\mathbb{R}^2$ and $\mathbb{R}^3$) show that V represents a line or a plane containing the origin.

First two methods are general, while the last two are available for some special cases.

Methods for disproving subspace

We have the following *five* methods for showing a subset V of $\mathbb{R}^n$ to be not a subspace of $\mathbb{R}^n$.

1. Show that zero vector $\mathbf{0}$ is not in V.
2. Find $\boldsymbol{u}, \boldsymbol{v} \in V$ such that $\boldsymbol{u} + \boldsymbol{v} \notin V$.
3. Find $\boldsymbol{u} \in V$ and a scalar $c \in \mathbb{R}$ such that $c\boldsymbol{u} \notin V$.
4. Show that V is a solution set of some nonhomogeneous linear system.
5. (For $\mathbb{R}^2$ and $\mathbb{R}^3$) show that V is not a line or a plane containing the origin.

First three methods are general, while the last two are available for some special cases.

Operations of subspaces

Let V and W be subspaces of $\mathbb{R}^n$.

- Define $V + W = \{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W\}$.

  Then $V + W$ is a subspace of $\mathbb{R}^n$. See Exercise 3.20 (a).
- $V \cap W$ is a subspace of $\mathbb{R}^n$. See Exercise 3.24 (a).
- $V \cup W$ is a subspace of $\mathbb{R}^n$ if and only if $V \subseteq W$ or $W \subseteq V$. See Exercise 3.24 (c).

Difference between $V + W$ and $V \cup W$

Take V to be the x-axis, and W to be the y-axis in $\mathbb{R}^2$. It is clear that V and W are subspaces in $\mathbb{R}^2$.

- $V + W = \mathbb{R}^2$.
- While $V \cup W$ is the union of the x-axis and the y-axis, which is not a subspace.

Properties of Linear Independence

Some properties of linear independence are listed in this section.

You are encouraged to prove them as exercises.

1. Suppose that S is a set of vectors with $\mathbf{0} \in S$, then S is linearly dependent.
2. Suppose that S is a linearly dependent set of vectors and $S \subseteq T$, where T is also a set of vectors, then T is linearly dependent.
3. Suppose that T is a linearly independent set of vectors and S is a nonempty subset of T, then S is also linearly independent.
4. Suppose that $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$, where each $\boldsymbol{u_i}$ is a vector in $\mathbb{R}^n$. If $k > n$, then S is linearly dependent.
5. Suppose that $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$ is linearly dependent, where each $\boldsymbol{u_i}$ is an n-vector in $\mathbb{R}^n$ with $\boldsymbol{u_i} = (u_{i1}, u_{i2}, \dots, u_{in})$ and $n \geq 2$. Let $\boldsymbol{v_i} = (u_{i1}, u_{i2}, \dots, u_{i,n-1})$ be a vector in $\mathbb{R}^{n-1}$ for each $1 \leq i \leq n$. Then the set $\{ \boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k} \}$ is also linearly dependent.
6. Suppose that $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$ is a linearly independent set, where each $\boldsymbol{u_i}$ is an n-vector in $\mathbb{R}^n$ with $\boldsymbol{u_i} = (u_{i1}, u_{i2}, \dots, u_{in})$ and $n \geq 2$. Let $\boldsymbol{w_i} = (u_{i1}, u_{i2}, \dots, u_{in}, x_i)$ be a vector in $\mathbb{R}^{n+1}$ with arbitrary number $x_i \in \mathbb{R}$ for each $1 \leq i \leq n$. Then the set $\{ \boldsymbol{w_1}, \boldsymbol{w_2}, \dots, \boldsymbol{w_k} \}$ is also linearly independent.
7. Suppose that $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$ is linearly independent and $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}, \boldsymbol{v} \}$ is linearly dependent. Then $\boldsymbol{v}$ can be uniquely written as a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}$.

Examples

1. Suppose that $\boldsymbol{v}$ cannot be written as a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}$.

   It is NOT necessarily true that the set $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}, \boldsymbol{v} \}$ is linearly independent.

   (Compare this with Theorem 3.4.4 of the textbook.)

   ✍ Example: $\boldsymbol{v} = (0, 1, 0), \boldsymbol{u_1} = (1, 0, 0), \boldsymbol{u_2} = (-1, 0, 0)$.

   This example also shows that if S is a linearly dependent set, it is NOT necessarily true that <u>every</u> vector in S can be written as a linear combination of other vectors in S.

2. It is true that if the set $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$ is linearly independent, then any two distinct vectors in this set are also linearly independent.

   However, suppose that any two distinct vectors in the set $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$ are linearly independent, it is NOT necessarily true that the set $\{ \boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k} \}$ is linearly independent.

   ✍ Example: $\boldsymbol{u_1} = (1, 0, 0), \boldsymbol{u_2} = (0, 1, 0), \boldsymbol{u_3} = (1, 1, 0)$.

Additional Material

Question 8 in Semester 2 Final Exam 2005-2006.

Let V and W be subspaces of $\mathbb{R}^n$.

Define $V + W = \{\, \boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W \,\}$.

(a) Show that $V + W$ is a subspace of $\mathbb{R}^n$.

(b) Let $V = \{\, (t, t, t) \mid t \in \mathbb{R} \,\}$ and $W = \{\, (t, -t, 0) \mid t \in \mathbb{R} \,\}$.

Write down the subspace $V + W$ explicitly.

Question 5 (b) in Semester 2 Final Exam 2007-2008.

Determine whether the following statement is true. Justify your answer.

