Student Number:

A/U/T*								

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2012-2013

MA1100 Fundamental Concepts of Mathematics

November/December 2012 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains a total of **EIGHT (8)** questions and comprises **NINETEEN (19)** printed pages.

3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.

4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Question 1 [10 marks]

(a) Use <u>mathematical induction</u> to prove that $1 + 4n < 2^n$ for all integers $n \geq 5$.

(b) A sequence is defined by

$$a_1 = 2, \quad a_2 = 4, \qquad a_{n+2} = 5a_{n+1} - 6a_n \quad \text{for all } n \geq 1.$$

Use a version of mathematical induction to prove that $a_n = 2^n$ for all $n \in \mathbb{Z}^+$.

Show your working below and on the next page.

(More working spaces for Question 1)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 2 [15 marks]

(a) Define sets A and B as follows:
$A = \{n \in \mathbb{Z} \mid n = 8r - 3 \text{ for some integer } r\}$ and
$B = \{m \in \mathbb{Z} \mid m = 4s + 1 \text{ for some integer } s\}$.

Prove $A \subseteq B$ using <u>element method</u>.

(b) Let A, B, C be three subsets of a universal set. Use <u>algebra of sets</u> to prove that

$$(A - B) \cap (A - C) = A - (B \cup C).$$

State the properties (laws) that you use for your steps.

(c) Let $A_n = \left\{ \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n} \right\}$ for integer $n \geq 2$.
Use <u>element method</u> to prove that

$$\bigcup_{n=2}^{\infty} A_n = \{q \in \mathbb{Q} \mid 0 < q < 1\}.$$

Show your working below and on the next page.

(More working spaces for Question 2)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 3 [10 marks]

Define a function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ by the formula $f(x) = \dfrac{x+3}{x}$.

(i) Prove that f is one-to-one (injective).

(ii) Replace the codomain of f with a subset of $\mathbb{R}$ so that f becomes a bijection. Justify your answer.

(iii) Write down the inverse function of the bijection in part (ii).

Show your working below and on the next page.

(More working spaces for Question 3)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 4 [15 marks]

(a) Let a and b be integers such that $a \equiv 5 \mod 7$ and $b \equiv 4 \mod 7$.

(i) Find integers s, t where $0 \leq s, t < 7$ such that $a + b \equiv s \mod 7$ and $ab \equiv t \mod 7$.

(ii) Is there an integer c such that $ac \equiv 2 \mod 14$? Justify your answer.

(b) Use congruence modulo to compute the remainder of 18^{199} when it is divided by 65.

(c) Let a, b and $n > 1$ be integers. Prove that if $m > 1$ is a divisor of n and $a \equiv b \mod n$, then $a \equiv b \mod m$.

Show your working below and on the next page.

(More working spaces for Question 4)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 5 [15 marks]

(a) Let $A = \{1, 2, 3, 4\}$. Let R be the relation on A given by $a\ R\ b$ if and only if $a + b$ is even.

(i) Express R as an ordered pair representation. (List all the ordered pairs in R.)

(ii) Show that R is an equivalence relation.

(iii) Find the distinct equivalence classes of R.

(iv) Is it possible to remove one ordered pair from R so that the resulting relation on A is reflexive, symmetric but not transitive? Justify your answer.

(b) Determine whether the following relation S on $\mathbb{Z}$ is reflexive, symmetric and transitive? Justify your answers.

$$S = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid (a - b)^2 < 9\}.$$

Show your working below and on the next page.

(More working spaces for Question 5)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 6 [10 marks]

Determine whether the following sets are countable or uncountable. Justify your answers.

(a) $A = \{n \in \mathbb{Z} \mid n \text{ is a square free number}\}$.

(b) $B = \{x \in \mathbb{R} \mid x = \sqrt{n} \text{ for some integer } n\ \}$.

(c) $C = [0,1] - \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$.

Show your working below and on the next page.

(More working spaces for Question 6)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 7 [15 marks]

Let S be the set of all finite strings in 0's and 1's where the left most digit is 1.

(a) Define a function $g : S \to \mathbb{Z}$ as follows:
for each string s in S,

$$g(s) = \text{ the number of 1's in } s \text{ \underline{minus} the number of 0's in } s.$$

(i) What is $g(101011)$? $g(100100)$?

(ii) Is g one-to-one? Prove or give a counterexample.

(iii) Is g onto? Prove or give a counterexample.

(b) Define a function $h : S \to \mathbb{Z}^+$ as follows:
for each string $s = a_n a_{n-1} \dots a_1 a_0$ in S,

$$h(s) = a_n \times 2^n + a_{n-1} \times 2^{n-1} + \cdots + a_1 \times 2 + a_0.$$

Is h a bijection? Justify your answer. (You may use any result proven in class.)

Show your working below and on the next page.

(More working spaces for Question 7)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 8 [10 marks]

Let p_n be the nth prime number, i.e. $p_1 = 2, p_2 = 3, p_3 = 5 \ldots$ etc.

(a) Is it true that $p_1 p_2 \cdots p_n \equiv 2 \mod 4$ for all $n \in \mathbb{Z}^+$? Justify your answer.

(b) Prove that $p_n < 2^{2^n}$ for all $n \in \mathbb{Z}^+$.

Show your working below and on the next page.

(More working spaces for Question 8)

Continue on page 18-19 if you need more space. Please indicate clearly.

(Additional working spaces for ALL questions - indicate your question numbers clearly.)

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