Student Number:

A/U/T*								

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2011-2012

MA1100 Fundamental Concepts of Mathematics

November/December 2011 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper contains a total of **EIGHT (8)** questions and comprises **NINETEEN (19)** printed pages.
3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.
4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Question 1 [10 marks]

(a) Use <u>induction</u> to prove that 8 divides $9^n - 1$ for all $n \in \mathbb{N}$. the set of positive integers

(b) If you have proven:

(i) $P(1), P(3)$ are true; and

(ii) $(\forall\ k \in \mathbb{N} \text{ and } k \geq 2)\ \ P(k) \Rightarrow P(k+4)$ is true.

What is the <u>largest possible domain</u> S that you have proven $(\forall\ n \in S)\ P(n)$? Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 1)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 2 [15 marks]

Let $f : \mathbb{R} - \{-2\} \to \mathbb{R} - \{4\}$ be a function defined by $f(x) = \dfrac{4x}{x+2}$.

(i) Show that f is a bijection.

(ii) Write down the inverse function f^{-1}.

(iii) Compute the composite function $f^{-1} \circ f$. Show your working clearly.

(iv) Let $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ be the floor function. Determine the range of $\lfloor\ \rfloor \circ f$. Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 2)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 3 [10 marks]

(a) Construct the addition and multiplication tables for $\mathbb{Z}_5$.

(b) Consider the statement:
If $x^2 + y^2 \equiv xy \mod 5$, then $x \equiv 0 \mod 5$ and $y \equiv 0 \mod 5$.

(i) Rewrite this statement in terms of congruence classes.

(ii) Give a proof of part (i) using the tables in (a).

Show your working below and on the next page.

(More working spaces for Question 3)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 4 [15 marks]

(a) Given a set $A = \{a, b, c, d, e, f\}$ with a partition $\{\{a, b\}, X, Y\}$ where X and Y are subsets of A such that $|X| > |Y|$ and $c \in Y$.

Find X and Y, and write down the ordered pair representation of the equivalence relation R on A corresponding to this partition.

(b) A relation S on $\mathbb{R}$ is defined for all $x, y \in \mathbb{R}$ by $x \sim y \Leftrightarrow x + y \in \mathbb{Q}$.

Determine whether S is reflexive, symmetric and transitive. Justify your answers.

(c) An equivalence relation T on $\mathbb{N}$ is defined for all $x, y \in \mathbb{N}$ by

$$x \sim y \Leftrightarrow x = 2^n y \ \text{ or } \ y = 2^n x \ \text{ for some non-negative integer } \ n.$$

(i) Write down the equivalence class $[1]_T$ using any set notation.

(ii) How many square free integers are there in each equivalence class of T?

(You do not need to justify your answers for part (c).)

Show your working below and on the next page.

(More working spaces for Question 4)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 5 [15 marks]

(a) Find two subsets A and B of $\mathbb{R}$ such that they satisfy the following conditions simultaneously: $A \cup B$ is uncountable, $A - B$ is infinite, $B - A$ is denumerable, and $A \cap B$ is countable and nonempty. Briefly explain your answer. countably infinite

(b) Determine whether the following sets are finite, denumerable or uncountable. Justify your answers.

(i) $\left\{ \dfrac{m}{n} \in \mathbb{Q} \mid 7m = n \text{ with } m, n \in \mathbb{N} \right\}$. N is the set of all positive integers

(ii) $\{ r \in \mathbb{R} \mid r^2 \in \mathbb{Q} \}$.

(iii) $\left\{ x \in \mathbb{R} - \mathbb{Q} \mid x = (\sqrt{2} + 1)^n \text{ with } n \in \mathbb{N} \right\}$.

Show your working below and on the next page.

(More working spaces for Question 5)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 6 [10 marks]
In each of the following parts, construct a function with the required conditions. Justify your answers.

(a) An injective function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ such that the range is $\{m + n\sqrt{3} \mid m, n \in \mathbb{Z}\}$.

(b) A function $g : \{1, 2, 3\} \to \{1, 2, 3\}$ such that $g \neq g \circ g$ but $g^n = g \circ g$ for all $n \geq 2$. (Here $g^n = g \circ g \circ \ldots \circ g$ denote the composition of g with itself for n times.)

Show your working below and on the next page.

(More working spaces for Question 6)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 7 [15 marks]

(a) Use Euclidean Algorithm to find integers x and y such that

$$\gcd(88, 121) = 88x + 121y.$$

(b) Show that if m and $m+2$ are both primes with $m > 3$, then $m+1$ is divisible by 6.

(Hint: Consider cases based on an appropriate congruence modulo.)

(c) Let a, b be integers such that $\gcd(a, b) = 1$. Prove that $\gcd(ab, a+b) = 1$.

(Hint: Use proof by contradiction.)

Show your working below and on the next page.

(More working spaces for Question 7)

Continue on page 18-19 if you need more space. Please indicate clearly.

Question 8 [10 marks]

Let p be a prime number. Prove the following:

(i) For all non-zero $[a]_p \in \mathbb{Z}_p$, the equation $[a]_p \cdot [x]_p = [1]_p$ has a solution.

(ii) $(p-1)! + 1$ is divisible by p.

Show your working below and on the next page.

(More working spaces for Question 8)

Continue on page 18-19 if you need more space. Please indicate clearly.

(Additional working spaces for ALL questions - indicate your question numbers clearly.)

(Additional working spaces for ALL questions - indicate your question numbers clearly.)

[END OF PAPER]