(b) Let $\boldsymbol{u}, \boldsymbol{v}$ be non-zero vectors in some vector space.

If $\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \operatorname{span}\{\boldsymbol{u} + \boldsymbol{v}\}$, then $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly dependent.

Question 1 in Semester 1 Final Exam 2010-2011.

(b) Let $T_1 = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ be a set of *linearly independent* vectors in $\mathbb{R}^3$ and $T_2 = \{\boldsymbol{u} + 2\boldsymbol{v}, \boldsymbol{v} + 2\boldsymbol{w}, \boldsymbol{u} + \boldsymbol{w}\}$.

(i) Determine whether T_2 is also linearly independent. Justify your answer.

(ii) Is it true that $\operatorname{span}(T_1) = \operatorname{span}(T_2)$? Justify your answer.

(c) Let $X = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of linearly independent vectors in $\mathbb{R}^n$, $n > k$.

Suppose $\boldsymbol{v}$ is a vector in $\mathbb{R}^n$ and $\boldsymbol{v} \notin \operatorname{span}(X)$, show that the set $X \cup \{\boldsymbol{v}\}$ is linearly independent.

Question 3 (a) in Semester 2 Final Exam 2010-2011.

(i) Determine whether the subset $U = \{\, (1,0,1,0),\ (0,1,1,0),\ (0,0,0,1),\ (1,1,1,0) \,\}$ of $\mathbb{R}^4$ is linearly independent.

(ii) Express the subset $V = \{\, (a + b - 2c, 0, c - a, b) \mid a, b, c \in \mathbb{R} \,\}$ of $\mathbb{R}^4$ in linear span form if possible (show your working). Is V a subspace of $\mathbb{R}^4$?

(iii) Show that the subset $W = \{\, (a, b, a + b, ab) \mid a, b \in \mathbb{R} \,\}$ of $\mathbb{R}^4$ is <u>not</u> a subspace of $\mathbb{R}^4$.

Ranks and Nullities

In this section, we list some properties of the rank and the nullity of a matrix:

1. Let $\boldsymbol{A}$ be a matrix. Then $\text{rank}(\boldsymbol{A}) = 0$ if and only if $\boldsymbol{A}$ is a zero matrix.

2. Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then $\text{rank}(\boldsymbol{A}) \leq \min\{m, n\}$. See Remark 4.2.5.1 of the textbook.

3. For a square matrix $\boldsymbol{A}$ of order n, $\boldsymbol{A}$ is invertible if and only if $\text{rank}(\boldsymbol{A}) = n$.

   See Theorem 6.1.8 of the textbook.

4. Multiplying an invertible matrix on the left does not change the rank of a matrix.

   See Question 4.22 of the textbook.

   In fact, multiplying an invertible matrix on the right does not change the rank of a matrix either.

5. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices such that $\boldsymbol{AB}$ is well defined, then $\text{rank}(\boldsymbol{AB}) \leq \min\{\text{rank}(\boldsymbol{A}), \text{rank}(\boldsymbol{B})\}$.

   See Theorem 4.2.8 of the textbook.

6. Ranks satisfy the *subadditivity*: $\text{rank}(\boldsymbol{A} + \boldsymbol{B}) \leq \text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B})$ when $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of the same size.

   See Question 4.23 of the textbook.

7. Let $\boldsymbol{A}$ be a matrix and let $\boldsymbol{A}^{\mathrm{T}}$ be the transpose of $\boldsymbol{A}$.

   Then $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^{\mathrm{T}}) = \text{rank}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}) = \text{rank}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})$.

   See Remark 4.2.5.3 and Question 4.25 of the textbook.

8. (Dimension Theorem for Matrices)

   The rank of a matrix plus the nullity of the matrix equals the number of columns of the matrix.

   See Theorem 4.3.4 of the textbook.

9. For a matrix $\boldsymbol{A}$, $\text{nullity}(\boldsymbol{A}) = \text{nullity}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$. See Question 4.25 of the textbook.

Additional Material

Question 4.20 (Textbook).

Question 4.26 (Textbook).

Question 8 (a) in Semester 1 Final Exam 2006-2007.

Let $\boldsymbol{A}$ be a square matrix such that $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^2)$.

(i) Show that the nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}^2$.

(ii) Show that the nullspace of $\boldsymbol{A}$ and the column space of $\boldsymbol{A}$ intersect trivially, i.e.

(the nullspace of $\boldsymbol{A}$) $\cap$ (the column space of $\boldsymbol{A}$) $= \{\boldsymbol{0}\}$.

Hint.

(i) Let $\boldsymbol{u} \in$ (the nullspace of $\boldsymbol{A}$). Then

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0} \implies \boldsymbol{A}^2\boldsymbol{u} = \boldsymbol{A}\boldsymbol{0} = \boldsymbol{0}$$
$$\implies \boldsymbol{u} \in (\text{the nullspace of } \boldsymbol{A}^2)$$

So (the nullspace of $\boldsymbol{A}$) $\subseteq$ (the nullspace of $\boldsymbol{A}^2$).

Since $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^2)$, by the Dimension Theorem for Matrices, ...

(ii) Suppose $\boldsymbol{b} \in$ (the nullspace of $\boldsymbol{A}$) $\cap$ (the column space of $\boldsymbol{A}$). Then

1. $\boldsymbol{A}\boldsymbol{b} = \boldsymbol{0}$;
2. There exists a vector $\boldsymbol{u}$ such that $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$.

Then prove that $\boldsymbol{b} = \boldsymbol{0}$.

Question 1 in Semester 2 Final Exam 2009-2010.

