MOBIUS TRANSFORMATIONS IN SEVERAL DIMENSIONS

Lars V. Ahlfors

INTRODUCTION

These are day to day notes from a course that I gave as Visiting Ordway Professor at the University of Minnesota in the fall 1980. I had used some of the material earlier for a similar course at the University of Michigan and for a month each at the University of Vienna and the University of Graz. The present notes are a little more detailed, but still not in a definitive form.

My aim was to give a rather elementary course which would acquaint the hearers with the geometric and analytic properties of Möbius transformations in n real dimensions which could serve as an introduction to discrete groups of non-euclidean motions in hyperbolic space, especially from the viewpoint of solutions of the hyperbolically invariant Laplace equation as well as quasiconformal deformations.

I wish to thank the University of Minnesota and the Chairman of the Mathematics Department for having invited me, and above all my friend and former student Albert Marden for having initiated my visit. I also thank the many faculty members and students who had the patience to sit in at my lectures to the very end.

I am also indebted to Dr. Helmut Maier who helped me edit the notes and supervised the typing. The typist, unknown to me, has also my sympathy.

Lars V. Ahlfors

TABLE OF CONTENTS

I. The classical case.

1.1. Everybody is familiar with the fractional linear transformations

(1) $$\gamma(z) = \frac{az+b}{cz+d}$$

where $a,b,c,d \in \mathbb{C}$ and $ad-bc \neq 0$. They act on the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we identify with the 1-dimensional projective space $P_1(\mathbb{C})$.

In terms of homogeneous coordinates $w = \gamma(z)$ can be expressed through the matrix equation

(2) $$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

This has the advantage that the Möbius group of all γ can be represented as a matrix group either by means of the general linear group $GL_2(\mathbb{C})$ or by the unimodular or special linear group $SL_2(\mathbb{C})$. More precisely the Möbius group is isomorphic to

$$GL_2(\mathbb{C}) / \mathbb{C}^*$$

where $\mathbb{C}^*$ is the multiplicative group of nonzero complex numbers and to

$$PSL_2(\mathbb{C}) = SL_2(\mathbb{C}) / \{\pm I\}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In spite of this identification we shall denote the Möbius group by $M_2(\mathbb{C})$ rather than $PSL_2(\mathbb{C})$ for the simple reason that the identification does not carry over to higher dimensions.

We shall often simplify $\gamma(z)$ to γz and we think of γ as a matrix normalized by $ad-bc = 1$. We should be aware, of course, that γ and $-\gamma$

represent the same Möbius transformation. It is useful to memorize the formula for the inverse in SL_2 :

$$(3) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

1.2. There is a natural topology on $GL_2(\mathbb{C})$ and hence also on $SL_2(\mathbb{C})$ and $M_2(\mathbb{C})$. $GL_2(\mathbb{C})$ is <u>connected</u> for the space of all complex 2×2 matrices has eight real dimensions and the space of singular matrices has only six. The mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{a}{ad-bc} & \frac{b}{ad-bc} \\ c & d \end{pmatrix}$$

of $GL_2(\mathbb{C})$ on $SL_2(\mathbb{C})$ shows that $SL_2(\mathbb{C})$ is likewise connected. We can consider $SL_2(\mathbb{C})$ as a double covering of $M_2(\mathbb{C})$.

The subgroups with real coefficients, are denoted by $GL_2(\mathbb{R})$, $SL_2(\mathbb{R})$ and $M_2(\mathbb{R})$. There is also a subgroup $GL_2^+(\mathbb{R})$ with positive determinant. $GL_2^+(\mathbb{R})$ and $SL_2(\mathbb{R})$ are connected, but $GL_2(\mathbb{R})$ is not for one cannot pass continuously from a matrix with positive determinant to one with negative determinant.

Any $\gamma \in M_2(\mathbb{R})$ maps the real axis, the upper half-plane, and the lower half-plane on themselves. This is obvious from

$$(4) \qquad \gamma z - \overline{\gamma z} = \frac{z - \bar{z}}{|cz + d|^2} .$$

1.3. From (1) one computes

$$(5) \qquad \gamma(z) - \gamma(z') = \frac{(ad - bc)(z - z')}{(cz + d)(cz' + d)}$$

and in the limit for $z' \to z$

$$(6) \qquad \gamma'(z) = \frac{ad - bc}{(cz + d)^2} \quad .$$

In particular, $z \mapsto \gamma z$ is conformal. We rewrite (5) as

$$(7) \qquad \gamma(z) - \gamma(z') = \gamma'(z)^{1/2} \, \gamma'(z')^{1/2} \, (z - z') \quad ,$$

where it is understood that

$$\gamma'(z)^{1/2} = \frac{\sqrt{ad - bc}}{cz + d}$$

with a fixed choice of the square root.

1.4. The cross-ratio of four points z , z' , ξ , ξ' in that order is defined by

$$(8) \qquad (z , z' , \zeta, \zeta') = \frac{z - \zeta}{z - \zeta'} : \frac{z' - \zeta}{z' - \zeta'}$$

when this makes sense, i.e. when at most two points coincide. By use of (7) it follows that

$$(9) \qquad (\gamma z , \gamma z' , \gamma \zeta, \gamma \zeta') = (z , z' , \zeta, \zeta')$$

for the derivatives cancel against each other. In other words, the cross-ratio is invariant under all Möbius transformations.

The cross-ratio is real if and only if the four points lie on a circle or straight lines. Note that our definition makes

$$(z , 1 , 0 , \infty) = z \quad .$$

1.5. From (6) one obtains

(10) $$\frac{\gamma''(z)}{\gamma'(z)} = -\frac{2c}{cz+d}$$

and hence

(11) $$\frac{\gamma'(z)}{\gamma''(z)} = -\frac{1}{2}\left(z+\frac{d}{c}\right)$$

if $c \neq 0$. It follows that

(12) $$\frac{\gamma'(z)}{\gamma''(z)} - \frac{\gamma'(\xi)}{\gamma''(\xi)} = -\frac{1}{2}(z-\xi)$$

and for $\xi \to z$

(13) $$D\,\frac{\gamma'(z)}{\gamma''(z)} = -\frac{1}{2} \;.$$

More explicitly, this is equivalent to the vanishing of the Schwarzian derivative:

(14) $$\frac{\gamma'''}{\gamma'} - \frac{3}{2}\left(\frac{\gamma''}{\gamma'}\right)^2 = \left(\frac{\gamma''}{\gamma'}\right)' - \frac{1}{2}\left(\frac{\gamma''}{\gamma'}\right)^2 = 0 \;.$$

There is obviously a close link between the cross-ratio and the Schwarzian

(15) $$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

for arbitrary meromorphic f. For those who like computing, I recommend proving the formula

(16) $$(f(z+ta), f(z+tb), f(z+tc), f(z+td)) =$$
$$(a,b,c,d)\left(1+\frac{t^2}{6}(a-b)(c-d)S_f(z)\right) + O(t^3) \;.$$

1.6. The <u>half-plane</u>. We return to the action of $SL_2(\mathbb{R})$ on the upper half-plane $H^2 = \{z = x+iy \; ; \; y > 0\}$. We use the notation $G = SL_2(\mathbb{R})$ (or $M_2(\mathbb{R})$).

Let K denote the isotropy group of i. From

$$\frac{ai+b}{ci+d} = i$$

we get $a=d$, $b=-c$, $a^2+b^2=1$. Hence $g \in K$ has the form

$$g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{17}$$

In other words, $K \simeq SO(2)$. More precisely, θ and $\theta+\pi$ give different elements of K but the same element of $M_2(\mathbb{R})$.

Let A be the group of homothetic transformations $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $t > 0$, and N the group of translations $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. The group NA is the group of <u>similarities</u> and also the isotropy group of ∞. It is simply transitive on H^2 as seen from

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}(i) = u+t^2 i .$$

Hence $G = NAK$; this is the Iwasawa decomposition. Each $g \in G$ has a unique decomposition $g = nak$. Explicitly, if $g(i) = x+iy$, $y > 0$, then

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{18}$$

Every function f on H^2 can be lifted to a function $\underline{f}$ on G defined by $\underline{f}(g) = f(gi)$. Conversely, a function on G becomes a function on H^2 if it is constant on the left cosets gK. H^2 is identified with G/K. (many authors prefer $K\backslash G$).

1.7. The circles orthogonal to the real axis, including vertical lines, are mapped on each other by all $g \in G$. They are the straight lines in Poincaré's model of hyperbolic or non-euclidean geometry. Any two distinct points $z_1 , z_2 \in H^2$ lie on a unique n.e. line (Fig. 1)

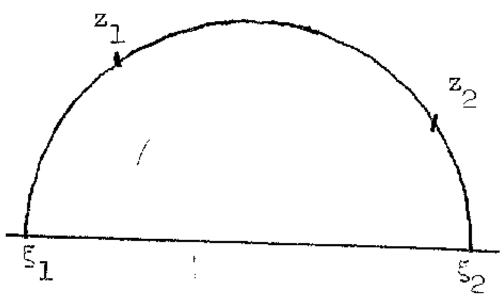


Fig. 1

A distance can be defined by the point-pair invariant

$$\delta(z_1 , z_2) = \left| \frac{z_1 - z_2}{z_1 - \bar{z}_2} \right| \quad ; \tag{19}$$

the invariance is seen from the relation

$$\delta(z_1 , z_2)^2 = (z_1 , \bar{z}_1 , z_2 , \bar{z}_2) \quad . \tag{20}$$

For $z_2 \to z_1$ we obtain the infinitesimal invariant

$$ds = \frac{|dz|}{y} \tag{21}$$

(we have dropped a factor 2). This defines the Poincaré metric on H^2 . It leads to a new distance function $d(z_1 , z_2)$ defined as the minimum

$$d(z_1 , z_2) = \min \int_{z_1}^{z_2} \frac{|dz|}{y} \tag{22}$$

over all paths from z_1 to z_2. The shortest curves are the <u>geodesics</u>. To find the geodesic we use a $g \in G$ which maps z_1, z_2 on iy_1, iy_2. It is readily seen that the vertical line segment from iy_1 to iy_2 is minimal, and thus

$$d(iy_1, iy_2) = \left| \log \frac{y_2}{y_1} \right| . \tag{23}$$

It follows that the geodesics are the orthogonal circles.

The relation between δ and d is determined by

$$\delta(iy_1, iy_2) = \frac{y_2 - y_1}{y_2 + y_1} = \frac{e^{d(iy_1, iy_2)} - 1}{e^{d(iy_1, iy_2)} + 1}$$

from which we conclude that

$$\delta = \tanh \frac{d}{2} . \tag{24}$$

The distance function d is additive on n.e. lines, i.e. $d(z_1, z_3) = d(z_1, z_2) + d(z_2, z_3)$ if z_2 lies between z_1 and z_3.

We can use (24) to verify that δ is a distance function, for if $d \leqq d_1 + d_2$ then

$$\delta = \tanh \frac{d}{2} \leqq \frac{\tanh \frac{d_1}{2} + \tanh \frac{d_2}{2}}{1 + \tanh \frac{d_1}{2} \tanh \frac{d_2}{2}} \leqq \delta_1 + \delta_2 .$$

1.8. There is a different way of computing $d(z_1, z_2)$ which shows the additivity at once. We refer to Fig. 1 and compute (z_2, z_1, ξ_1, ξ_2). If the circle is mapped on the imaginary axis the cross-ratio becomes

$$(iy_2, iy_1, 0, \infty) = \frac{y_2}{y_1} .$$

This proves the relation

(25) $$d(z_1 , z_2) = \log (z_2 , z_1 , \xi_1 , \xi_2) \quad .$$

In this form we may even regard $d(z_1 , z_2)$ as a signed distance which becomes negative when the order of z_1 , z_2 is reversed. The additivity on a geodesic follows at once from

$$(z_2 , z_1 , \xi_1 , \xi_2)\ (z_3 , z_2 , \xi_1 , \xi_2) = (z_3 , z_1 , \xi_1 , \xi_2) \quad .$$

1.9. The length of an arc is

$$\int_c \frac{|dz|}{y}$$

and the area of a measurable set E is

$$\int_E \frac{dx\,dy}{y^2} \quad .$$

It is a useful exercise to compute the area of a n.e. polygon P (Fig. 2)

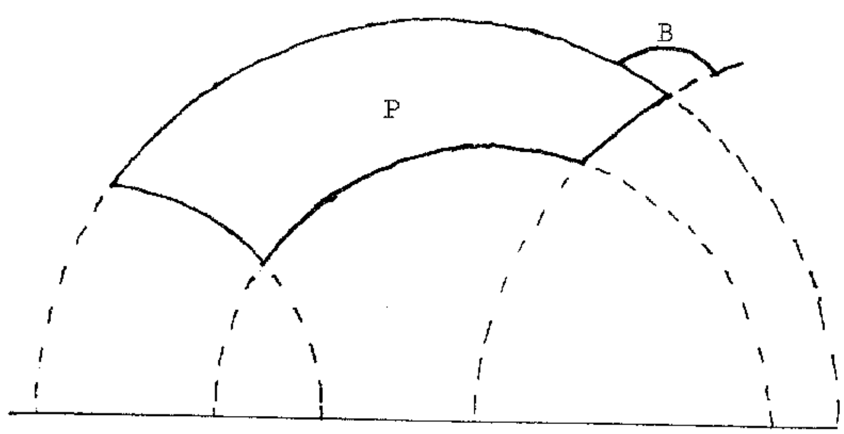


Fig. 2

By Stokes' formula

$$A = \int_P \frac{dx \wedge dy}{y^2} = \int_P d\left(\frac{1}{y}\right) \wedge dx = \int_{\partial P} \frac{dx}{y} \quad .$$

Each side is an arc of $z - a = re^{i\theta}$ and $\frac{dx}{y} = -d\theta$. $\int d\theta$ measures the change of the angle of the tangent. For a simply connected polygon the total change, including the jumps at the angles, is 2π . If the outer angles are β_ν and the inner angles α_ν one obtains

$$A = \Sigma\, \beta_\nu - 2\pi = (n-2)\pi - \Sigma\, \alpha_\nu \quad .$$

1.10. It is easy to pass from the half-plane H^2 to the unit disk $B^2 = \{\zeta \in \mathbb{C} \,;\, |\zeta| < 1\}$. We want to choose a canonical Möbius mapping $H^2 \to B^2$. A good choice is to let $z = 0, i, \infty$ correspond to $\zeta = -i, 0, i$. This gives

(26) $$\zeta = i\,\frac{z-i}{z+i} \,, \quad z = -i\,\frac{\zeta+i}{\zeta-i} \quad .$$

We introduce the notation $\zeta^* = \frac{1}{\bar{\zeta}} = \frac{\zeta}{|\zeta|^2}$ for the symmetric point of ζ with respect to the unit circle. By the reflection principle $z, \bar{z}$ go over into ζ, ζ^* and by the invariance of the cross-ratio

(27) $$(z_1, \bar{z}_1, z_2, \bar{z}_2) = (\zeta_1, \zeta_1^*, \zeta_2, \zeta_2^*) = \frac{|\zeta_1 - \zeta_2|^2}{|1 - \zeta_1 \bar{\zeta}_2|^2} \quad .$$

This shows that the point-pair invariant in B^2 is

$$\delta(\zeta_1, \zeta_2) = \frac{|\zeta_1 - \zeta_2|}{|1 - \bar{\zeta}_1 \zeta_2|}$$

and the corresponding Poincaré metric is

(28) $$ds = \frac{2\,|d\zeta|}{1-|\zeta|^2}$$

(remember that we had suppressed a factor 2).

It is again trivial that diameters are geodesics and hence the same is true of all circles orthogonal to the unit circle. The distance from 0 to $r > 0$ is

$$d = d(0,r) = \int_0^r \frac{2dt}{1-t^2} = \log\frac{1+r}{1-r}\ , \quad r = \tanh\frac{d}{2}\ .$$

Note that $\delta(0,r) = r$.

Another observation is that

$$(r\,,0\,,-1\,,1) = \frac{1+r}{1-r}\ .$$

If the geodesic through ζ_1, ζ_2 meets the unit circle in ω_1, ω_2 we have thus

$$d(\zeta_1\,,\zeta_2) = \log(\zeta_2\,,\zeta_1\,,\omega_1\,,\omega_2)\ .$$

From now on we prefer to use the variable z in the unit disk, and with the notation of Fig. 3 we have thus

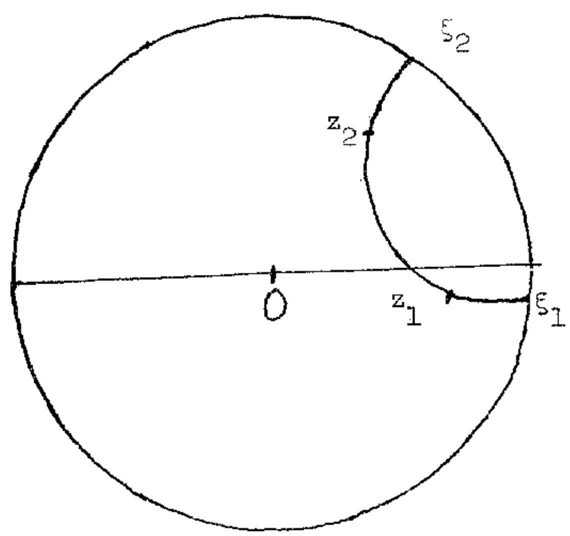


Fig. 3

(29) $$d(z_1, z_2) = \log(z_2, z_1, \xi_1, \xi_2) .$$

1.11. The self-mappings of B^2 which take a into 0 are of the form

(30) $$\gamma z = e^{i\theta} \frac{z-a}{1-\bar{a}z}$$

We want to derive this formula in a geometric way.

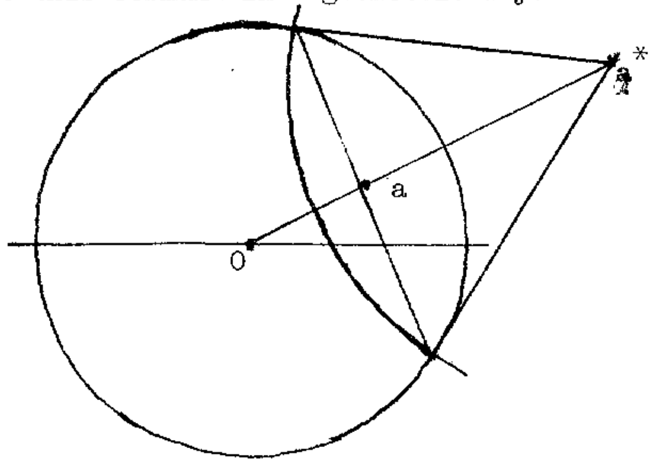


Fig. 4

For this purpose we construct a^* and the orthogonal circle with center a^* as shown in the picture. The reflection in this orthogonal circle carries a into 0 and an arbitrary point z into

$$\sigma_a z = a^* + (|a^*|^2 - 1)(z - a^*)^* = -\frac{a}{\bar{a}} \frac{\bar{z}-\bar{a}}{1-a\bar{z}} .$$

To obtain a sense-preserving mapping we let σ_a be followed by reflection in the line through the origin perpendicular to a. If $\arg a = \alpha$ this reflection is expressed by $w \mapsto -e^{2i\alpha}\bar{w}$ and we end up with the mapping

(31) $$z \mapsto T_a z = \frac{z-a}{1-\bar{a}z} .$$

The most general mapping is T_a followed by a rotation about the origin, and it is hence of the form (30). We make a note of the formulas

$$T_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}$$

(32) $$1-|T_a z|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}$$

$$\frac{|T_a'(z)|}{1-|T_a z|^2} = \frac{1}{1-|z|^2} \quad ,$$

all well known to conformal mappers. The last formula expresses the invariance of the Poincaré metric. The normalization of T_a is such that $T_a'(a) > 0$ and $T_a'(0) > 0$.

Note that $T_a 0 = -a$ and hence $T_{-a} = T_a^{-1}$.

II. The general case.

2.1. Before passing to the general case we shall pay special attention to the group $M(H^3)$ of Möbius transformations acting on the upper half-space $H^3 : (x = (x_1, x_2, x_3), x_3 > 0)$. Already Poincaré was well aware that the action of $M(\mathbb{C})$ can be lifted to H^3 or if one prefers to all of $\mathbb{R}^3$.*) In fact, any $\gamma \in M(\mathbb{C})$ is a product of reflections in circles (or straight lines), the circle determines an orthogonal hemisphere (or half-plane) and the reflection extends to a reflection in the hemisphere. One has to show, of course, that the end result does not depend on the particular factorization of γ.

The three-dimensional upper half-space is very special because of the fact that one can make use of quaternions in a very elegant manner.

The quaternions can be identified with matrices

(1) $$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

where $u, v \in \mathbb{C}$. In fact, if we write $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $u = u_1 + iu_2$, $v = v_1 + iv_2$ we obtain

(2) $$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} = u_1 + u_2 i + v_1 j + v_2 k$$

$$= u + vj \quad .$$

The conjugate of $z = u + vj$ is $\bar{z} = \bar{u} - vj$ and the absolute value $|z|$ is given by $|z|^2 = z\bar{z} = |u|^2 + |v|^2$. When computing the product we have used the rule $aj = j\bar{a}$ for arbitrary complex a.

---

*) We have replaced the notation $M_2(\mathbb{C})$ by $M(\mathbb{C})$.

Points in $\mathbb{R}^3$ will now be denoted by $z = x + yj$ with $x \in \mathbb{C}$, $y \in \mathbb{R}$ ($y > 0$ if $z \in H^3$). Suppose $\gamma \in SL_2(\mathbb{C})$ is given by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 .$$

We define the action by

$$\text{(3)} \qquad \gamma z = (az+b)(cz+d)^{-1} = (zc+d)^{-1}(za+b) .$$

We need to show that these expressions are equal, and that γz is a special quaternion of the same form as z.

In the first place, the two expressions (3) are equal if and only if

$$(zc+d)(az+b) - (za+b)(cz+d) = 0 .$$

This works out to

$$zcb + daz - zad - bcz = 0$$

which is true because $ad - bc$ is real.

The next step is to actually compute γz. To begin with

$$\gamma z = \frac{(az+b)(\overline{cz+d})}{|cz+d|^2}$$

Here

$$cz+d = cx+d + cyj$$

$$\overline{cz+d} = \bar{c}\bar{x}+\bar{d} - cyj$$

$$az+b = ax+b + ayj$$

$$(az+b)(\overline{cz+d}) = (ax+b)(\bar{c}\bar{x}+\bar{d}) + a\bar{c}y^2 + [-(ax+b)\ \ cy + ay(cx+d)]\,j$$

$$= (ax+b)(\bar{c}\bar{x}+\bar{d}) + a\bar{c}y^2 + yj$$

and thus

$$\gamma z = \frac{(ax+b)(\bar{c}\bar{x}+\bar{d}) + a\bar{c}y^2 + yj}{|cz+d|^2} \tag{4}$$

which is of the desired form. Also, $|cz+d|^2 = |cx+d|^2 + |c|^2 y^2$.

2.2. We shall also compute a formula for the difference $\gamma z - \gamma z'$. We write it in the form

$$\begin{aligned} \gamma z - \gamma z' &= (az+b)(cz+d)^{-1} - (z'c+d)^{-1}(z'a+b) \\ &= (z'c+d)^{-1}[(z'c+d)(az+b) - (z'a+b)(cz+d)](cz+d)^{-1} \\ &= (z'c+d)^{-1}(z-z')(cz+d)^{-1} \quad . \end{aligned} \tag{5}$$

This is at least similar to the corresponding formula (1.4) in the complex case.

On passing to the absolute values we obtain

$$|\gamma z - \gamma z'| = \frac{|z-z'|}{|cz+d|\,|cz'+d|} \tag{6}$$

and infinitesimally

$$|d\gamma(z)| = \frac{|dz|}{|cz+d|^2} \quad . \tag{7}$$

Comparison with (4) shows that

$$ds = \frac{|dz|}{y} \tag{8}$$

is invariant.

The cross-ratio should be defined by

$$(z_1, z_2, z_3, z_4) = (z_1 - z_3)(z_1 - z_4)^{-1}(z_2 - z_4)(z_2 - z_3)^{-1} .$$

By use of (5) one obtains

$$(\gamma z_1, \gamma z_2, \gamma z_3, \gamma z_4) = (z_3 c + d)^{-1}(z_1, z_2, z_3, z_4)(z_3 c + d) .$$

Thus the matrices corresponding to the quaternions $(\gamma z_1, \gamma z_2, \gamma z_3, \gamma z_4)$ and (z_1, z_2, z_3, z_4) are similar. Because the absolute value is the square root of the determinant and the real part is half of the trace of the matrix (1) it follows that the absolute value and the real part of the cross-ratio are invariant.

The stabilizer of j is the group of unitary matrices

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} , \quad |a|^2 + |b|^2 = 1 .$$

The full group $M(H^3)$ is transitive, for $\gamma j = u + vj$, $v > 0$, by taking

$$\gamma = \begin{pmatrix} v^{1/2} & uv^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix} .$$

2.3. The quaternion technique works only for $M(H^3)$ and even if it looks elegant it is not particularly useful. We shall now again deal with actions on the entire space $\mathbb{R}^n$. We shall use the notation $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and when operating with matrices we treat x as a column vector. The group of <u>similarities</u> consists of all mappings

(9) $$x \mapsto mx + b$$

where $b \in \mathbb{R}^n$ and m is a conformal matrix, i.e. $m = \lambda k$ with $\lambda > 0$ and $k \in O(n)$.

Reflection in the unit sphere is defined by

$$(10) \qquad x \mapsto x^* = Jx = \frac{x}{|x|^2} \qquad (J0 = \infty \ , \ J\infty = 0)$$

where of course $|x|^2 = x_1^2 + \ldots + x_n^2$. We use mostly the notation x^* , but sometimes one needs a letter symbol for the mapping, and then we use J . Obviously, $J^2 = I$, the identity mapping. I will also stand for the unit matrix I_n .

We adopt the following definition:

Definition. The full Möbius group $\hat{M}(\mathbb{R}^n)$ is the group generated by all similarities together with J . The Möbius group $M(\mathbb{R}^n)$ is the subgroup whose elements contain an even number of factors J and sense-preserving similarities.

In other words, $M(\mathbb{R}^n)$ is the sense-preserving Möbius group. Note that I do not use $M(\overline{\mathbb{R}}^n)$ as being too pedantic.

The derivative of a differentiable mapping f from one open set in $\mathbb{R}^n$ to another is the Jacobian matrix

$$f'(x) \quad \text{or} \quad Df(x)$$

with the elements

$$(11) \qquad f'(x)_{ij} = \frac{\partial f_i}{\partial x_j} = D_j f_i(x) \quad .$$

The derivative of a similarity $\gamma x = mx + b$ is the constant conformal matrix m . The matrix $J'(x)$ has the components

(12) $$J'(x)_{ij} = \frac{1}{|x|^2}\left(\delta_{ij} - \frac{2x_i x_j}{|x|^2}\right)$$

for $x \neq 0$.

It is almost indispensable to adopt a special notation for the ubiquitous matrix $Q(x)$ with the entries

(13) $$Q(x)_{ij} = \frac{x_i x_j}{|x|^2} \quad .$$

This enables us to write (12) in the form

(14) $$J'(x) = \frac{1}{|x|^2}(I - 2Q(x)) \quad .$$

This is perhaps the most important formula in the whole theory of Möbius transformations.

From $Q^2 = Q$ we obtain

$$(I-2Q)^2 = I$$

which means that $I-2Q \in O(n)$. Thus $J'(x)$ is a conformal matrix for each $x \neq 0$.

It now follows by the chain rule that $\gamma'(x)$ is a conformal matrix for any $\gamma \in \hat{M}(\mathbb{R}^n)$. In other words, all Möbius transformations are conformal with a suitable interpretation at ∞ and $\gamma^{-1}\infty$.

Definition. For any $\gamma \in \hat{M}(\mathbb{R}^n)$ we denote by $|\gamma'(x)|$ the positive number such that $\frac{\gamma'(x)}{|\gamma'(x)|} \in O(n)$. In other words, $|\gamma'(x)|$ is the linear change of scale at x, the same in all directions.

Another application of the chain rule proves that

$$(15) \qquad |\gamma x - \gamma y| = |\gamma'(x)|^{1/2} \, |\gamma'(y)|^{1/2} \, |x - y| \quad .$$

In fact, this is trivial when γ is a similarity, and for J we obtain

$$|Jx - Jy|^2 = \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right|^2 = \frac{1}{|x|^2} + \frac{1}{|y|^2} - \frac{2xy}{|x|^2 \, |y|^2} =$$

$$\frac{|x - y|^2}{|x|^2 \, |y|^2} = |J'(x)| \, |J'(y)| \, |x - y|^2 \quad \text{by (14)} \; .^{*)}$$

If γ_1 and γ_2 satisfy (15) then it is also true for $\gamma_1 \gamma_2$:

$$|\gamma_1 \gamma_2 x - \gamma_1 \gamma_2 y| = |\gamma_1'(\gamma_2 x)|^{1/2} \, |\gamma_1'(\gamma_2 y)|^{1/2} \, |\gamma_2 x - \gamma_2 y| =$$

$$|\gamma_1'(\gamma_2 x)|^{1/2} \, |\gamma_2'(x)|^{1/2} \, |\gamma_1'(\gamma_2 y)|^{1/2} \, |\gamma_2'(y)|^{1/2} \, |x-y| = |(\gamma_1\gamma_2)'(x)|^{1/2} \, |(\gamma_1\gamma_2)'(x)|^{1/2} \, |x-y| \quad .$$

As in the complex case we wish to use (15) to prove the invariance of the cross-ratio, but this time there is only an <u>absolute cross-ratio</u>

$$(16) \qquad |a,b,c,d| = \frac{|a-c|}{|a-d|} : \frac{|b-c|}{|b-d|}$$

which is obviously invariant in the sense that

$$(17) \qquad |\gamma a \, , \gamma b \, , \gamma c \, , \gamma d| = |a \, , b \, , c \, , d| \quad .$$

We can use the invariance to prove that circles are mapped on circles. Indeed, it is a classical theorem in $\mathbb{R}^2$ and $\mathbb{R}^3$, extendable to $\mathbb{R}^n$,

---

*) $xy = x_1 y_1 + \ldots + x_n y_n$

that a, b, c, d lie on a circle in cyclic order if and only if $|a-b||c-d| + |b-c||d-a| = |a-c||b-d|$. This condition can be written in the invariant form $|a, d, b, c| + |c, d, b, a| = 1$.

