Student Number: ______________________

NATIONAL UNIVERSITY OF SINGAPORE

MA1100 - Fundamental Concepts of Mathematics

(Semester 1 : AY2013/2014)

Name of examiner : Assoc Prof Tan Victor

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number clearly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination.
2. Please write your matriculation/student number only. Do not write your name.
3. This examination paper contains **SEVEN** questions and comprises **SEVENTEEN** printed pages.
4. Answer **ALL** questions.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. You are allowed to use two A4 size helpsheets.
7. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations)

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
7	
Total	

Question 1 [15 marks]

(a) Use mathematical induction to prove that $n^3 < n!$ for all integers $n \geq 6$.
(Hint: You need to show and use the inequality $(k+1)^2 \leq k^3$ for $k \geq 3$.)

(b) The Fibonacci sequence is defined by

$$f_1 = 1, \quad f_2 = 1, \qquad f_{n+2} = f_{n+1} + f_n \quad \text{for all } n \geq 1.$$

Use strong mathematical induction to prove that $2f_n + 3f_{n+1} = f_{n+4}$ for all $n \in \mathbb{Z}^+$.

(c) Suppose you want to prove $P(n)$ is true (only) for all integers $n \geq 7$ that are not divisible by 4 using a version of mathematical induction as follow:

(i) Basis step: $P(a), P(b), P(c)$ are true; and

(ii) Inductive step: $(\forall\, k \in \mathbb{Z}^+)\ \ P(k) \to P(k+d)$ is true.

Write down the values for a, b, c, d.

Show your working below and on the next page.

(More working spaces for Question 1)

Continue on page 16-17 if you need more space. Please indicate clearly.

Question 2 [15 marks]

(a) Define sets A and B as follows:
$A = \{n \in \mathbb{Z} \mid n = 6r - 1 \text{ for some integer } r\}$ and
$B = \{m \in \mathbb{Z} \mid m = 3s + 2 \text{ for some integer } s\}$.

Prove $A \subseteq B$ using <u>element method</u>.

(b) Let A, B, C be three subsets of a universal set. Use <u>algebra of sets</u> to prove that

$$(A - B) \cup (C - B) = (A \cup C) - B.$$

State the properties (laws) that you use for your steps.

(c) Let $S = \{1, 2, 3\}$ and $T = \{a, b\}$.

List down all the elements in the set $\{A \in \mathcal{P}(S \times T) \mid |A| = 5\}$.

(Here $\mathcal{P}(S \times T)$ denotes the power set of the Cartesian product $S \times T$.)

Show your working below and on the next page.

(More working spaces for Question 2)

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Question 3 [20 marks]

(a) Define a function $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{1\}$ by the formula $f(x) = \dfrac{x+2}{x-2}$.
Prove that f is a bijection and find the inverse function f^{-1}.

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = x^2$ and $G : \mathbb{R} \to \mathbb{R}$ be defined by $G(x) = \sin(x)$.
Find the ranges of $F \circ G$ and $G \circ F$, and the inverse image of 0 under $G \circ F$.

(c) Let $A = \{u, v\}$ and $B = \{x, y, z\}$. How many pairs of functions $p : A \to B$ and $q : B \to A$ are there such that $u \in A$ has two preimages in B under q and $q \circ p = I_A$ (the identity function on A)?
Use arrow diagrams to list all such pairs.

Show your working below and on the next page.

(More working spaces for Question 3)

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Question 4 [15 marks]

(a) Find relations R_1 and R_2 on the set $A = \{a, b\}$ such that

(i) R_1 is not reflexive, not symmetric, but is transitive;

(ii) R_2 is not reflexive, not transitive, but is symmetric.

Give your answers as ordered pair representation and briefly justify your answers.

(b) Given a partition $P = \{\{1, 4\}, \{2, 3, 5\}\}$ of the set $B = \{1, 2, 3, 4, 5\}$. Write down the equivalence relation R on B induced by the partition P. How many different equivalence classes does R have?

(c) Determine whether the relation S on $\mathbb{Z}$ defined by

$$x \; S \; y \text{ if and only if } x^2 \geq y$$

is reflexive, symmetric and transitive. Justify your answers.

Show your working below and on the next page.

(More working spaces for Question 4)

Continue on page 16-17 if you need more space. Please indicate clearly.

Question 5 [15 marks]

(a) Find $\gcd(378, 144)$ using Euclidean algorithm. Show your steps clearly.

(b) Find the remainder of 44^{100} when it is divided by 7. Show your working clearly.

(c) Rewrite the congruence equation $x^2 + 2x + 3 \equiv 0 \mod 6$ in terms of congruence classes of $\mathbb{Z}_6$. Use that to solve the above congruence equation.

Show your working below and on the next page.

(More working spaces for Question 5)

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Question 6 [10 marks]

(a) If $\gcd(a, b) = 1$, what are the possible values for $\gcd(a + b, a - b)$? Justify your answer.

(b) Prove that for any integer a, if a and 35 are relatively prime, then $a^{12} \equiv 1 \mod 35$.

Show your working below and on the next page.

(More working spaces for Question 6)

Continue on page 16-17 if you need more space. Please indicate clearly.

Question 7 [10 marks]

Let $A_1 = \mathbb{Z}^+$, $A_2 = \{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \ldots\}$, $A_3 = \{\sqrt{3}, 2\sqrt{3}, 3\sqrt{3}, \ldots\}$ etc.

(i) Is A_n countable for every n?

(ii) Is $\bigcup_{n=1}^{100} A_n$ countable?

(iii) Is $\bigcup_{n=1}^{\infty} A_n$ countable?

Justify your answers.

Show your working below and on the next page.

(More working spaces for Question 7)

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(Additional working spaces for ALL questions - indicate your question numbers clearly.)

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