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

(i) Write down a basis for the row space of $\boldsymbol{A}$, and a basis for the column space of $\boldsymbol{A}$.

(ii) Find a basis for the nullspace of $\boldsymbol{A}$. Show your working.

(iii) Find $\text{rank}(\boldsymbol{A})$, $\text{nullity}(\boldsymbol{A})$ and $\text{nullity}(\boldsymbol{A}^{\text{T}})$.

(iv) Extend the basis for the nullspace of $\boldsymbol{A}$ found in (ii) to a basis for $\mathbb{R}^5$.

(v) Find a non-zero vector that is contained in both the row space and the nullspace of $\boldsymbol{A}$.

(vi) Is it possible to find a matrix $\boldsymbol{B}$ such that $\boldsymbol{AB}$ is an invertible matrix?

Justify your answers for parts (iv) to (vi).

Matrix	Rank	Nullity	the Row Space	the Column Space	the Nullspace
$\boldsymbol{I_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	0	$\text{span}\{(1,0),\ (0,1)\} = \mathbb{R}^2$	$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2$	$\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$
$\boldsymbol{0}_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0	2	$\{(0,0)\}$	$\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2$
$\boldsymbol{A_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	1	1	$\text{span}\{(1,0)\}$	$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$
$\boldsymbol{A_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	1	1	$\text{span}\{(0,1)\}$	$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$
$\boldsymbol{A_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	1	1	$\text{span}\{(1,0)\}$	$\text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$
$\boldsymbol{A_4} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	1	1	$\text{span}\{(0,1)\}$	$\text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned}
A_1^2 = A_1A_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A_1 \\
A_1A_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = A_2 \\
A_1A_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 2} \\
A_1A_4 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 2}
\end{aligned}$$

$$\begin{aligned}
A_3A_1 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = A_3 \\
A_3A_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A_4 \\
A_3^2 = A_3A_3 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 2} \\
A_3A_4 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 2}
\end{aligned}$$

$$\begin{aligned}
A_2A_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 2} \\
A_2^2 = A_2A_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 2} \\
A_2A_3 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A_1 \\
A_2A_4 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = A_2
\end{aligned}$$

$$\begin{aligned}
A_4A_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 2} \\
A_4A_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}_{2\times 2} \\
A_4A_3 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = A_3 \\
A_4^2 = A_4A_4 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A_4
\end{aligned}$$

The Dot Product

The dot product of two vectors $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ is defined to be the value

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Properties of the dot product:

1. Commutative: $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.
2. Distributive over vector addition: $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$ and $\boldsymbol{w} \cdot (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{u} + \boldsymbol{w} \cdot \boldsymbol{v}$.
3. $(c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$.
4. $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$ with equality only for $\boldsymbol{u} = \boldsymbol{0}$.

Orthogonal and Orthonormal Bases

Let V be a subspace of $\mathbb{R}^n$ and $\boldsymbol{w}$ a vector in V.

- If $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\}$ is a basis for V, then

  $$\boldsymbol{w} = a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2 + \cdots + a_k\boldsymbol{w}_k,$$

  where we need to solve linear system to get $a_1, a_2, \ldots, a_k$.

- If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an **orthogonal basis** for V, then

  $$\boldsymbol{w} = \frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1}\boldsymbol{u}_1 + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{\boldsymbol{u}_2 \cdot \boldsymbol{u}_2}\boldsymbol{u}_2 + \cdots + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_k}{\boldsymbol{u}_k \cdot \boldsymbol{u}_k}\boldsymbol{u}_k.$$

- If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is an **orthonormal basis** for V, then

  $$\boldsymbol{w} = (\boldsymbol{w} \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{w} \cdot \boldsymbol{v}_2)\boldsymbol{v}_2 + \cdots + (\boldsymbol{w} \cdot \boldsymbol{v}_k)\boldsymbol{v}_k.$$

- Let V be a subspace of $\mathbb{R}^n$.

  A vector $\boldsymbol{u} \in \mathbb{R}^n$ is said to be orthogonal (or perpendicular) to V if $\boldsymbol{u}$ is orthogonal to all vectors in V.

- Let V be a plane in $\mathbb{R}^3$ defined by the equation $ax + by + cz = 0$.

  Then $\boldsymbol{n} = (a, b, c)$ is orthogonal to V.

  The vector $\boldsymbol{n}$ is called a normal vector of V.

- If $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is a subspace of $\mathbb{R}^n$, then a vector $\boldsymbol{v}$ is orthogonal to V if and only if $\boldsymbol{v} \cdot \boldsymbol{u}_i = 0$ for $i = 1, 2, \ldots, k$.

Orthogonal Complement

Let V be a subspace of $\mathbb{R}^n$.

Define $V^\perp = \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{u} \text{ is orthogonal to } V\}$. Then we have

1. $V^\perp$ is a subspace of $\mathbb{R}^n$.
2. $V \cap V^\perp = \{\boldsymbol{0}\}$.
3. $V + V^\perp = \mathbb{R}^n$.
4. $\dim(V) + \dim(V^\perp) = \dim(\mathbb{R}^n) = n$.
5. $(V^\perp)^\perp = V$.
6. If $V \subseteq W$, then $V^\perp \supseteq W^\perp$.
7. For a matrix $\boldsymbol{A}$, the orthogonal complement of the row space of $\boldsymbol{A}$ is equal to the null space of $\boldsymbol{A}$. Similarly, the orthogonal complement of the column space of $\boldsymbol{A}$ is equal to the null space of $\boldsymbol{A}^{\mathrm{T}}$.
8. If $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n\}$ is an orthogonal basis for $\mathbb{R}^n$, where $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis for V, then $\{\boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n\}$ is an orthogonal basis for $V^\perp$.
9. If $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis for V, then it can be extended to an orthogonal basis for $\mathbb{R}^n$: $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n\}$. Thus $\{\boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n\}$ is an orthogonal basis for $V^\perp$.