As a second application of (17) we prove the following important lemma.

Lemma 1. If γ leaves ∞ fixed, then γ is a similarity.

Proof. If $\gamma\infty = \infty$ then $\gamma x - \gamma 0$ leaves 0 and ∞ fixed and it is sufficient to prove that $\gamma 0 = 0$, $\gamma\infty = \infty$ implies $\gamma x = mx$ with a constant conformal matrix m . In other words, we need to show that $\gamma'(x)$ is constant.

First,

$$|\gamma x, \gamma y, 0, \infty| = |x, y, 0, \infty|$$

or

$$\frac{|\gamma x|}{|\gamma y|} = \frac{|x|}{|y|} \quad , \quad \frac{|\gamma x|}{|x|} = \lambda = \text{const.}$$

Next $|\gamma x, 0, \gamma y, \infty| = |x, 0, y, \infty|$ gives $\frac{|\gamma x - \gamma y|}{|\gamma y|} = \frac{|x-y|}{|y|}$ or

$$|\gamma x - \gamma y|^2 = \lambda^2 |x-y|^2$$

which implies $(\gamma x, \gamma y) = \lambda^2 (x,y)$.

It follows that

$$|\gamma(x+y) - \gamma x - \gamma y|^2 = \lambda^2 |(x+y) - x - y|^2 = 0$$

so that

$$\gamma(x+y) = \gamma x + \gamma y$$

$$\gamma'(x+y) = \gamma'(x) = \text{const.}$$

Corollary. If $a \neq b$ are both finite, then the most general $\gamma \in \hat{M}(\mathbb{R}^n)$ with $\gamma a = 0$, $\gamma b = \infty$ is of the form

$$\gamma x = m[(x-b)^* - (a-b)^*] \tag{18}$$

where m is a constant conformal matrix.

In fact, it is clear that $(x-b)^* - (a-b)^*$ takes a into 0 and b into ∞.

2.4. We denote by $\hat{M}(B^n)$ and $M(B^n)$ the subgroups which keep $B^n = \{x ; |x| < 1\}$ invariant. Because Möbius transformations are bijective the unit sphere $S^{n-1} = \{x ; |x| = 1\}$ and the exterior of B^n are also invariant.

Lemma 2. If $\gamma \in \hat{M}(B^n)$ and $\gamma 0 = 0$, then γ is a rotation (i.e. $\gamma x = kx$, $k \in O(n)$).

Proof. If $\gamma\infty = \infty$ we know by Lemma 1 that $\gamma = mx$ and because $|mx| = 1$ for $|x| = 1$ it follows that $m = k \in O(n)$.

Assume now that $\gamma^{-1}\infty = b \neq \infty$. Then (18) implies

$$|(x-b)^* + b^*| = \text{const.}$$

for $|x| = 1$. But

$$|(x-b)^* + b^*)| = \frac{|x|}{|x-b|\,|b|} .$$

Hence $|x-b| = \text{const.}$ for $|x| = 1$, and that is impossible since $b \neq 0$.

2.5. We shall now determine the most general $\gamma \in \hat{M}(B^n)$. This is done almost exactly as in 1.11 for $n = 2$ except that we do not have the complex notation.

We begin by proving a non-trivial identity:

Lemma 3.

(19) $$(x-y^*)^* + y = |y|^2(I-2Q(y))(x^*-y)^*$$

Proof. As it stands (19) does not make sense for $y = 0$, but it is obvious that both sides tend to 0 for $y \to 0$. We assume now that $y \neq 0$ and write

(20)
$$Ax = (x-y^*)^*$$
$$Bx = |y|^2(I-2Q(y))(x^*-y)^* - y .$$

In other words, we treat y as a constant and have to show that $Ax = Bx$.

It is immediate that

$$Ay^* = By^* = \infty$$
$$A\infty = B\infty = 0 .$$

Therefore, 0 and ∞ are fixed points of AB^{-1} and by Lemma 1 $(AB^{-1})'$ is constant. But then $A'(x)\,B'(x)^{-1}$ is also constant, for

$$(AB^{-1})'(x) = A'(B^{-1}x) \cdot (B^{-1})'(x)$$

$$= A'(B^{-1}x) \cdot B'(B^{-1}x)^{-1} =$$

$$(A' B'^{-1}) \circ Bx = \text{const.}$$

From (20) and (14)

$$A'(x) = \frac{I - 2Q(x - y^*)}{|x - y^*|^2}$$

(21)

$$B'(x) = \frac{|y|^2(I - 2Q(y))(I - 2Q(x^* - y))(I - 2Q(x))}{|x^* - y|^2|x|^2}$$

and for $x = y$ we find

$$A'(y) = B'(y) = \frac{(I - 2Q(y))|y|^2}{(1 - |y|^2)^2} .$$

Hence $A'(x) = B'(x)$ for all x, and since $A0 = B0 = -y$ it follows that $Ax = Bx$.

Direct comparison of $A'(x)$ and $B'(x)$ as given by (21) yields

(22) $$(I - 2Q(y))(I - 2Q(x - y^*)) = (I - 2Q(x^* - y))(I - 2Q(x)) .$$

This is an important identity.

2.6. We repeat the construction that was given in 1.11 and refer to the same figure. Given $a \in B^n$ $(a \neq 0)$ we construct a^* and the sphere $S^{n-1}(a^*, (|a^*|^2 - 1)^{1/2})$ with center a^* and radius $\sqrt{1 - |a|^2} / |a|$ which intersects S^{n-1} orthogonally. The reflection in this sphere is given by

(23) $$\sigma_a x = a^* + (|a^*|^2 - 1)(x - a^*)^* \quad .$$

We let σ_a be followed by reflection in the plane through the origin perpendicular to a . It is easy to see that this second reflection amounts to multiplication with the matrix $I - 2Q(a)$. In fact

$$y' = (I - 2Q(a))y = y - \frac{2(ya)a}{|a|^2}$$

and Fig. 5 shows the location of y' .

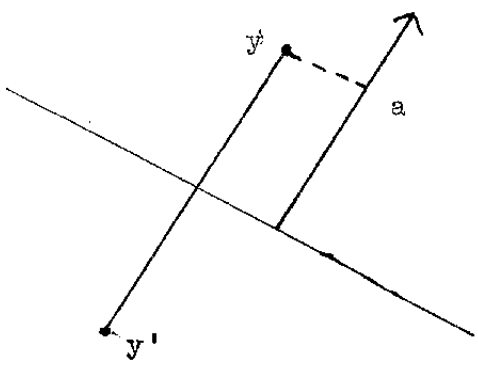


Fig. 5

We now define the canonical mapping

(24) $$T_a x = (I - 2Q(a))\,\sigma_a x$$

and conclude that the most general $\gamma \in \hat{M}(\mathbb{B}^n)$ with $\gamma a = 0$ is of the form kT_a with $k \in O(n)$.

The explicit expression for $T_a x$ becomes much simpler if we use the identity (19). It is clear that $(I - 2Q(a))\mathbf{a}^* = -a^*$ and thus we obtain

$$T_a x = -a^* + (|a^*|^2 - 1)(I - 2Q(a))(x - a^*)^* \quad .$$

Replace y by a in (19) and multiply by $I - 2Q(a)$. We obtain

$$(I - 2Q(a))(x - a^*)^* = a + |a|^2(x^* - a)^*$$

and

$$T_a x = -a^* + (|a^*|^2 - 1)a + (1 - |a|^2)(x^* - a)^* \quad .$$

Finally,

$$T_a x = -a + (1 - |a|^2)(x^* - a)^* \quad . \tag{25}$$

This is as simple as one can wish, but if one wants to avoid the x^* notation it can easily be brought to the form

$$T_a x = \frac{(1 - |a|^2)(x - a) - |x - a|^2 a}{[x,a]^2} \tag{26}$$

where $[x,a] = |x||x^* - a| = |a||x - a^*|$ and

$$[x,a]^2 = 1 + |x|^2 |a|^2 - 2xa \quad .$$

2.7. We collect the various expressions for $T_y x$ that we have derived.

a) $$T_y x = (I - 2Q(y))(y^* + (|y^*|^2 - 1)(x - y^*)^*)$$

b) $$T_y x = -y + (1 - |y|^2)(x^* - y)^*$$

c) $$T_y x = \frac{(1 - |y|^2)x - (1 - 2xy + |y|^2)y}{[x,y]^2}$$

d) $$T_y x = \frac{(1 - |y|^2)(x - y) - |x - y|^2 y}{[x,y]^2}$$

Recall that $[x,y] = |x||y - x^*| = |y||x - y^*|$.

a) and b) are easy to differentiate and yield

(27)
$$T_y'(x) = \frac{1-|y|^2}{[x,y]^2}(I-2Q(y))(I-2Q(x-y^*))$$
$$= \frac{1-|y|^2}{[x,y]^2}(I-2Q(x^*-y))(I-2Q(x)) \ .$$

We have already remarked on the identity of the two matrix products. We choose to introduce the notation

$$\Delta(x,y) = (I-2Q(y))(I-2Q(x-y^*))$$

which leads to

$$\Delta(x,y) = \Delta(y,x)^T \quad .$$

Note that this can also be written

(28)
$$\Delta(x,y)\,\Delta(y,x) = I$$

It is helpful to view the formula

(29)
$$T_y'(x) = \frac{1-|y|^2}{[x,y]^2}\ \Delta(x,y)$$

as a representation in polar coordinates:

(30)
$$|T_y'(x)| = \frac{1-|y|^2}{[x,y]^2}$$

is the "absolute" value and $\Delta(x,y)$ is the "argument" of $T_y'(x)$.

We make a note of the special values

(31) $$T_y'(0) = 1 - |y|^2 \ , \quad T_y'(y) = \frac{1}{1-|y|^2} \ .$$

Also, $T_y 0 = -y$, and hence $T_{-y} = T_y^{-1}$.

Direct computation of $|T_y x|$ from any of the formulas a) - d) is very laborious. Fortunately, we can use the difference formula to obtain

$$|T_y x| = |T_y x - T_y y| = |T_y'(x)|^{1/2} |T_y'(y)|^{1/2} |x-y|$$

(32) $$|T_y x| = \frac{|x-y|}{[x,y]} \ .$$

Now a short computation leads to

(33) $$1 - |T_y x|^2 = \frac{(1-|x|^2)(1-|y|^2)}{[x,y]^2} \ .$$

Together with (29) we have thus

(34) $$\frac{|T_y'(x)|}{1-|T_y x|^2} = \frac{1}{1-|x|^2} \ .$$

We have thereby proved the invariance of the Poincaré metric

(35) $$ds = \frac{2|dx|}{1-|x|^2} \ .$$

It is again seen that every orthogonal circle is a geodesic. The hyperbolic distance from the origin is

(36) $$d(0,x) = \log \frac{1+|x|}{1-|x|}$$

and more generally

(37) $$d(x,y) = \log |y, x, \xi, \eta|$$

where ξ, η are the end points of the directed geodesic from x to y.

2.8. There is a close relation between $T_y x$ and $T_x y$. To derive it we shall first prove:

LEMMA. For any $\gamma \in \hat{M}(B^n)$

(38) $$T_{\gamma y}(\gamma x) = \frac{\gamma'(y)}{|\gamma'(y)|} T_y x .$$

<u>Proof</u>: Let Lx and Rx be the left and right hand side of (38). Clearly, $Ly = Ry = 0$ and hence $LR^{-1}(0) = 0$. But

$$L'(y) = \frac{\gamma'(y)}{1 - |\gamma y|^2} = R'(y)$$

where the left hand equation is obvious, whereas the right hand equation follows by application of (34). Thus

$$(LR^{-1})'(0) = L'(y)R'(y)^{-1} = 1$$

which proves that $LR^{-1} = I$. Apply the Lemma with $\gamma = T_x$. The left hand side is

$$T_{T_x y}(0) = -T_x y .$$

and one obtains

$$T_x y = -\frac{T_x'(y)}{|T_x'(y)|} T_y x$$

or

(39)
$$T_x y = -\Delta(y,x) T_y x$$
$$T_y x = -\Delta(x,y) T_x y$$

by (29), (30).

<u>Remark</u>. T_y is not defined for $|y| = 1$ but by formula b) we should write $T_y x = -y$, by continuity. According to (39) it follows that

(40)
$$T_x y = \Delta(y,x) y \quad \text{for} \quad |y| = 1$$
$$T_y x = \Delta(x,y) x \quad \text{for} \quad |x| = 1$$

One more formula. Differentiation of (38) with respect to x yields

$$T'_{\gamma y}(\gamma x)\, \gamma'(x) = \frac{\gamma'(y)}{|\gamma'(y)|}\; T'_y(x) \quad .$$

On equating the arguments in (29) one obtains

(41)
$$\Delta(\gamma x\,, \gamma y)\, \frac{\gamma'(x)}{|\gamma'(x)|} = \frac{\gamma'(y)}{|\gamma'(y)|}\, \Delta(x,y) \quad .$$

2.9. Although the formulas for the selfmappings of B^n are simple enough it is sometimes more advantageous to concentrate on the geometric picture.

Recall that every $\gamma \in M(B^n)$, $\gamma \neq I$, has a canonical representation

(42)
$$\gamma = k\, T_a$$

where $a = \gamma^{-1} 0$ and $k \in SO(n)$. Since $|k| = 1$ we have

(43)
$$|\gamma'(x)| = |T'_a(x)| = \frac{1 - |a|^2}{[x,a]^2} \quad .$$

The set $\{x ; |\gamma'(x)| = 1\}$ is nothing else than the orthogonal sphere with center a^*. Nothing prevents us from considering the full sphere rather than only the part inside B^n. It is called the <u>isometric sphere</u> of γ, and we denote it by $K(\gamma)$. Moreover, we denote the inside of $K(\gamma)$ by $I(\gamma)$ and the exterior by $E(\gamma)$. Observe that the identity has no isometric sphere.

The chain rule applied to $\gamma[\gamma^{-1}x] = x$ yields

$$|\gamma'[\gamma^{-1}x]| \; |(\gamma^{-1})'(x)| = 1$$

and of course just as well

$$|(\gamma^{-1})'(\gamma x)| \; |\gamma'(x)| = 1 \; .$$

From this we draw the following conclusions:

γ maps $I(\gamma)$ on $E(\gamma^{-1})$ and $E(\gamma)$ on $I(\gamma^{-1})$

γ^{-1} maps $I(\gamma^{-1})$ on $E(\gamma)$ and $E(\gamma^{-1})$ on $I(\gamma)$.

Of course this does not determine γ completely, but only up to rotations about the common axis of the unit sphere and the isometric sphere.

Furthermore, the restriction $\gamma : K(\gamma) \to K(\gamma^{-1})$ is an isometry and hence a congruence mapping both with respect to the euclidean and non-euclidean geometry.

III. Hyperbolic geometry

3.1. This is a short chapter devoted to two separate tasks:

1° Derive some formulas of hyperbolic trigonometry.

2° Show how to represent Möbius groups as matrix groups.

3.2. Let ABC be a non-euclidean triangle in B^n. We denote the angles by A, B, C and the n.e. lengths of the opposite sides by a, b, c. We shall prove:

I. The hyperbolic law of cosines:

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C \tag{1}$$

II. The hyperbolic law of sines

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c} . \tag{2}$$

3.3. Proof of I. We may assume that $C = 0$, that a falls along the positive x_1-axis and that b lies in the x_1x_2-plane. It is thus sufficient to consider the case $n = 2$ and we may use the complex notation.

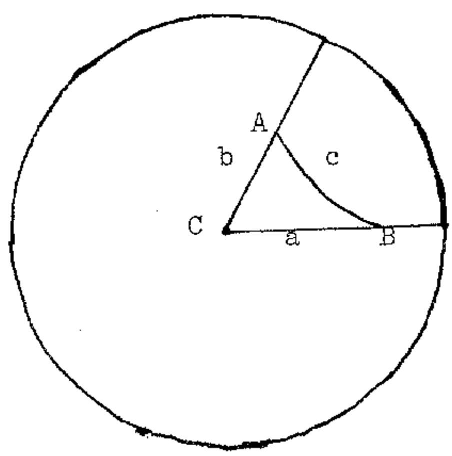


Fig. 6

The points B and A are at euclidean distance $\tanh\frac{a}{2}$ and $\tanh\frac{b}{2}$ from O. Hence B and A are represented by $\tanh\frac{a}{2}$ and $\tanh\frac{b}{2}\ e^{iC}$. The point-pair invariant is

$$\delta(A,B) = \frac{\left|\tanh\frac{a}{2} - \tanh\frac{b}{2}\ e^{iC}\right|}{\left|1-\tanh\frac{a}{2}\ \tanh\frac{b}{2}\ e^{-iC}\right|} = \tanh\frac{c}{2} \quad . \tag{3}$$

Just as in ordinary trigonometry all the hyperbolic functions can be expressed rationally through $\tanh\frac{x}{2}$. The formulas are

$$\cosh x = \frac{1+\tanh^2\frac{x}{2}}{1-\tanh^2\frac{x}{2}}$$

$$\sinh x = \frac{2\tanh\frac{x}{2}}{1-\tanh^2\frac{x}{2}} \quad .$$

From (3)

$$\cosh c = \frac{(1+th^2\frac{a}{2})(1+th^2\frac{b}{2}) - 4th\frac{a}{2}\ th\frac{b}{2}\cos C}{(1-th^2\frac{a}{2})(1-th^2\frac{b}{2})}$$

$$= \cosh a \cosh b - \sinh a \sinh b \cos C \ .$$

3.4. Proof of II. From (1)

$$\cos C = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}$$

$$\sin^2 C = \frac{(ch^2 a - 1)(ch^2 b - 1) - (ch\,a\ ch\,b - ch\,c)^2}{sh^2 a\ sh^2 b}$$

$$\frac{\sin^2 C}{sh^2 c} = \frac{1 - ch^2 a - ch^2 b - ch^2 c + 2ch\,a\ ch\,b\ ch\,c}{sh^2 a\ sh^2 b\ sh^2 c}$$

The symmetry proves (II).

3.5. In this section we want to show that $M(R^{n-1})$ is isomorphic with $M(B^n)$. For $n = 2$ this was already done in Ch.I by observing (tacitly) that $M(R^1)$ and $M(H^2)$ are both equal to $PSL_2(R)$, while on the other hand $M(H^2)$ and $M(B^2)$ are isomorphic because there is a conformal mapping from H^2 to B^2.

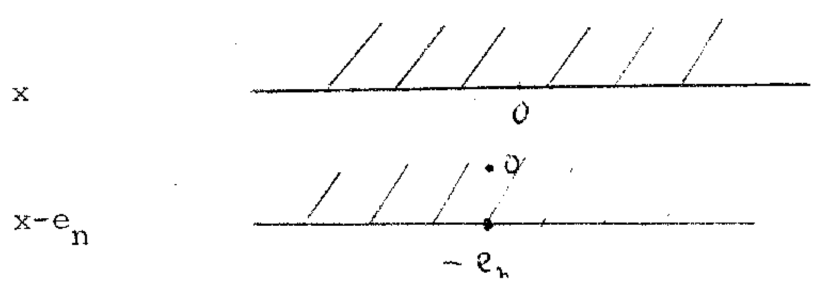
In the general case we begin by extending an arbitrary $\gamma \varepsilon M(R^{n-1})$ to a mapping in $M(H^n)$. For this purpose we identify R^{n-1} with the subspace $\{x_n=0\}$ of R^n. Recall (Corollary 2.5) that any $\gamma \varepsilon M(R^{n-1})$ with $\gamma a = 0$, $\gamma b = \infty$, $b \neq \infty$ has a unique representation of the form

$$\gamma x = m[(x-b)^* - (a-b)^*] \tag{4}$$

where $m = \lambda k$, $\lambda > 0$, $k \varepsilon O(n-1)$. If we replace m by $\tilde{m}$ we obtain a mapping of R^n whose restriction to H^n is an extension of γ. (In case $b = \infty$ (4) should be replaced by $\gamma x = m(x-a)$). This construction shows that $M(R^{n-1})$ is isomorphic to $M(H^n)$.

It remains to prove that $M(H^n)$ is isomorphic to $M(B^n)$. For this it suffices to find a standard Möbius transformation $\sigma: H^n \to B^n$ which we choose so that $x = (0, e_n, \infty)$ correspond to $y = \sigma x = (-e_n, 0, e_n)$ where e_n is the last coordinate vector. The restriction of σ to R^{n-1} is the usual stereographic projection.

The following pictures illustrate the choice better than words:



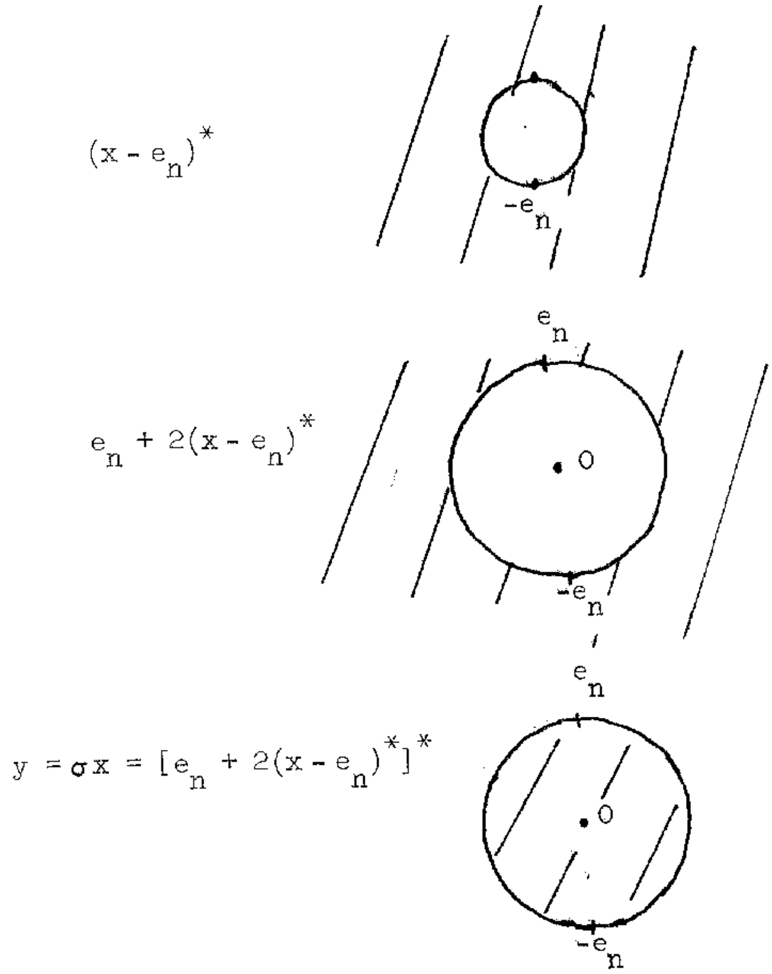


Fig. 7

The correspondence is thus given by

$$(5)\qquad y = (e_n + 2(x - e_n)^*)^*$$

$$x = e_n + 2(y^* - e_n)^*$$

In terms of coordinates one finds

$$(6)\qquad \begin{aligned} y_i^* &= \frac{2x_i}{|x - e_n|^2} \qquad (i = 1, \ldots, n-1) \\ y_n^* &= \frac{|x|^2 - 1}{|x - e_n|^2} \end{aligned}$$

and

$$x_i = \frac{2y_i}{|y-e_n|^2} \qquad (i = 1, \dots, n-1)$$

(7)

$$x_n = \frac{1-|y|^2}{|y-e_n|^2}$$

When $x_n = 0$ one verifies that $|y|^2 = 1$, $y^* = y$, and (6) reduces to

(8) $$y_i = \frac{2x_i}{|x|^2+1} \quad , \quad y_n = \frac{|x|^2-1}{|x|^2+1}$$

and for $|y| = 1$

(9) $$x_i = \frac{y_i}{1-y_n} \quad , \quad x_n = 0 \; .$$

One recognizes these formulas.

3.6. The stereographic projection (8) maps $B^{n-1} = \{x_1, \dots, x_{n-1}, 0\}$ with $x_1^2 + \dots + x_{n-1}^2 < 1$ on the lower hemisphere of S^{n-1}, i.e. $\{y \in \mathbb{R}^n ; |y| = 1, y_n < 0\}$. Let it be followed by the projection $(y_1, \dots, y_n) \mapsto (y_1, \dots, y_{n-1}, 0)$. The result is a mapping of B_{n-1} on itself defined by

(10) $$y = \frac{2x}{|x|^2+1} \quad ,$$

$x \in R^{n-1}$, $|x| < 1$. This is of course not a Möbius transformation, for the mapping $(y_1, \dots, y_n) \mapsto (y_1, \dots, y_{n-1}, 0)$ is not conformal.

What is the nature of the mapping (10)? The stereographic projection is conformal and leaves the sphere $S^{n-2} = \{x_n = 0 , |x| = 1\}$ fixed. Therefore it maps orthogonal circles of B^{n-1}, i.e. circles orthogonal to S^{n-2} on circles on the hemisphere, which are also orthogonal to S^{n-2}.

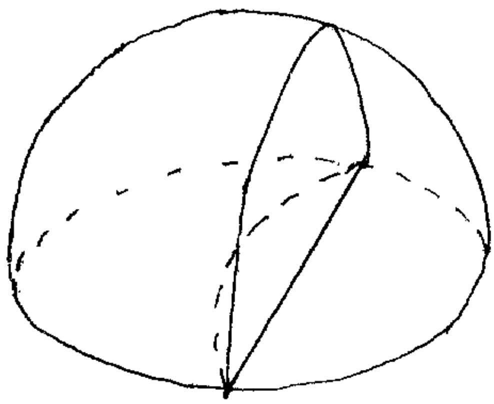


Fig. 8

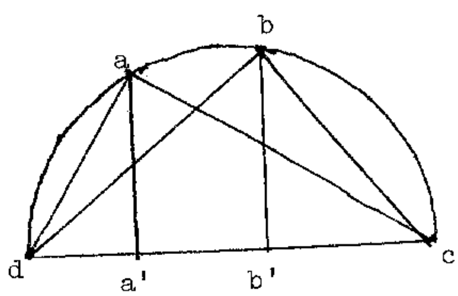
But an orthogonal circle on the hemisphere lies in a plane parallel to e_n and its projection is a straight line segment. In other words, (10) maps every geodesic in B^{n-1} on the segment between its end points. This is also obvious by a simple computation. The equation of an orthogonal circle is of the form

$$|x - a|^2 = |a|^2 - 1 \qquad (|a| > 1)$$

or

$$|x|^2 + 1 = 2x\,a$$

and thus (10) is equivalent to $ay = 1$, the equation of a straight line. This can be used to construct the Klein model of hyperbolic space. In this model the noneuclidean lines are the line segments in B^{n-1}. The distance of two points y', y'' is defined as $d(x', x'')$ between the corresponding points in the Poincaré model.



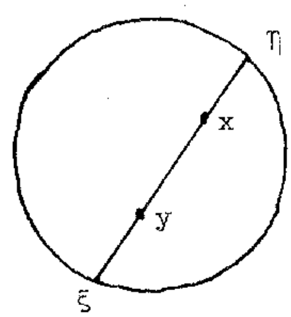


Fig. 9

A glance at Fig.9 shows that $|a',b',c,d|=|a,b,c,d|^2$. Thus the n.e. distance between x,y in the Klein model is $\frac{1}{2}\log|x,y,\xi,\eta|$.

3.7. We pass now to the matrix representation. Recall that the group $O(n,1)$ acts on R^{n+1} and leaves the quadratic and bilinear forms

$$<x,x> = x_o^2 - x_1^2 - \dots - x_n^2$$

$$<x,y> = x_o y_o - x_1 y_1 - \dots - x_n y_n$$

invariant. In matrix language $A \varepsilon O(n,1)$ if

(11) $$A^T \begin{pmatrix} 1 & 0 \\ 0 & -1_n \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & -1_n \end{pmatrix} .$$

It follows from (11) that $(\det A)^2 = 1$. The subgroup with $\det A = 1$ is denoted by $SO(n,1)$.

Geometrically, the lightcone $\{<x,x>=0\}$ and its interior $\{<x,x> >0\}$ are invariant under all $A \varepsilon O(n,1)$. The same is true of the hyperboloid $\{<x,x> = 1\}$ with two mantles, one with $x_o>0$ and one with $x_o<0$.

Theorem 1. The groups $M(R^{n-1})$ and $SO(n,1)$ are isomorphic.

Proof: The upper hyperboloid

$$U = \{x \varepsilon R^{n+1} |\ <x,x>= 1,\ x_o>0\}$$

and the plane section

$$V = \{x' \varepsilon R^{n+1} \mid \langle x', x' \rangle > 0,\ x'_o = 1\}$$

are in bijective correspondence through the central projection $x' = x/x_o$. Fig.20 shows $x, y \varepsilon U$ and the intersections ξ, η of the line through x and y with the lightcone together with the projections x', y' and ξ', η'.

By elementary geometry the cross-ratios $|x, y, \xi, \eta|$ and $|x', y', \xi', \eta'|$ are equal. To determine their common value we observe that ξ and η are both of the form $\frac{x+ty}{1+t}$ with real t satisfying $\langle x+ty, x+ty \rangle = 0$. This leads to the equation

$$\langle x,x \rangle + 2t\langle x,y \rangle + t^2 \langle y,y \rangle = 0 \tag{12}$$

which consequently has two real roots t_1, t_2 corresponding to ξ, η. The cross-ratio turns out to be t_1/t_2 which is positive.

The section V can be identified with the Klein model of B^n. The straight line segments are the geodesics and $d(x', y') = \frac{1}{2}|\log t_1/t_2|$. By conjugation with the central projection each $A \varepsilon SO(n,1)$ determines a bijective mapping A' of V on itself. Because the coefficients of (12) are invariant the same is true of the cross-ratios and hence also of the n.e. distances $d(x', y')$. This means that each A' is a n.e. motion, and it is obviously sense-preserving.

