Student Number:

A/U/T*								

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2010-2011

MA1100 Fundamental Concepts of Mathematics

November/December 2010 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper contains a total of **EIGHT (8)** questions and comprises **TWENTY (20)** printed pages.
3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.
4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Question 1 [10 marks]

Let A, B, C be subsets of some universal set U.

(a) Use Element-Chasing method to show that $(A \times C) - (B \times C) = (A - B) \times C$.

(b) Is it true that $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$? Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 1)

Continue on page 18-20 if you need more space. Please indicate clearly.

Question 2 [15 marks]

(a) Let $f_1, f_2, \ldots, f_n, \ldots$ denote the Fibonacci sequence defined by

$$f_1 = 1, \quad f_2 = 1, \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3.$$

Show by induction that, for all $n \in \mathbb{N}$, $f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$.

(b) If you have proven: (i) $P(3)$ is true; and (ii) $P(k) \Rightarrow P(k+4)$ for all $k \geq 3$ is true.

For which universal set U have you proven $(\forall n \in U)\ P(n)$? Justify your answer.

positive integer

(c) Suppose you want to prove $P(n)$ is true for every natural number $n \geq 5$ by induction and you manage to show the inductive step $P(k) \wedge P(k+2) \Rightarrow P(k+4)$ for all $k \geq 7$. What are the base cases that you need to establish (i.e. which additional values a that you need to show that $P(a)$ is true separately)? Justify your answer.

base case = basis step

Show your working below and on the next page.

(More working spaces for Question 2)

Continue on page 18-20 if you need more space. Please indicate clearly.

Question 3 [10 marks]

(a) Find <u>all</u> the congruence classes $[x]_7$ in $\mathbb{Z}_7$ such that $[5]_7 \cdot [x]_7 = [3]_7$. Show your working.

(b) Show that for any integers a and b, if $a + b \equiv ab \mod 4$, then <u>either</u> (both $a \equiv 0$ and $b \equiv 0 \mod 4$) <u>or</u> (both $a \equiv 2$ and $b \equiv 2 \mod 4$).
(Hint: Consider $\mathbb{Z}_4$, the set of integers modulo 4.)

Show your working below and on the next page.

(More working spaces for Question 3)

Continue on page 18-20 if you need more space. Please indicate clearly.

Question 4 [15 marks]

(a) Let $C = \{\{1,3\},\{2,4\}\}$ be a partition of $A = \{1,2,3,4\}$. Write down the equivalence relation R on A that is corresponding to the partition C in terms of ordered pair representation.

(b) (i) Show that the relation S on $\mathbb{N}$ defined below is an equivalence relation.

$$a \sim b \text{ if and only if } \quad ab \text{ is a perfect square.}$$

(ii) Determine all the equivalence classes of S. Justify your answers.

Show your working below and on the next page.

(More working spaces for Question 4)

Continue on page 18-20 if you need more space. Please indicate clearly.

Question 5 [10 marks]

countably infinite

(a) Give two examples of $\boxed{\text{denumerable}}$ sets (i.e. countable sets that are infinite) $S_1, S_2 \subseteq \mathbb{R}$ that satisfy the following conditions:

(i) all the elements in S_1 are irrational numbers;

(ii) all the elements in S_2 are within the open interval $(0, 1)$.

For each of your examples, briefly explain why it is denumerable.

(b) True or false: If the intersection of two uncountable sets is infinite, then the intersection is uncountable. Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 5)

Continue on page 18-20 if you need more space. Please indicate clearly.

Question 6 [15 marks]

(a) Determine whether the following function is a bijection. Justify your answer.

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \quad \text{where} \quad f(x, y) = (2x + 1, 2y - 1).$$

(b) Let $h : \mathbb{R} - \{2\} \to \mathbb{R} - \{2\}$ be a function defined by $h(x) = \dfrac{2x}{x - 2}$.

(i) Determine the composite function $h \circ h$ and inverse function h^{-1}. Show your working.

(ii) Suppose the domain of h is replaced by $\mathbb{Q} - \{2\}$. What would be the range of the new function? Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 6)

Continue on page 18-20 if you need more space. Please indicate clearly.

Question 7 [15 marks]

(a) Use Euclidean Algorithm to find integers x and y such that $\gcd(33, 54) = 33x + 54y$.

(b) Show that, if p is an odd prime, then $\gcd(p^2 + 1, (p + 1)^2) = 2$.

(c) Show that we cannot find three consecutive odd numbers that are all primes, except 3, 5 and 7.

Show your working below and on the next page.

(More working spaces for Question 7)

Continue on page 18-20 if you need more space. Please indicate clearly.

Question 8 [10 marks]

Let $\mathcal{C} = \{A_1, A_2, \ldots, A_i, \ldots\}$ be an indexed collection of subsets of $\mathbb{Q}^+$ such that

this denote the set of all positive integers

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{Q}^+ \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i = \boxed{\mathbb{N}} \quad (*)$$

(a) Is it possible to find functions $f_i : A_i \to \mathbb{N}$ that are bijections for all $i \in \mathbb{N}$? Justify your answer.

(b) Give an example of a collection $\mathcal{C}$ that satisfies $(*)$ and such that $A_i \nsubseteq A_j$ for any two distinct sets A_i, A_j in $\mathcal{C}$. Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 8)

Continue on page 18-20 if you need more space. Please indicate clearly.

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