(Orthogonal) Projections

Let V be a subspace of $\mathbb{R}^n$. Every vector $\boldsymbol{u} \in \mathbb{R}^n$ can be written uniquely as $\boldsymbol{u} = \boldsymbol{n} + \boldsymbol{p}$ such that $\boldsymbol{n}$ is a vector orthogonal to V and $\boldsymbol{p}$ is a vector in V.

The vector $\boldsymbol{p}$ is called the (orthogonal) projection of $\boldsymbol{u}$ onto V.

Let V be a subspace of $\mathbb{R}^n$ and $\boldsymbol{w}$ a vector in $\mathbb{R}^n$.

1. If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis for V, then
$$\frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1}\boldsymbol{u}_1 + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{\boldsymbol{u}_2 \cdot \boldsymbol{u}_2}\boldsymbol{u}_2 + \cdots + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_k}{\boldsymbol{u}_k \cdot \boldsymbol{u}_k}\boldsymbol{u}_k$$
is the projection of $\boldsymbol{w}$ onto V.
2. If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is an orthonormal basis for V, then
$$(\boldsymbol{w} \cdot \boldsymbol{v}_1)\,\boldsymbol{v}_1 + (\boldsymbol{w} \cdot \boldsymbol{v}_2)\,\boldsymbol{v}_2 + \cdots + (\boldsymbol{w} \cdot \boldsymbol{v}_k)\,\boldsymbol{v}_k$$
is the projection of $\boldsymbol{w}$ onto V.

The Uniqueness of (Orthogonal) Projection

Show that the representation $\boldsymbol{u} = \boldsymbol{n} + \boldsymbol{p}$ is unique, where $\boldsymbol{n}$ is a vector orthogonal to V and $\boldsymbol{p}$ is a vector in V.

Proof.

Assume $\boldsymbol{u} = \boldsymbol{n}_1 + \boldsymbol{p}_1 = \boldsymbol{n}_2 + \boldsymbol{p}_2$ where $\boldsymbol{n}_1, \boldsymbol{n}_2$ are orthogonal to V and $\boldsymbol{p}_1, \boldsymbol{p}_2 \in V$.

It follows that $\boldsymbol{n}_1 - \boldsymbol{n}_2 = \boldsymbol{p}_2 - \boldsymbol{p}_1$.

Since $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are orthogonal to V, so is $\boldsymbol{n}_1 - \boldsymbol{n}_2$. Hence $\boldsymbol{p}_2 - \boldsymbol{p}_1$ is also orthogonal to V.

Note that $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ belong to V, we have $\boldsymbol{p}_2 - \boldsymbol{p}_1 \in V$.

Therefore $\boldsymbol{p}_2 - \boldsymbol{p}_1$ is orthogonal to itself, i.e. $(\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_1) = 0$.

Hence $\boldsymbol{p}_2 - \boldsymbol{p}_1 = \boldsymbol{0}$. This implies $\boldsymbol{p}_1 = \boldsymbol{p}_2$ and $\boldsymbol{n}_1 = \boldsymbol{n}_2$. ∎

Gram-Schmidt Process

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a basis for a vector space V. Let

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{u}_1, \\
\boldsymbol{v}_2 &= \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1, \\
\boldsymbol{v}_3 &= \boldsymbol{u}_3 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2, \\
&\vdots \\
\boldsymbol{v}_k &= \boldsymbol{u}_k - \frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 - \cdots - \frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_{k-1}}{\boldsymbol{v}_{k-1} \cdot \boldsymbol{v}_{k-1}}\boldsymbol{v}_{k-1}.
\end{aligned}$$

Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is an orthogonal basis for V.

Let $\boldsymbol{w}_i = \frac{1}{\|\boldsymbol{v}_i\|}\boldsymbol{v}_i$ for $i = 1, 2, \ldots, k$. Then $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\}$ is an orthonormal basis for V.

Least Squares Solution

Let $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ be a linear system where $\boldsymbol{A}$ is an $m \times n$ matrix. A vector $\boldsymbol{u} \in \mathbb{R}^n$ is called the least squares solution to the linear system if $\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{u}\| \leq \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{v}\|$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

The following statements are equivalent:

- $\boldsymbol{u}$ is the least squares solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.
- $\boldsymbol{A}\boldsymbol{u}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.
- $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$.

Eigenvalues and Eigenvectors

Let $\boldsymbol{A}$ be a square matrix of order n.

A *nonzero* column vector $\boldsymbol{u}$ in $\mathbb{R}^n$ is called an eigenvector of $\boldsymbol{A}$ if $\boldsymbol{Au} = \lambda \boldsymbol{u}$ for some scalar λ.

The scalar λ is called an eigenvalue of $\boldsymbol{A}$, and $\boldsymbol{u}$ is said to be an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ.

The equation $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ is called the characteristic equation of $\boldsymbol{A}$ and the polynomial $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ is called the characteristic polynomial of $\boldsymbol{A}$.

Let λ be an eigenvalue of $\boldsymbol{A}$. Then the solution space of the linear system $(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$ is called the eigenspace of $\boldsymbol{A}$ associated with the eigenvalue λ and is denoted by $E_\lambda = \{\boldsymbol{x} \in \mathbb{R}^n \mid (\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}\}$.