Conversely, if A' is a n.e. motion it can be lifted to a mapping A of U which carries collinear points (x, y, ξ, η) into collinear points $(Ax, Ay, A\xi, A\eta)$ with the same cross-ratio. Because of this the coefficients of (12) will remain proportional to each other, and since $\langle Ax,Ax\rangle = \langle x,x\rangle = 1$ it follows that $\langle Ax,Ay\rangle=\langle x,y\rangle$ as well.

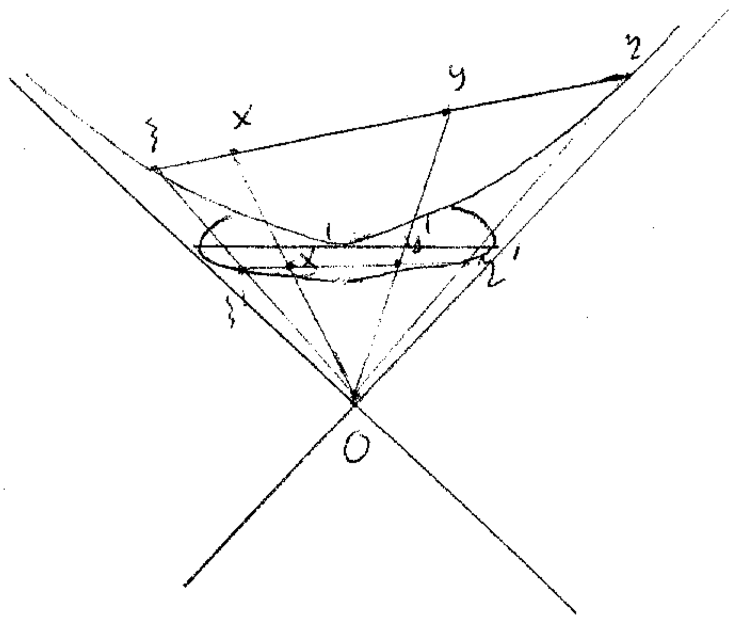


Fig.10

The mapping A can be extended to all x inside the double cone by defining $Ax = \langle x,x\rangle^{\frac{1}{2}} A(\langle x,x\rangle^{-\frac{1}{2}} x)$ in the upper cone and subsequently $Ax = -A(-x)$ in the lower cone. One verifies that this extension still satisfies $\langle Ax,Ay\rangle = \langle x,y\rangle$ as well as

$$\langle A(\alpha x)-\alpha Ax, A(\alpha x)-\alpha Ax\rangle = 0$$

$$\langle A(x+y)-Ax-Ay, A(x+y)-Ax-Ay\rangle = 0 \ .$$

It follows that $A(\alpha x) = \alpha Ax$ and $A(x+y) = Ax + Ay$. In other words, A is linear, at least inside the double cone. Because every $x \in R^{n+1}$ is a linear combination of vectors in the cone it is clear that A extends to a linear mapping of R^{n+1} on itself, and because $\langle Ax,Ay\rangle = \langle x,y\rangle$ it is represented by a matrix $A \in O(n,1)$; since the upper cone maps on itself it is even in $SO(n,1)$.

It is evident from the construction that the correspondence between A and A' is bijective and product preserving. We have thus proved, so far, that $M_n(B)$ is isomorphic to $SO(n,1)$. In (3.5) we showed that $M_n(B)$ is isomorphic to $M(R^{n-1})$, and this completes the proof of Theorem 1.

The passage from $M(R^{n-1})$ to $O(n,1)$ means a loss of two dimensions. For instance, the classical Möbius group $M(C)$ is represented by the Lorentz group of 4×4 matrices. For practical purposes this is much too complicated, but for theoretical reasons it is important to know that every $M(R^n)$ can be written as a matrix group.

IV. Elements of differential geometry.

4.1. I want to take a very unsophisticated look at differential geometry and regard it simply as a set of rules for changing coordinates.

A differentiable n-manifold is a set covered by coordinate patches such that overlapping parts are connected by coordinate changes of the form

$$(1) \qquad \bar{x}_i = \bar{x}_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \ .$$

The change of coordinates shall be reversible, and this means that the Jacobian matrix

$$\left\| \frac{\partial \bar{x}_i}{\partial x_j} \right\|$$

is non-singular. The coordinate changes will always be C^∞.

The typical contravariant vector is a differential

$$(2) \qquad d\bar{x}^i = \sum_j \frac{\partial \bar{x}_i}{\partial x_j} \, dx^j \ .$$

Observe that we are using upper indices for the differentials, but not for the variables.

An arbitrary contravariant vector is a system of vector-valued functions, one for each coordinate system, connected by the same relation as in (2), namely

$$(3) \qquad \bar{a}^i = \frac{\partial \bar{x}_i}{\partial x_j} \, a^j$$

where we are using the summation convention, as always from now on.

Similarly, a covariant vector is typified by a derivative (or gradient)

(4) $$\frac{\partial f}{\partial \bar{x}_i} = \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial \bar{x}_i} .$$

We use lower indices for covariant vectors, and the general rule is

(5) $$\bar{b}_i = b_j \frac{\partial x_j}{\partial \bar{x}_i} .$$

The idea carries over to <u>tensors</u> of higher order. Here are the rules for contravariant, covariant and mixed tensors of second order:

$$\bar{a}^{ij} = \frac{\partial \bar{x}_i}{\partial x_h} \frac{\partial \bar{x}_j}{\partial x_k} a^{hk}$$

$$\bar{a}_{ij} = a_{hk} \frac{\partial x_h}{\partial \bar{x}_i} \frac{\partial x_k}{\partial \bar{x}_j}$$

$$\bar{a}^i_j = \frac{\partial \bar{x}_i}{\partial x_h} a^h_k \frac{\partial x_k}{\partial \bar{x}_j} .$$

It makes sense to speak of symmetric and skew-symmetric tensors, for if $a^{hk} = a^{kh}$, then $\bar{a}^{ij} = \bar{a}^{ji}$.

The Kronecker δ^i_j is a mixed tensor with the same components in all coordinate systems. In fact,

$$\bar{\delta}^i_j = \frac{\partial \bar{x}_i}{\partial x_h} \delta^h_k \frac{\partial x_k}{\partial \bar{x}_j} = \frac{\partial \bar{x}_i}{\partial x_h} \frac{\partial x_h}{\partial \bar{x}_j} = \delta^i_j$$

because the matrices $\frac{\partial \bar{x}_i}{\partial x_j}$ and $\frac{\partial x_j}{\partial \bar{x}_i}$ are inverses.

4.2. An important way to form new tensors is by <u>contraction</u>. Example: if a^k_{ij} is a twice covariant and once contravariant tensor, then

$$a_i = a^j_{ij} \qquad \text{(summation!)}$$

is a covariant vector, for

$$\bar{a}^{j}_{ij} = \frac{\partial \bar{x}_j}{\partial x_h} a^h_{kh} \frac{\partial x_k}{\partial \bar{x}_i} \frac{\partial x_h}{\partial \bar{x}_j} = a^h_{kh} \frac{\partial x_k}{\partial \bar{x}_i} \quad .$$

4.3. A differentiable manifold becomes a Riemannian space by the choice of a metric tensor whose matrix is positive definite. It serves several purposes:

1. It defines arc length and volume.
2. It serves to push indices up and down.
3. It defines covariant differentiation.
4. It defines parallel displacement.

Arc length is defined by setting

$$ds = (g_{ij}\, dx^i\, dx^j)^{\frac{1}{2}}$$

and defining the length of an arc γ by

$$\ell(\gamma) = \int_{\gamma} ds \quad .$$

Because g_{ij} is covariant of order two and the differentials dx^i are contravariant, ds has a meaning independent of the local coordinates.

The determinant of g_{ij} is denoted by g. $\sqrt{g}$ transforms like a <u>density</u> in the sense that

$$\sqrt{\bar{g}} = (\det \frac{\partial(x_1, \dots, x_n)}{\partial(\bar{x}_1, \dots, \bar{x}_n)}) \sqrt{g} \quad .$$

This follows by the determinant multiplication rule. This defines an invariant volume by

$$V = \int \sqrt{g}\ \ dx^1 \dots dx^n$$

(we consider only sense preserving changes of coordinates.)

4.4. The inverse of the matrix g_{ij} is denoted by g^{ij}. It is clearly a contravariant tensor of order two. Multiplication with g_{ij} or g^{ij} followed by contraction pushes indices up and down. For instance

$$g_{ij}\, a^j = a_i$$

$$g^{ij}\, b_j = b^i \quad .$$

The mixed components of g_{ij} are

$$g^i_j = g^{ik} g_{kj} = \delta^i_j \; .$$

4.5. The gradient $\frac{\partial f}{\partial x_i}$ of a scalar function is a covariant vector. For other tensors differentiation of the components does not by itself lead to a new tensor. It has to be replaced by covariant differentiation, which we proceed to define.

The Christoffel symbols are

$$\text{(6)} \qquad \Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right)$$

and

$$\text{(7)} \qquad \Gamma^h_{ij} = g^{hk}\, \Gamma_{ij,k} \quad .$$

In order to find out how they transform we start from

$$g_{ij} = \overline{g}_{ab}\, \frac{\partial \overline{x}_a}{\partial x_i}\, \frac{\partial \overline{x}_b}{\partial x_j}$$

and differentiate using the chain rule

$$\frac{\partial g_{ij}}{\partial x_k} = \frac{\partial \bar{g}_{ab}}{\partial \bar{x}_c} \frac{\partial \bar{x}_a}{\partial x_i} \frac{\partial \bar{x}_b}{\partial x_j} \frac{\partial \bar{x}_c}{\partial x_k}$$

$$+ \bar{g}_{ab} \left(\frac{\partial^2 \bar{x}_a}{\partial x_i \partial x_k} \frac{\partial \bar{x}_b}{\partial x_j} + \frac{\partial^2 \bar{x}_a}{\partial x_j \partial x_k} \frac{\partial \bar{x}_b}{\partial x_i} \right) ,$$

where we have interchanged a , b in the last term.

Permutation of the subscripts leads to

(8) $$\Gamma_{ij,k} = \overline{\Gamma}_{ab,c} \frac{\partial \bar{x}_a}{\partial x_i} \frac{\partial \bar{x}_b}{\partial x_j} \frac{\partial \bar{x}_c}{\partial x_k} + \bar{g}_{ab} \frac{\partial^2 \bar{x}_a}{\partial x_i \partial x_j} \frac{\partial \bar{x}_b}{\partial x_k} .$$

The extra term on the right shows that $\Gamma_{ij,k}$ is not a tensor.

Multiply both sides of (8) by

$$g^{hk} = \frac{\partial x_h}{\partial \bar{x}_d} \frac{\partial x_k}{\partial \bar{x}_c} \cdot \bar{g}^{cd} .$$

This gives

(9) $$\Gamma^h_{ij} = \overline{\Gamma}^d_{ab} \frac{\partial \bar{x}_a}{\partial x_i} \frac{\partial \bar{x}_b}{\partial x_j} \frac{\partial x_h}{\partial \bar{x}_d} + \frac{\partial^2 \bar{x}_d}{\partial x_i \partial x_j} \frac{\partial x_h}{\partial \bar{x}_d} .$$

Again, there is an extra term.

We use (9) to find the mixed derivatives $\frac{\partial^2 \bar{x}_k}{\partial x_i \partial x_j}$. For that purpose multiply (9) by $\frac{\partial \bar{x}_k}{\partial x_h}$ to obtain

(10) $$\frac{\partial^2 \bar{x}_k}{\partial x_i \partial x_j} = \Gamma^h_{ij} \frac{\partial \bar{x}_k}{\partial x_h} - \overline{\Gamma}^k_{ab} \frac{\partial \bar{x}_a}{\partial x_i} \frac{\partial \bar{x}_b}{\partial x_j} .$$

4.6. If v^j is a contravariant vector we shall show that

(11) $$\nabla_i v^j = \frac{\partial v^j}{\partial x_i} + \Gamma^j_{ik} v^k$$

is a mixed tensor. From

$$\bar{v}^j = \frac{\partial \bar{x}_j}{\partial x_h} v^h$$

we obtain

$$\frac{\partial \bar{v}^j}{\partial \bar{x}_i} = \frac{\partial v^h}{\partial x_k} \frac{\partial x_k}{\partial \bar{x}_i} \frac{\partial \bar{x}_j}{\partial x_h} + \frac{\partial^2 \bar{x}_j}{\partial x_h \partial x_k} \frac{\partial x_k}{\partial \bar{x}_i} v^h .$$

Use (10) to eliminate the cross-derivative:

$$\frac{\partial \bar{v}^j}{\partial \bar{x}_i} = \frac{\partial v^h}{\partial x_k} \frac{\partial x_k}{\partial \bar{x}_i} \frac{\partial \bar{x}_j}{\partial x_h} + \left(\Gamma^a_{hk} \frac{\partial \bar{x}_j}{\partial x_a} - \overline{\Gamma^j_{ab}} \frac{\partial \bar{x}_a}{\partial x_h} \frac{\partial \bar{x}_b}{\partial x_k} \right) \frac{\partial x_k}{\partial \bar{x}_i} v^h .$$

The very last term simplifies to $\overline{\Gamma^j_{ai}}\, \bar{v}^a$. We pull it over to the left hand side which becomes

$$\frac{\partial \bar{v}^j}{\partial \bar{x}_i} + \overline{\Gamma^j_{ai}}\, \bar{v}^a = \nabla_i \bar{v}^j .$$

On the right we are left with

$$\frac{\partial v^h}{\partial x_k} \frac{\partial x_k}{\partial \bar{x}_i} \frac{\partial \bar{x}_j}{\partial x_h} + \Gamma^a_{hk} \frac{\partial \bar{x}_j}{\partial x_a} \frac{\partial x_k}{\partial \bar{x}_i} v^h .$$

Interchange h and a in the second term. It becomes

$$\left(\frac{\partial v^h}{\partial x_k} + \Gamma^h_{ak} v^a \right) \frac{\partial x_k}{\partial \bar{x}_i} \frac{\partial \bar{x}_j}{\partial x_h}$$

and we have shown that

$$\nabla_i \bar{v}^j = \nabla_k v^h \frac{\partial x_k}{\partial \bar{x}_i} \frac{\partial \bar{x}_j}{\partial x_h}$$

which is the rule for a mixed tensor.

The formula for the covariant derivative of a covariant vector is similar:

$$\nabla_i u_j = \frac{\partial u_j}{\partial x_i} - \Gamma^a_{ij} u_a .$$

We skip the invariance proof.

The general rule is to add one term for each contravariant index and subtract one term for each covariant index.

Example:

$$\nabla_h v^k_{ij} = \frac{\partial v^k_{ij}}{\partial x_h} + \Gamma^k_{ha} v^a_{ij} - \Gamma^a_{ih} v^k_{aj} - \Gamma^a_{jh} v^h_{ai} .$$

One verifies that the covariant differentiation of a product follows the usual rules. Another practical rule is that the metric tensors g_{ij} , g^{ij} , δ^i_j behave like constants. For instance

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x_k} - \Gamma^a_{ki} g_{aj} - \Gamma^b_{kj} g_{ib} =$$

$$\frac{\partial g_{ij}}{\partial x_k} - \Gamma_{ki,j} - \Gamma_{kj,i} = 0 .$$

It is customary to write $\nabla^k = g^{kh} \nabla_h$.

4.7. If f is a scalar, then

$$\nabla_k \nabla_j f = \nabla_k \frac{\partial f}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k} - \Gamma^a_{jk} \frac{\partial f}{\partial x_a}$$

is symmetric in j and k , so that

$$(\nabla_k \nabla_j - \nabla_j \nabla_k) f = 0 .$$

For other tensors this is no longer true. For instance, operating on a covariant vector one obtains

$$(\nabla_k \nabla_j - \nabla_j \nabla_k) v_i = R^a_{ijk} v_a$$

where R^h_{ijk} is a tensor, the curvature tensor of the metric g_{ij}. It is given by the somewhat complicated formula

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ki}}{\partial x_j} - \frac{\partial \Gamma^h_{ji}}{\partial x_k} + \Gamma^a_{ki} \Gamma^h_{aj} - \Gamma^a_{ji} \Gamma^h_{ak} \quad . \tag{12}$$

We make no attempt to carry out the computation. Contraction with respect to h and k leads to the Ricci curvature

$$R_{ij} = R^a_{ija} \tag{13}$$

and double contraction to the scalar curvature

$$R = R^a_a = g^{ab} R_{ab} \quad . \tag{14}$$

4.8. Vectors and tensors are sitting in the tangent space attached to each point of a differentiable manifold, but there is no automatic way of comparing vectors and tensors at different points. The notion of parallel displacement, or connection, serves this purpose. We shall use only the simplest connection, known as the Riemannian connection.

Let $x = x(t)$ represent an arc γ in a Riemannian space with the metric tensor g_{ij} ; we assume that $x(t) \in C^\infty$. Let $\xi(t)$ be a contravariant vector at $x(t)$.

Definition. We say that $\xi(t)$ remains parallel along γ if

(15) $$\nabla_k \xi^i(t) x'_k(t) = 0$$

for all t.

More explicitly, (15) reads

(16) $$\left(\frac{\partial\xi^i}{\partial x_k} + \Gamma^i_{kj}\xi^j\right)\frac{dx_k}{dt} = 0$$

or

(17) $$\frac{\partial\xi^i}{dt} + \Gamma^i_{kj}\frac{dx_k}{dt}\xi^j = 0 \; .$$

This is a system of first order linear differential equations. Because the Γ^i_{kj} and the $\frac{dx_k}{dt}$ are C^∞ it has a unique solution with given initial values $\xi^i(t_0)$. In other words, any vector $\xi(t_0)$ determines a vector $\xi(t)$ obtained by parallel displacement along γ.

4.9. The length of a vector ξ, measured in the Riemannian metric, is

$$\langle\xi,\xi\rangle^{1/2} = \left(\sum_{i,j} g_{ij}\,\xi^i\xi^j\right)^{1/2}$$

Similarly, the angle between two vectors ξ, η is given by

$$\langle\xi,\eta\rangle = \sum_{ij} g_{ij}\,\xi^i\,\eta^j$$

$$\cos\theta = \frac{\langle\xi,\eta\rangle}{\langle\xi,\xi\rangle^{1/2}\,\langle\eta,\eta\rangle^{1/2}} \; .$$

Theorem. The inner product $\langle\xi,\eta\rangle$, and hence length and angle, remain constant under parallel displacement.

The proof is a straight forward computation making use of (16) and the identity

$$\Gamma_{ki,j} + \Gamma_{kj,i} = \frac{\partial g_{ij}}{\partial x_k} \; .$$

4.10.

Definition. An arc $x(t)$, $0 \leq t \leq t_0$, is a geodesic if the tangent vector $x'(t)$ remains parallel to itself. If we put $\xi^i = \frac{dx_i}{dt}$ in (16) we obtain the condition

$$\frac{d^2x_i}{dt^2} + \Gamma^i_{kj} \frac{dx_k}{dt} \frac{dx_j}{dt} = 0 \tag{18}$$

for $x(t)$ to be a geodesic.

It is not only the shape of the arc but also the parametrization that makes an arc geodesic. By the theorem the length of $x'(t)$ remains constant along a geodesic. If we replace the parameter t by τ, the length would be multiplied by $\frac{dt}{d\tau}$ and would no longer be constant unless the change were linear: $t = a\tau + b$.

We can regard (18) as a system of differential equations for the functions $x_i(t)$. According to the general theory we can prescribe the initial values $x_i(0)$ and the initial derivatives $x_i'(0)$. In other words, there is a geodesic from every point in every direction. We may even normalize so that the tangent vector has length one, in which case t becomes arc length, usually denoted by s.

The existence theorem produces $x(t)$ only in some small interval $[0,t_0)$, but we can start anew from the point $x(t_0)$. Thus a geodesic can be continued forever unless it "tends to the ideal boundary".

A geodesic arc is, at least locally, the shortest arc between its end points. This is proved in calculus of variations.

4.11. We shall now apply these notions of differential geometry to the unit ball B^n with the Poincaré metric

$$\text{(19)} \qquad ds^2 = \frac{4(dx_1^2 + \ldots + dx_n^2)}{(1 - |x|^2)^2} \quad .$$

We are in the unusual situation where we can use the same coordinate system in the whole space B^n, namely by identifying each point $x \in B^n$ with its coordinates $(x_1, \ldots, x_n)$. Any diffeomorphism of B^n would lead to another coordinate system, but we choose to consider only coordinate changes of the form

$$\text{(20)} \qquad \bar{x} = \gamma x$$

where $\gamma \in M(B^n)$. This means that we are interested in the conformal structure of B^n.

The metric (19) corresponds to the metric tensor

$$\text{(21)} \qquad g_{ij} = \frac{4\delta_{ij}}{(1 - |x|^2)^2} \quad , \quad g^{ij} = \frac{(1 - |x|^2)^2}{4} \; \delta_{ij}$$

A little more generally we consider an arbitrary conformal metric

$$ds^2 = \rho^2 |dx|^2$$

and have then

$$g_{ij} = \rho^2 \delta_{ij} \; , \quad g^{ij} = \rho^{-2} \delta_{ij}$$

(in this connection δ_{ij} is an element of the unit matrix and not a tensor.)

We wish to compute the curvature tensor. It is convenient to use the notation

$$u = \log \rho \quad , \quad u_i = \frac{\partial u}{\partial x_i} \quad , \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

One obtains

$$\frac{\partial g_{ij}}{\partial x_k} = 2\rho^2 \delta_{ij} u_k$$

$$\Gamma_{ij,k} = \rho^2(\delta_{ik} u_j + \delta_{jk} u_i - \delta_{ij} u_k)$$

$$\Gamma^h_{ij} = \delta_{ih} u_j + \delta_{jh} u_i - \delta_{ij} u_h$$

$$\frac{\partial \Gamma^h_{ij}}{\partial x_k} = \delta_{ih}\delta_{jk} + \delta_{jh}\delta_{ik} - \delta_{ij}\delta_{hk}$$

$$\begin{aligned} R^h_{ijk} = {} & \delta_{hk}(u_{ij} - u_i u_j) - \delta_{ik}(u_{hj} - u_h u_j) \\ & + \delta_{ij}(u_{hk} - u_h u_k) - \delta_{hj}(u_{ik} - u_i u_h) \\ & + (\delta_{ij}\delta_{hk} - \delta_{ik}\delta_{hj})|\nabla u|^2 \end{aligned}$$

$$R_{ij} = \delta_{ij}\,\Delta u + (n-2)(u_{ij} - u_i u_j + \delta_{ij}|\nabla u|^2) \; .$$

For the Poincaré metric

$$u = \log\rho = \log 2 - \log(1 - |x|^2)$$

$$u_i = \frac{2x_i}{1 - |x|^2}$$

$$u_{ij} = \frac{2\delta_{ij}}{1 - |x|^2} + \frac{4x_i x_j}{(1 - |x|^2)^2}$$

$$u_{ij} - u_i u_j = \frac{2\delta_{ij}}{1 - |x|^2}$$

$$|\nabla u|^2 = \frac{4|x|^2}{(1 - |x|^2)^2}$$

$$R^h_{ijk} = \frac{4}{(1-|x|^2)^2}(\delta_{hk}\delta_{ij} - \delta_{ik}\delta_{hj})$$

$$R_{hijk} = \frac{16}{(1-|x|^2)^4}(\delta_{hk}\delta_{ij} - \delta_{ik}\delta_{hj}) = g_{hk}g_{ij} - g_{ik}g_{jh}$$

$$R_{ij} = \frac{4(n-1)\delta_{ij}}{(1-|x|^2)^2}$$

$$R = g^{ij}R_{ij} = n(n-1) \quad .$$

The scalar curvature is constant, but not $= -1$. However, the formula for R_{hijk} shows that the sectional curvature is constantly equal to -1 .

4.12. The Beltrami parameter

There are three important invariant differential operators which generalize the gradient, the divergence and the Laplacian.

a) The gradient maps functions on vectors. If f is a scalar

$$\nabla_i f = \frac{\partial f}{\partial x_i}$$

is a covariant vector. The corresponding contravariant vector is

$$\nabla^i f = g^{ij}\nabla_j f = g^{ij}\frac{\partial f}{\partial x_j} \quad .$$

The square length can be written either as a dot product $\nabla^i f \cdot \nabla_i f$ or as

$$g_{ij}\nabla^i f\, \nabla^j f = g^{ij}\nabla_i f\, \nabla_j f =$$

(22) $$|\nabla f|^2 = \sum_{i,j} g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \quad .$$

This is the first Beltrami parameter and it used to be called $\nabla_1 f$. It obviously generalizes the square of the gradient.

---

*See L.P. Eisenhart: Riemannian Geometry, Princeton University Press, 1960, p.81, (25.9).

b) The divergence leads from vectors to scalars. If v is a vector-valued function we define

$$\text{div}\, v = \nabla_i v^i = \nabla^i v_i \quad . \tag{23}$$

These expressions are identical for g^{ij} behaves like a constant and we obtain

$$\nabla_i v^i = \nabla_i g^{ij} v_j = g^{ij} \nabla_i v_j = \nabla^j v_j \quad .$$

From

$$\nabla_i v^j = \frac{\partial v^j}{\partial x_i} + \Gamma^j_{ik} v^k$$

we have

$$\nabla_i v^i = \frac{\partial v^i}{\partial x_i} + \Gamma^i_{ik} v^k \quad . \tag{24}$$

There is a simple formula for Γ^i_{ik}. By the well known rule for differentiation, of a determinant.

$$\frac{\partial}{\partial x_k} \log g = \text{tr}\,(g^{-1} \frac{\partial g}{\partial x_k}) = g^{ij} \frac{\partial g_{ij}}{\partial x_k} \quad .$$

But

$$\frac{\partial g_{ij}}{\partial x_k} = \Gamma_{ki,j} + \Gamma_{kj,i}$$

and hence

$$g^{ij} \frac{\partial g_{ij}}{\partial x_k} = g^{ij} \Gamma_{ki,j} + g^{ij} \Gamma_{kj,i} = 2\, \Gamma^i_{ki}$$

and thus

$$\Gamma^i_{ki} = \frac{\partial}{\partial x_k} \log \sqrt{g} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_k} \sqrt{g} \tag{25}$$

From (24) and (25) we get

(26) $$\operatorname{div} v = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} (\sqrt{g}\, v^i) .$$

c) When both operators are combined we obtain the <u>second Beltrami parameter</u>

(27) $$\Delta_2 f = \operatorname{div} \operatorname{grad} f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x_j}\right) .$$

This is a direct generalization of the Laplacian and it is frequently referred to as the Laplace-Beltrami operator.

Its importance lies in the fact that it is independent of the local coordinates. For instance, if f is defined on B^n and if we use the Poincaré metric, then for $\gamma \in M(B^n)$,

(28) $$\Delta_2(f \circ \gamma) = (\Delta_2 f) \circ \gamma .$$

For short, a solution of $\Delta_2 f = 0$ on B^n will be called <u>hyperbolically harmonic</u> (h-h). We see from (24) that if f is h-h, so is $f \circ \gamma$. This is not true for ordinary harmonic functions, except, as we shall see, when $n = 2$.

On a Riemannian space a solution of $\Delta_2 f$ will be called <u>harmonic</u> since this does not make sense in any other meaning.

4.13. We compute $\Delta_2 f$ for a conformal metric $ds = \rho|dx|$. One gets simply

(29) $$\Delta_2 f = \rho^{-n} \frac{\partial}{\partial x_i} \left(\rho^{n-2} \frac{\partial f}{\partial x_i}\right) .$$

In the special case $\rho = \dfrac{2}{1 - |x|^2}$ one finds

(30) $$\Delta_2 f = \frac{(1 - |x|^2)^2}{4} \left[\Delta f + \frac{2(n-2)}{1 - |x|^2} x_i \frac{\partial f}{\partial x_i} \right] .$$

With the customary notation $|x| = r$

$$x_i \frac{\partial f}{\partial x_i} = r \frac{\partial f}{\partial r}$$

(31) $$\Delta_2 f = \frac{(1 - r^2)^2}{4} \left[\Delta f + \frac{2(n-2)}{1 - r^2} r \frac{\partial f}{\partial r} \right] .$$

For the half-space $\rho = \dfrac{1}{x_n}$ and

(32) $$\Delta_2 f = x_n^2 \left[\Delta f - (n-2) \frac{1}{x_n} \frac{\partial f}{\partial x_n} \right] .$$

V. Hyperharmonic functions

5.1. We shall now study hyperharmonic functions on B^n. Our first task will be to find all solutions $u(x)$ of $\Delta_2 u = 0$ that depend only on $r = |x|$. For a function $u(r)$ one obtains

$$\frac{\partial u}{\partial x_i} = u'(r)\,\frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = u''(r)\,\frac{x_i x_j}{r^2} + u'(r)\left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right)$$

and thus

$$\Delta u = u''(r) + (n-1)\,\frac{u'(r)}{r} \quad .$$

According to equation (32) in 4.13 $u(r)$ will satisfy

$$(1) \qquad u''(r) + (n-1)\,\frac{u'(r)}{r} + \frac{2(n-2)}{1-r^2}\, r\, u'(r) = 0 \quad .$$

If $u'(r) \neq 0$ this can be written as

$$\frac{u''(r)}{u'(r)} + \frac{(n-1)}{r} + \frac{2(n-2)r}{1-r^2} = 0$$

or

$$\frac{d}{dr}\left[\log u'(r) + (n-1)\log r - (n-2)\log(1-r^2)\right] = 0$$

from which we conclude that

$$u'(r)\,\frac{r^{n-1}}{(1-r^2)^{n-2}} = \text{const.}$$

This leads to the general solution

$$(2) \qquad u(r) = a\int^{r} \frac{(1-t^2)^{n-2}}{t^{n-1}}\,dt + b \quad .$$

We see at once that no solution can stay finite for $r = 0$. As a normalized solution we introduce

$$(3) \qquad g(r) = \int_r^1 \frac{(1-t^2)^{n-2}}{t^{n-1}}\, dt \quad .$$

For $n=2$ $g(r) = \log \frac{1}{r}$. For $n > 2$ $g(r) \sim \frac{1}{n-2} r^{2-n}$ for $r \to 0$ and $g(r) = O((1-r)^{n-1})$ for $r \to 1$. For $n = 3$ $g(r) = (\sqrt{r} - \frac{1}{\sqrt{r}})^2 = \frac{1}{r} + r - 2$.

