Student Number: | U/T* | – | | | – | | | | | – | |

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NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA1100 Fundamental Concepts of Mathematics

November/December 2009 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains a total of **NINE (9)** questions and comprises **TWENTY THREE (23)** printed pages.

3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.

4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
Total	

Question 1 [10 marks]

Let S be the statement: 3 divides $n^3 + 2n$ for every positive integer n.

(a) Prove S directly by considering three cases in terms of congruence modulo 3.

(b) Prove S using mathematical induction.

Show your working below and on the next page.

(More working spaces for Question 1)

Continue on page 20-23 if you need more space. Please indicate clearly.

Question 2 [10 marks]

(a) Let R be the relation on $\mathbb{Z}$ defined by $a \sim b$ if and only if $|a - b| > 3$. Determine whether R is reflexive, symmetric and transitive.

(b) List all possible pairs of classes $[a]_{12}$ and $[b]_{12}$ in $\mathbb{Z}_{12}$ such that $[a]_{12} \cdot [b]_{12} = [0]_{12}$.

Show your working below and on the next page.

(More working spaces for Question 2)

Continue on page 20-23 if you need more space. Please indicate clearly.

Question 3 [15 marks]

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2 - 4$.

(i) Show that f is not an injective map.

(ii) The function f can be made into a bijection $\hat{f}$ by replacing the domain and codomain with some intervals A and B in $\mathbb{R}$. Find the largest possible intervals A and B so that the function is a bijection. (Hint: look at the graph of f.)

(iii) For your bijection $\hat{f}$ in (ii), determine its inverse function $\hat{f}^{-1}$.

(iv) Show that there is no function $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f$ is one-to-one.

(v) Can you find a function $h : \mathbb{R} \to \mathbb{R}$ such that the range of $f \circ h$ contains only irrational numbers? Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 3)

Continue on page 20-23 if you need more space. Please indicate clearly.

Question 4 [15 marks]

(a) (i) Use Euclidean Algorithm to find $\gcd(124, 262)$.

(ii) Find two integers x and y such that $\gcd(124, 262) = 124x + 262y$. Show your working.

(b) (i) Find three distinct integers a_1, b_1, c_1 such that $\gcd(a_1, b_1) = 2$, $\gcd(a_1, c_1) = 2$ and $\gcd(a_1,\ b_1c_1) = 2$.

(ii) Find three distinct integers a_2, b_2, c_2 such that $\gcd(a_2, b_2) = 2$, $\gcd(a_2, c_2) = 2$ and $\gcd(a_2,\ b_2c_2) = 4$.

(iii) Show that there are no integers a, b, c such that $\gcd(a, b) = 2$, $\gcd(a, c) = 2$ and $\gcd(a,\ bc) \neq 2$ and 4.

Show your working below and on the next page.

(More working spaces for Question 4)

Continue on page 20-23 if you need more space. Please indicate clearly.

Question 5 [10 marks]

Let F_n be the sequence of Fibonacci numbers:

$$F_1 = 1,\ F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for all} \quad n \geq 3$$

and L_n be the sequence of Lucas numbers:

$$L_1 = 1,\ L_2 = 3 \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for all} \quad n \geq 3$$

(a) List down the first 10 Lucas numbers.

(b) Show that $L_n = F_{n-1} + F_{n+1}$ for all integers $n > 1$.

(c) Explain clearly what is wrong with the following "proof" of the false statement:

$$L_n \leq F_n \quad \text{for all} \quad n \in \mathbb{N}.$$

"Proof"
Base case: When $n = 1$, $L_1 = 1$ and $F_1 = 1$. So $L_1 \leq F_1$.
Inductive step: Assume $L_i \leq F_i$ for all $1 \leq i \leq k$.
Since $L_{k+1} = L_k + L_{k-1}$ and by hypothesis, $L_k \leq F_k$ and $L_{k-1} \leq F_{k-1}$, we have

$$L_{k+1} = L_k + L_{k-1} \leq F_k + F_{k-1} = F_{k+1}.$$

By strong mathematical induction, we have proven that $L_n \leq F_n$ for all $n \in \mathbb{N}$.

Show your working below and on the next page.

(More working spaces for Question 5)

Continue on page 20-23 if you need more space. Please indicate clearly.

Question 6 [10 marks]

(a) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and R be the equivalence relation on A:

$$n \sim m \quad \text{if and only if} \quad n \equiv m \mod 4.$$

(i) Write down the distinct equivalence classes determined by R by listing the elements in each class.

(ii) By regarding R as a subset of $A \times A$, what is the cardinality of R? Justify your answer.

(b) Let $B = \{a, b, c, d, e\}$.

(i) How many different equivalence relations on B are there? Justify your answer.

(ii) Let S be an equivalence relation on B such that

- S has three distinct equivalence classes $[a]_S$, $[b]_S$ and $[c]_S$;
- the ordered pair $(a, d) \in S$ but $(e, a) \notin S$;
- the cardinality of $[b]_S$ is 1.

Write down the relation S as a subset of $B \times B$.

Show your working below and on the next page.

(More working spaces for Question 6)

Continue on page 20-23 if you need more space. Please indicate clearly.

Question 7 [10 marks]

(a) Prove that if n is a positive odd integer of the form $4k+3$, then n has a prime factor of this form as well.

(b) Suppose $2^p - 1$ is a prime number. Prove that $2^{p-1} + 2^p + \cdots + 2^{2p-2}$ is a perfect number.

(Recall a positive integer n is called a perfect number if the sum of all its positive proper divisors is equal to n itself.)

Show your working below and on the next page.

(More working spaces for Question 7)

Continue on page 20-23 if you need more space. Please indicate clearly.

Question 8 [10 marks]

(a) Is it possible to find a partition C of $\mathbb{N}$ such that C is infinite and S is infinite for every $S \in C$? Justify your answer.

(b) Let A be the set of all functions with domain $\{0, 1\}$ and codomain $\mathbb{N}$. i.e.

$$A = \{f \mid f : \{0, 1\} \to \mathbb{N}\}.$$

Is A a countable set? Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 8)

Continue on page 20-23 if you need more space. Please indicate clearly.

Question 9 [10 marks]

Let R be the equivalence relation on $\mathbb{N} \times \mathbb{N}$ defined by:

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

Let C be the set of equivalence classes determined by R.

(a) Construct a bijection $f : C \to \mathbb{Q}^+$ where $\mathbb{Q}^+$ is the set of positive rational numbers.

(b) Let $S \in C$. Is $|S| = |C|$?

Justify your answers above.

Show your working below and on the next page.

(More working spaces for Question 9)

Continue on page 20-23 if you need more space. Please indicate clearly.

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