Basic Properties

- If λ is an eigenvalue of $\boldsymbol{A}$, then $c\lambda$ is an eigenvalue of $c\boldsymbol{A}$.

  (This gives us a hint for Question 6.6 (c).)

- Let $\boldsymbol{A}$ be a square matrix of order n. If $\boldsymbol{A}$ has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\operatorname{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} \lambda_i$ and $\det(\boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i$.

  (See Question 6.2 (a).)

- Cayley-Hamilton Theorem

  If $\varphi(\lambda)$ is the characteristic polynomial, then $\varphi(\boldsymbol{A}) = \boldsymbol{0}$.

  (See Question 6.2 (b).)

- Let λ_1 and λ_2 be two *distinct* eigenvalues of $\boldsymbol{A}$.

  Let $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ be two eigenvectors associated with λ_1 and λ_2 respectively.

  Show that $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are linearly independent.

  Proof.

  Set up vector equation $a\boldsymbol{x_1} + b\boldsymbol{x_2} = \boldsymbol{0}$. Then

  $$\boldsymbol{0} = \boldsymbol{A0} = \boldsymbol{A}(a\boldsymbol{x_1} + b\boldsymbol{x_2}) = a\boldsymbol{Ax_1} + b\boldsymbol{Ax_2} = \underline{a\lambda_1\boldsymbol{x_1} + b\lambda_2\boldsymbol{x_2}}$$

  $$\boldsymbol{0} = \lambda_1\boldsymbol{0} = \lambda_1(a\boldsymbol{x_1} + b\boldsymbol{x_2}) = \underline{a\lambda_1\boldsymbol{x_1} + b\lambda_1\boldsymbol{x_2}}$$

  So we obtain $b(\lambda_1 - \lambda_2)\boldsymbol{x_2} = \boldsymbol{0}$. Thus $b = 0$.

  Similarly, $a = 0$. So $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are linearly independent. ∎

Additional Material

Question 5.19 (Textbook).

Hint:

(a) Express the dot products in terms of matrix multiplication.

(b) Show $\boldsymbol{A}\boldsymbol{w}$ is the projection of $\boldsymbol{w}$ onto V by showing $\boldsymbol{w} - \boldsymbol{A}\boldsymbol{w}$ is orthogonal to V.

Question 4 in Semester 1 Final Exam 2006-2007.

Let $V = \text{span}\{(1,1,1),(1,p,p)\}$ where p is a real number.

(a) Find an orthonormal basis for V.

(b) Compute the projection of $(5,3,1)$ onto V.

Question 4 in Semester 2 Final Exam 2007-2008.

Consider the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

(i) Show that the linear system is inconsistent.

(ii) Find the least squares solution for the system.

(iii) Find the projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

(iv) Find the smallest possible value of $\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|$ for some $\boldsymbol{x} \in \mathbb{R}^2$.

(v) Find the projection of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$ onto the column space of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

(vi) Write down an explicit set notation for the set of all $\boldsymbol{c} \in \mathbb{R}^3$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$ has the same least squares solution as $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Question 4 (b) in Semester 2 Final Exam 2009-2010.

For an $m \times n$ matrix $\boldsymbol{A}$ and $m \times 1$ matrix $\boldsymbol{b}$, let $\boldsymbol{p}$ be the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$. Show that $\boldsymbol{b} - \boldsymbol{p}$ is a solution of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{0}$.

Question 3 (a) in Semester 1 Final Exam 2011-2012.

Let $\boldsymbol{v}_1 = (1, 2, 0, 1)$, $\boldsymbol{v}_2 = (1, 0, 1, 1)$, $\boldsymbol{v}_3 = (-1, -4, 1, 0)$.

(i) Write down a 3×4 matrix whose row space is $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(ii) Find a basis for $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and determine $\dim(V)$.

(Hint: Use part (i).)

(iii) Is it possible to find a vector $\boldsymbol{v}_4$ such that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is a basis for $\mathbb{R}^4$?

If it is possible, find one such vector $\boldsymbol{v}_4$.

(iv) Find a nonzero subspace W of $\mathbb{R}^4$ such that $\boldsymbol{w} \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{w} \in W$, $\boldsymbol{v} \in V$.

Similar Matrices

Two *square* matrices A and B are said to be similar if there exists an invertible matrix P such that $B = P^{-1}AP$.

Similar matrices share the following properties:

(1) Trace; (2) Determinant; (3) Rank; (4) Characteristic polynomial; (5) Eigenvalues.

(1) **Similar matrices share trace.**

Let $B = P^{-1}AP$.

$$\text{tr}(B) = \text{tr}(P^{-1}AP) \overset{\clubsuit}{=} \text{tr}(APP^{-1}) = \text{tr}(A)$$

The step $\clubsuit$ follows from the result of Question 2.11 (d), i.e. $\text{tr}(CD) = \text{tr}(DC)$.

(2) **Similar matrices share determinant.**

Let $B = P^{-1}AP$.

$$\begin{aligned}\det(B) &= \det(P^{-1}AP)\\ &= \det(P^{-1})\det(A)\det(P)\\ &= \det(A)\det(P^{-1})\det(P)\\ &= \det(A)\end{aligned}$$

Here $\det(P^{-1})$ and $\det(P)$ are two real numbers satisfying $\det(P^{-1})\det(P) = 1$.

(3) We have shown that

if P is invertible, then $\text{rank}(PA) = \text{rank}(A)$ and $\text{rank}(AP) = \text{rank}(A)$.

Similar matrices share rank.