5.2. Together with $g(r)$ any $g(|\gamma x|)$ with $\gamma \in M(B^n)$ is again h - h, because by (29) in 4.13 γ commutes with the Laplace-Beltrami operator. In particular

$$(4) \qquad g(x,y) = g(|T_y x|) = g(\frac{|x-y|}{[x,y]})$$

has a singularity at y and will be regarded as the Green's function with pole at y.

5.3. The fact that a h - h function that depends only on r is either a constant or has a rather strong singularity suggests that any isolated singularity of a h - h function should be equally strong. In function theory we are used to study such questions by use of integral formulas, especially Cauchy's integral formula and for ordinary harmonic functions Green's formula.

For h - h functions there are two ways of approaching the question. Either one can derive Green's formula in the general setting of Riemannian spaces or, in the hyperbolic case, one can apply the classical Green's formula to suitably chosen functions that are naturally connected with the hyperbolic metric.

We shall use both methods, but we begin with the second which is by far the simplest.

5.4. As a preparation it is useful to collect a few facts about multiple integrals especially these connected with spheres.

Recall Euler's Γ-function and B-function

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt\,, \qquad \mathrm{Re}\, a > 0 \tag{5}$$

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt\,, \qquad \mathrm{Re}\, a > 0\,, \quad \mathrm{Re}\, b > 0\,. \tag{6}$$

They are closely related. The substitution $t = x^2$ gives

$$\Gamma(a) = 2\int_0^\infty x^{2a-1} e^{-x^2} dx \tag{7}$$

and $t = \sin^2\varphi$ in $B(a,b)$ gives

$$B(a,b) = 2\int_0^{\pi/2} \sin^{2a-1}\varphi\, \cos^{2b-1}\varphi\, d\varphi\,. \tag{8}$$

From (7)

$$\Gamma(a)\,\Gamma(b) = 4\int_0^\infty\int_0^\infty x^{2a-1} y^{2b-1} e^{-(x^2+y^2)}\, dx\, dy$$

and in polar coordinates

$$\Gamma(a)\,\Gamma(b) = 4\int_0^\infty r^{2a+2b-1} e^{-r^2} dr \int_0^{\pi/2} \cos^{2a-1}\varphi\, \sin^{2b-1}\varphi\, d\varphi$$

$$= \Gamma(a+b)\, B(a,b)$$

and thus

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,. \tag{9}$$

In particular, since $B(\frac{1}{2}, \frac{1}{2}) = \pi$ and $\Gamma(1) = 1$,

(10) $$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad .$$

We shall denote the "surface area" or $(n-1)$ - dimensional euclidean measure of S^{n-1} by ω_n (some authors prefer ω_{n-1}). The area of $S^{n-1}(r)$ is then $\omega_{n-1} r^{n-1}$ and the volume of $B^n(r)$ is

$$V_n(r) = \int_0^r \omega_n r^{n-1} dr = \frac{\omega_n}{n} r^n \quad .$$

To compute the area of a spherical cap of radius φ (measured along the unit sphere) we project on the equatorial plane $x_n = 0$ and note that the area element on the sphere is

$$d\sigma = \frac{dx_1 \ldots dx_{n-1}}{x_n}$$

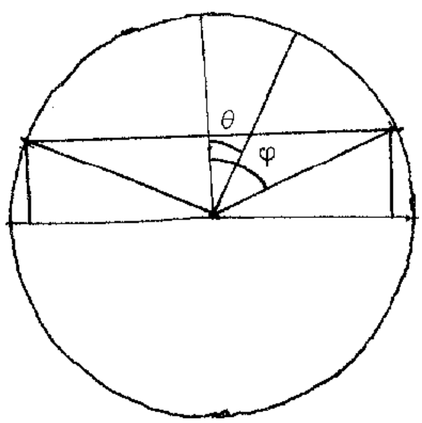


Fig. 11

The area of the cap is thus

$$A(\varphi) = \omega_{n-1} \int_0^{\sin\varphi} \frac{r^{n-2}}{x_n} dr \quad .$$

With $r = \sin\theta$, $dr = \cos\theta\, d\theta$, $x_n = \cos\theta$ we find

$$A(\varphi) = \omega_{n-1} \int_0^{\varphi} \sin^{n-2}\theta\, d\theta \ . \tag{11}$$

In particular,

$$\omega_n = 2A(\tfrac{\pi}{2}) = 2\omega_{n-1} \int_0^{\pi/2} \sin^{n-2}\theta\, d\theta$$

and thus

$$\frac{\omega_n}{\omega_{n-1}} = B(\frac{n-1}{2}, \frac{1}{2}) = \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})} \ .$$

This makes $\omega_n \Gamma(\frac{n}{2})\Gamma(\frac{1}{2})^{-n}$ independent of n and since $\omega_2 = 2\pi$ it follows that

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \ . \tag{12}$$

For comparison we shall also compute the area of a sphere and volume of a ball in hyperbolic space. Recall that a hyperbolic radius s corresponds to the euclidean radius $r = \tanh\frac{s}{2}$ of a ball with center O . The n.e. area of the sphere is then

$$\omega_n \frac{2^{n-1} r^{n-1}}{(1-r^2)^{n-1}} = \omega_n \sinh^{n-1} s$$

and the volume is

$$V_h(s) = \omega_n \int_0^s \sinh^{n-1} t\, dt \ . \tag{13}$$

(V_h stands for n.e. volume). The formula is thus the same as before with sinh instead of sin and of course $n+1$ in the place of n .

5.5 We return to h-h functions. Recall the classical Green's formula

(14) $$\int_D u\,\Delta v\,dx = \int_{\partial D} u \frac{\partial v}{\partial n}\,d\sigma - \int_D (\nabla u \cdot \nabla v)\,dx \quad .$$

Here D is a region in $\mathbb{R}^n$, $dx = dx_1 \dots dx_n$, $\frac{\partial}{\partial n}$ is the derivative in the direction of the outer normal of the smooth boundary ∂D and $(\nabla u \cdot \nabla v)$ is the inner product of the gradients.

Let us now assume that D lies in B^n. We shall replace everything by its hyperbolic counterpart, using the following notations: for the hyperbolic

volume element: $dx_h = \dfrac{2^n dx}{(1-|x|^2)^n}$

the area element: $d\sigma_h = \dfrac{2^{n-1} d\sigma}{(1-|x|^2)^{n-1}}$

the normal derivative: $\dfrac{\partial v}{\partial n_h} = \dfrac{1-|x|^2}{2}\,\dfrac{\partial v}{\partial n}$

the gradient: $\nabla_h u = \dfrac{1-|x|^2}{2}\,\nabla u$

the Laplacian: $\Delta_h v = \Delta_2 v$.

With these notations we claim that

<u>Lemma 1</u>:

(15) $$\int_D u\,\Delta_h v\,dx_h = \int_{\partial D} u \frac{\partial v}{\partial n_h}\,d\sigma_h - \int_D (\nabla_h u \cdot \nabla_h v)\,dx_h \quad .$$

It is easy to check that this is exactly the same as (14) with u replaced by $\bar{u} = 2^{n-2}(1-|x|^2)^{2-n} u$. As customary, one can eliminate the "Dirichlet integral" on the right by interchanging u and v and subtracting. The resulting formula

$$(16) \qquad \int_D (u\,\Delta_h v - v\Delta_h u)\,dx_h = \int_{\partial D} \left(u\,\frac{\partial v}{\partial n_h} - v\,\frac{\partial u}{\partial n_h}\right) d\sigma_h$$

is the one in most common use.

The special case $v = 1$ leads to

$$(17) \qquad \int_D \Delta_h u\,dx = \int_{\partial D} \frac{\partial u}{\partial n_h}\,d\sigma_h$$

and it shows above all that an $h-h$ function has vanishing flux

$$(18) \qquad \int_{\partial D} \frac{\partial u}{\partial n_h}\,d\sigma_h = 0\ .$$

In euclidean terms (18) reads

$$(19) \qquad \int_{\partial D} \frac{\partial u}{\partial n}\,\frac{d\sigma}{(1-|x|^2)^{n-2}} = 0\ .$$

If $D = B(r)$ we obtain *)

$$\int_{S(r)} \frac{\partial u}{\partial r}\,d\sigma = 0\ .$$

In other words, the mean value

$$m(r) = \frac{1}{\omega_n}\int_{S(r)} u\,d\sigma$$

is constant. In particular, if u is $h-h$ in a full neighborhood of the origin, then

$$(20) \qquad u(0) = \frac{1}{\omega_n}\int_{S(r)} u\,d\sigma$$

---

*) From now on n will be fixed and we write B for B^n and S for S^{n-1}.

for all sufficiently small r. This is the mean-value property. The mean-value can also be taken with respect to volume, and because of the rotational symmetry it does not matter whether we use euclidean or non-euclidean measure. However, the n.e. average is invariant and we obtain:

Lemma. The value of a h-h function at the center of a n.e. sphere or ball is equal to the n.e. average.

Remark. The mean-value formula can also be proved without any computation whatsoever. In fact, it is possible to write

$$m(|x|) = \int_{O(n)} u(kx)dk$$

where dk is the Haar measure of $O(n)$. Because $u(kx)$ is h-h for every k it follows that $m(|x|)$ is a h-h function which depends only on $|x|$ and hence must be a constant. As a result of the mean-value property h-h functions satisfy the maximum-minimum principle:

A non-constant h-h function cannot have a relative maximum or minimum on an open connected set.

The proof is the same as for $n = 2$.

5.6. The question of removable singularities has the following answer:

Theorem. Suppose that $D \subset B$ is open and $a \in D$. If $u(x)$ is h-h in $D \setminus \{a\}$ and if

$$\lim_{x \to a} u(x) \, |x - a|^{n-2} = 0 \quad \text{if} \quad n > 2$$

$$\lim_{x \to a} u(x) \frac{1}{\log \frac{1}{|x-a|}} = 0 \quad \text{if} \quad n = 2 ,$$

then u has an $h-h$ extension to D .

Proof: Without loss of generality we may assume that $a = 0$ and $D = B(\rho)$, $\rho > 0$. We choose a fixed $y \in B(\rho)$, $y \neq 0$ and apply Lemma 1 to $u = u(x)$, $v = g(x,y)$ in the region $B(\rho) \setminus (B(r) \cup B(y,r))$ with $r < \min(|y|, \rho - |y|)$. Since $\Delta_h u = \Delta_h v = 0$ we obtain

$$(21) \qquad \int_{S(\rho)+S(r)+S(y,r)} \left(u(x) \frac{\partial g(x,y)}{\partial n_h} - g(x,y) \frac{\partial u(x)}{\partial n_h}\right) d\sigma_h(x) = 0 .$$

We look for the limit as $r \to 0$ and start by considering the integral $S(y,r)$. It is evident that the integral of the second term will tend to 0 , for $g(x,y) = O\left(\frac{1}{r^{n-2}}\right)$ (or $O\left(\log \frac{1}{r}\right)$) while $\frac{\partial u}{\partial n_h}$ is bounded and $d\sigma_h$ contains the factor r^{n-1} . The integral of the first term can be written as

$$(22) \qquad -\int_{|x-y|=r} (1-|x|^2)^{2-n} u(x) \frac{\partial g(x,y)}{\partial r} r^{n-1} d\omega$$

where $d\omega$ is the element of "solid angle" (or the measure on S^{n-1}).

We recall that $g(x,y) = g(|T_y x|)$ and $|T_y x| = \frac{|x-y|}{[x,y]}$ so that

$$-\frac{\partial g(x,y)}{\partial r} = -g'(|T_y x|) \frac{\partial |T_y x|}{\partial r} =$$

$$\frac{(1-|T_y x|^2)^{n-2}}{|T_y x|^{n-1}} |T_y x| \frac{\partial}{\partial r} \log |T_y x| =$$

$$\left(\frac{(1-|x|^2)(1-|y|^2)}{[x,y]^2}\right)^{n-2} \left(\frac{[x,y]}{|x-y|}\right)^{n-2} \frac{\partial}{\partial r} \log |T_y x|$$

$$= \left(\frac{(1-|x|^2)(1-|y|^2)}{[x,y]}\right)^{n-2} \frac{1}{r^{n-2}} \frac{\partial}{\partial r} \log |T_y x| .$$

Here $[x,y] \to |1-|y||^2$ as $x \to y$ and

$$r \frac{\partial}{\partial r} \log |T_y x| = r \frac{\partial}{\partial r} \log \frac{r}{[x,y]} \to 1 .$$

It follows at once that (22) tends to $\omega_n u(y)$.

We have now to investigate the part of (21) that refers to $S(r)$. We make first the observation that by Green's formula the integral

$$(23) \qquad \int_{S(r)} \left(u(x) \frac{\partial g(x,y)}{\partial n_h} - g(x,y) \frac{\partial u}{\partial n_h}\right) d\sigma_h(x)$$

is in fact independent of r and hence defined for all $y \in B(\rho) \setminus \{0\}$. Moreover, because $g(x,y) = g(y,x)$ it is evidently a $h-h$ function of y . We wish to show that the boundedness condition of Theorem 1 makes it identically zero.

Let us rewrite (23) as

$$(24) \qquad \frac{r^{n-1}}{(1-r^2)^{n-2}} \int_{S(1)} \left(u(r\xi) \frac{\partial g(r\xi,y)}{\partial r} - g(r\xi,y) \frac{\partial u(r\xi)}{\partial r}\right) d\omega(\xi) .$$

Observe that g and $\frac{\partial g}{\partial r}$ remain bounded when $r \to 0$. Hence the condition $u(x)|x|^{n-2} \to 0$ makes the limit of the first term zero.

For the second term we write

$$\mu(r) = \int_{S(1)} g(r\xi,y) \frac{\partial u(r\xi)}{\partial r} d\omega(\xi)$$

and obtain

$$\int_r^{2r} \mu(t)\,dt = \int_{S(1)} d\omega \int_r^{2r} g(t\xi,y) \frac{\partial u(t\xi)}{\partial t} dt$$

$$= \int_{S(1)} \left[g(t\xi,y)\,u(t\xi) \Big|_r^{2r} - \int_r^{2r} u(t\xi) \frac{\partial g(t\xi,y)}{\partial t} dt \right] d\omega(\xi) .$$

This is of the order $o(\frac{1}{r^{n-2}})$. Hence there is a t between r and $2r$ with $\mu(t) = o(\frac{1}{t^{n-1}})$ and since $\lim r^{n-1}\mu(r)$ exists it must be zero.

We can now conclude from (21) that

$$u(y) = \frac{1}{\omega_n} \int_{S(\rho)} (u(x) \frac{\partial g(x,y)}{\partial n_h} - g(x,y) \frac{\partial u}{\partial n_h}) d\sigma_h \quad .$$

The right hand side is a h-h function in $S(\rho)$ and defines the desired extension of u .

5.7. Boundary values. If u is h-h in B and has a continuous extension to the closure $\bar{B}$, then the mean value property implies

$$u(0) = \frac{1}{\omega_n} \int_{S^{n-1}} u(x) d\omega \quad . \tag{25}$$

We obtain a more general formula if we apply (25) to $u \circ \gamma$ where $\gamma \in M(B^n)$. Indeed, $u \circ \gamma$ is again h-h and extends continuously to the boundary. Hence

$$u(\gamma 0) = \frac{1}{\omega_n} \int_{S^{n-1}} u(\gamma x)\, d\omega(x) \quad . \tag{26}$$

We specialize to $\gamma = T_y^{-1}$ for a fixed $y \in B$. Because $T_y^{-1} 0 = y$ we get

$$u(y) = \frac{1}{\omega_n} \int_S u(T_y^{-1} x) d\omega(x)$$

and replace x by $T_y x$ in the integral. This leads to

$$u(y) = \frac{1}{\omega_n} \int_S u(x) |T_y'(x)|^{n-1} d\omega(x)$$

or explicitly

$$u(y) = \frac{1}{\omega_n} \int_S \left(\frac{1 - |y|^2}{|x - y|^2}\right)^{n-1} u(x) d\omega \tag{27}$$

where we have used the simplification $[x,y] = |x-y|$ when $|x| = 1$.

This is the Poisson formula for the ball. Note that the kernel is just the $(n-1)$st power of the Poisson kernel for $n = 2$.

5.8. We shall use the notation

$$k(x,y) = \frac{1-|y|^2}{|x-y|^2}$$

with the understanding that $|x| = 1$ and $|y| < 1$. Formula (27) lets us suspect that $k(x,y)$ is always a $h-h$ function of y .

The direct verification of this is not difficult, but it is much quicker to pass to the half-space H^n . Recall that the Laplacian for the half space is

$$\Delta_h = x_n^2 \left(\Delta - \frac{n-2}{x_n} \frac{\partial}{\partial x_n} \right) .$$

It follows without computation that

$$\Delta_h x_n^\alpha = \alpha(\alpha - n + 1) x_n^\alpha .$$

Thus every x_n^α is an eigen-function and x_n^{n-1} is $h-h$.

We recall (see 3.5) that the canonical mapping $B^n \to H^n$ is such that

$$x_n = \frac{1-|y|^2}{|y-e_n|^2} = k(e_n, y) ,$$

where $x \in H^n$, $y \in B^n$. We conclude that $k(e_n, y)^\alpha$ is an eigenfunction of Δ_h for B^n and that $k(e_n, y)^{n-1}$ is $h-h$. Given any $x \in S^{n-1}$ (apologies for using x twice) there exists a rotation β such that $\beta x = e_n$, But it is obvious that $k(x,y) = k(\beta^{-1} e_n, y) = k(e_n, \beta y)$. Because of the invariance we conclude that

$$\Delta_h \, k(x,y)^{\alpha} = \alpha(\alpha - n + 1)\, k(x,y)^{\alpha} \tag{28}$$

and $\Delta_h \, k(x,y)^{n-1} = 0$.

5.9. Suppose now that $f(x)$ is of class $L_1(S)$, i.e.

$$\int_S |f(x)| \, d\omega(x) < \infty \quad .$$

Then we can form

$$u(y) = \frac{1}{\omega_n} \int_{S^{n-1}} k(x,y)^{n-1} f(x) \, d\omega(x) \, , \tag{29}$$

which is obviously $h - h$.

<u>Theorem</u>. $u(y)$ has radial limits $f(x)$ a.e.

<u>Proof</u>. It is known that

$$\lim_{\delta \to 0} \frac{\int_{B(\xi,\delta)} f(x) \, d\omega(x)}{\int_{B(\xi,\delta)} d\omega(x)} = f(\xi) \tag{30}$$

for a.e. $\xi \in S$ (see e.g. Rudin, Real and Complex Analysis, Thm. 8.8). We shall prove that $u(r\xi) \to f(\xi)$ for $r \to 1$ whenever (30) is fulfilled. We may assume that $\xi = e_n$ so that

$$u(re_n) = \frac{1}{\omega_n} \int_S k(x \, , re_n)^{n-1} f(x) \, d\omega(x) \; .$$

We introduce the colatitude φ defined by $x_n = \cos \varphi$ and change the notation of the kernel to

$$K(r,\varphi) = \frac{1}{\omega_n} \left(\frac{1 - r^2}{1 - 2r \cos \varphi + r^2} \right)^{n-1} \quad .$$

Moreover, we shall write

$$F(\varphi) = \int_{x_n > \cos\varphi} f(x)\,d\omega(x)$$

for the integral of f over the polar cap of radius φ . Clearly, the formula for $u(\mathbf{r}e_n)$ can be written in the form

$$(31) \qquad u(re_n) = \int_0^\pi K(r,\varphi)\,F'(\varphi)\,d\varphi$$

where $F'(\varphi)$ exists a.e. and the measure $F'(\varphi)d\varphi$ is absolutely continuous.

Integration by parts yields

$$u(r,e_n) = K(r,\pi)\,F(\pi) - \int_0^\pi F(\varphi)\,\frac{\partial K(r,\varphi)}{\partial\varphi}\,d\varphi \quad .$$

We wish to show that $u(re_n) \to f(e_n)$. Without loss of generality we may assume that $f(e_n) = 0$, for otherwise we need only replace f by $f - f(e_n)$.

It is immediate that K and $\frac{\partial K}{\partial\varphi}$ tend uniformly to zero for $r \to 1$ in any interval $\delta \leq \varphi \leq \pi$, $\delta > 0$. Hence $u(re_n)$ has the same limit as

$$u_\delta(r) = -\int_0^\delta F(\varphi)\,\frac{\partial}{\partial\varphi}\,K(r,\varphi)d\varphi \quad .$$

Because of (30) we can choose δ so small that

$$|F(\varphi)| \leq \epsilon\,A(\varphi) = \epsilon\,\omega_{n-1}\int_0^\varphi \sin^{n-2}\theta\,d\theta \quad .$$

It follows that

$$|u_\delta(r)| \leq -\epsilon\int_0^\delta A(\varphi)\,\frac{\partial K}{\partial\varphi}\,d\varphi \quad ,$$

for $\frac{\partial K}{\partial\varphi} > 0$. Another integration by parts leads to

$$|u_\delta(r)| \leq -\epsilon A(\delta) K(\delta) + \epsilon \int_0^\delta K A'(\varphi)\, d\varphi$$

$$< \epsilon \int_0^\pi K A'(\varphi) d\varphi = \epsilon \int_S K(r,\varphi) d\omega = \epsilon$$

by virtue of the fact that

$$\int_S K(r,\varphi)\, d\omega = 1$$

as a special case of the Poisson formula. We conclude that $\lim_{r\to 1} u(re_n) = \lim_{\delta\to 0} u_\delta(r) = 0$.

5.10. A little more generally $u(y) \to f(e_n)$ in any "Stolz cone" characterized by

(32) $$|y - e_n| \leq M(1 - |y|) .$$

Write $y' = |y| e_n$ for short. By (32)

$$|x - y| \leq |x - y'| + |y - e_n| + |y' - e_n| \leq$$

$$|x - y'| + (M+1)(1 - |y|) .$$

If $|x| = 1$ this gives

$$|x - y| \leq (M+2)|x - y| .$$

Similarly,

$$|x - y'| \leq |x - y| + |y - e_n| + |y' - e_n| \leq (M+2)|x - y| .$$

Hence the ratio $\frac{k(x,y)}{k(x,y')}$ lies between fixed bounds and we conclude that the limit of $u(y) - f(e_n)$ is also zero when $y \to e_n$ inside the cone.

VI. The geodesic flow.

6.1. We shall now pass from $B = B^n$ to its unit tangent space $T_1(B)$. The points of $T_1(B)$ consist of a point $x \in B$ and a direction at that point. The direction will be given by a unit vector $\xi \in S = S^{n-1}$. Thus $T_1(B)$ can be identified with $B \times S$, but we prefer to think of it as the space of directed line elements (x,ξ) . Because of connections with dynamics $T_1(B)$ is sometimes referred to as the phase space.

The Möbius group $M(B^n)$ acts in an obvious way on $T_1(B)$. If $\gamma \in M(B^n)$ it is clear that x should be mapped on γx . At the same time ξ should be transformed by the matrix $\gamma'(x)$ to give the new direction $\gamma'(x)\xi$ for the line element at γx , but in order to obtain a unit vector we have to divide by $|\gamma'(x)|$.

Hence we define the action of γ by

$$\gamma : (x,\xi) \to (\gamma x, \frac{\gamma'(x)}{|\gamma'(x)|}\xi) . \tag{1}$$

There is an obvious invariant volume element, namely,

$$dm = dx_h \, d\omega(\xi) , \tag{2}$$

where $\omega(\xi)$ denotes the solid angle. In fact, the Poincaré metric is invariant under γ and ξ undergoes a rotation which keeps the spherical measure invariant.

6.2. It is of interest to introduce a point-pair invariant for the action (1). Let (x,ξ) and (y,η) be two line elements. We claim that

$$(3) \qquad \delta[(x,\xi),(y,\eta)] = |\eta - \Delta(x,y)\xi|$$

is a suitable invariant.

We observe first that this expression is symmetric. In fact, multiplication by $\Delta(y,x)$ does not change length and

$$\Delta(y,x)(\eta - \Delta(x,y)\xi) = \Delta(y,x)\eta - \xi$$

by virtue of the identity $\Delta(x,y)\,\Delta(y,x) = I$. Thus

$$|\eta - \Delta(x,y)\xi| = |\xi - \Delta(y,x)\eta| .$$

Now let us check the invariance. Simultaneous application of γ on (x,ξ) and (y,η) changes (3) to

$$(4) \qquad \left| \frac{\gamma'(y)}{|\gamma'(y)|}\,\eta - \Delta(\gamma x, \gamma y)\,\frac{\gamma'(x)}{|\gamma'(x)|}\,\xi \right| .$$

But we have proved (see (41) in (2.8)) that

$$\Delta(\gamma x, \gamma y)\,\frac{\gamma'(x)}{|\gamma'(x)|} = \frac{\gamma'(y)}{|\gamma'(y)|}\,\Delta(x,y) .$$

Thus (4) becomes

$$\left| \frac{\gamma'(y)}{|\gamma'(y)|}\,(\eta - \Delta(x,y)\xi) \right| = |\eta - \Delta(x,y)\xi|$$

and the invariance is established.

6.3. We shall now determine the infinitesimal form of (3). In other words, we want to compute

$$(5) \qquad |\xi + d\xi - \Delta(x, x+dx)\xi| .$$

Here is the computation:

$$Q(x)_{ij} = \frac{x_i x_j}{|x|^2}$$

$$dQ(x)_{ij} = \frac{x_i dx_j + x_j dx_i}{|x|^2} - \frac{2x_i x_j (xdx)}{|x|^4}$$

(6) $$(QdQ)_{ij} = \frac{x_i x_k}{|x|^2} \left(\frac{x_k dx_j + x_j dx_k}{|x|^2} - \frac{2x_j x_k (xdx)}{|x|^4} \right)$$

$$= \frac{x_i dx_j}{|x|^2} - \frac{x_i x_j (xdx)}{|x|^4}$$

where we made use of the summation convention.

(7) $$\left[(I - 2Q(x))\, dQ(x)\right]_{ij} = \frac{x_j dx_i - x_i dx_j}{|x|^2}$$

$$\Delta(x, x + dx) = (I - 2Q(x^* - x - dx))(I - 2Q(x)) .$$

Here

$$Q(x^* - x - dx) = Q\left(\frac{1 - |x|^2}{|x|^2}\, x - dx \right)$$

$$= Q\left(x - \frac{|x|^2}{1 - |x|^2}\, dx\right)$$

$$= Q(x) - \frac{|x|^2}{1 - |x|^2}\, d\,Q(x)$$

and thus

$$\Delta(x, x + dx) = \left(I - 2Q(x) + \frac{2|x|^2}{1 - |x|^2}\, d\,Q(x)\right)(I - 2Q(x))$$

$$= I - \frac{2|x|^2}{1 - |x|^2}\, (I - 2Q(x))\, d\,Q(x) \qquad *)$$

*) Note that $(I - 2Q)^2 = I$ implies $d\,Q\,(I - 2Q) = -(I - 2Q)dQ$.

and by (7)

$$\Delta(x\,,x+dx)_{ij} = \delta_{ij} + \frac{2(x_i dx_j - x_j dx_i)}{1-|x|^2} \quad . \tag{8}$$

Now we substitute (8) in (5) to obtain

$$[(\xi + d\xi - \Delta(x\,,x+dx)\xi)_i =$$

$$= d\xi_i - 2\,\frac{(dx\xi)x_i - (x\xi)dx_i}{1-|x|^2} \quad .$$

The infinitesimal invariant is hence

$$|d\xi - 2\,\frac{(\xi dx)x - (\xi x)dx}{1-|x|^2}\,| \quad . \tag{9}$$

An invariant Riemannian metric on $T_1(B)$ can now be introduced by

$$ds^2 = \frac{4|dx|^2}{(1-|x|^2)^2} + |d\xi - 2\,\frac{(\xi dx)x - (\xi x)dx}{1-|x|^2}\,|^2 \quad . \tag{10}$$

6.4. We shall now define the geodesic flow which is a one parameter group of diffeomorphisms V_t of $T_1(B)$ which satisfy $V_s V_t = V_{s+t}$.

Every line element (x,ξ) determines a geodesic ray which starts from x in the direction ξ . Fix a real number t . Let x move along the geodesic from x to a point x' at distance t from x ; distances are counted positive in the direction of the ray, negative in the opposite direction. At the same time we let the vector ξ slide to the positive tangent vector ξ' at x' ; (see Fig. 15)

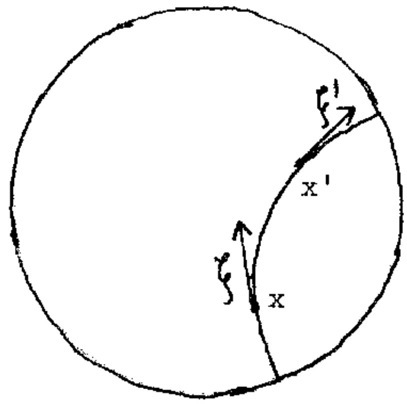


Fig. 12

We define

(11) $$V_t(x,\xi) = (x', \xi') .$$

It is quite evident that $V_s V_t = V_{s+t}$ and $V_t^{-1} = V_{-t}$. Moreover, the construction is clearly invariant with respect to Möbius transformations in the sense that

(12) $$V_t \circ \gamma = \gamma \circ V_t$$

for each $\gamma \in M(B)$.

6.5. The most important property of the V_t is that they define a flow in the sense that each V_t leaves the volume element dm , defined by (2) , invariant. We shall prove the invariance by introducing an other volume element $d\bar{m}$ on $T_1(B)$ which is at the same time invariant under Möbius transformations and under the transformations V_t .

