

Б. П. 吉米多维奇
Б. П. ДЕМИДОВИЧ

数学分析

习题集题解

(六)

费定晖 周学圣 编演
郭大钧 邵品琮 主审



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出版说明

吉米多维奇(Б. П. Д ЕМИД ОВИЧ)著《数学分析习题集》一书的中译本,自 50 年代初在我国翻译出版以来,引起了全国各大专院校广大师生的巨大反响。凡从事数学分析教学的师生,常以试解该习题集中的习题,作为检验掌握数学分析基本知识和基本技能的一项重要手段。二十多年来,对我国数学分析的教学工作是甚为有益的。

该书四千多道习题,数量多,内容丰富,由浅入深,部分题目难度大。涉及的内容有函数与极限,单变量函数的微分学,不定积分,定积分,级数,多变量函数的微分学,带参变量积分以及重积分与曲线积分、曲面积分等等,概括了数学分析的全部主题。当前,我国广大读者,特别是肯于刻苦自学的广大数学爱好者,在为四个现代化而勤奋学习的热潮中,迫切需要对一些疑难习题有一个较明确的回答。有鉴于此,我们特约作者,将全书 4462 题的所有解答汇编成书,共分六册出版。本书可以作为高等院校的教学参考用书,同时也可作为广大读者在自学微积分过程中的参考用书。

众所周知,原习题集,题多难度大,其中不少习题如果认真习作的话,既可以深刻地巩固我们所学到的基本概念,又可以有效地提高我们的运算能力,特别是有些难题还可以迫使我们学会综合分析的思维方法。正由于这样,我们殷切期望初学数学分析的青年读者,一定要刻苦钻研,千万不要轻易查抄

本书的解答,因为任何削弱独立思索的作法,都是违背我们出版此书的本意。何况所作解答并非一定标准,仅作参考而已。如有某些误解、差错也在所难免,一经发觉,恳请指正,不胜感谢。

本书蒙潘承洞教授对部分难题进行了审校。特请郭大钧教授、邵品琮教授对全书作了重要仔细的审校。其中相当数量的难度大的题,都是郭大钧、邵品琮亲自作的解答。

参加编演工作的还有黄春朝同志。

本书在编审过程中,还得到山东大学、山东工业大学、山东师范大学和曲阜师范大学的领导和同志们的大力支持,特在此一并致谢。

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第八章 重积分和曲线积分

§ 1. 二 重 积 分

1° 二重积分的直接算法 所谓连续函数 $f(x, y)$ 展布在有限封闭可求积二维域 Ω 内的二重积分乃是指的数

$$\iint_{\Omega} f(x, y) dx dy = \lim_{\substack{\max |\Delta x_i| \rightarrow 0 \\ \max |\Delta y_j| \rightarrow 0}} \sum_i \sum_j f(x_i, y_j) \Delta x_i \Delta y_j,$$

其中 $\Delta x_i = x_{i+1} - x_i, \Delta y_j = y_{j+1} - y_j$, 而其和为对所有 i, j 使 $(x_i, y_j) \in \Omega$ 的那些值来求的。

若域 Ω 由下面的不等式所给出

$$a \leq x \leq b, y_1(x) \leq y \leq y_2(x),$$

其中 $y_1(x)$ 和 $y_2(x)$ 为在闭区间 $[a, b]$ 上的连续函数, 则对应的二重积分可按下面的公式来计算

$$\iint_{\Omega} f(x, y) dx dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy.$$

2° 二重积分中的变量代换 若可微分的连续函数

$$x = x(u, v), y = y(u, v)$$

把平面 Oxy 上的有限闭域 Ω 单值唯一地映射为平面 Ouv 上的域 Ω' 及雅哥比式

$$I = \frac{D(x, y)}{D(u, v)} \neq 0,$$

则下之公式正确:

$$\iint_{\Omega} f(x, y) dx dy = \iint_{\Omega'} f(x(u, v), y(u, v)) |I| du dv.$$

特别是,根据公式 $x = r\cos\varphi, y = r\sin\varphi$ 变换为极坐标 r 和 φ 的情形有

$$\iint_D f(x, y) dx dy = \iint_D f(r\cos\varphi, r\sin\varphi) r dr d\varphi.$$

3901. 把积分 $\iint_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1}} xy dx dy$, 当作积分和的极限, 用直线

$$x = \frac{i}{n}, y = \frac{j}{n} (i, j = 1, 2, \dots, n-1)$$

把积分域分为许多正方形, 并选取被积函数在这些正方形之右顶点的值, 计算所论积分的值.