Let $B = P^{-1}AP$.

Then $\text{rank}(B) = \text{rank}(P^{-1}AP) = \text{rank}(P^{-1}A) = \text{rank}(A)$.

(4) **Similar matrices share characteristic polynomial and eigenvalues.**

Let $B = P^{-1}AP$.

Since P is invertible, we have $\det(P^{-1})\det(P) = 1$.

The characteristic polynomials of A and B are equal, i.e.

$$\begin{aligned}\det(\lambda I - A) &= \det(P^{-1})\det(\lambda I - A)\det(P)\\ &= \det(P^{-1}(\lambda I - A)P)\\ &= \det(\lambda P^{-1}IP - P^{-1}AP)\\ &= \det(\lambda I - B)\end{aligned}$$

Moreover, A and B have the same eigenvalues.

Diagonalization

Let $\boldsymbol{A}$ be a square matrix of order n.

- $\boldsymbol{A}$ is called diagonalizable if there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix. The matrix $\boldsymbol{P}$ is said to diagonalize $\boldsymbol{A}$.
- $\boldsymbol{A}$ is diagonalizable if and only if

$$\boldsymbol{A} \text{ has } n \text{ linearly independent eigenvectors.}$$

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_n \end{pmatrix}$$

- If $\boldsymbol{A}$ has n distinct eigenvalues, then $\boldsymbol{A}$ is diagonalizable; while the converse is not necessarily true. If $\boldsymbol{A}$ is diagonalizable, $\boldsymbol{A}$ may have some same eigenvalues. For instance, $\boldsymbol{I_n}$.

How to determine whether a square matrix is diagonalizable?

Given a square matrix $\boldsymbol{A}$ of order n, we want to determine whether $\boldsymbol{A}$ is diagonalizable.

Also, if $\boldsymbol{A}$ is diagonalizable, find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

1. Solve the characteristic equation $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$ to find all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$.
2. For each eigenvalue λ_i, find a basis S_{λ_i} for the eigenspace E_{λ_i}.
3. Let $S = S_{\lambda_1} \cup S_{\lambda_2} \cup \cdots \cup S_{\lambda_k}$.
   (a) If $|S| < n$, then $\boldsymbol{A}$ is not diagonalizable.
   (b) If $|S| = n$, say $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, then the square matrix
   $$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_n \end{pmatrix}$$
   diagonalizes $\boldsymbol{A}$.

Examples Related to Eigenvalues and Diagonalization

Here is a list of examples illustrating some false statements.

To avoid confusion, the statements in this section are stated in such a way that they are all true.

1. Two matrices are row equivalent does not imply that they have the same eigenvalues.

   ✎ Example $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

2. Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size, λ is an eigenvalue of $\boldsymbol{A}$, and μ is an eigenvalue of $\boldsymbol{B}$. Then $\lambda\mu$ need not be an eigenvalue of $\boldsymbol{AB}$.

   Question: What is wrong with the following "proof"?

   Proof. (It is wrong!)
   Suppose that $\boldsymbol{Av} = \lambda\boldsymbol{v}$ and $\boldsymbol{Bv} = \mu\boldsymbol{v}$, then $\boldsymbol{ABv} = \boldsymbol{A}(\boldsymbol{Bv}) = \boldsymbol{A}\mu\boldsymbol{v} = \mu\boldsymbol{Av} = \mu\lambda\boldsymbol{v} = \lambda\mu\boldsymbol{v}$, which proves that $\lambda\mu$ is an eigenvalue of $\boldsymbol{AB}$. ∎

3. A matrix $\boldsymbol{A}$ is row equivalent to a diagonalizable matrix does not imply that $\boldsymbol{A}$ is diagonalizable.

   ✎ Example $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

   We can check that $\boldsymbol{A}$ is <u>**not**</u> diagonalizable. However, $\boldsymbol{B}$ is clearly diagonalizable.

4. The product of two diagonalizable matrices is not necessarily diagonalizable.

   ✎ Example $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

   We can check that both $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable.

   However, $AB = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ is <u>**not**</u> diagonalizable.

5. The sum of two diagonalizable matrices is not necessarily diagonalizable.

   ✎ Example $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$.

   We can check that both $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable.

   However, $AB = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ is <u>**not**</u> diagonalizable.

Orthogonal Matrices

Let A be a square matrix of order n. The following are equivalent:

1. A is orthogonal, i.e. $A^{-1} = A^{\mathrm{T}}$.
2. $AA^{\mathrm{T}} = I$.
3. $A^{\mathrm{T}}A = I$.
4. The rows of A form an orthonormal basis for $\mathbb{R}^n$.
5. The columns of A form an orthonormal basis for $\mathbb{R}^n$.
6. $\|Au\| = \|u\|$ for any vector $u \in \mathbb{R}^n$. (Exercise 5.32)
7. $Au \cdot Av = u \cdot v$ for any vectors $u, v \in \mathbb{R}^n$. (Exercise 5.32)

Basic Properties of Orthogonal Matrices

1. Any orthogonal matrix is invertible, with $A^{-1} = A^{\mathrm{T}}$.

   If A is orthogonal, so are A^{T} and A^{-1}.

2. The product of orthogonal matrices is orthogonal.

   If $AA^{\mathrm{T}} = I$ and $BB^{\mathrm{T}} = I$, then $(AB)(AB)^{\mathrm{T}} = (AB)(B^{\mathrm{T}}A^{\mathrm{T}}) = A(BB^{\mathrm{T}})A^{\mathrm{T}} = AA^{\mathrm{T}} = I$.