If dm and $d\bar{m}$ are both invariant under $M(B)$, then their quotient is an automorphic function. But $M(B)$ is transitive in the sense that any line element (x,ξ) can be mapped on any other. But this means that the ratio $d\bar{m} : dm$ is constant. Since $d\bar{m}$ is invariant under every V_t the same is then true of dm .

To construct $d\overline{m}$ we introduce a new set of parameters on $T_1(B)$. As we have already pointed out every line element (x,ξ) determines an oriented geodesic through it. This geodesic has an initial point u and a terminal point v. Clearly, u and v can be calculated explicitly from x and ξ (by a complicated formula).

Conversely, u and v determine the geodesic, but we need still another parameter to find the position of x on the geodesic. Let the midpoint of the geodesic from u to v be denoted by $\alpha = \alpha(u,v)$. To locate x we use the signed non-euclidean distance s from α to x (see Fig. 16).

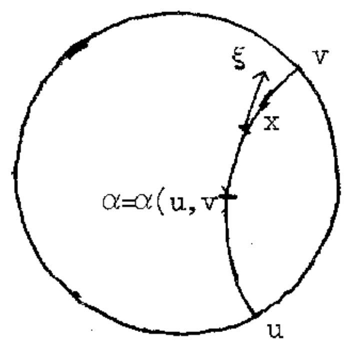


Fig. 13

It is now clear that there is a bijective correspondence between the pairs (x,ξ) and the triples (u,v,s). This correspondence is in fact a diffeomorphism between $B \times S$ and $S \times S \times \mathbb{R}$.

The action of V_t on (u,v,s) is quite obvious: (u,v,s) is replaced by $(u,v,s+t)$. As a result the volume element

$$d\omega(u)\, d\omega(v)\, ds$$

and more generally any element of the form

$$f(u,v)\, d\omega(u)\, d\omega(v)\, ds$$

is invariant.

We want to find a $\overline{dm}$ that is invariant also against all $\gamma \varepsilon M(B)$. γ moves x to a point γx at distance s from $\gamma\alpha$ and thus to the directed distance $s + d(\gamma\alpha, \alpha(\gamma u, \gamma v)$ from the midpoint $\alpha(\gamma u, \gamma v)$. This means that s is invariant up to an added term that depends only on u and v. Because $|\gamma u - \gamma v|^{2n-2} = |\gamma'(u)|^{n-1} |\gamma'(v)|^{n-1} |u-v|^{2n-2}$ it follows that

$$\overline{dm} = \frac{d\omega(u)\, d\omega(v)\, ds}{|u-v|^{2n-2}}$$

is invariant with respect to γ. The conclusion is that

$$dm = c \, \frac{d\omega(u)\, d\omega(v)\, ds}{|u-v|^{2n-2}}$$

with constant c. Actually, one shows by suitable specialization that $c = 1$.

VII. Discrete Subgroups.

7.1. As seen in 2.6 the most general $\gamma \in M(B^n)$ is of the form kT_a, $(k \in SO(n), a \in B^n)$. Therefore $M(B^n)$ can be topologized by $SO(n) \times B^n$. A subgroup $\Gamma \subset M(B^n)$ is discrete if the identity I has a neighbourhood whose intersection with Γ reduces to I. From now on Γ will always denote a discrete subgroup.

We know that any $\gamma \in M(B^n)$ which fixes the origin is in $SO(n)$. Because $SO(n)$ is compact the subgroup of Γ which stabilizes 0 is a finite subgroup of $SO(n)$

The points $\gamma 0$, $\gamma \in \Gamma$, are isolated. If not there would exist an infinite sequence of distinct $\gamma_n \in \Gamma$ such that $\gamma_n^{-1} 0 = a_n \to a \in B$. We know that $\gamma_n = \beta_n T_{a_n}$ with $\beta_n \in O(n)$. By passing to a subsequence we may assume that $\beta_n \to \beta$ and hence $\gamma_n \to \beta T_a$. But then $\gamma_m \gamma_n^{-1} \to I$ when $m, n \to \infty$ which implies $\gamma_m = \gamma_n$ except for finitely many pairs m,n. This is a contradiction. We conclude that $\gamma 0$ tends to S^{n-1} when γ runs through an infinite group Γ. It is equally true that γa tends to S^{n-1} for any $a \in B$. In fact, since $d(\gamma a, \gamma b) = d(a,b)$ is a finite constant the n.e. distance from 0 to γa will also tend to infinity.

More generally, if $K \subset B$ is compact there are only finitely many $\gamma \in \Gamma$ such that $K \cap \gamma K \neq \emptyset$.

7.2. Two points a and γa, $\gamma \in \Gamma$, are called equivalent, and one can pass to the quotient B/Γ by identification of equivalent points; we shall denote it by $\mathfrak{M}(\Gamma)$. It is definitely a Hausdorff space, and for $n=2$ and $n=3$ it is known to be a manifold; for $n>3$ it is still referred to as the quotient-manifold, but it probably need not be a manifold.

The difficulty comes from the fixed points. Suppose a is a fixed point of γ . Then $|\gamma'(a)| = 1$ as seen from

$$\frac{|\gamma'(a)|}{1-|\gamma a|^2} = \frac{1}{1-|a|^2} .$$

In other words, a lies on the isometric sphere $K(\gamma)$ (see 2.9), provided that $a \neq 0$. If $a = 0$ then $\gamma \in SO(n)$ and the fixed points of γ fill some k-dimensional plane with $0 \leq k \leq n-2$. Hence an arbitrary γ fixes some k-dimensional geodesic subspace. As is seen by the above equation we have $|\gamma'(a)| \to 0$ uniformly for all a from any compact set inside B^n . Therefore these subspaces tend to the boundary. We conclude that every point in B has a neighbourhood which meets only a finite number of these fixed subspaces.

7.3. We adopt the following definition:

<u>Definition:</u> A point $b \in \bar{B}$ is called a <u>limit point</u> of Γ if there exists an infinite sequence of $\gamma_\nu \in \Gamma$ and a point $a \in \bar{B}$ such that $\gamma_\nu a \to b$.

The set of all limit points of Γ is the <u>limit set</u> $\Lambda = \Lambda(\Gamma)$. The set of accumulation points of Γa is denoted by $\Lambda(a)$. Clearly, $\Lambda = \cup \Lambda(a)$ The following is true:

<u>Theorem.</u> $\Lambda = \Lambda(a)$ for all $a \in \bar{B}$ (with a few trivial exceptions).

<u>Proof.</u> If $a, b \in B$ it is trivial that $\Lambda(a) = \Lambda(b)$ because the distance $d(\gamma a, \gamma b) = d(a,b)$ is fixed. In particular $\Lambda(a) = \Lambda(0)$.

Consider now the case $a \in S$. We shall assume that $\Gamma a \neq \{a\}$, i.e. a is not a fixed point for the whole group. There is then a $b = \gamma_0 a \neq a$, $\gamma_0 \in \Gamma$.

1) We prove first that $\Lambda(0) \subset \Lambda(a)$. For this purpose let c be an interior point of the geodesic (a,b) . We know that $\Lambda(c) = \Lambda(o)$. For every $e \in \Lambda(o)$ there is thus

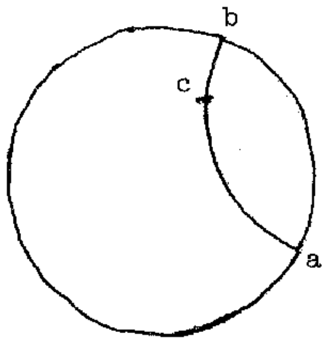


Fig. 14

a sequence of $\gamma_\nu \in \Gamma$ with $\gamma_\nu c \to e$. We can pass to a subsequence for which $\gamma_\nu a$ and $\gamma_\nu b$ converge to limits a' and b' . If $a' = b'$ it is clear that $e = a'$. On the other hand, if $a' \neq b'$ the $\gamma_\nu c$ must tend to the geodesic (a',b') but also to S , and hence either to a' or b' . Because $b = \gamma_0 a$ both a' and b' belong to $\Lambda(a)$. Thus $e \in \Lambda(a)$ and we have proved that $\Lambda(o) \subset \Lambda(a)$.

2) For the opposite conclusion choose an arbitrary $d \in \Lambda(a)$ and a sequence $\{\gamma_\nu\}$ such that $\gamma_\nu a \to d$ while $\gamma_\nu 0 \to e$ and $\gamma_\nu^{-1} 0 \to e'$. Recall that $K(\gamma)$, $E(\gamma)$, $I(\gamma)$ are characterized by $|\gamma'(x)| = 1$, $|\gamma'(x)| < 1$ and $|\gamma'(x)| > 1$. (see 2.9). Because $0 \in E(\gamma_\nu)$ its image $\gamma_\nu 0$ lies in $I(\gamma_\nu^{-1})$ and $\gamma_\nu^{-1} 0$ lies in $I(\gamma_\nu)$. It follows that $I(\gamma_\nu^{-1})$ tends to e and $I(\gamma_\nu)$ to e' .

If $a \neq e'$ then $a \in E(\gamma_\nu)$ for large ν and hence $\gamma_\nu a \in I(\gamma_\nu^{-1}) \to e$, i.e. $d = e \in \Lambda(o)$. But if $a = e' \in \Lambda(o)$ then $\gamma_\nu a \in \Lambda(0)$ because $\gamma_\nu \Lambda(0) = \Lambda(0)$. But $\Lambda(0)$ is closed and we find again that $d \in \Lambda(0)$. Since d was an arbitrary point in $\Lambda(a)$ we have proved that $\Lambda(a) \subset \Lambda(0)$.

The proof fails when a is a fixed point for all of Γ . There are at most two such points and all such groups can be classified (the elementary groups).

7.4. We list the following properties of Λ .

1.) Λ is closed because $\Lambda(0)$ is closed. The complement $\Omega = \bar{B} \setminus \Lambda$ is open and is called the <u>set of discontinuity</u>.

2.) Λ is invariant under Γ, and every invariant closed set contains Λ.

3.) Either $\Lambda = S$ or Λ is nowhere dense on S. Γ is said to be of the first kind if $\Lambda = S$, otherwise of the second kind.

4.) Λ has no isolated points; it is a perfect set.

5.) Let A and B be compact sets in Ω. Then A meets only a finite number of $\gamma B, \gamma \in \Gamma$.

The proofs of these statements are all very easy and are left to the reader.

7.5. Assume that the identity is the only element of $\Gamma \cap SO(n)$; in other words, O is not a fixed point. Then every $\gamma \in \Gamma \setminus I$ has an isometric sphere $K(\gamma)$ with corresponding $E(\gamma)$ and $I(\gamma)$. In what follows these notations will refer only to the part contained in B. Thus $K(\gamma)$ is a non-euclideon hyperplane, and $E(\gamma)$, $I(\gamma)$ are hyperbolic half-spaces.

We shall consider the set

$$P = \bigcap_{\gamma \in \Gamma \setminus I} E(\gamma)$$

and prove that it has the following properties:

(i) P is open

(ii) $\partial P \subset \bigcup K(\gamma)$ (here ∂P is the boundary in B).

(iii) every $x \in B$ is equivalent to either a single point in P or to at least one and at most finitely many points on ∂P.

(i) and (ii) follow from the fact that every intersection $P \cap B(r)$ is automatically contained in all but a finite member of the $E(\gamma)$ and is thus a finite intersection.

Recall that $K(\gamma), E(\gamma)$ and $I(\gamma)$ are characterized by $|\gamma'(x)| = 1$, $|\gamma'(x)| < 1$ and $|\gamma'(x)| > 1$ respectively, and that $|\gamma'(x)|/(1-|\gamma x|^2) = 1/(1-|x|^2)$.

It follows that $x \in E(\gamma)$ is equivalent to $|\gamma x| > |x|$ while $|\gamma x| = |x|$ if and only if $x \in K(\gamma)$. In other words, a point is in P if and only if it is strictly closer to 0 than all its images, and it lies on ∂P if there are several closest points (including the case of a closest fixed point). Since the points in an orbit are isolated the existence of at least one and at most finitely many closest points is obvious, and property (iii) follows. Observe the rather remarkable fact that equivalent points on ∂P are all equidistant from 0.

Property (iii) characterizes P as a fundamental set. As an intersection of half-spaces it is convex in the n.e. sense. P is a n.e. polyhedron and it is referred to as the Poincare'(or Dirichlet) fundamental polyhedron of Γ with respect to the origin. The faces of P are the $(n-1)$-dimensional intersections $\partial P \cap K(\gamma)$. The faces on $K(\gamma)$ and $K(\gamma^{-1})$ are equivalent and congruent in the euclidean and non-euclidean sense.

7.6. Convergence and divergence. We are interested to know how fast the points in an orbit tend to S or, which is the same thing, how fast the orbits tend to infinity in the hyperbolic sense.

The first observation is that any two orbits Γa and Γb are comparable in the sense that the ratios

$$\frac{1-|\gamma a|}{1-|\gamma b|}$$

lie between finite limits.

In fact, from $d(\gamma a, \gamma b) = d(a,b)$ it follows that

$$d(0,\gamma b) \leq d(0,\gamma a) + d(a,b)$$

or

$$\log \frac{1+|\gamma b|}{1-|\gamma b|} \leqq \log \frac{1+|\gamma a|}{1-|\gamma a|} + d(a,b)$$

$$\therefore \quad \frac{1+|\gamma b|}{1-|\gamma b|} \leqq e^{d(a,b)} \frac{1+|\gamma a|}{1-|\gamma a|} .$$

from which we deduce that

$$1-|\gamma a| \leqq 2e^{d(a,b)}(1-|\gamma b|)$$

for all γ .

A good way to study the density of an orbit is to investigate the divergence or convergence of series of the form

$$\sum_{\gamma\in\Gamma} (1-|\gamma a|)^{\alpha}$$

for different powers α .

We prove first:

Lemma 1. Every discrete group Γ satisfies

(1) $$\sum_{\gamma\in\Gamma} (1-|\gamma a|)^{\alpha} < \infty$$

for all $\alpha > n-1$.

Proof. It is sufficient to consider the case $a=0$. As before, P denotes the Poincare polyhedron of Γ .

Because $\alpha > n-1$

$$\int_B (1-|x|^2)^{\alpha-n} dx = c < \infty \quad .$$

Since $B = \cup \gamma P$ up to a null-set

$$c = \sum_{\gamma \in \Gamma} \int_{\gamma P} (1 - |x|^2)^{\alpha - n} dx = \sum_{\gamma} \int_{P} (1 - |\gamma x|^2)^{\alpha - n} |\gamma'(x)|^n dx \quad .$$

As usual we write $\gamma^{-1} 0 = a$. Then

$$|\gamma'(x)| = |T_a'(x)| = \frac{1 - |a|^2}{[x,a]^2}$$

and

$$(1 - |\gamma x|^2) = (1 - |T_a x|^2) = \frac{(1 - |a|^2)(1 - |x|^2)}{[x,a]^2} \quad .$$

This leads to

$$c = \sum_{\gamma \in \Gamma} \int_P \frac{(1 - |x|^2)^{\alpha - n} (1 - |a|^2)^{\alpha}}{[x,a]^{2\alpha}} dx \quad .$$

Here $[x,a] \leq 2$ as seen from $[x,a]^2 = 1 + |x|^2 |a|^2 - 2xa \leq 4$.

Therefore we obtain

$$(2) \qquad \sum_{\gamma \in \Gamma} (1 - |a|^2)^{\alpha} \leq \frac{2^{2\alpha} c}{\int_P (1 - |x|^2)^{\alpha - n} dx} < \infty$$

and the Lemma is proved.

Observe that (2) even gives a **computable** upper bound for the series. Moreover, because

$$1 - |\gamma x|^2 = (1 - |x|^2) |\gamma'(x)|$$

we conclude that

(3) $$\sum_{\gamma\in\Gamma} |\gamma'(x)|^{\alpha} < \infty \quad ,$$

7.7. For $\alpha = n-1$ the series may or may not converge, and accordingly the group Γ is said to be of <u>convergence type</u> or <u>divergence type</u>. There is a nice geometric characterization of the two types.

Consider the orbit of a ball B_o, for instance one that is so small that the images γB_o are disjoint.

Let a be the n.e. center and ρ the n.e. radius of B_o. For present and future use we shall determine the euclidean radius and center of B_o.

The diametrically opposite points of B_o on the line from 0 through a are at the (signed) euclidean distances

$$\tan h \ \frac{1}{2}\left(\log \frac{1+|a|}{1-|a|} \pm \rho\right) = \frac{|a| \pm \operatorname{th}\frac{\rho}{2}}{1 \pm |a| \operatorname{th}\frac{\rho}{2}}$$

from 0 . By use of these expressions we see that the euclidean radius is

$$r = \frac{1}{2}\left(\frac{|a| + \operatorname{th}\frac{\rho}{2}}{1+|a| \operatorname{th}\frac{\rho}{2}} - \frac{|a| - \operatorname{th}\frac{\rho}{2}}{1-|a| \operatorname{th}\frac{\rho}{2}}\right)$$

(4) $$= \frac{(1-|a|^2)\operatorname{th}\frac{\rho}{2}}{1-|a|^2 \operatorname{th}^2\frac{\rho}{2}}$$

and that the center c is given by

(5) $$|c| = \frac{|a|(1-\operatorname{th}^2\frac{\rho}{2})}{1-|a|^2 \operatorname{th}^2\frac{\rho}{2}} \quad .$$

As $|a| \to 1$ we see from (4) that r behaves asymptotically like a constant times $(1-|a|)$ and the surface area of B_0 behaves like a constant times $(1-|a|)^{n-1}$.

We conclude:

Γ <u>is of convergence type if and only if the sum of the surface areas of the</u> γB_0 <u>is finite.</u>

Instead of looking at the areas of the γB_0 one can also look at the areas of their central projections or "shadows" on $S(1)$.

7.8. Recall that Γ is said to be of the second kind if $S \setminus \Lambda$ is not empty. We prove now:

<u>Lemma 2.</u> Every group of the second kind is of convergence type.

<u>Proof.</u> There exists a spherical cap C with $\bar{C} \subset \Omega$. Because $\bar{C}$ is compact C meets only a finite number of γC . It follows that

(6) $$\sum_{\gamma} \int_{C} |\gamma'(x)|^{n-1} d\omega(x) < \infty \quad .$$

With the usual notation $\gamma^{-1}0 = a$ we know that

(7) $$|\gamma'(x)| = \frac{1-|a|^2}{[x,a]^2} \geq \frac{1-|a|^2}{4} \quad .$$

From (6) and (7) we conclude at once that $\sum_{a=\gamma 0} (1-|a|)^{n-1} < \infty$.

7.9. The type of a group Γ is closely related to the existence of a Green's function on $\mathfrak{M}(\Gamma)$. Any function on $\mathfrak{M}(\Gamma)$ can be viewed as the projection of a function on B which is automorphic with respect to Γ . The Green's function on $\mathfrak{M}(\Gamma)$ with pole at the projection of x_0 is defined to be the projection of a function $g_{\Gamma}(x)$ with the following properties:

1° g_Γ is h-h for $x \in B \setminus \Gamma x_0$

2° g_Γ is automorphic: $g_\Gamma(\gamma x) = g_\Gamma(x)$

3° $\lim_{x \to x_0} (g_\Gamma(x) - g(x,x_0)) =$ exists

4° g_Γ is the smallest positive function with these properties.

For simplicity we specialize to the case $x_0 = 0$. The function on $\mathfrak{m}(\Gamma)$ which corresponds to g_Γ will be denoted by $\tilde{g}_\Gamma$.

Theorem 1. Γ is of convergence type if and only if $\mathfrak{m}(\Gamma)$ has a Green's function.

Proof:

1) We assume that $\sum_\Gamma (1-|a|)^{n-1} < \infty$ where, as before, $a = \gamma^{-1}0$. We show that

$$g_\Gamma(x) = \Sigma\, g(x,a) = \Sigma\, g(|T_a x|)$$

converges and possesses the properties 1° - 4°. The convergence follows from

$$g(|T_a x|) = O\,((1-|T_a x|^2)^{n-1})$$

and

$$1-|T_a x|^2 = \frac{(1-|a|^2)(1-|x|^2)}{[x,a]^2} \leq \frac{(1+|x|)}{1-|x|}\;(1-|a|^2) \quad .$$

It is obvious that g_Γ has the properties 1° - 3°. To prove 4° let h be any function with the properties 1° - 2° and let g_Γ^N be a partial sum of the series g_Γ . We have $g_\Gamma^N = 0$ at the boundary. By the maximum principle for h - h functions $g_\Gamma^N \leqq h$, and hence also $g_\Gamma(x) \leqq h(x)$.

2) Assume now that g_Γ exists. We denote by $n(r)$ the number of points

$a_\nu = \gamma_\nu 0$ in the ball $B(r)$; we choose r so that no $|a_\nu| = r$.

Choose ρ so small that the balls $B(a_\nu,\rho)$ do not intersect and are contained in $B(r)$. We apply (17) in (5.5) to conclude that

$$(8) \qquad \int_{S(r)\cup S(a_\nu,\rho)} \frac{\partial g_T}{\partial n_h}\, d\sigma_h = 0 \quad .$$

We let $\rho \to 0$ and observe that the integral over $S(a_\nu,\rho)$ is the same as

$$\int_{S(a_\nu,\rho)} \frac{\partial g(x,a_\nu)}{\partial n_h}\, d\sigma_h$$

and this in turn has the same limit as

$$(9) \qquad \frac{2^{n-2}}{(1-|a_\nu|^2)^{n-2}} \int_{S(a_\nu,\rho)} \frac{\partial g(x,a_\nu)}{\partial n}\, d\sigma \quad .$$

Here $g(x,a_\nu) = g(|T_{a_\nu} x|)$ and

$$\frac{\partial g(x,a_\nu)}{\partial n} = g'(|T_{a_\nu} x|)\, \frac{\partial |T_{a_\nu} x|}{\partial n} = \left(\frac{1-|T_{a_\nu} x|^2}{|T_{a_\nu} x|}\right)^{n-2} \frac{\partial}{\partial\rho} \log |T_{a_\nu} x| =$$

$$\left(\frac{(1-|a_\nu|^2)(1-|x|^2)}{[x,a_\nu]}\right)^{n-2} \frac{1}{\rho^{n-2}}\, \frac{\partial}{\partial\rho} \log \frac{\rho}{[x,a_\nu]}$$

$$\sim (1-|a_\nu|^2)^{n-2}\, \frac{1}{\rho^{n-1}} \quad .$$

It follows that (9) tends to ω_n and we conclude from (8) that

$$n(r) = -\frac{1}{\omega_n} \int_{S(r)} \frac{\partial g_T}{\partial n_h}\, d\sigma_h = -\frac{1}{\omega_n}\, \frac{r^{n-1}}{(1-r^2)^{n-2}} \int_{S(r)} \frac{\partial g_T}{\partial r}\, d\omega \quad .$$

We can now integrate to obtain

$$\frac{1}{\omega_n} \int_{S(1)} (g_\Gamma(r_o\xi) - g_\Gamma(r\xi))\, d\omega(\xi) = \int_{r_o}^{r} \frac{(1-t^2)^{n-2}}{t^{n-1}}\, n(t)dt \quad .$$

Here $g_\Gamma(r\xi) > 0$ while $g_\Gamma(r_o\xi)$ is independent of r . We have proved that the integral

$$\int_{r_o}^{1} \frac{(1-t^2)^{n-2}}{t^{n-1}}\, n(t)dt$$

converges. This means that

$$(10) \qquad \int_0^1 (1-t)^{n-2}\, n(t)dt < \infty \quad .$$

It is a very familiar fact, proved by integration by parts, that the integral (10) converges together with

$$\int_0^1 (1-t)^{n-1} dn(t)$$

which is nothing else than

$$\Sigma(1 - |a_\nu|)^{n-1} < \infty$$

The theorem is proved.

7.10. In the theory of Riemann surfaces it is customary to say that a surface is of class O_G if it has no Green's function. There is no reason why this terminology should not be carried over to arbitrary Riemannian manifolds and even to those quotient manifolds $\mathfrak{M}(\Gamma)$ which are perhaps no manifolds. Our theorem states that Γ is of divergence type if and only if $\mathfrak{M}(\Gamma) \subset O_G$.

Similarly, a Riemann surface is said to be of class O_{HB} if it carries no bounded harmonic functions other than the constants. An important theorem of P. J. Myrberg states that

$$O_G \subset O_{HB} \quad .$$

In other words, if there is a non-trivial bounded harmonic function, then there is also a Green's function with an arbitrary pole. The opposite inclusion is not true.

The terminology and the theorem carry over to arbitrary Riemannian spaces. The proof remains essentially the same as for Riemann surfaces. We refer to the proof in Ahlfors - Sario, Riemann surfaces, p. 204 - 206.

We can therefore state:

Theorem 2. If Γ is of divergence type there are no non-constant bounded harmonic functions on $\mathfrak{M}(\Gamma)$.

7.11. Let G be a transformation group acting on a measure space $\mathfrak{M}$. One says that G acts ergodically on $\mathfrak{M}$ if every invariant subset of $\mathfrak{M}$ is either a null-set or the complement of a null-set.

Theorem 3. If Γ is of divergence type, then Γ acts ergodically on S .

Proof. If not there would exist an invariant measurable set $E \subset S$ with $0 < m(E) < m(S)$. Use the Poisson formula (27) in (5.7) to construct the harmonic function whose radial boundary values are given by the characteristic function χ of E . The function is

$$u(y) = \frac{1}{\omega_n} \int_S \left(\frac{1 - |y|^2}{|x - y|^2} \right)^{n-1} \chi(x) \, d\omega(x)$$

Because $\chi(\gamma x) = \chi(x)$ for $\gamma \in \Gamma$ one finds

$$u(\gamma y) = \frac{1}{\omega_n} \int_S \left(\frac{1 - |\gamma y|^2}{|\gamma x - \gamma y|^2} \right)^{n-1} \chi(\gamma x) |\gamma'(x)|^{n-1} \, d\omega(x)$$

$$= \frac{1}{\omega_n} \int_S \left(\frac{1-|y|^2}{|x-y|^2} \right)^{n-1} \chi(x)\; d\omega(x) = u(y)$$

where we have used the identities $1-|\gamma y|^2 = |\gamma'(y)|(1-|y|^2)$ and $|\gamma x - \gamma y|^2 = |\gamma'(x)|\,|\gamma'(y)|\,|x-y|^2$.

The function u is automorphic, h - h , and non-constant. This contradicts Theorem 2. Hence the set E cannot exist.

7.12. Consider a ball $B_o = B(r_o) = B_h(\rho_o)$ about the origin. We denote by $L(B_o)$ the set of all $\xi \in S$ such that the radius to ξ meets infinitely many γB_o , $\gamma \in \Gamma$. Another way of saying the same thing is that $\xi \in L(B_o)$ if there are infinitely many points $\gamma 0$ at non-euclidean distance $< \rho_o$ from the radius $(0,\xi)$. Geometrically, this means that there are infinitely many $\gamma 0 = a_\nu$ inside the lens-shaped region shown in Fig.15 arbitrarily close to ξ . In

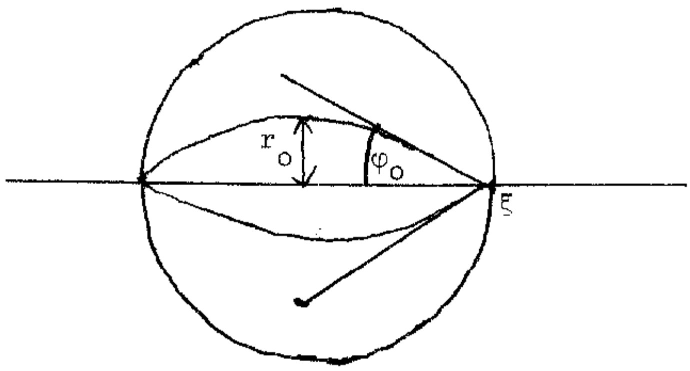


Fig. 15

particular, there are then infinitely many a_ν in the Stolz cone with the opening $\varphi_o = 2$ arc tan r_o . If this is true for some B_o then ξ is said to be a <u>conical limit point</u> and the set

$$L = \cup\, L(B_o)$$

is the <u>conical limit set</u> of Γ . Obviously, $L \subset \Lambda$.

If ξ is a conical limit point for the orbit ΓO, then it is also a conical limit for any orbit Γa. Indeed, if γO is at n.e. distance $\leq \rho_0$ from the radius (O,ξ), then γa is at distance $\leq \rho_0 + d(o,a)$ from the same radius.

This leads to another characterization of conical limits. The radius (O,ξ) is a geodesic and it projects to a geodesic on $\mathfrak{M}(\Gamma)$. If ξ is a conical limit this geodesic will come again and again within some fixed distance from any given point. On the contrary, if ξ is not a conical limit, then the geodesic will eventually leave any compact set (the geodesic tends to the "ideal boundary"). In particular, if $\mathfrak{M}(\Gamma)$ is compact, then all $\xi \in S$ are conical limits.

<u>Lemma 3</u>. If Γ is of convergence type, then $m(L) = 0$.