解 由于

$$\sum_{i=1}^n \sum_{j=1}^n \frac{i}{n} \cdot \frac{j}{n} \cdot \frac{1}{n^2} = \frac{n^2(n+1)^2}{4n^4} \longrightarrow \frac{1}{4} \quad (n \rightarrow \infty),$$

其中

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{j=1}^n j = \frac{n(n+1)}{2},$$

故

$$\iint_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1}} xy dx dy = \frac{1}{4}.$$

3902. 用直线

$$x = 1 + \frac{i}{n}, y = 1 + \frac{2j}{n} (i, j = 0, 1, \dots, n)$$

把域 $1 \leq x \leq 2, 1 \leq y \leq 3$ 分为许多矩形. 作出函数 $f(x, y) = x^2 + y^2$ 在此域内的积分下和 $\underline{S}$ 与积分上和 $\bar{S}$. 当 $n \rightarrow \infty$ 时, 上和与下和的极限等于什么?

解 下和

$$\begin{aligned}
S &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\left(1 + \frac{i}{n} \right)^2 + \left(1 + \frac{2j}{n} \right)^2 \right] \cdot \frac{1}{n} \cdot \frac{2}{n} \\
&= \frac{2n}{n^2} \left[n + \frac{2}{n} \sum_{i=0}^{n-1} i + \frac{1}{n^2} \sum_{i=0}^{n-1} i^2 + n + \frac{4}{n} \sum_{j=0}^{n-1} j \right. \\
&\quad \left. + \frac{4}{n^2} \sum_{j=0}^{n-1} j^2 \right] \\
&= \frac{40}{3} + \frac{11}{n} + \frac{5}{3n^2},
\end{aligned}$$

其中

$$\begin{aligned}
\sum_{i=0}^{n-1} i^2 &= \frac{(n-1)n(2n-1)}{6}, \\
\sum_{j=0}^{n-1} j^2 &= \frac{(n-1)n(2n-1)}{6},
\end{aligned}$$

而上和

$$\begin{aligned}
\bar{S} &= \sum_{i=1}^n \sum_{j=1}^n \left[\left(1 + \frac{i}{n} \right)^2 + \left(1 + \frac{2j}{n} \right)^2 \right] \cdot \frac{1}{n} \cdot \frac{2}{n} \\
&= \frac{40}{3} + \frac{11}{n} + \frac{5}{3n^2}.
\end{aligned}$$

当 $n \rightarrow \infty$ 时, $\underline{S}$ 与 $\bar{S}$ 的极限均等于 $\frac{40}{3} = 13\frac{1}{3}$.

3903. 用一系列内接正方形作为积分域的近似域, 这些正方形的顶点 A_n 在整数点, 并取被积函数在每个正方形距原点的最远的顶点之值. 近似地计算积分

$$\iint_{x^2+y^2 \leq 25} \frac{dxdy}{\sqrt{24+x^2+y^2}},$$

并与精确的值加以比较。

解 由题意知, 应取的正方形顶点为 $(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2),$

$(3,3), (3,4), (4,1), (4,2), (4,3)$, 故利用对称性知

$$\begin{aligned} \frac{1}{4} \iint_{x^2+y^2 \leq 25} \frac{dxdy}{\sqrt{24+x^2+y^2}} &= \frac{1}{\sqrt{26}} + \frac{2}{\sqrt{29}} + \frac{2}{\sqrt{34}} \\ &+ \frac{2}{\sqrt{41}} + \frac{1}{\sqrt{32}} + \frac{2}{\sqrt{37}} + \frac{2}{\sqrt{44}} + \frac{1}{\sqrt{42}} + \frac{2}{\sqrt{49}} \\ &\doteq 0.196 + 0.371 + 0.343 + 0.312 + 0.177 \\ &+ 0.329 + 0.302 + 0.154 + 0.285 \\ &\doteq 2.470, \end{aligned}$$

$$\text{即 } \iint_{x^2+y^2 \leq 25} \frac{dxdy}{\sqrt{24+x^2+y^2}} \doteq 9.880.$$