3. The determinant of an orthogonal matrix is equal to 1 or -1.

   $1 = \det(I) = \det(AA^{\mathrm{T}}) = \det(A)\det(A^{\mathrm{T}}) = \det(A)^2$.

Orthogonal Diagonalization

- A square matrix A is called orthogonally diagonalizable if there exists an orthogonal matrix P such that $P^{\mathrm{T}}AP$ is a diagonal matrix.
- A square matrix is orthogonally diagonalizable if and only if it is symmetric.

Examples Related to Orthogonal Matrices

Here we list some examples illustrating some false statements.

To avoid confusion, the statements in this section are stated in such a way that they are all true.

1. The determinant of any orthogonal matrix is 1 or -1.

   However, a matrix with determinant 1 or -1 is not necessarily orthogonal.

   ✎ Example The matrix $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ has determinant 1, but it is **not** an orthogonal matrix.

2. A matrix with orthogonal rows and orthogonal columns is not necessarily an orthogonal matrix.

   ✎ Example The matrix $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ has orthogonal rows and orthogonal columns, but it is **not** an orthogonal matrix.

3. The product of two $n \times n$ orthogonal matrices is an orthogonal matrix.

   However, the sum of two $n \times n$ orthogonal matrices is not necessarily an orthogonal matrix.

   ✎ Example The matrices $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are two orthogonal matrices.

   However, $\boldsymbol{A} + \boldsymbol{B}$ is a zero matrix, which is clearly **not** an orthogonal matrix.

4. A square matrix whose row vectors are orthogonal may not have orthogonal column vectors.

   ✎ Example $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

   Similarly, a square matrix whose column vectors are orthogonal may not have orthogonal row vectors.

5. A diagonalizable matrix is not necessarily orthogonally diagonalizable.

   ✎ Example $\begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix}$.

Additional Material

Question 6.18 (Textbook).

Hint:

Let a_n, b_n and c_n be the percentage of customers choosing brand A, B and C, and set up a system with 3 equations involving a_n, b_n and c_n. (Refer to Example 6.1.1).

Find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

We express $\boldsymbol{A}$ in the form $\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ where $\boldsymbol{D}$ is a diagonal matrix. Then we compute $\boldsymbol{A}^n$.

Question 5 (i)–(iii) in Semester 1 Final Exam 2005-2006.

In this question, all vectors $\boldsymbol{x}$ in $\mathbb{R}^n$ are written in column form.

Let $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ be an orthonormal basis for $\mathbb{R}^n$.

(i) Show that $(\boldsymbol{x}_i + \boldsymbol{x}_j)$ and $(\boldsymbol{x}_i - \boldsymbol{x}_j)$ are orthogonal to each other for all $i \neq j$.

(ii) Show that if $\boldsymbol{x} = a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2 + \cdots + a_n\boldsymbol{x}_n$, then

$$\|\boldsymbol{x}\|^2 = {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2.$$

(iii) If $\boldsymbol{A}$ is an $n \times n$ diagonalizable matrix and $\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} = 0$ for every eigenvector $\boldsymbol{x}$ of $\boldsymbol{A}$, show that $\boldsymbol{A}$ is the zero matrix.

Question 3 (b) in Semester 1 Final Exam 2009-2010.

For $n \geq 2$, let $\boldsymbol{B}_n = (b_{ij})$ be a square matrix of order n such that

$$b_{ij} = \begin{cases} 0 & \text{if } i > j \text{ or } j > i+1 \\ 1 & \text{if } j = i+1 \\ k & \text{if } i = j \end{cases}$$

where k is a real number.

(i) Write down $\boldsymbol{B}_2$ and $\boldsymbol{B}_3$.

(ii) Find all the eigenvalues of $\boldsymbol{B}_n$.

(iii) Prove that $\boldsymbol{B}_n$ is <u>not</u> diagonalizable for all $n \geq 2$.

Question 2 (b) in Semester 2 Final Exam 2009-2010.

Let B be a 2×2 matrix such that $B\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$ and $B\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$.

(i) Write down an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.

Briefly explain how you obtain the answers.

(ii) Let n be a positive integer. Write $\boldsymbol{B}^n$ in the form $\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$ where the entries b_{ij} are in terms of n.

(iii) Is it possible to find a non-zero column vector $\boldsymbol{v}$ such that $\boldsymbol{B}\boldsymbol{v} = \boldsymbol{v}$? Justify your answer.

Question 2 (b) in Semester 2 Final Exam 2010-2011.

Let $B = \begin{pmatrix}a & 1 & 0\\ 0 & b & 1\\ 0 & 0 & a\end{pmatrix}$. Show that B is <u>not</u> diagonalizable for any values a and b.

Hint: consider two cases, $a = b$ and $a \neq b$.

Linear Transformations

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Question. How to completely determine T?

- Find the standard matrix for T.
- Find the formula of T.
- Determine the images $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$ of the basis vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$.

  (In particular, we can take the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$.)

Theorem 7.1.4

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1. $T(\boldsymbol{0}) = \boldsymbol{0}$.
2. If $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \in \mathbb{R}^n$ and $c_1, c_2, \ldots, c_k \in \mathbb{R}$, then

   $T(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k) = c_1T(\boldsymbol{u}_1) + c_2T(\boldsymbol{u}_2) + \cdots + c_kT(\boldsymbol{u}_k)$.