<u>Proof.</u> As usual the orbit ΓO will be written as a sequence $\{a_\nu\}$. Assuming convergence there exists, for every $\epsilon > 0$, a ν_0 such that

$$(11) \qquad \sum_{\nu > \nu_0} (1 - |a_\nu|)^{n-1} < \epsilon \quad .$$

Assume that $\xi \in L(B_0)$. There is then an a_ν with $\nu > \nu_0$ inside the Stolz cone with angle φ_0 arbitrarily close to ξ. Let α_ν be the angle between ξ and a_ν. The fact that a_ν is in the Stolz cone translates by trigonometry to

$$\frac{2 \sin \frac{\alpha_\nu}{2}}{1 - |a_\nu|} \leq \frac{\sin (\varphi_0 + \alpha_\nu)}{\cos (\varphi_0 + \frac{\alpha_\nu}{2})}$$

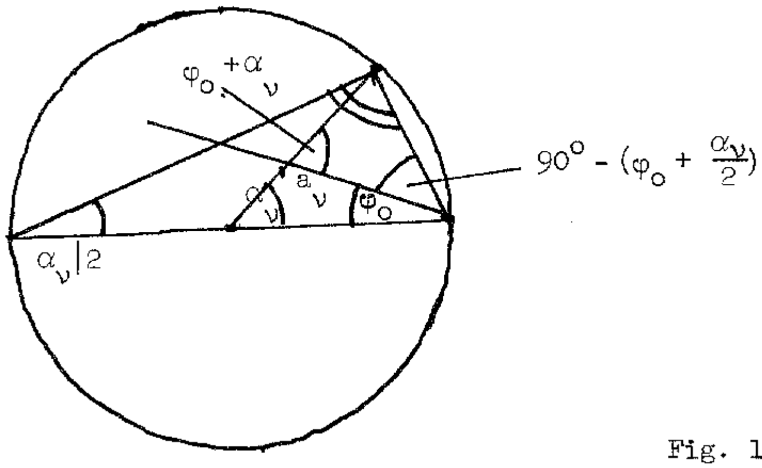


Fig. 16

In the limit

(12) $$\lim \frac{\alpha_\nu}{1 - |a_\nu|} \leq \tan \varphi_o \quad .$$

The point ξ is contained in the spherical cap with center $a_\nu / |a_\nu|$ and radius α_ν . This ξ is covered by all these caps with $\nu > \nu_o$. The area of each cap is asymptotically proportional to α_ν^{n-1} which is majorized by $(1 - |a_\nu|)^{n-1}$. Hence the measure of $L(B_o)$ is $<$ constant times ϵ by (11), and consequently equal to zero. Finally, L is a union of countably many $L(B_o)$, and we have proved that $m(L) = 0$.

7.13. The conical limit set is in any case invariant under Γ . For if $\xi \in L$ and $\gamma \in \Gamma$ then γ maps a Stolz cone at ξ on a set contained in a slightly larger Stolz cone at $\gamma \xi$.

If Γ is of divergence type we conclude that either $m(L) = 0$ or $m(L) = \omega_n$. Thus, for <u>any</u> group Γ only the extreme cases can occur.

We shall eventually prove that $m(L) = \omega_n$ for all Γ of divergence type so that the distinction between convergence and divergence is the same as between $m(L) = 0$ and $m(L) = \omega_n$. However, it is appropriate at this time to say something about the history of the problem.

Originally the problem dealt with the geodesics on a closed Riemann surface. In the 1930's important groundwork was done by Morse, Hedlund, Myrberg and many others. A very decisive step was taken by Eberhard Hopf (1936) who proved that Γ acts ergodically on $S \times S$ in the case of a compact Riemann surface with or without punctures. The action is the one that takes (ξ, η) to $(\gamma\xi, \gamma\eta)$. This is of course very much stronger than ergodicity on S.

It did not take long for Hopf to realize that his theorem can be extended to several dimensions in a much more general form, and that finite volume is not the right condition. In 1939 he proved the following:

Theorem. (E. Hopf) Γ acts ergodically on $S \times S$ if and only if $m(L) = \omega_n$.

The proof makes essential use of Birkhoff's individual ergodic theorems and cannot be considered elementary.

The most recent development of the theory is due to Dennis Sullivan (1978).

Theorem. (D. Sullivan) Γ acts ergodically on $S \times S$ if and only if Γ is of divergence type.

His proof uses Markov chains, and I must confess that I do not understand it. However, there are now in existence relatively elementary proofs of the fact that every Γ of divergence type satisfies $m(L) = \omega_n$. Such proofs have been given both by Sullivan and Thurston. In what follows we shall give a rather detailed version of Thurston's proof.

7.14. The action of Γ on $S \times S$ is defined by $\gamma(\xi, \eta) = (\gamma\xi, \gamma\eta)$. This action is said to be dissipative if there exists a measurable set $\Delta \subset S \times S$ such that

1) $\Delta \cap \gamma\Delta = \emptyset$ for all $\gamma \in \Gamma \setminus I$

2) $m(\bigcup_{\gamma \in \Gamma} \gamma\Delta) = m(S \times S) = \omega_n^2$

Briefly, Δ is a measurable fundamental set.

Lemma 4. If $m(L) = 0$ then the action of Γ on $S \times S$ is dissipative.

Proof. Suppose that $(\xi, \eta) \in (S - L) \times (S - L) \setminus$ diagonal. Then

(13) $$\lim_{\nu\to\infty} \frac{1 - |a_\nu|}{|\xi - a_\nu|} \to 0 \quad , \quad \lim_{\nu\to\infty} \frac{1 - |a_\nu|}{|\eta - a_\nu|} \to 0$$

by the definition of L .

As before $\gamma_\nu^{-1} 0 = a_\nu$ and hence

$$|\gamma_\nu'(\xi)||\gamma_\nu'(\eta)| = \frac{(1 - |a_\nu|^2)^2}{|\xi - a_\nu|^2 |\eta - a_\nu|^2} \quad \text{by (43) in (2.9).}$$

Either $|\xi - a_\nu| \geqq \frac{1}{2} |\xi - \eta|$ or $|\eta - a_\nu| \geqq \frac{1}{2} |\xi - \eta|$.

This implies

$$|\gamma_\nu'(\xi)||\gamma_\nu'(\eta)| \leqq \frac{4}{|\xi - \eta|^2} \max \left(\frac{1 - |a_\nu|}{|\xi - a_\nu|}\right)^2 , \left(\frac{1 - |a_\nu|}{|\eta - a_\nu|}\right)^2$$

Therefore, by (13), $|\gamma_\nu'(\xi)| \cdot |\gamma_\nu'(\eta)| \to 0$ and it follows that there is $\gamma_0 \in \Gamma$ for which $|\gamma_0'(\xi)| \cdot |\gamma_0'(\eta)|$ is a maximum.

Equivalently, since

$$|\gamma\xi - \gamma\eta| = |\gamma'(\xi)|^{1/2} |\gamma'(\eta)|^{1/2} |\xi - \eta| \quad .$$

this also means that $|\gamma_0 \xi - \gamma_0 \eta|$ is a maximum. In other words, if $\xi_0 = \gamma_0 \xi$, $\eta_0 = \gamma_0 \eta$ then

(14) $$|\xi_0 - \eta_0| \geqq |\gamma\xi_0 - \gamma\eta_0|$$

for all $\gamma \in \Gamma$, and this is equivalent to

(15) $$|\gamma'(\xi_0)||\gamma'(\eta_0)| \leqq 1 \quad .$$

For simplicity, let us assume that the origin is not a fixed point. We claim that the set

$$\Delta = \{(\xi,\eta) \mid |\gamma'(\xi)|\cdot|\gamma'(\eta)| < 1 \text{ for } \gamma\in\Gamma\setminus I\}$$

has properties 1) and 2). In the first place if (ξ,η) and (ξ_o,η_o) are equivalent they cannot both be in Δ for that would imply $|\xi_o-\eta_o| > |\xi-\eta|$ and $|\xi-\eta| > |\xi_o-\eta_o|$. Secondly, our construction has shown that every (ξ,η) is equivalent to a $(\xi_o,\eta_o)\in\Delta$ except in the following cases:

a) $\xi=\eta$

b) $\xi\in L$ or $\eta\in L$

c) $|\gamma'(\xi)|\cdot|\gamma'(\eta)| = 1$ for some $\gamma\in\Gamma\setminus I$.

It is clear that a) and c) are true only on a null-set, and if $m(L)=0$ the same is the case of b). The proof is complete.

(The idea of this proof is due to Sullivan)

7.15. We embark now on Thurston's proof of the following fact:

Theorem 4. If $m(L)=0$, then Γ is of convergence type.

Remark. This is a consequence of Sullivan's theorem. On the other hand, Theorem 4 makes Sullivan's theorem a consequence of Hopf's theorem.

Proof. (Thurston). We consider again a ball $B_o = B(\rho_o)$ centered at the origin and we denote its images $\gamma_\nu^{-1}B_o$ with B_ν and the shadow of B_ν with B'_ν. The measure of a set $E\subset S$ will now be denoted by $|E|$ rather than $m(E)$.

The condition $m(L)=0$ can be written in the form

(16) $$\lim_{\nu_o\to\infty} \Big|\bigcup_{\nu>\nu_o} B'_\nu\Big| = 0 .$$

Indeed, let χ_{ν_0} denote the characteristic function of $\bigcup_{\nu > \nu_0} B'_\nu$. Then (16) means that

$$\lim_{\nu_0 \to \infty} \int_S \chi_{\nu_0}(\xi) d\omega(\xi) = 0$$

while $|L| = 0$ means that

$$\int_S \lim_{\nu_0 \to \infty} \chi_{\nu_0}(\xi) d\omega(\xi) = 0 \quad .$$

These conditions are simultaneously fulfilled by virtue of Fatou's lemma.

On the other hand, the assertion that Γ is of convergence type is, as we have seen, equivalent to

$$(17) \qquad \sum_{\nu=0}^{\infty} |B'_\nu| < \infty \quad .$$

It is evident that (16) implies (17) provided that the B'_ν do not overlap too much. The idea of the proof is to show that this is the case.

Step 1. As a first simplification we show that one can discard many of the B_ν. For this purpose we choose a number $\rho > 0$, which will ultimately be large, and use it to define a subsequence $\{\nu_k\}_0^\infty$ as follows:

1° Choose $\nu_0 = 0$.

2° Suppose that $\nu_0, \ldots, \nu_k$ have been chosen so that the mutual distances $d(a_{\nu_i}, a_{\nu_j})$ are all $> \rho$. Then choose ν_{k+1} as the smallest ν such that $d(a_{\nu_i}, a_{\nu_{k+1}}) > \rho$ for $i = 0, \ldots, k$.

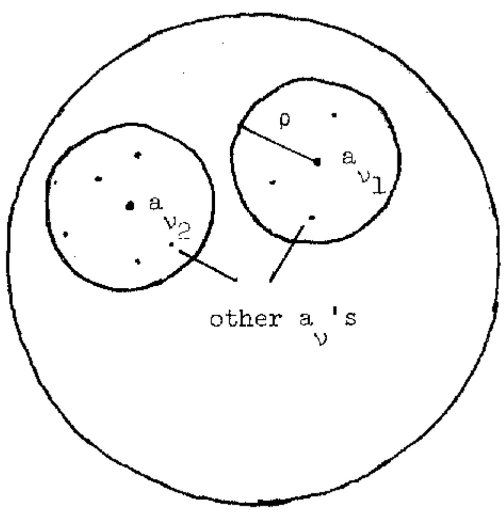


Fig. 17

This choice is always possible and we see that the following is true:

a) $d(a_{\nu_h}, a_{\nu_k}) > \rho$ for all $h \neq k$.

b) to every ν there exists a ν_k such that $d(a_\nu, a_{\nu_k}) \leqq \rho$.

There are only a finite number N of a_ν with $d(a_\nu, 0) \leqq \rho$ and hence also only N points a_ν with $d(a_\nu, a_{\nu_k}) \leqq \rho$. When this is the case then

$$\frac{1 - |a_\nu|}{1 - |a_{\nu_k}|} \leqq M$$

where M depends only on ρ. If $\Sigma (1 - |a_{\nu_k}|)^{n-1} < \infty$ it follows that $\Sigma (1 - |a_\nu|)^{n-1} < \infty$.

For simplicity we can return to the notation a_ν for (16) remains true and it is sufficient to prove (17) for the subsequence.

Step 2. We choose ρ so large that the B_ν do not overlap. We place an observer at the origin and speak of total or partial eclipses when two shadows overlap. The B_ν will be subdivided into classes depending on the number of times they are eclipsed.

The class I_0 consists only of B_0. We remove B_0 and define I_1 as the class of all B_ν that are completely visible from 0. Next we remove all $B_\nu \in I_1$ and define I_2 as the class of those B_ν which are now completely visible, and so on. Clearly, every B_ν will belong to a class I_m and the shadows of the $B_\nu \in I_m$ are disjoint.

It will be shown that

$$\sum_{I_{m+1}} |B'_\nu| \leqq \varkappa \sum_{I_m} |B'_\nu|$$

where $\varkappa < 1$. This will obviously imply (17).

Step 3. Every $B_j \in I_{m+1}$ is partially or totally eclipsed by a $B_i \in I_m$. Let r_i, r_j be the euclidean radii of B_i, B_j. We shall need an upper bound for the ratio r_j/r_i .

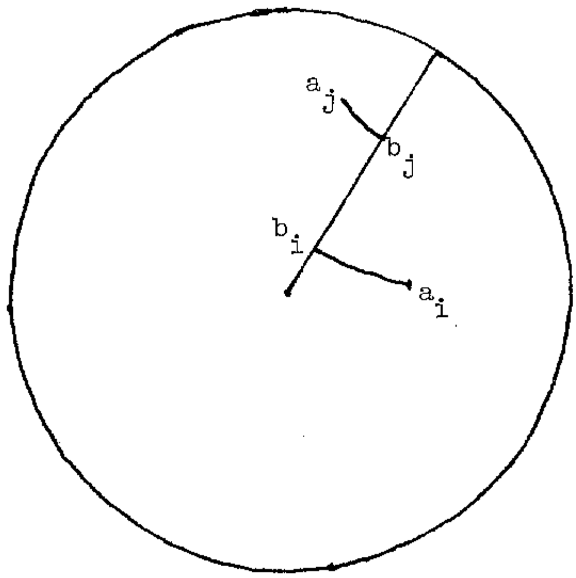


Fig. 18

We refer to the picture in which a_i and a_j are at n.e. distance $< \rho_o$ from the same radius. Let b_i, b_j be the n.e. orthogonal projections of a_i, a_j on the radius. We have then

$$\rho < d(a_i, a_j) < d(b_i, b_j) + 2\rho_o = d(O, b_j) - d(O, b_i) + 2\rho_o < d(O, a_j) - d(O, a_i) + 4\rho_o$$

or

$$d(O, a_j) > d(O, a_i) + \rho - 4\rho_o$$

and

$$\frac{1+|a_j|}{1-|a_j|} > \frac{1+|a_i|}{1-|a_i|} e^{\rho - 4\rho_0}$$

which implies

$$1-|a_j| < 2e^{-\rho + 4\rho_0}(1-|a_i|) \quad .$$

By use of formula (4) we finally find

$$\text{(18)} \qquad \frac{r_j}{r_i} < 4e^{-\rho+4\rho_0} \cosh \frac{2\rho_0}{2} \quad .$$

The main thing is that this quotient can be made arbitrarily small by choosing ρ large enough.

Consider now all the $B_j \in I_{m+1}$ which are partially eclipsed by $B_i \in I_m$. Their shadows B_j' are disjoint and they lie asymptotically within distance r_j from the rim of B_i' . Their total area is therefore asymptotically at most

$$A(r_i + r_j) - A(r_i - r_j) \sim \frac{r_j}{r_i} \ |B_i'| \quad .$$

(We have again used the notation $A(r)$ for the area of a spherical cap with radius r). In view of the estimate we can thus choose ρ so that, for sufficiently large m , the total area of the shadows B_j' of all $B_j \in I_{m+1}$ that are partially but not totally eclipsed by some $B_i \in I_m$ will be less than

$$\frac{1}{3} \sum_{B_i \in I_m} |B_i'| \quad .$$

Step. 4. We pass now to the $B_j \in I_{m+1}$ which are totally eclipsed by a $B_i \in I_m$. We need an auxiliary lemma.

Lemma 5. If $\xi \in B_i' \cap B_j'$ with $B_i \in I_m$, $B_j \in I_{m+1}$, then the geodesic (a_i, ξ) intersects the ball $\overline{B}_j$ with center a_j and n.e. radius $2\rho_o$, where ρ_o is the radius of B_o.

Proof. We map the unit ball conformally on the half-space H^n so that 0 goes to e_n and ξ to ∞. We keep the names of the points a_i, a_j. The geodesic $(0,\xi)$ becomes a vertical line through e_n and (a_i,ξ) a vertical through a_i whose intersection with $\mathbb{R}^{n-1}$ we shall denote by c.

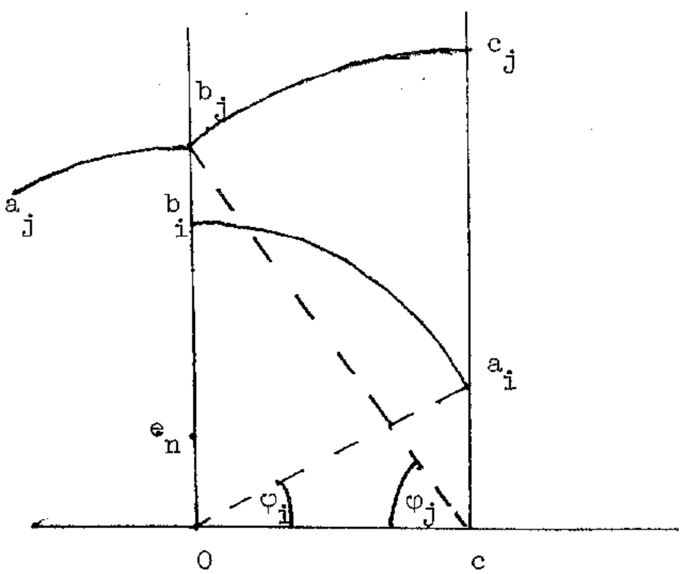


Fig. 19

Let b_i, b_j be the closest points to a_i, a_j on the vertical through 0

and let c_j be the closest point to b_j on the vertical through c (the picture is misleading in as much as a_j need not be in the same plane as the two verticals). The non-euclidean distances are computed by

$$d(a_i,b_i) = \int_{\varphi_i}^{\frac{\pi}{2}} \frac{d\varphi}{\sin\varphi} = \log\cot\frac{\varphi_i}{2}$$

$$d(b_j,c_j) = \int_{\varphi_j}^{\frac{\pi}{2}} \frac{d\varphi}{\sin\varphi} = \log\cot\frac{\varphi_j}{2} \quad .$$

But

$$\cos\varphi_i = \frac{|c|}{|a_i|} = \frac{|c|}{|b_i|}$$

$$\cos\varphi_j = \frac{|c|}{|c-b_j|} < \frac{|c|}{|b_j|} < \frac{|c|}{|b_i|} = \cos\varphi_i \quad \text{since} \quad |b_j| > |b_i|$$

so that $\varphi_j > \varphi_i$ and

$$d(b_j,c_j) < d(a_i,b_i) \leqq \rho_0 \quad .$$

It follows that

$$d(a_j,c_j) \leqq d(a_j,b_j) + d(b_j,c_j) \leqq 2\rho_0$$

and the lemma is proved.

The next picture shows on the left a B_i and some totally eclipsed B_j together with their shadows. On the right the whole configuration has been transformed by γ_i .

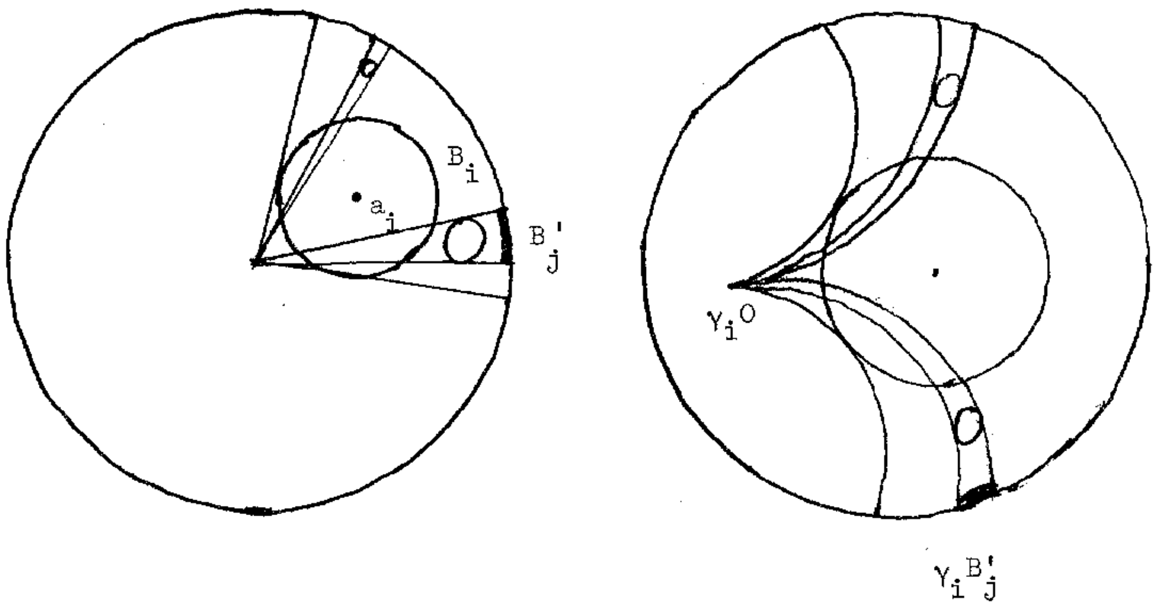


Fig. 20

The image $\gamma_i B_j'$ is not the same as $(\gamma_i B_j)'$. However, the lemma tells us that $\gamma_i B_j'$ is contained in $(\gamma_i \bar{B}_j)'$.

It is clear that condition (16) remains true when B_o is replaced by $\bar{B}_o$ and each B_j by $\bar{B}_j$. Therefore, as soon as m is big enough,

(19) $$\sum_j |\gamma_i B_j'| \leqq \sum_j |(\gamma_i \bar{B}_j)'| < \epsilon \quad .$$

On the other hand, if $\rho \nearrow \infty$ then $\gamma_i O$ approaches the boundary while the area of the shadow of B_o viewed from $\gamma_i O$ decreases to a positive limit α_o . In other words,

(20) $$|\gamma_i B_i'| > \alpha_o \quad .$$

But

$$|\gamma_i B_j'| > (\min |\gamma_i'(x)|)^{n-1} |B_j'|$$

(21) $$|\gamma_i B_i'| < (\max_i |\gamma_i'(x)|)^{n-1} |B_i'| .$$

where the minimum and maximum may be taken with respect to B_i' .

From (19), (20) and (21) it follows that

$$\sum_j |B_j'| < \frac{\max |\gamma_i'(x)|^{n-1}}{\min |\gamma_i'(x)|^{n-1}} \frac{\epsilon}{\alpha_o} |B_i'| .$$

Here the maximum of

$$|\gamma_i'(x)| = \frac{1 - |a_i|^2}{|x - a_i|^2}$$

is taken at the center and the minimum on the rim. The ratio tends to the same limit as $r_i/(1 - |a_i|)$ and we know this limit to be finite There is thus a constant K such that

$$\sum_j |B_j'| \leqq \frac{K \epsilon}{|\alpha_o|} |B_i'| .$$

We choose ϵ so small that the factor on the right is $< \frac{1}{3}$.

This lets us conclude that the shadows of the totally eclipsed $B_j \in I_{m+1}$ make up at most one third of the shadows of the $B_i \in I_m$. Together with our previous result for the partially eclipsed B_j we have proved that

$$\sum_{B_j \in I_{m+1}} |B_j'| < \frac{2}{3} \sum_{B_i \in I_m} |B_i'|$$

for all sufficiently large m . This in turn proves that the sum of all the $|B_i'|$ is finite, and hence that

$$\sum_{\nu=0}^{\infty} (1-|a_\nu|)^{n-1} < \infty \quad .$$

The theorem is proved.

8. Quasiconformal Deformations.

8.1 By way of motivation we shall first recall the basic properties of quasicomformal (q.c.) mappings in n-dimensions. A q.c. mapping is first of all a homeomorphism.

$$F: \ \Omega \to \Omega'$$

from one open set in $\mathbb{R}^n$ to another. One of several ways to impose the right regularity conditions is to require that F is absolutely continuous on lines (a.c.l.) . This means that the restriction of F to a.e. coordinate parallel line is absolutely continuous.

Under these conditions the partial derivatives $D_i F_j$ exist a.e. and we can form the Jacobian matrix

$$DF = || D_i F_j ||$$

a.e. The Jacobian determinant will be denoted by JF and

(1) $$XF = (JF)^{-\frac{1}{n}} DF$$

is the normalized Jacobian, while

(2) $$MF = (XF)^T . XF$$

is the symmetrized and normalized Jacobian. (X^T is the transpose of X) .
F is called K— q.c. if

(3) $$||XF||^2 = \operatorname{tr} MF \leq nK^2 \ .$$

It is possible to write

(4) $$MF = U^T \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix} U$$

where $U \in SO(n)$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$.

Clearly,

$$(5) \qquad XF = V\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} U$$

where V is another orthogonal matrix, although U and V are not **uniquely** determined in the case of equal eigen-values. Observe that $\lambda_1\lambda_2\cdots\lambda_n = 1$ and that

$$(6) \qquad nK^2 \geqq ||XF||^2 = \lambda_1^2 + \cdots + \lambda_n^2 \geqq n \quad .$$

This implies upper bounds on $\lambda_1, \lambda_n^{-1}$ and λ_1/λ_n , and sometimes K is used to denote one of these bounds.

8.2. Because q.c. mappings in n dimensions are difficult to handle it is reasonable to linearize by passing to the infinitesimal case. Let $F(x,t)$ be a one-parameter family of a.e. differentiable mappings which for $t \to 0$ has the development

$$(7) \qquad F(x,t) = x + tf(x) + o(t) \quad .$$

We denote differentiation with respect to t by a dot and write

$$(8) \qquad \dot{F}(x) = \dot{F}(x,0) = f(x) \quad .$$

We assume that (7) can be differentiated to yield

$$DF(x,t) = I + t\,Df(x) + o(t)$$

or

$$(9) \qquad (DF)^{\bullet}(x) = Df(x) \quad .$$

Furthermore,

$$\dot{J}(x) = \mathrm{tr}Df$$

(10) $$(XF)^{\cdot} = Df - \frac{1}{n}\,\mathrm{tr}Df \cdot I$$

$$(MF)^{\cdot} = Df + Df^T - \frac{2}{n}\,\mathrm{tr}Df \cdot I \quad .$$

All this motivates introducing the matrix

(11) $$Sf = \frac{1}{2}(Df + Df^T) - \frac{1}{n}\,\mathrm{tr}Df \cdot I$$

or, in terms of indices,

(12) $$(Sf)_{ij} = \frac{1}{2}\left(\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i}\right) - \frac{\delta_{ij}}{n}\sum_1^n \frac{\partial f_k}{\partial x_k} \quad .$$

Observe that $\mathrm{tr}Sf = 0$. Thus Sf is obtained by first symmetrizing Df and then subtracting the multiple of I which makes the trace 0 . The space of symmetric $n \times n$ matrices with zero trace will,in these lectures, be denoted by SM^n . We shall use the square norm defined by

(13) $$||A||^2 = \mathrm{tr}(A^TA) \quad .$$

<u>Definition 1.</u> f is called a k-q.c. deformation if $||Sf|| \leqq k\sqrt{n}$ a.e. in Ω .

For $n=2$ we can use the complex notation $f = u+iv$, $z = x+iy$. It turns out that

(14) $$Sf = \left\| \begin{matrix} \mathrm{Re}f_{\bar z} & \mathrm{Im}f_{\bar z} \\ \mathrm{Im}f_{\bar z} & -\mathrm{Re}f_{\bar z} \end{matrix} \right\|$$

and $||Sf||^2 = 2|f_{\bar z}|^2$ so that f is k-q.c. if and only if $|f_{\bar z}| \leqq k$ a.e.

In view of (14) it is reasonable to regard Sf as a natural generalization of the complex derivative $f_{\bar{z}}$.

8.3. Our definition is still deficient because it does not spell out the regularity conditions. We replace it by the following:

Definition 1'. A homeomorphism $f : \Omega \to \mathbb{R}^n$ is called a q.c. deformation if f has locally integrable distributional derivatives $D_j f_i$ and if the matrix-valued function Sf formed with these derivatives has norm $\|Sf\| \in L^\infty$. It is k-q.c. if the L^∞ norm of $\|Sf\|$ is $\leq k\sqrt{n}$.

For $n = 2$ a 0-q.c. mapping is conformal (this is Weyl's lemma) and for $n > 2$ all conformal mappings are Möbius transformations. It can be expected that for $n > 2$ there are only a few deformations with $Sf = 0$.

Lemma 1. If $n > 2$ then $Sf = 0$ if and only if f is of the form

(15) $$f = a + Sx + b|x|^2 - 2(bx)x$$

where a and b are constant vectors and S is a constant matrix which is skew outside and constant on the diagonal.

Proof. We assume first that $f \in C^2$. For this proof only we denote components by superscripts and derivatives by subscripts. The hypothesis means that $f^i_j = -f^j_i$ for $i \neq j$ and $f^i_i = f^j_j$. If i , j , k are all different $f^i_{jk} = -f^j_{ik} = -f^j_{ki} = f^k_{ji} = -f^i_{jk}$ and hence $f^i_{jk} = 0$.

If $j \neq k$ we can find $h \neq j,k$ and obtain $f^i_{ijk} = f^h_{hjk} = 0$. Also, $f^k_{kjj} = -f^j_{jkk} = -f^h_{hkk} = f^k_{khh} = f^j_{jhh} = -f^h_{hjj} = -f^k_{kii} = 0$. Similarly, $f^j_{iii} = 0$ both for $j = i$ and $j \neq i$.

Consequently all third and higher derivatives are zero, and it is easy to check that the lower terms have to be as in (15).

If f has merely distributional derivatives one uses the trick of convolving with a radial function δ_ϵ with support in $B(\epsilon)$. Since $S(f * \delta_\epsilon) = Sf * \delta_\epsilon = 0$ we conclude that $f * \delta_\epsilon$ has the form (16) and in the limit for $\epsilon \mapsto 0$ f has the same form.

8.4. Let φ be a differentiable function with values in SM^n. We define $S^* \varphi$ to be a vector-valued function with the components

$$(16) \qquad (S^* \varphi)_i = D_j \varphi_{ij} \quad ,$$

where we are again using the summation convention.

It is readily seen by the Green-Stokes' formula that

$$(17) \qquad \int (Sf \cdot \varphi)\, dx = -\int (f \cdot S^* \varphi)\, dx$$

provided that either f or φ has compact support. The dot products are abbreviations of $Sf_{ij} \varphi_{ij}$ and $f_i (S^* \varphi)_i$. Because of (17) we regard S^* as the adjoint of S. Note that the formula remains valid as soon as φ is continuous with locally integrable distributional derivatives, and this will henceforth be the required degree of regularity.