下面计算积分的精确值:

$$\begin{aligned} &\iint_{x^2+y^2 \leq 25} \frac{dxdy}{\sqrt{24+x^2+y^2}} \\ &= 4 \int_0^5 \ln(y + \sqrt{24+x^2+y^2}) \Big|_0^{\sqrt{25-x^2}} dx \\ &= 4 \int_0^5 \ln(\sqrt{25-x^2} + 7) dx - 2 \int_0^5 \ln(24+x^2) dx. \end{aligned}$$

由于

$$\begin{aligned} \int \ln(24+x^2) dx &= x \ln(24+x^2) - \int \frac{2x^2}{24+x^2} dx \\ &= x \ln(24+x^2) - 2x + \frac{24}{\sqrt{6}} \operatorname{arctg} \frac{x}{\sqrt{24}} + C, \end{aligned}$$

从而

$$\begin{aligned} &2 \int_0^5 \ln(24+x^2) dx \\ &= \left[2x \ln(24+x^2) - 4x + \frac{48}{\sqrt{6}} \operatorname{arctg} \frac{x}{\sqrt{24}} \right] \Big|_0^5 \end{aligned}$$

$$= 20\ln 7 - 20 + 8\sqrt{6} \operatorname{arctg} \frac{5}{\sqrt{24}};$$

又

$$\begin{aligned} & 4 \int_0^5 \ln(\sqrt{25-x^2} + 7) dx \\ &= 4(x \ln(\sqrt{25-x^2} + 7)) \Big|_0^5 \\ &\quad + 4 \int_0^5 \frac{x^2 dx}{(\sqrt{25-x^2} + 7) \sqrt{25-x^2}} \\ &= 20\ln 7 + 4 \int_0^5 \frac{x^2 dx}{(\sqrt{25-x^2} + 7) \sqrt{25-x^2}}, \end{aligned}$$

再令 $x = 5\sin t$, 有

$$\begin{aligned} & \int_0^5 \frac{x^2 dx}{(\sqrt{25-x^2} + 7) \sqrt{25-x^2}} = \int_0^{\frac{\pi}{2}} \frac{25\cos^2 t + 25}{5\cos t + 7} dt \\ &= - \int_0^{\frac{\pi}{2}} (5\cos t - 7) dt - \int_0^{\frac{\pi}{2}} \frac{24}{5\cos t + 7} dt \\ &= (7t - 5\sin t) \Big|_0^{\frac{\pi}{2}} - 24 \left[\frac{1}{\sqrt{6}} \operatorname{arctg} \left(\frac{1}{\sqrt{6}} \operatorname{tg} \frac{t}{2} \right) \right] \Big|_0^{\frac{\pi}{2}} \\ &= \frac{7\pi}{2} - 5 - 4\sqrt{6} \operatorname{arctg} \frac{2}{\sqrt{24}}, \end{aligned}$$

从而

$$\begin{aligned} & 4 \int_0^5 \ln(\sqrt{25-x^2} + 7) dx \\ &= 20\ln 7 + 14\pi - 20 - 16\sqrt{6} \operatorname{arctg} \frac{2}{\sqrt{24}}. \end{aligned}$$

注意到

$$2\operatorname{arctg} \frac{2}{\sqrt{24}} = \operatorname{arctg} \frac{5}{\sqrt{24}} = \frac{\pi}{2},$$

最后使得到

$$\begin{aligned}
& \iint_{x^2+y^2 \leq 25} \frac{dxdy}{\sqrt{24+x^2+y^2}} \\
&= 14\pi - 4\sqrt{24} \left(2\operatorname{arctg} \frac{2}{\sqrt{24}} + \operatorname{arctg} \frac{5}{\sqrt{24}} \right) \\
&= 2\pi(7 - \sqrt{24}) \doteq 13.19.
\end{aligned}$$

将精确值与近似值作比较, 显见误差较大, 其原因在于有不少不是正方形的域都被忽略, 因而产生较大的绝对误差 4.31 及较大的相对误差 $\frac{4.31}{13.19} \doteq 32.7\%$.