Linear Transformation vs Subspace

$T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. (*Linearity conditions*)	U is a subspace of $\mathbb{R}^n$. (*Closure Properties*)
T preserves addition: Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. Then $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$.	U is closed under addition: Let $\boldsymbol{u}, \boldsymbol{v} \in U$. Then $\boldsymbol{u} + \boldsymbol{v} \in U$.
T preserves scalar multiplication: Let $\boldsymbol{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$.	U is closed under scalar multiplication: Let $\boldsymbol{u} \in U$ and $c \in \mathbb{R}$. Then $c\boldsymbol{u} \in U$.
T preserves zero vector: $T(\boldsymbol{0}) = \boldsymbol{0}$.	U contains the zero vector: $\boldsymbol{0} \in U$.

Linear Transformation vs Standard Matrix

Linear transformation	Standard matrix
$T : \mathbb{R}^n \to \mathbb{R}^m$	$\boldsymbol{A}$ is an $m \times n$ matrix
$T(\boldsymbol{u})$	$\boldsymbol{Au}$
$T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \ldots, T(\boldsymbol{e}_n)$	the columns of $\boldsymbol{A}$
$\mathrm{R}(T)$	the column space of $\boldsymbol{A}$
$\mathrm{Ker}(T)$	the nullspace of $\boldsymbol{A}$
$\mathrm{rank}(T)$	$\mathrm{rank}(\boldsymbol{A})$
$\mathrm{nullity}(T)$	$\mathrm{nullity}(\boldsymbol{A})$
$S \circ T$	$\boldsymbol{BA}$

Map of Linear Algebra

Suppose $\boldsymbol{A}$ is an $n \times n$ matrix, and $\boldsymbol{A}$ is the standard matrix for a linear transformation T.

$\boldsymbol{A}$ is invertible	Chapter 2	$\boldsymbol{A}$ is not invertible
$\det(\boldsymbol{A}) \neq 0$	Chapter 2	$\det(\boldsymbol{A}) = 0$
The RREF of $\boldsymbol{A}$ is an identity matrix	Chapter 1	The RREF of $\boldsymbol{A}$ has a zero row
$\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution	Chapter 1	$\boldsymbol{Ax} = \boldsymbol{0}$ has non-trivial solutions
$\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution	Chapter 1	$\boldsymbol{Ax} = \boldsymbol{b}$ has no solution or infinitely many solutions
rows (columns) of $\boldsymbol{A}$ are linearly independent	Chapter 3	rows (columns) of $\boldsymbol{A}$ are linearly dependent
$\operatorname{rank}(\boldsymbol{A}) = n$	Chapter 4	$\operatorname{rank}(\boldsymbol{A}) < n$
$\operatorname{nullity}(\boldsymbol{A}) = 0$	Chapter 4	$\operatorname{nullity}(\boldsymbol{A}) > 0$
0 is not an eigenvalue of $\boldsymbol{A}$	Chapter 6	0 is an eigenvalue of $\boldsymbol{A}$
$\mathrm{R}(T) = \mathbb{R}^n$	Chapter 7	$\mathrm{R}(T) \neq \mathbb{R}^n$
$\operatorname{Ker}(T) = \{\boldsymbol{0}\}$	Chapter 7	$\operatorname{Ker}(T) \neq \{\boldsymbol{0}\}$

Additional Material

Question 6 in Semester 2 Final Exam 2005-2006.

For any vectors $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^n$, the vector $(\boldsymbol{u}, \boldsymbol{v})$ is a vector in $\mathbb{R}^{2n}$.

Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be two linear transformations.

Define $V = \{(\boldsymbol{u}, \boldsymbol{v}) \mid S(\boldsymbol{u}) = T(\boldsymbol{v})\}$.

Show that V is a subspace of $\mathbb{R}^{2n}$.

Question 3 (a) in Semester 2 Final Exam 2006-2007.

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}3\\6\\9\end{pmatrix}, \quad T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}-1\\-2\\-3\end{pmatrix}.$$

(i) Find $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right)$ and $T\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$. Write down the standard matrix for T.

(ii) Find a basis for the range $\mathrm{R}(T)$ of T.

(iii) Find a basis for the kernel $\mathrm{Ker}(T)$ of T.

(iv) Is it possible to find a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^2$ such that the standard matrix for the composition map $S \circ T$ is an invertible matrix? Justify your answer.

Question 5 (g) in Semester 2 Final Exam 2007-2008.

Determine whether the following statement is true or false. Justify your answer.

(g) Let $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations with $m > n$.

Then the composition $S \circ T : \mathbb{R}^m \to \mathbb{R}^m$ has a non-zero vector in the kernel.

Question 4 (a) in Semester 1 Final Exam 2010-2011.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}3\\2\\2\end{pmatrix}, \quad T\left(\begin{pmatrix}2\\1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

(i) Write down the standard matrix for T.

(ii) Find the kernel of T.

(iii) Find the kernel of $T \circ T$.

(iv) Find a basis for the subspace V given by $V = \mathrm{R}(T) \cap \mathrm{Ker}(T)$.

Exam Advice

1. Know all of the definitions given in this module, know the statements of all the results proven in lecture or in tutorial.
2. Manage your time during the exam.
3. Try to have an intuition about how to approach problems. When trying to solve a problem, think about what you are given and what the givens imply, think about what you are trying to do and how you could do it, think about what techniques we have and which might be useful, and think about which tools (i.e. results and theorems) you have and how they might be applied.
4. Do not give up on a problem. Most problems in this module do not involve an absurdly difficult trick and can be solved by reasoning using the definitions and results done in lecture.
5. Get a good night sleep the night before the exam.
6. Bring a jacket along in case you find the air con in MPSH too cold.

Wish the very best to all of you for your final examinations!