It is again instructive to look at the case $n = 2$. If

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{12} & -\varphi_{11} \end{pmatrix}$$

we write $\varphi = u + iv$ with $u = \varphi_{11}$, $v = \varphi_{12}$ and we let $z = x + iy = x_1 + ix_2$ stand for the independent variable. If we identify the vector $S^* \varphi$ with

$(S^*\varphi)_1 + i(S^*\varphi)_2$ we find that

$$(18) \qquad S^*\varphi = 2\varphi_2 \quad .$$

Together with $Sf = f_{\bar{z}}$ it follows that $S^*Sf = \frac{1}{2}\Delta f$.

In the general case

$$(19) \qquad S^*Sf = \frac{1}{2}\Delta f + (\frac{1}{2} - \frac{1}{n}) \text{ grad div } f \quad .$$

Operators of this type are familiar from the theory of elasticity.

8.5. It is natural to look for fundamental solutions of $S^*Sf = 0$. We start with solutions of $S^*\gamma = 0$ where γ shall have values in SM^n . It turns out that there are n linearly independent solutions γ^k , $k = 1, \ldots, n$, with the weakest possible singularity at 0 . They are given explicitly by

$$(20) \qquad \gamma^k_{ij}(x) = \frac{\delta_{ik}x_j + \delta_{jk}x_i - \delta_{ij}x_k}{|x|^n} + (n-2)\frac{x_i x_j x_k}{|x|^{n+2}} \quad .$$

A short computation reveals that each term is separately annihilated by S^* . The reason for the linear combination is to make the trace equal to zero. The singularity is weak enough to make $\gamma^k(x)$ integrable, but the derivatives are not.

For $n = 2$ one finds that γ^1 and γ^2 are represented by $1/\bar{z}$ and $i/\bar{z}$ respectively. The connection with the Cauchy kernel is obvious and we shall find that γ^k_{ij} does indeed play very much the same role as the Cauchy kernel.

8.6. We shall now solve the equation

$$(21) \qquad Sg^k = \gamma^k$$

which would imply $S^{*}Sg^{k}=0$. It is a good guess that the components of g^{k} should be of the form

(22) $$g_i^k = a(r)\delta_{ik} + b(r)x_i x_k \quad ,$$

where $r=|x|$ as usual.

Elementary computation gives

$$(Sg^k)_{ij} = \frac{1}{2}\left(\frac{a'(r)}{r} + b(r)\right)(\delta_{ik}x_j + \delta_{jk}x_i) + \frac{b'(r)}{r}\, x_i x_j x_k$$
$$- \frac{1}{n}\,\delta_{ij}x_k\left(\frac{a'(r)}{r} + b(r) + rb'(r)\right) \quad .$$

Comparison with (20) gives the conditions

$$\frac{1}{2}\left(\frac{a'(r)}{r} + b(r)\right) = \frac{1}{r^n} \quad .$$

$$\frac{b'(r)}{r} = \frac{n-2}{r^{n+2}}$$

$$-\frac{1}{n}\left(\frac{a'(r)}{r} + b(r) + rb'(r)\right) = \frac{-1}{r^n}$$

The last is a consequence of the first two and one finds

(23) $$g_i^k(x) = \frac{-3n-2}{n(n-2)}\,\frac{\delta_{ik}}{r^{n-2}} - \frac{n-2}{n}\,\frac{x_i x_k}{r^n}$$

if $n>2$ and $g_i^k = (\log r)\,\delta_{ik}$ for $n=2$.

8.7. The following problem arises naturally:

Problem. If one knows Sf , how can one find f , and what conditions must the matrix-valued function Sf satisfy?

In other words, when does the inhomogeneous equation

(24) $$Sf = \nu$$

have a solution, and how can one find it? For $n=2$ it is known that the equation is always solvable and the solution is given by the generalized Cauchy integral formula, also known as Pompeiu's formula.

For simplicity we shall treat only the case where ν is of class $L^\infty(R^n)$ and has compact support. From 8.3 we know that the solution of (24), if it exists, is unique up to functions of the form (15).

The first theorem is a generalization of Pompeiu's formula which connects f and Sf .

<u>Theorem 1.</u> Every quasiconformal deformation f satisfies

$$c_n f_k(y) = -\int_{B(y,r)} Sf(x)\cdot\gamma^k(x-y)\,dx$$
$$+ \int_{S(y,r)} [(I+(n-2)Q(x-y))f(x)]_k\, d\omega(x)$$

with $c_n = \dfrac{2(n-1)\omega_n}{n}$. If f has compact support the formula reduces to

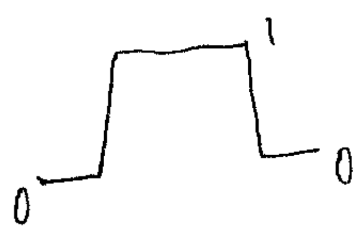
$$c_n f_k(y) = -\int_{R^n} Sf(x)\cdot\gamma^k(x-y)\,d\omega(x) \quad .$$

<u>Proof.</u> It is sufficient to consider $y=0$. Integration by parts *) gives

*) If f has only distributional derivatives one has to consider integrals of the form

$$\int D_i f_j(x)\lambda(|x|)\gamma^k_{ij}(x)\,dx$$

where λ has a graph like



$$\int_{B(r)-B(r_o)} Sf_{ij}\gamma^k_{ij}\, dx = \int_{B(r)-B(r_\nu)} D_i f_j \gamma^k_{ij}\, dx =$$

$$\Big|_{r_o}^{r} \int_{S(r)} f_j \gamma^k_{ij} \frac{x_i}{r}\, d\sigma =$$

$$\Big|_{r_o}^{r} \int_{S(r)} f_j \frac{x_i}{r} \left(\frac{\delta_{ik}x_j + \delta_{jk}x_i - \delta_{ij}x_k}{r^n} + (n-2)\frac{x_i x_j x_k}{r^{n+2}}\right) r^{n-1} d\omega$$

$$= \Big|_{r_o}^{r} \int_{S(r)} \left(f_k + (n-2)\frac{(fx)}{r^2} x_k\right) d\omega \quad .$$

In the limit for $r_o \to 0$ the integral over $S(r_o)$ tends to

$$\omega_n f_k(0) + (n-2) f_i(0) \int_{S(1)} x_i x_k \, d\omega \quad .$$

Obviously,

$$\int_{S(1)} x_i x_k \, d\omega = \delta_{ik} \frac{\omega_n}{n}$$

so that the limit adds up to $\frac{2(n-1)}{n} \omega_n f_k(0)$.
Because $\frac{(fx)x}{r^2} = Q(x)f$ the lemma follows.

8.8. A little more generally we shall define

$$(25) \qquad I^k\nu(g) = \int_{R^n} \nu_{ij}(x)\ \gamma^k_{ij}(x-y)\,dx$$

to be the k th component of the potential $I\nu$ of ν ; here ν is supposed to be of class L^1 with compact support.

<u>Lemma 2.</u> $I^k\nu$ has the distributional derivatives

(26) $$D_h I^k \nu(y) = -b_n \nu_{hk}(y) - \text{p.r.v.} \int \nu_{ij}(x) D_h \gamma^k_{ij}(x-y)\,dx$$

and $$D_h I^k \in L^p \quad \text{for every} \quad p > 1 \quad , \quad b_n = \frac{4\omega n}{n+2} \quad .$$

Before proving this lemma we write down the explicit expression for

(27) $$\begin{aligned} D_h \gamma^k_{ij}(x) = {} & \frac{\delta_{ik}\delta_{jh} + \delta_{jk}\delta_{ih} - \delta_{ij}\delta_{hk}}{|x|^n} \\ & - n\, \frac{\delta_{ik}x_j x_h + \delta_{jk}x_i x_h - \delta_{ij}x_k x_h}{|x|^{n+2}} \\ & + (n-2)\, \frac{\delta_{ih}x_j x_k + \delta_{jh}x_i x_k + \delta_{hk}x_i x_j}{|x|^{n+2}} \\ & - (n^2-4)\, \frac{x_i x_j x_h x_k}{|x|^{n+4}} \quad . \end{aligned}$$

We claim that this is a Calderon-Zygmund kernel. Since it is obviously homogeneous of degree $-n$ we need only check that

(28) $$\int_{S(1)} D_h \gamma^k_{ij}(x)\, d\omega(x) = 0 \quad .$$

This can be checked by computation with only a little bit of trouble being caused by the last term in (27).

Without any computation at all we can also reason as follows: By Stokes' formula

$$\int_{B(r_2)-B(r_1)} D_h \gamma^k_{ij}(x)\,dx = \int_{r_1}^{r_2} \int_{S(r)} \gamma^k_{ij}(x)\, \frac{x_h}{r}\, d\sigma(x) \quad .$$

Here the integrals over $S(r_1)$ and $S(r_2)$ are equal simply because the integrand is homogeneous of order 0 . It follows that the volume integral on the left vanishes, and this implies (28).

The Calderon-Zygmund theory informs us that the principal value in (26) exists a.e. and represents a function in L^p for any $p > 1$.

Proof of the Lemma. We assume first that $\nu \in C^\infty$ and write

$$I_\rho^k \nu(y) = \int_{|x-y|>\rho} \nu_{ij}(x) \gamma_{ij}^k(x-y)dx$$

$$= \int_{|x|>\rho} \nu_{ij}(x+y) \gamma_{ij}^k(x)dx$$

for the translated potential. It follows at once that

$$D_h I_\rho^k(y) = \int_{|x|>\rho} D_h \nu_{ij}(x+y) \gamma_{ij}^k(x)dx =$$

(29) $$- \int_{|x|>\rho} \nu_{ij}(x+y) D_h \gamma_{ij}^k(x)dx - \int_{S(\rho)} \nu_{ij}(x+y)\gamma_{ij}^k(x) \frac{x_h}{\rho} d\sigma(x) \quad .$$

For $\rho \to 0$ the surface integral tends to

$$\nu_{ij}(y) \int_{S(1)} \gamma_{ij}^k(x)x_h d\omega(x) \quad .$$

The evaluation of this integral requires a formula for

$$\int_{S(1)} x_i x_j x_h x_k \, d\omega(x) \quad .$$

The easiest way to get this formula is to substitute (27) in (28). One obtains

$$\int_{S(1)} x_i x_j x_h x_k \, d\omega = \frac{\omega_n}{n(n+2)} (\delta_{ij}\delta_{hk} + \delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh})$$

and finally

$$\int_{S(1)} \gamma_{ij}^k(x)\, x_h \, d\omega = \frac{2\omega_n}{n+2} (\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) - \frac{4\omega_n}{n(n+2)} \delta_{ij}\delta_{hk}$$

It now follows from (29) that

$$\lim_{\rho \to 0} D_h I_\rho^k(\nu_h) = -b_n \nu_{hk}(y) - \text{pr.v.} \int \nu_{ij}(x)\, D_h \gamma_{ij}^k(x-y)\,dx \quad .$$

It is known from the Calderon-Zygmund theory of singular integral operators that the truncated integral converges to its pr. v. in L^p for every $p > 1$. From this it is trivial to conclude that the limit of $D_h I_\rho^k$ is the distributional derivative $D_h I^k$, and the lemma is proved for all $\nu \in C^\infty$.

To prove it for general ν we pass again to a convolution $\nu * \delta_\varepsilon$ where $\delta_\varepsilon \in C^\infty$ is supported in $B(\varepsilon)$ and $\int \delta_\varepsilon dx = 1$. From what we have proved

$$D_h I^k(\nu * \delta_\varepsilon)(y) = -b_n(\nu_{hk} * \delta_\varepsilon)(y)$$

$$\text{(30)} \qquad - \text{pr.v.} \int (\nu_{ij} * \delta_\varepsilon)(x) D_h \gamma_{ij}^k(x-y)\,dx \quad .$$

The pr.v. is a bounded linear operator on L^p, $p > 1$. Therefore the right hand side of (30) tends in L^p to the right hand side of (26). Hence the left hand side has a limit in L^p and that limit is the distributional derivative. The lemma is proved.

8.9. We regard $I\nu$ as a vector-valued function with the components $I^k \nu$. It makes sense to form $S(I\nu)_{hk}$ and by Lemma 2 we find at once:

(31) $$SI\nu(y)_{hk} = -b_n \nu_{hk}(y) - \text{pr.v.} \int \nu_{ij}(x) S\gamma_{ij,hk}(x-y)\,dx \quad .$$

where

$$S\gamma_{ij,hk}(x) = (S\gamma^{\cdot}_{ij})_{hk}$$

in the sense of $\gamma^{\cdot}_{ij}$ standing for a vector with the components γ^{k}_{ij} .

The explicit expression is

$$S\gamma_{ij,hk}(x) = (\delta_{ik}\delta_{jh} + \delta_{jk}\delta_{ih} - \frac{n+2}{2} \delta_{ij}\delta_{hk}) \frac{1}{|x|^n}$$

$$+ n(\delta_{ij}x_h x_k + \delta_{hk} x_i x_j) \frac{1}{|x|^{n+2}} - (\delta_{ik}x_j x_h + \delta_{ih}x_j x_k + \delta_{jh}x_i x_k + \delta_{jk}x_i x_h) \frac{1}{|x|^{n+2}}$$

$$- (n^2-4)\, x_i x_j x_h x_k \frac{1}{|x|^{n+4}} \quad .$$

Observe that $S\gamma_{ij,hk} = S\gamma_{hk,ij}$. It is also a C.- Z . kernel.

Suppose that we apply (31) to $\nu = Sf$. Because $ISf = -c_n f$ we have $SI\nu = -c_n Sf = -c_n \nu$ and thus

(32) $$a_n \nu_{hk}(y) = \text{pr.v.} \int \nu_{ij}(x) S\gamma_{ij,hk}(x-y)\,dx$$

with

$$a_n = c_n - b_n = \frac{2(n+1)(n-2)}{n(n+2)} \quad \omega_n \quad .$$

Conversely, if ν (with compact support) satisfies (32) then (31) implies

$$SI\nu(y)_{hk} = - c_n \nu_{hk}(y)$$

so that $Sf = \nu$ is satisfied by $f = -\frac{1}{c_n} I\nu$. We have proved:

Theorem 2. If $\nu \in L^\infty$ has compact support a necessary and sufficient condition that the equation $Sf = \nu$ have a solution f with compact support is that ν satisfies (32).

As for the sufficiency we are not claiming that $f = -\frac{1}{c_n} I\nu$ has compact support, but since $Sf = 0$ in a neighbourhood of infinity f^n is trivial of the form (15) and by subtracting that trivial solution of $Sf = 0$ we obtain a solution of $Sf = \nu$ with compact support.

8.10 We shall now apply Theorem 1 to show that every q.c. deformation satisfies a near-Lipschitz condition.

Lemma 3. Every q.c. deformation f satisfies a condition of the form

$$(33) \qquad |f(y) - f(y')| \leqq A(R)|y-y'|(1 + \overset{+}{\log} \frac{1}{|y-y'|})$$

for $\qquad |y| , |y'| \leqq R < \infty$.

Proof. We may choose $y' = 0$. From Theorem 1 we obtain an obvious estimate of the form

$$|f_k(y) - f_k(0)| \leqq M \int_{B(2R)} ||\gamma^k(x-y) - \gamma^k(x)|| dx$$

$$+ C(R) |y| \quad .$$

Replace x by $|y|x$ in the integral. Because $\gamma^k(x)$ is homogeneous of degree $1-n$ we obtain

$$\int_{B(2R)} \| \dot{\gamma}^k(x-y) - \gamma^k(x) \| \, dx = |y| \int_{B\left(\frac{2R}{|y|}\right)} \left\| \gamma^k\left(x - \frac{y}{|y|}\right) - \gamma^k(x) \right\| \, dx \quad .$$

If $|y| > R$ the integral on the right is smaller than the same integral over $B(2)$ which is a finite constant by rotational symmetry. If $|y| \leq R$ we must also estimate the integral over $2 \leq |x| \leq \frac{2R}{|y|}$.
It is readily seen that $\|\gamma^k(x - \frac{y}{|y|}) - \gamma^k(x)\| = O(|x|^{-n})$ as $x \to \infty$. Therefore the integral is bounded by a constant times $\log \frac{R}{|y|}$ and we conclude that (33) is true for a suitable $A(R)$.

8.11. We shall now prove that every q.c. deformation $f(x)$ with compact support gives rise to a one-parameter group of conformal mappings $F_t(x) = F(x,t)$ such that $F_s \circ F_t = F_{s+t}$ and $\dot{F}(x,o) = f(x)$. More precisely:

Theorem 3. If f is a k-q.c. deformation then F_t is a $e^{k|t|}$-q.c. mapping.

(This theorem was first proved by M. Reimann).

Proof. Assume first that f is C^∞. For fixed x we consider the differential equation

$$\dot{F}(x,t) = f(F(x,t)) \tag{34}$$

with initial condition $F(x,0) = x$. The existence of a solution is classical and the uniqueness follows because f satisfies (33) which is an "Osgood Condition". Actually, the solution will exist for all f because f is bounded. Moreover,

$$F(x, s+t) = F(F(x,s), t)$$

for both sides are solutions of (34) with initial value $F(x,s)$. This means that $F_t \circ F_s = F_{s+t}$. In particular, $F_t \circ F_{-t} = x$ and since $F(x,t)$ is continuous in x it is a homeomorphism.

Differentiation of (34) with respect to x yields

$$(\dot{DF})(x,t) = (Df \circ F) DF(x,t) \quad .$$

Also

$$(\log JF)^{\cdot} = tr(DF)^{-1}(DF)^{\cdot} = tr(Df \circ F) \quad .$$

We apply this to $XF = (JF)^{-\frac{1}{n}} DF$ to obtain

$$(XF)^{\cdot} = \left(-\frac{1}{n}\, tr(Df \circ F) + Df \circ F \right) XF$$

and

$$[(XF)^T (XF)]^{\cdot} = (XF)^T [Df \circ F + Df^T \circ F - \frac{2}{n}\, tr(Df \circ F)] XF$$

from which it follows that

$$(||XF||^2)^{\cdot} = 2 tr[(Sf \circ F) XF (XF)^T] \quad .$$

By the Cauchy-Schwarz inequality

$$|tr(Sf).XX^T| \leqq ||Sf||.||XX^T|| \leqq ||Sf||.||X||^2 \; .$$

Together with $||Sf|| \leqq k$ we now obtain

$$\frac{d}{dt}||XF||^2 \leqq 2k||XF||^2$$

and by integration

$$||XF(x,t)|| \leqq \sqrt{n}\ e^{k|t|}$$

where we have used the initial condition

$$||XF(x,0)|| = \sqrt{n} \quad .$$

If f is not differentiable we form the convolution $f_\varepsilon = f * \delta_\varepsilon$ and use it to generate F_ε. Because $||Sf_\varepsilon|| \leqq k$ it follows as before that $F_\varepsilon(x,t)$ is K-q.c. with $K = e^{k|t|}$.

We write the differential equation for F_ε in integrated form

$$F_\varepsilon(x,t) = x + \int_0^t f_\varepsilon(F_\varepsilon(x,t))dt \quad .$$

Because the F_ε are K-q.c. with a fixed K they are equicontinuous on every compact set. Hence one can choose a sequence $\varepsilon(N) \to 0$ such that $F_{\varepsilon(N)}$ converges to a limit $F_0(x,t)$ which satisfies

$$F_0(x,t) = x + \int_0^t f(F_0(x,t))dt \quad .$$

This means that $F_0(x,t)$ is the unique solution of (34) and hence equal to $F(x,t)$. Since it is a homeomorphism and limit of K-q.c. mappings with $K = e^{k|t|}$ it is itself K-q.c. and the theorem is proved.

8.12. The quantity MF undergoes a very simple and predictable change when F is composed with a Möbius transformation. The corresponding rules for the operators S and S^* are not quite as simple, but of vital importance.

Because we have used the letter γ in an important role we shall now use capital letters A, B, etc. for Möbius transformations. Recall that a Möbius transformation A defines a change of coordinates that we prefer to write as $x = A\bar{x}$ (rather than $\bar{x} = Ax$). A contravariant vector $f(x)$ expressed in the $\bar{x}$ - coordinates acquires the components

$$\bar{f}^i(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j} f^j(x)$$

which we can also write as

$$\bar{f}(\bar{x}) = A'(\bar{x})^{-1} f(x) = A'(\bar{x})^{-1} f(A\bar{x}) \quad .$$

We shall denote the function $\bar{f}$ by f_A. Explicitly,

(35) $$f_A(x) = A'(x)^{-1} f(Ax) \quad .$$

The formula (35) defines a <u>group representation</u>, for

(36) $$f_{AB} = (f_A)_B$$

as seen by the computation

$$(f_A)_B(x) = B'(x)^{-1} f_A(Bx) = B'(x)^{-1} A'(Bx)^{-1} f(ABx)$$

$$= (AB)'(x)^{-1} f(ABx) = f_{AB}(x) \quad .$$

The basic transformation formula for Sf reads:

Lemma 4.

(37) $$S(f_A) = (DA)^{-1}(SfoA)DA \quad .$$

Proof. We remark first that differentiation of the identity $X^{-1}X = I$ yields

$$(D_j X^{-1})X + X^{-1}(D_j X) = 0$$

and hence

$$D_j X^{-1} = - X^{-1}(D_j X)X^{-1} .$$

Therefore, differentiation of

$$f_A = (DA)^{-1}(foA)$$

leads to

$$(Df_A)_{ij} = - [(DA)^{-1}D_j\ DA]_{ik}(foA)_k$$

(38) $$+((DA)^{-1}(DfoA)DA)_{ij} \quad .$$

In the first term on the right

$$[(DA)^{-1}D_j DA]_{ik} = [(DA)^{-1}D_k DA]_{ij} = (JA)^{\frac{-2}{n}} [(DA)^T D_k DA]_{ij}$$

where we have used the identity

(39) $$(DA)^T DA = (JA)^{\frac{2}{n}} I \quad .$$

Differentiation of (39) gives

$$(D_k DA)^T DA + (DA)^T D_k DA = \frac{2}{n} (JA)^{\frac{2}{n}} (D_k \log JA) I = \frac{2}{n} (JA)^{\frac{2}{n}-1} (tr(DA)^{-1} D_k DA)\ I$$

$$= \frac{2}{n} (JA)^{\frac{2}{n}} (tr(DA)^T D_k DA) I \quad .$$

This shows that the symmetrization and normalization of the first term in (38) contributes nothing. On account of (39)

$$[(DA)^{-1}(DfoA)DA]^T = (DA)^{-1}(DfoA)^T DA$$

while

$$tr[(DA)^{-1} DfoA\ DA] = tr\ DfoA \quad .$$

Therefore the second term of (38) contributes exactly $(DA)^{-1}(SfoA)DA$.

Observe that this computation has made essential use of the fact that DA is a conformal matrix. More generally, we shall say that a function $\nu(x)$ with values in SM^n transforms like a mixed tensor if

(40) $$\nu_A = (DA)^{-1}(\nu oA)DA \quad .$$

Note that ν_A has again values in SM^n .

8.13. In order to continue this discussion of the invariance properties we want to take a good look at the relation

(41) $$\int Sf.\varphi\ dx = - \int f.S^* \varphi\ dx \quad .$$

We would like these inner products to be invariant when f is replaced by f_A and ϕ by ϕ_A. But this is not so if Sf and ϕ are both transformed by (40). Indeed, we would have

$$(Sf)_A \cdot \phi_A = \mathrm{tr}(Sf)_A \phi_A = \mathrm{tr}[DA^{-1}(Sf\circ A)(\phi\circ A)DA] = (Sf\cdot\phi)\circ A ,$$

and the integral in (41) would not stay invariant. To remedy the situation we regard the integral as an inner product between the function Sf and the measure ϕdx, and we transform the measure according to the rule

$$(\phi dx)_A = DA^{-1}(\phi\circ A)DA\cdot JA(x)dx .$$

This amounts to transforming ϕ as a <u>mixed tensor density</u> according to the rule

(42) $$\phi_A(x) = A'(x)^{-1}\phi(Ax)A'(x)|A'(x)|^n .$$

Similarly, the right hand side of (41) makes sense only if we regard $S^*\phi$ as a contravariant vector density. In order to derive the transformation rule for $S^*\phi$ let f be a test-function and consider that on one hand

$$\int f\cdot S^*\phi\, dx = \int (f\circ A)\cdot(S^*\phi\circ A)|A'(x)|^n dx$$

and on the other hand

$$\int f\cdot S^*\phi\, dx = -\int Sf\cdot\phi\, dx = -\int (Sf\circ A)(\phi\circ A)|A'(x)|^n dx =$$

$$= -\int (SfoA)\ .\ A'(x)\varphi_A(x)A'(x)^{-1}dx$$

$$= -\int A'(x)^{-1}(SfoA)A'(x)\ .\ \varphi_A(x)dx$$

$$= -\int S(f_A)\ .\ \varphi_A(x)dx$$

$$= \int f_A\ .\ S*\varphi_A(x)dx = \int f_A^T\ S*\varphi_A(x)dx$$

$$= \int (foA)^T(A'(x)^{-1})^T S*\varphi_A dx$$

$$= \int (foA)\ .\ (A'(x)^{-1})^T S^*\varphi_A dx \quad .$$

The comparison shows that

$$S^*(\varphi_A) = |A'(x)|^n A'(x)^T(S^*\varphi oA) \quad .$$

Here $A'(x)^T = |A'(x)|^2 A'(x)^{-1}$ and if we were to regard $S*\varphi$ as a contravariant vector, then the transformation rule would read

(43) $$S^*(\varphi_A) = |A'(x)|^{n+2}(S^*\varphi)_A \quad .$$

However, it is more natural to regard $v = S^*(\varphi_A)$ as a covariant vector density which transforms according to

$$v_A(x) = |A'(x)|^n A'(x)^T v(Ax) \quad ,$$

for then $S*$ satisfies $S*(\varphi_A) = (S*\varphi)_A$. To sum up: The pairings

$$\langle f,v\rangle = \int f.vdx, \langle \nu,\varphi\rangle = \int \nu.\varphi\ dx$$

are defined and Möbius invariant when f is a contravariant vector, v is a covariant vector density, ν is a mixed tensor, φ is a mixed tensor density which transform according to the following rules;

$$f_A(x) = A'(x)^{-1}f(Ax)$$

$$v_A(x) = |A'(x)|^n A'(x)^T v(Ax)$$

$$\nu_A(x) = A'(x)^{-1}\nu(Ax)A'(x)$$

$$\varphi_A(x) = |A'(x)|^n A'(x)^{-1}\varphi(Ax)A'(x).$$

The invariance is in the sense that

$$\langle f_A , v_A\rangle = \langle f, v\rangle$$

$$\langle \nu_A , \varphi_A\rangle = \langle \nu,\varphi\rangle$$

The operators S , S^* satisfy

$$S(f_A) = (Sf)_A$$

$$S (\varphi_A) = |A'(x)|^{n+2}(S \varphi)_A$$

Sf is a mixed tensor, $S^*\varphi$ is a covariant vector density. We do <u>not</u> define Sv or $S^*\nu$.

8.14. The fact that Sf is not a density while S^*v is defined only for densities makes the operator S^*S meaningless except in the euclidean case. To rescue the situation we need an invariant density. In hyperbolic space such a density is given by the Poincaré metric. We use the notation $ds = \rho|dx|$ with

$$\rho = \frac{2}{1-|x|^2} \quad ,$$

the density itself being given by ρ^n .

Now we can pass from tensors and vectors to densities simply through multiplication by ρ^n . For instance $\rho^n Sf$ is a mixed tensor density, $S^* \rho^n Sf$ is a covariant vector density, and $\rho^{-n-2} S^* \rho^n Sf$ is a contravariant vector. To account for these changes it is convenient to define two new operators, namely $P = \rho^n S$ and $P^* = \rho^{-n-2} P$. They satisfy the relations

$$P(f_A) = (Pf)_A$$

$$P^*(\varphi_A) = (P^* \varphi)_A$$

and thus also

$$P^* P(f_A) = (P^* Pf)_A \quad .$$

The basic invariant second order differential operator is P^*P and in this connection we shall say that a vector-valued function f is <u>harmonic</u> if $P^*Pf = 0$. It is convenient to split this into two pieces, namely

(44) $\qquad \varphi = Pf \ , \ P^* \varphi = 0$

or $S^* \varphi = 0$. For $n = 2$ it turns out that φ is a holomorphic quadratic differential.

How does $P^* P$ compare with the Laplace-Beltrami Δ_h . In the first place Δ_h applies to scalars and P^*P to vectors. But in the Hodge-DeRham theory there are two invariant operators $d\delta$ and δd which apply to differential forms of any order and in particular to first order differentials

$$\alpha = f_1 dx_1 + \ldots + f_n dx_n$$

which may be identified with f . A differential is harmonic if $(d\delta + \delta d)\alpha = 0$ whereas in our terminology f is harmonic if

$$\left(\frac{1}{n} - 1\right) d\,\delta\alpha - \frac{1}{2}\,\delta d\alpha - R\alpha = 0 \quad .$$

Here

$$R\alpha = R^i_j\, f_i\, dx_j$$

where R^i_j is the Ricci curvature tensor.