注意, 求 $\iint_{x^2+y^2 \leq 25} \frac{dxdy}{\sqrt{24+x^2+y^2}}$ 的精确值若采用极坐标则较为简单:

$$\begin{aligned}
\iint_{x^2+y^2 \leq 25} \frac{dxdy}{\sqrt{24+x^2+y^2}} &= \int_0^{2\pi} d\theta \int_0^5 \frac{rdr}{\sqrt{24+r^2}} \\
&= 2\pi(7 - \sqrt{24}).
\end{aligned}$$

但按原习题集的安排, 似应在 3937 题以后才开始使用极坐标, 故本题仍用直角坐标进行计算.

3904. 用直线 $x = \text{常数}$, $y = \text{常数}$, $x + y = \text{常数}$ 把域 S 分为四个相等的三角形, 并取被积函数在每个三角形的中线交点之值, 近似地计算积分

$$\iint_S \sqrt{x+y} dS,$$

其中 S 表由直线 $x = 0$, $y = 0$ 及 $x + y = 1$ 所围成的三角形.

解 我们只须以 $x = \frac{1}{2}$, $y = \frac{1}{2}$ 及 $x + y = \frac{1}{2}$ 分域 S , 即得四个相等的三角形, 它们的面积均为 $\frac{1}{8}$, 重心为

$\left(\frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{6}\right)$ 及 $\left(\frac{1}{6}, \frac{2}{3}\right)$. 于是, 得此积分的近似值为

$$\begin{aligned} \iint_D \sqrt{x+y} dS & \doteq \frac{1}{8} \left[\sqrt{\frac{1}{6} + \frac{1}{6}} + \sqrt{\frac{1}{3} + \frac{1}{3}} + 2\sqrt{\frac{2}{3} + \frac{1}{6}} \right] \\ & \doteq \frac{1}{8} (0.577 + 0.816 + 2.0913) \doteq 0.402, \end{aligned}$$

其精确值为

$$\begin{aligned} \iint_D \sqrt{x+y} dS &= \int_0^1 dx \int_0^{1-x} \sqrt{x+y} dy \\ &= \frac{2}{3} \int_0^1 (1-x^{\frac{3}{2}}) dx = \frac{2}{5} = 0.4. \end{aligned}$$

3905. 把域 $S \{x^2 + y^2 \leq 1\}$ 分为有限个直径小于 δ 的可求积的子域 $\Delta S_i (i = 1, 2, \dots, n)$. 对于什么样的值 δ 能保证不等式:

$$\left| \iint_S \sin(x+y) dS - \sum_{i=1}^n \sin(x_i + y_i) \Delta S_i \right| < 0.001$$

成立? 其中 $(x_i, y_i) \in \Delta S_i$.

解 记函数 $\sin(x+y)$ 在 ΔS_i 中的振幅为 ω_i , 则

$$\begin{aligned} & \left| \iint_S \sin(x+y) dS - \sum_{i=1}^n \sin(x_i + y_i) \Delta S_i \right| \\ &= \left| \sum_{i=1}^n \iint_{\Delta S_i} [\sin(x+y) - \sin(x_i + y_i)] dS \right| \\ &\leq \sum_{i=1}^n \iint_{\Delta S_i} |\sin(x+y) - \sin(x_i + y_i)| dS \end{aligned}$$

$$\leq \sum_{i=1}^n \iint_{\Delta S_i} \omega_i dS = \sum_{i=1}^n \omega_i \Delta S_i.$$

由于域 $S\{x^2 + y^2 \leq 1\}$ 的面积等于 π , 故只要

$$\omega_i < \frac{0.001}{\pi},$$

便能满足原不等式的要求。但因为

$$\begin{aligned} \omega_i &= \sup_{\substack{(x_i', y_i') \in \Delta S_i \\ (x_i, y_i) \in \Delta S_i}} |\sin(x_i' + y_i') - \sin(x_i + y_i)| \\ &\leq \sup_{\substack{(x_i', y_i') \in \Delta S_i \\ (x_i, y_i) \in \Delta S_i}} |(x_i' + y_i') - (x_i + y_i)| \\ &\leq \sup_{\substack{(x_i', y_i') \in \Delta S_i \\ (x_i, y_i) \in \Delta S_i}} [|x_i' - x_i| + |y_i' - y_i|] \\ &\leq \sup_{\substack{(x_i', y_i') \in \Delta S_i \\ (x_i, y_i) \in \Delta S_i}} \sqrt{2[(x_i' - x_i)^2 + (y_i' - y_i)^2]}^{*}) \\ &= \sqrt{2} \delta_i, \end{aligned}$$

