National University of Singapore
Department of Mathematics

2009/2010 Sem I | MA4247 Complex Analysis II | Notes for E-Learning Week

Note that the following notes refers to the Lecture notes available from the workbin, specifically Parts V and VI of the lecture notes (LN5 and LN6). To make up for the lecture missed on Thursday, all of you should go through the following:

Things to go through in LN5:

1. Go through the statement and the proof of Theorem 3.1.8 in LN5. In particular, for the "if" part, note that since we are assuming regularity properties of the first order derivatives, we only need to show that the Cauchy Riemann equations hold. Notice how the geometric property is being used to prove this essentially analytic property.
2. Fill in the diagrams for section 3.2 on the mappings by elementary functions. In particular, draw a suitable orthonormal net in the domain, and draw the image of this orthonormal net in the range, using the arguments in the notes.
3. For section 3.3, do the exercise to show that equation (1) on page 14 holds. Go through proposition 3.3.1 and fill in the details.
4. Go through section 3.4.2 and fill in the details of the proofs for the various properties of the LFT's listed there.
5. Go through pages 18-20 detailing a more conceptual approach to understanding the group properties of the LFT's. In particular, think about how the extended complex plane $\hat{\mathbb{C}}$ is being represented by homogeneous coordinates, that is 2×1 non-zero vectors in $\mathbb{C}^2$, up to multiplication by a non-zero constant. Consider also the map from the set of non-singular 2×2 matrices to the set of LFT's, show that it is a homomorphism (with multiplication operation for the matrices and composition for the LFT's) and find the kernel of this homomorphism.

For LN6, go through the following:

1. Go through Theorem 3.5.1 and its proof. The main point about this theorem is that it shows that any LFT can be decomposed into the composition of basic transformations, consisting of translations, rotations, dilations and inversions. It is easy to understand what translations, rotations and dilations do to geometric objects in the plane. It is not so clear what inversion does, but since it is fairly simple, can can actually do calculations in many cases. Lemma 3.5.3 is an example, where we show the important property that inversion takes circles or extended lines bijectively to circles or extended lines. Note that the other basic transformations (translations, rotations and dilations) obviously do the same, in fact, they map circles to circles and extended lines to extended lines bijectively.
2. Go through the proofs of Lemma 3.5.2 and 3.5.3

3. Go through the proof of Theorem 3.5.5. Note structure of this proof. In order to prove that some property holds, we show a decomposition result for LFT's first, into basic transformations, then we show that each basic transformation has this property, and finally we put it together to show that any LFT has this property. We could have done this directly too, by just considering an arbitrary LFT and showing that it maps circles or extended lines to circles or extended lines, but the sort of "divide and conquer" idea used in this proof is used very often in mathematics.

4. Go through Theorem 3.5.6. Note that the fact that a non-identity LFT fixes only one or two points in $\hat{\mathbb{C}}$ is fairly trivial and easy to prove, by elementary calculations. Parts 2 and 3 of the theorem are important, another way to phrase part 3 is that the Mobius group acts 3-transitively on $\hat{\mathbb{C}}$ (recall that a group G acts transitively on a set S if for every $x, y \in S$, there exists $g \in G$ such that $gx = y$). You should be able to see why we call this action 3-transitive. Part 3 takes this one step further by determining the unique LFT which takes an arbitrary ordered triple of 3 distinct points (z_1, z_2, z_3) to a standard triple $(\infty, 0, 1)$.