8.15. We shall now study the theory of functions f that satisfy $P^* Pf = 0$ in some detail.

Lemma 5. If $P^* Pf = 0$ then

$$\text{a)}\quad \int_{S(r)} Sf_{ij} x_j \, d\omega(x) = 0$$

$$(45)\quad \text{b)}\quad \int_{S(r)} Sf_{ij} x_i x_j \, d\omega(x) = 0$$

$$\text{c)}\quad \int_{S(r)} Sf_{ij} x_i x_j x_k \, d\omega(x) = 0$$

Proof. By Green's formula

$$\int_{S(r)} \rho^n Sf_{ij} \frac{x_j}{r}\, d\sigma = \int_{B(r)} (S^* \rho^n Sf)_i \, dx = 0$$

and this implies (45a). Similarly,

$$\int_{S(r)} \rho^n Sf_{ij} x_i \frac{x_j}{r} d\sigma = \int_{B(r)} D_j(\rho^n Sf_{ij} x_i) dx = \int_{B(r)} \rho^n Sf_{ij} \delta_{ij} dx = 0$$

and

$$\int \rho^n Sf_{ij} \frac{x_i x_j x_k}{r} d\sigma = \int_{B(r)} D_j(\rho^n Sf_{ij} x_i \; x_k) dx$$
$$= \int_{B(r)} \rho^n Sf_{ij}(\delta_{ij} x_k + \delta_{kj} x_i) dx$$
$$= \int_{B(r)} \rho^n Sf_{ik} x_i \; dx = 0 \quad \text{by (45a)} \quad .$$

Cor. $F^* Pf = 0$ implies

$$\int_{S(r)} Sf_{ij} \; \gamma_{ij}^k(x) \; d\sigma(x) = 0 \quad .$$

In fact,

$$Sf_{ij} \gamma_{ij}^k(x) = \frac{2Sf_{kj}}{r^n} \; x_j + \frac{(n-2)}{r^{n+2}} \quad Sf_{ij} x_i x_j x_k \quad .$$

Now Theorem 1 (8.2) leads at once to the

Center formula.

(46) $$c_n f(0) = \int_{S(r)} (I + (n-2)Q(x)) f(rx) d\omega(x) \quad , \quad 0 \leqq r < 1 \quad .$$

Recall that $c_n = \frac{2(n-1)}{n} \; \omega_n$.

If f has a continuous extension to $S(1)$, or even if it has radial limits a.e., the formula ramains true for $r = 1$:

(46') $$c_n f(0) = \int_S (I + (n-2)Q(x)) f(x) d\omega(x) \quad .$$

There is a simplification if $f(x)$ is tangential for $|x| = 1$ for then

$$(Q(x)f(x))_i = x_i x_j f_j(x) = 0$$

and the formula reduces to

$$(46'') \qquad f(0) = \frac{1}{c_n} \int_S f(x)d\omega \quad .$$

8.16. Just as in the case of Poisson's formula for harmonic functions we can use (46') to express $f(y)$ in terms of the boundary values. Clearly, we need only apply (46') to the function

$$f_{T_y^{-1}}(x) = (T_y^{-1})'(x)^{-1} f(T_y^{-1}x) \quad .$$

Because $T_y^{-1}(0) = y$ we obtain

$$c_n (T_y^{-1})'(0)^{-1} f(y) = \int_S (I + (n-2)Q(x))(T_y^{-1})'(x)^{-1} f(T_y^{-1}x)d\omega(x) \quad .$$

On the left

$$(T_y^{-1})'(0)^{-1} = T_y'(y) = (1 - |y|^2)^{-1}$$

and on the right we replace x by $T_y x$ to obtain

$$c_n f(y) = (1 - |y|^2) \int_S (I + (n-2)Q(T_y x))T_y'(x) f(x) |T_y'(x)|^{n-1} d\omega(x) \quad .$$

Recall that $T_y'(x) = |T_y'(x)| \; \Delta(x,y)$ and

$$|T_y'(x)| = \frac{1 - |y|^2}{[x,y]^2}$$

or $\dfrac{1 - |y|^2}{|x-y|^2}$ because $|x| = 1$. We have also shown (see 2.8, (40)) that

$T_y x = \Delta(x,y)x$ when $|x| = 1$. Thus

$$Q(T_y x)_{ij} = \Delta(x,y)_{ik}\ \Delta(x,y)_{jk} x_h x_k$$

or

$$Q(T_y x) = \Delta(x,y)Q(x)\Delta(y,x)$$

$$Q(T_y x)\ \Delta(x,y) = \Delta(x,y)Q(x) \quad .$$

Taking these simplifications into account we end up with

Theorem 4.

(47) $$c_n f(y) = \int_S \frac{(1-|y|^2)^{n+1}}{|x-y|^{2n}}\ \Delta(x,y)(I+(n-2)Q(x))f(x)d\omega(x) \quad .$$

There is again a simplification if f is tangential, for then $Q(x)f(x) = 0$. Moreover, $\Delta(x,y) = (I-2Q(x,y))(I-2Q(x))$ so that, if we prefer, (47) can be written as

(47)' $$c_n f(y) = \int_S \frac{(1-|y|^2)^{n+1}}{|x-y|^{2n}}(I-2Q(x-y))f(x)d\omega(x) .$$

8.17. Evidently we can get a corresponding formula for $Sf(y)$ by differentiating (47) or (47'). For arbitrary y the computation becomes almost prohibitive, but we can use the device of computing the derivatives only at $y=0$.

Ignoring quadratic and higher terms in y we have

$$\frac{(1-|y|^2)^{n+1}}{|x-y|^{2n}} \sim \frac{1}{(1-2xy)^n} \sim 1+2n(xy)$$

and

$$Q(x-y)_{ij} = \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2} \sim (x_i x_j - x_i y_j - x_j y_i)(1+2xy)$$

$$\sim x_i x_j + 2x_i\ x_j (xy) - x_i y_j - x_j y_i$$

so that

$$\frac{(1-|y|^2)^{n+1}}{|x-y|^{2n}} \quad (I - 2Q(x-y))_{ij} \sim$$

$$(I-2Q(x))_{ij} + 2n(I-2Q(x))_{ij}(xy) - 4x_i x_j (xy) + 2(x_i y_j + x_j y_i) \quad .$$

It follows that

$$\frac{\partial}{\partial y_k} \left[\frac{(1-|y|^2)^{n+1}}{|x-y|^{2n}} (I-2Q(x-y))_{ij} \right]_{y=0} = 2n(I-2Q(x))_{ij} x_k - 4x_i x_j x_k$$

$$+ 2(\delta_{jk} x_i + \delta_{ik} x_j) = 2n\ \delta_{ij} x_k + 2\delta_{jk} x_i + 2\delta_{ik} x_j - 4(n+1) x_i x_j x_k \quad .$$

For simplicity we shall do only the tangential case. By virtue of $x_j f_j = 0$ on the boundary the result reduces to

$$c_n D_k f_i(0) = \int_S (2n f_i x_k + 2 f_k x_i) d\omega(x) \quad .$$

Symmetrization leads to

$$c_n (D_k f_i(0) + D_i f_k(0)) = 2(n+1) \int_S (f_j x_k + f_k x_j) d\omega \quad .$$

The trace on the right is already zero and we find

$$(48) \qquad c_n Sf(0)_{ik} = (n+1) \int_S (f_i x_k + f_k x_i) d\omega \ .$$

This is already quite neat, but we get an even nicer formula if we replace the integral by a volume integral. By the Green-Stokes formula

$$\int_S (f_i x_k + f_k x_i)\, d\omega = \int_B (D_k f_i + D_i f_k)\, dx \quad ,$$

and similarly

$$0 = \int_S f_h x_h\, d\omega = \int_B D_h f_h\, dx \quad .$$

Accordingly, we conclude from (48) that

(49) $$c_n Sf(0) = 2(n+1) \int_B Sf(x)\, dx \quad .$$

It is now very easy to pass to the general formula. As before, we apply (49) to $f_{T_y^{-1}}$. $Sf(x)$ is then replaced by

$$(T_y^{-1})'(x)^{-1} Sf(T_y^{-1} x)(T_y^{-1})'(x) \quad .$$

For $x = 0$ the left and right factors cancel against each other and the middle factor is $Sf(y)$. In the integral on the right we replace the integration variable by $T_y x$. If we observe that $(T_y^{-1})'(T_y x)^{-1} = T_y'(x)$ the integral becomes

$$\int_B T_y'(x) Sf(x) T_y'(x)^{-1} \quad . \quad |T_y'(x)|^n\, dx \quad .$$

The scalar factors in $T_y'(x)$ and $T_y'(x)^{-1}$ cancel, and we obtain

(50) $$c_n Sf(y) = 2(n+1) \int_B \Delta(x,y) Sf(x) \Delta(y,x) \quad . \quad \frac{(1-|y|^2)^n}{[x,y]^{2n}}\, dx \quad .$$

It is again preferable to focus the attention on

$$\varphi(x) = \rho^n Sf(x) = \left(\frac{2}{1-|x|^2}\right)^n Sf(x) \quad .$$

We obtain the reproducing formula for φ in the form:

Theorem 5. If $\varphi = \rho^n S f$ with tangential f and $S^*\varphi = 0$, then

$$c_n \varphi(y) = 2(n+1) \int_B \frac{\Delta(x,y)\, \varphi(x)\, \Delta(y,x)}{[x,y]^{2n}} (1-|x|^2)^n dx \quad . \tag{51}$$

It is instructive to compare (51) with the Bergman kernel formula for analytic functions in the unit disk. If $\varphi(z)$ is complex analytic for $|z| < 1$, then

$$\varphi(\zeta) = \frac{3}{\pi} \iint_{|z|<1} \frac{\varphi(z)\, (1-|z|^2)^2}{(1-\zeta\bar{z})^4} \, dxdy$$

for all $|\zeta| < 1$, provided that

$$\iint_{|z|<1} |\varphi(z)| (1-|z|^2)^2 dxdy < \infty \quad .$$

One checks that

$$\frac{2(n+1)}{c_n} = \frac{(n+1)n}{(n-1)\omega_n}$$

does indeed reduce to $\frac{3}{\pi}$ when $n = 2$. Clearly, the matrices $\Delta(x,y)$ and $\Delta(y,x)$ account for the argument of the kernel.

Remark. We have proved Theorem 5 under much weaker conditions, but it does remain true as soon as

$$\int_B \|\varphi(x)\| (1-|x|^2)^n \, dx < \infty \quad . \tag{52}$$

8.18. Because of the invariance properties of P and P^* all our considerations are easily adaptable to discrete groups.

Let Γ be a discrete subgroup of $M(B^n)$. We introduce the following definitions:

Def.1. If the vector-valued function $f(x)$ satisfies

$$f_A(x) = A'(x)^{-1} f(Ax) = f(x)$$

for all $A \in \Gamma$, then f is said to be automorphic under Γ.

Def.2. A mixed tensor density $\varphi(x)$ is automorphic under Γ if

$$\varphi_A(x) = |A'(x)|^n \, A'(x)^{-1} \varphi(Ax) \, A'(x) = \varphi(x)$$

for all $A \in \Gamma$.

Def.3. A mixed tensor $\nu(x)$ is a Beltrami differential if

$$\nu_A(x) = A'(x)^{-1} \nu(Ax) \, A'(x)^{-1} = \nu(x)$$

for all $A \in \Gamma$.

If f is automorphic, then Sf is a Beltrami differential and $\rho^n Sf$ is an automorphic mixed density. Conversely, if $\varphi = Pf$ is automorphic it does not follow that f is automorphic, but only that $S(f_A - f) = 0$. In other words, $f_A - f$ is trivial, i.e. the components are quadratic polynomials. We write $f_A - f = p_A f$ and call $p_A f$ the period of f under the mapping A.

The periods satisfy the cocycle condition

$$p_{AB} f = (p_A f)_B + p_B f$$

for

$$p_{AB}f = (f_A)_B - f_B + (f_B - f) = (f_A - f)_B + (f_B - f) \quad .$$

A vector function with zero periods defines a vector function on the quotient manifold M_Γ .

8.19. As far as boundedness is concerned there are two important conditions.

Condition 1. A mixed tensor $\nu(x)$ is said to be of class $L^\infty(\Gamma)$ if it is measurable, if $||\nu(x)||$ is (essentially) bounded and $\nu_A = \nu$ for all $\nu \in \Gamma$. We can use the same terminology for tensor densities $\varphi(x)$ when $\nu = \varphi(x)\left(\frac{1 - |x|^2}{2}\right)^n$ is in $L^\infty(\Gamma)$.

Condition 2. The tensor density $\varphi(x)$ is said to be of class $L^1(\Gamma)$ if

$$\int_{B/\Gamma} ||\varphi(x)|| d\mathbf{x} < \infty$$

Here the integrand is invariant so that the integral can be taken over any fundamental set , for instance the Poincare' fundamental polyhedron $P(\Gamma)$

Remark. Recall that $||\varphi(x)||$ refers to the square norm of the matrix. In particular, $\varphi \in L^1(I)$ if

$$\int_B ||\varphi|| \, dx < \infty$$

without any automorphy condition. In this situation the Poincare' θ-series applies and leads to $\theta\varphi \in L^1(\Gamma)$.

Theorem 6. If $\varphi \in L^1(I)$ then

$$\theta\varphi = \sum_{A\in\Gamma} \varphi_A$$

converges absolutely a.e. and belongs to $L^1(\Gamma)$.

Proof. Let P be a fundamental set for Γ . By assumption

$$\int_B ||\varphi|| dx = \sum_{A\in\Gamma} \int_{AP} ||\varphi|| dx < \infty \quad .$$

But

$$\int_{AP} ||\varphi|| dx = \int_{P} ||\varphi(Ax)|| |A'(x)|^n dx$$

and

$$\varphi_A(x) = |A'(x)|^n A'(x)^{-1} \varphi(Ax)\ A'(x)$$

$$||\varphi_A(x)|| = |A'(x)|^n ||\varphi(Ax)||$$

From this we conclude that

(53) $$\int_P \Sigma ||\varphi_A|| dx < \infty \quad .$$

and the convergence of the series follows. The same estimate shows that

$$\int_P || \Theta\ \varphi || dx < \infty$$

Finally, $\theta\ \varphi$ is automorphic because

$$(\Theta \varphi)_B = \Sigma_A (\varphi_A)_B = \Sigma_A \varphi_{AB} = \Theta \varphi \quad .$$

8.20. It is convenient to use the inner product

$$(\varphi, \psi)_\Gamma = \int_{B/\Gamma} \varphi \cdot \psi\ (1 - |x|^2)^n dx$$

when $\varphi \in L^1(\Gamma)$, $\psi \in L^\infty(\Gamma)$ (i.e. $\nu = \psi(1 - |x|^2)^n \in L^\infty$) . The invariance of the inner product is obvious. Also, if $\varphi \in L^1(I)$, $\psi \in L^\infty(\Gamma)$ then

$$(\Theta \varphi, \psi)_\Gamma = (\varphi, \psi)_I \quad .$$

<u>Proof</u>. Let χ be the characteristic function of P . Then

$$(\Theta \varphi, \psi)_\Gamma = (\Theta\ \varphi, \chi\psi)_I = \sum_{A \in \Gamma} (\varphi_A, \chi\psi)_I = \sum_{A \in \Gamma} (\varphi, (\chi\psi)_{A^{-1}})_I = \sum_{A \in \Gamma} (\varphi, (\chi \circ A^{-1})\psi)_I$$

$$= (\varphi, \psi)_I \quad .$$

Remark. If φ and $\psi \in L^1 \cap L^\infty$ then $(\Theta\varphi, \psi)_\Gamma = (\varphi, \Theta \Psi)_\Gamma$.

8.21. We return to Theorem 4 and Poisson's formula (47). Let us fix k and x, $|x| = 1$, and consider the vector-valued kernel $K(x,y)$ with the components

$$(54) \qquad K_i(y) = \frac{(1 - |y|^2)^{n+1}}{|x - y|^{2n}} \quad \Delta(x,y)_{ik} \; .$$

There is good reason to believe that $K(y)$ satisfies $P^* PK=0$. Again, direct computation is almost prohibitive.

Let us first drop the condition $|x| = 1$ and replace (54) by

$$(55) \qquad K_i(y) = \frac{(1 - |y|^2)^{n+1}}{[x,y]^{2n}} \quad \Delta(x,y)_{ik} \; .$$

Then

$$(1 - |x|^2)^n K_i(y) = \frac{(1 - |x|^2)^{n+1} (1 - |y|^2)^{n+1}}{[x,y]^{2n+2}} \cdot \left(\frac{(1 - |x|^2}{[x,y]^2} \right)^{-1} \Delta(x,y)_{ik} \; .$$

By our usual formulas

$$1 - |T_x y|^2 = \frac{(1 - |x|^2) \; (1 - |y|^2)}{[x,y]^2}$$

and

$$T_x'(y) = \frac{1 - |x|^2}{[x,y]^2} \quad \Delta(y,x) \quad ,$$

$$T_x'(y)^{-1} = \left(\frac{1 - |x|^2}{[xy]^2} \right)^{-1} \quad \Delta(x,y)$$

so that

$$(1-|x|^2)^n K_i(y) = (1-|T_x y|^2)^{n+1} \; T'_x(y)^{-1}_{ik} \, .$$

The right hand side is nothing else than $[(1-|y|^2)^{n+1} e_k]_{T_x}$ so that

$$K_i(y) = (1-|x|^2)^{-n} \; [(1-|y|^2)^{n+1} e_k]_{T_x}$$

where e_k is the unit vector with components δ_{ik} .

Almost without computation one obtains

$$P((1-|y|^2)e_k)_{ij} = \text{const.}(\delta_{ik}y_j + \delta_{jk}y_i - \tfrac{2}{n}\,\delta_{ij}y_k)$$
$$(S^* P((1-|y|^2)e_k)_i = \text{const.}\;\delta_{ik}$$
$$(P^* P((1-|y|^2)e_k)_i = \text{const.}\;(1-|y|^2)^{n+2}\;\delta_{ik}$$

where the values of the constants are irrelevant. By the fact that P^*P is an invariant operator it now follows that

$$P^* PK(y)_i = (1-|x|^2)^{-n} \; \{P^* P[(1-|\;\;|^2)^{n+1} e_k]_{T_x}\}_i =$$

$$\text{const.}(1-|x|^2)^{-n}(1-|T_x y|^2)^{n+2} T'_x(y)^{-1}_{ik} =$$

$$\text{const.}\;\frac{(1-|x|^2)(1-|y|^2)^{n+1}}{[x,y]^{2n+4}} \qquad \Delta(x,y)_{ik} \, .$$

This is an algebraic identity which must stay in force when $|x|=1$. It follows that $P^*PK(x,y) = 0$ for $|x| = 1$, and hence any integral of the form

$$f(y) = \int_S \frac{(1-|y|^2)^{n+1}}{[x,y]^{2n}} \quad \Delta(x,y)h(x)\,dx$$

is a harmonic function of y .

8.22. There is a similar fact for Theorem 5 . More explicitly, (51) reads

$$c_n\varphi_{ij}(y) = 2(n+1)\int_B \frac{\Delta(x,y)_{ih}}{[x,y]^{2n}}\,\Delta(x,y)_{jk}\,\varphi_{hk}(x)(1-|x|^2)^n dx \quad .$$

One can show that

$$\frac{\partial}{\partial y_j}\;\frac{\Delta(x,y)_{ih}\Delta(x,y)_{jk}}{[x,y]^{2n}}$$

is of the form τ_{hk} with $\tau_{hk} = -\tau_{kh}$ for $h \neq k$ and $\tau_{hh} = \tau_{kk}$ (skew-symmetric outside the diagonal, constant on the diagonal). This has the effect that for <u>any</u> $\nu(x)$ the function

$$L\nu(y) = \int_B \frac{\Delta(x,y)\nu(x)\Delta(y,x)}{[x,y]^{2n}}dx$$

satisfies $S^* L\nu(y) = 0$.

The operator L has other interesting properties. 1) If $\nu(x)$ is automorphic (as a mixed tensor) then $L\nu$ is automorphic as a mixed tensor density). 2) If $\nu \in L^\infty(\Gamma)$ then the same is true of $L\nu$. 3) If $\varphi = \frac{\nu}{(1-|x|^2)^n}$ is in $L^1(\Gamma)$ so is $L\nu$ ..

<u>Proof of 3</u>. It is clear that

$$||L\nu(y)|| \leq \int_B \frac{||\nu(x)||}{[x,y]^{2n}} \quad dx$$

and hence

$$\int_P ||L\nu(y)||dy \leqq \int_P dy \int_B \frac{||\nu(x)||}{[x,y]^{2n}} \quad dx \quad .$$

On partitioning B into copies AP one sees that

$$\int_P ..dy \int_B ..dx = \int_P ..dx \int_B ..dy \quad .$$

In this way

$$\int_P ||L\nu(y)||dy \leqq \int_P ||\nu(x)||dx \int_B \frac{dy}{[x,y]^{2n}} .$$

But

$$\int_B \frac{(1-|x|^2)^n}{[x,y]^{2n}} dy = \frac{\omega_n}{n}$$

and we obtain

$$\int_P ||L\nu(y)||dy \leqq \frac{\omega_n}{n} \int_P \frac{||\nu(x)||}{(1-|x|^2)^n} dx = \frac{\omega_n}{n} \int_P ||\varphi(x)|| dx .$$

8.22. A weak finiteness theorem.

We shall now introduce a rather special class $Q(\Gamma)$ defined as follows:

Definition $\varphi \in Q(\Gamma)$ if the following is true:

I. $S^*\varphi = 0$ and $\varphi = Pf$ for some f with $xf = 0$ on S . *)

II. $\varphi_A = \varphi$ for all $A \in \Gamma$.

III. $\varphi \in L^1(\Gamma) \cap L^\infty(\Gamma)$

(In other words, $\int_{B/\Gamma} ||\varphi||dx < \infty$ and $||\varphi(x)||(1-|x|^2)^n$ is bounded.)

IV. φ has a smooth extension to $S \cap \Omega$ and this extension satisfies

$$Q(x)\varphi(x) = \varphi(x) \; Q(x) .$$

More explicitly,

$$\varphi_{ij}x_j x_k = x_i x_j \varphi_{jk}$$

on $S \setminus \Lambda$.

---

*) It is understood that f has a continuous extension to S .

Condition IV needs a motivation. For $n=2$ $\varphi\, dz^2$ is a holomorphic quadratic differential. The condition means that $\varphi\, dz^2$ is real on $|z|=1$ outside the limit set. When this is so φ can be extended to $|z|>1$ by symmetry.

For arbitrary n the symmetric extension of φ would be defined by $\varphi_J=\varphi$ where $Jx=x^*$. This condition reads

$$\varphi(x^*) = |x|^{2n}(I-2Q(x))\varphi(x)(I-2Q(x))$$

and for $|x|=1$ it reduces to $Q(x)\varphi(x)=\varphi(x)Q(x)$.

The following is an analytic finiteness theorem whose topological and geometric consequences for the quotient manifold $M(\Gamma)$ are not clear.

Theorem 7. If Γ is finitely generated, then the dimension of the linear space $Q(\Gamma)$ is finite.

Sketch of proof. As already mentioned (see 8.18) if $\varphi_A=\varphi$ then $P_A f = f_A - f$ is a trivial vector function. As such each component of $P_A f$ is a quadratic polynomial in the x_i determined by a finite number of coefficients.

Suppose that Γ is generated by $A_1,\ldots,A_N$. There is a linear map of $Q(\Gamma)$ on a finite dimensional vector space which takes each $\varphi\in Q(\Gamma)$ into the coefficients of $P_{A_1}f,\ldots,P_{A_n}f$. The kernel of this homomorphism consists of all φ such that the corresponding f is automorphic with respect to Γ . The theorem will be proved if we show that such a φ must be identically zero.

For this purpose we study the integral

$$(\varphi,\varphi) = \int_P ||\varphi||^2\left(\frac{1-|x|^2}{2}\right)^n dx = \int_P \varphi_{ij}(x)\ Sf_{ij}\,dx$$

which can also be written as

$$(56) \qquad \int_P D_j f_i \; \varphi_{ij} dx = - \int_P f_i D_j \varphi_{ij} dx + \int_{\partial P} f_i \varphi_{ij} n_j d\sigma$$

where n_j is the outer normal of ∂P *.)

The boundary ∂P consists of pairwise congruent faces of the polyhedron, part of Ω and points on Λ . Let Δ and $A\Delta$ be a pair of corresponding faces. Then

$$\int_{A\Delta} f_i \varphi_{ij} n_j d\sigma = \int_{\Delta} f_i(Ax) \varphi_{ij}(Ax) n_j(Ax) |A'(x)|^{n-1} d\sigma \quad .$$

On the right we substitute

$$f_i(Ax) = (A'(x)f(x))_i = A'(x)_{ia} f_a(x)$$

$$\varphi_{ij}(Ax) = |A'(x)|^{-n} (A'(x)\varphi(x)A'(x)^{-1})_{ij}$$

$$= |A'(x)|^{-n} \; A'(x)_{ib} \varphi_{bc}(x) A'(x)^{-1}_{cj}$$

$$n_j(Ax) = - \frac{(A'(x)n(x))_j}{|A'(x)|} = - |A'(x)|^{-1} A'(x)_{jd} n_d(x) \quad .$$

The minus sign is because A maps the outer normal at x on the inner normal at Ax .

The integrand becomes

$$- |A'(x)|^{-2} A'(x)_{ia} f_a(x) A'(x)_{ib} \varphi_{bc}(x) A'(x)^{-1}_{cj} \; A'(x)_{jd} n_d(x) \; .$$

But $A'(x)_{ia} A'(x)_{ib} = \delta_{ab} |A'(x)|^2$ and $A'(x)^{-1}_{cj} \; A'(x)_{jd} = \delta_{cd}$ and we are left with

*) We are temporarily assumming that Green's formula is applicable.

$$- f_a(x)\phi_{ac}(x)n_c(x)$$

and thus

$$\int_{A\Delta} f_i\phi_{ij}n_j d\sigma = -\int_{\Delta} f_i\phi_{ij}n_j d\sigma \ .$$

The integrals cancel against each other.

Suppose now that P is a finite polyhedron and that no points of Λ belong to ∂P. The remaining part of the boundary integral in (56) is then

$$\int_{\Omega\cap P} f_i\phi_{ij}x_j d\sigma \ .$$

Now we make use of Condition IV to write

$$f_i\phi_{ij}x_j d\omega = f_i\phi_{ij}x_j x_k^2 d\omega = f_i x_i x_j \phi_{jk} x_k d\omega$$

which is zero because $f_i x_i = 0$ by assumption. This proves $(\phi,\phi) = 0$ and hence $\phi = 0$ under strong regularity assumptions.

In order to get rid of the extra assumptions we use the same mollifier technique as in the standard proof of the finiteness theorem for Kleinian groups. Suppose that $\lambda \in C^\infty(\overline{P})$, $\lambda \geq 0$, has compact support on $\overline{P} \setminus \Lambda$ and satisfies $\lambda(Ax) = \lambda(x)$ at equivalent points of ∂P. The earlier reasoning is sufficient to prove that

(57) $$(\phi,\lambda\phi) = -\int_P f_i \frac{\partial\lambda}{\partial x_j}\phi_{ij}\,dx .$$

The theorem will follow if we can find a sequence of λ_ν such that $\lambda_\nu \to 1$ and $f_i(\partial\lambda_\nu/\partial x_j) \to 0$ boundedly on P.

Let $\delta(x)$ be the euclidean distance from x to Λ. At corresponding points x and $Ax \in \partial P$ the distances $\delta(x)$ and $\delta(Ax)$ are equal. To see this observe that $|x| = |Ax|$ and hence $|A'(x)| = (1-|Ax|^2)/(1-|x|^2) = 1$. Furthermore, for any y

$$|Ax-Ay| = |x-y|\,|A'(x)|^{\frac{1}{2}}|A'(y)|^{\frac{1}{2}} = |x-y|\,|A'(y)|^{\frac{1}{2}}$$

$$|Ax-y| = |Ax-AA^{-1}y| = |x-y|\,|A'(y)|^{-\frac{1}{2}} .$$

and hence $\min(|Ax-Ay|,|Ax-y|) \leqq |x-y|$. As y varies over Λ it follows that $\delta(Ax) \leqq \delta(x)$ and by symmetry $\delta(Ax) = \delta(x)$ as asserted. δ is not smooth, but $|\delta(x)-\delta(y)| \leqq |x-y|$ and $|\operatorname{grad}\delta| = 1$ a.e.

Let $h_\nu(t)$ be 0 in $[0,1/\nu]$, linear in $[1/\nu,2/\nu]$ and 1 for $t \geqq 2/\nu$. We shall choose

$$\lambda_\nu(x) = h_\nu\left[\left(\log\log\frac{2e}{\delta(x)}\right)^{-1}\right] .$$

Although this function is not smooth it is not hard to see that (57) (57) remains in force when λ is replaced by λ_ν.

It is evident that λ_ν tends boundedly to 1 as $\nu\to\infty$. As for the derivative one finds that

$$|\partial\lambda_\nu/\partial x_j| \leq \frac{2\nu e}{\delta(x)\log\frac{2e}{\delta(x)}\left(\log\log\frac{2e}{\delta(x)}\right)^2}$$

a.e. Actually, the derivative is zero except when

$$\nu/2 < \log\log\frac{2e}{\delta(x)} < \nu .$$

This leads to the estimate

$$(58) \qquad |\partial\lambda_\nu/\partial x| < 8e/\nu\delta(x)\log\frac{2e}{\delta(x)} .$$

Let us now extend f to all of R^n by the symmetry relation $f_J = f$, that is to say according to the rule $|x|^2 f(x^*) = (I-2Q(x))f(x)$. Because of Condition I $Q(x)f(x) = 0$ on S and the extension is consequently continuous. Moreover, $Sf(x^*) = (I-2Q(x))Sf(x)(I-2Q(x))$ so that $||Sf||$ renains bounded. By Lemma 3 (8.10) we may therefore conclude that f satisfies a near-Lipschitz condition $|f(x)-f(y)| = O(|x-y|\log\frac{1}{|x-y|})$.

We observe next that f is identically zero on Λ . In fact, for any $A \varepsilon \Gamma$, $f(A0) = A'(0)f(0)$ and hence $|f(A0)|<|A'(0)||f(0)|= (1-|A0|^2)|f(0)|$. Choose a sequence of A such that $A0$ tends to $x \varepsilon \Lambda$. It follows by continuity that $f(x) = 0$.

Together with the near-Lipschitz condition we conclude that $|f(x)|= O(\delta(x)\log 1/\delta(x))$. In view of (58) this implies that $f_i\delta\lambda_\nu/\partial x_j$ tends boundedly to 0 as $\nu\to\infty$. This is precisely what was still needed in order to conclude from (57) that $(\phi,\phi) = 0$, and Theorem 7 is proved.

Remark. The theorem is of course rather meaningless unless one can show, in the opposite direction, that groups with a finite dimensional $Q(\Gamma)$ have some rather special properties, for instance with respect to the topology of B/Γ .