Student Number: | U/T/NT* | – | | | – | | | | | – | |

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2008-2009

MA1100 Fundamental Concepts of Mathematics

November/December 2008 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper contains a total of **Eight (8)** questions and comprises **Nineteen (19)** printed pages.
3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.
4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Question 1 [15 marks]

(a) Use Mathematical Induction to prove the following statement:

$$\text{For all integers } n \geq 1, \quad 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 3}{2}.$$

(b) A sequence $s_1, s_2, \ldots, s_n, \ldots$ is defined recursively as follows:

$$\begin{aligned} s_1 &= 4 \\ s_2 &= 8 \\ s_n &= 5s_{n-1} + (s_{n-2})^2 \quad \text{for all integers } n \geq 3 \end{aligned}$$

Use a version of Mathematical Induction to prove that s_n is divisible by 4 for all $n \in \mathbb{Z}^+$.

Show your working below and on the next page.

(More working spaces for Question 1)

Continue on page 18 - 19 if you need more space. Please indicate clearly.

Question 2 [15 marks]

Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation $\sim$ on A as follows:
For all $x, y \in A$, $x \sim y$ if and only if $3 \mid (x - y)$.

(a) Is $7 \sim 2$? Is $2 \sim 5$? Is $8 \sim 8$? Explain your answers briefly.

(b) Write down the ordered pair representation of the relation $\sim$.

(c) Show that $\sim$ is an equivalence relation.

(d) Write down explicitly all the equivalence classes of $\sim$.

Show your working below and on the next page.

(More working spaces for Question 2)

Continue on page 18 - 19 if you need more space. Please indicate clearly.

Question 3 [15 marks]

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 16x - 5$
and $g : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $g(x) = 16x - 5$.

(a) Show that f is a bijection.

(b) Find the inverse function f^{-1}.

(c) Is g an injection? Justify your answer.

(d) Is g a surjection? Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 3)

Continue on page 18 - 19 if you need more space. Please indicate clearly.

Question 4 [15 marks]

(a) Use Euclidean Algorithm to find $\gcd(284, 168)$.

(b) Rewrite the following set $\{n \in \mathbb{Z} \mid n = 284x + 168y \text{ for some } x, y \in \mathbb{Z}\}$ with a set builder notation in terms of congruence modulo.

(c) Find the smallest positive integer x such that $284x + 168y = 4$ for some integer y. Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 4)

Continue on page 18 - 19 if you need more space. Please indicate clearly.

Question 5 [10 marks]

(a) Determine all possible congruence classes in $\mathbb{Z}_7$ that are squares $[n]_7^2$ of some congruence class $[n]_7$ in $\mathbb{Z}_7$.

(b) Use part (a) to prove that, for all $n, m \in \mathbb{Z}$, if $n^2 + m^2 \equiv 0 \mod 7$, then n and m are both divisible by 7.

(c) Is it true that, for all $a, b, c \in \mathbb{Z}$, if $a^2 + b^2 + c^2 \equiv 0 \mod 7$, then a, b, c are all divisible by 7? Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 5)

Continue on page 18 - 19 if you need more space. Please indicate clearly.

Question 6 [10 marks]

Let U be a non-empty set and $\mathcal{S}(U)$ be the set of all non-empty subsets of U. Define a relation $\sim$ on $\mathcal{S}(U)$ as follows:
For $A, B \in \mathcal{S}(U)$, $A \sim B$ if and only if there exists a bijection $f : A \to B$.

(a) Show that $\sim$ is an equivalence relation.

(b) If $U = \mathbb{Z}_3$, how many different equivalence classes on $\mathcal{S}(U)$ are there determined by $\sim$? Justify your answer.

Show your working below and on the next page.

(More working spaces for Question 6)

Continue on page 18 - 19 if you need more space. Please indicate clearly.

Question 7 [10 marks]

Let A and B be two non-empty subsets of some universal set U. Let $f : A \to A$ and $g : B \to B$ be two functions such that $f(x) = g(x)$ for all $x \in A \cap B$.
Define the function $h : (A \cup B) \to (A \cup B)$ by $h(a) = f(a)$ for all $a \in A$ and $h(b) = g(b)$ for all $b \in B$.

(a) Suppose f and g are surjections. Is it necessary that h is a surjection?

(b) Suppose f and g are injections. Is it necessary that h is an injection?

Justify your answers clearly.

Show your working below and on the next page.

(More working spaces for Question 7)

Continue on page 18 - 19 if you need more space. Please indicate clearly.

Question 8 [10 marks]

(a) Show that, if p is a prime number greater than 3, then $2p+1$ and $4p+1$ cannot be both prime numbers.

(b) For any positive integer n, let $d(n)$ be the number of positive divisors of n.
Show that $d(n) \leq 2\sqrt{n}$ for all $n \in \mathbb{Z}^+$.

Show your working below and on the next page.

(More working spaces for Question 8)

Continue on page 18 - 19 if you need more space. Please indicate clearly.

(Additional working spaces for ALL questions - indicate your question numbers clearly.)

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