故只要取

$$\delta < \frac{1}{\sqrt{2}\pi} \times 0.001 \doteq 0.00022,$$

则有

$$\left| \iint_S \sin(x + y) dS - \sum_{i=1}^n \sin(x_i + y_i) \Delta S_i \right| < 0.001.$$

*) 对于任意非负实数 a, b 有

$$2ab \leq a^2 + b^2 \text{ 或 } (a + b)^2 \leq 2(a^2 + b^2),$$

从而

$$a + b \leq \sqrt{2(a^2 + b^2)}.$$

计算积分:

$$3906. \int_0^1 dx \int_0^1 (x+y) dy.$$

$$\text{解} \quad \int_0^1 dx \int_0^1 (x+y) dy = \int_0^1 \left(x + \frac{1}{2} \right) dx = 1.$$

$$3907. \int_0^1 dx \int_{x^2}^x xy^2 dy.$$

$$\text{解} \quad \int_0^1 dx \int_{x^2}^x xy^2 dy = \int_0^1 \left(\frac{x^4}{3} - \frac{x^7}{3} \right) dx = \frac{1}{40}.$$

$$3908. \int_0^{2\pi} d\varphi \int_0^a r^2 \sin^2 \varphi dr.$$

$$\begin{aligned} \text{解} \quad \int_0^{2\pi} d\varphi \int_0^a r^2 \sin^2 \varphi dr &= \frac{a^3}{3} \int_0^{2\pi} \sin^2 \varphi d\varphi \\ &= \frac{a^3}{3} \left(\frac{\varphi}{2} - \frac{1}{4} \sin 2\varphi \right) \Big|_0^{2\pi} = \frac{\pi a^3}{3}. \end{aligned}$$

3909. 设 R 为矩形

$$a \leq x \leq A, \quad b \leq y \leq B.$$

证明等式

$$\iint_R X(x)Y(y) dx dy = \int_a^A X(x) dx \int_b^B Y(y) dy.$$

证 根据在矩形域的情况下化二重积分为逐次积分的计算方法,不妨先对 y 后对 x 积分,即得

$$\begin{aligned} \iint_R X(x)Y(y) dx dy &= \int_a^A dx \int_b^B X(x)Y(y) dy \\ &= \int_a^A X(x) dx \int_b^B Y(y) dy. \end{aligned}$$

3910. 设:

$$f(x, y) = F''_{xy}(x, y),$$

计算

$$I = \int_a^A dx \int_b^B f(x, y) dy.$$

解 不妨按先对 y 后对 x 积分的顺序计算, 即得

$$\begin{aligned} I &= \int_a^A [F(x, B) - F(x, b)] dx \\ &= F(x, B) \Big|_a^A - F(x, b) \Big|_a^A \\ &= F(A, B) - F(a, B) - F(A, b) + F(a, b). \end{aligned}$$

3911. 设 $f(x)$ 为在闭区间 $a \leq x \leq b$ 内的连续函数, 证明不等式

$$\left(\int_a^b f(x) dx \right)^2 \leq (b-a) \int_a^b f^2(x) dx,$$

此处仅当 $f(x) = \text{常数}$ 时等号成立.

证 因为

$$\begin{aligned} 0 &\leq \int_a^b dx \int_a^b [f(x) - f(y)]^2 dy \\ &= (b-a) \int_a^b f^2(x) dx - 2 \left(\int_a^b f(x) dx \right)^2 \\ &\quad + (b-a) \int_a^b f^2(y) dy, \end{aligned}$$

故有

$$\left(\int_a^b f(x) dx \right)^2 \leq (b-a) \int_a^b f^2(x) dx.$$

当 $f(x) = \text{常数}$ 时, 显然上式中等号成立. 反之, 设上式中等号成立, 则

$$\int_a^b dx \int_a^b [f(x) - f(y)]^2 dy = 0.$$

由于函数 $F(x) = \int_a^b [f(x) - f(y)]^2 dy$ 是 $a \leq x \leq b$ 上的非负连续函数, 故 $F(x) \equiv 0 (a \leq x \leq b)$. 特别 $F(a) = 0$, 即 $\int_a^b [f(a) - f(y)]^2 dy = 0$. 又由于函数

$$G(y) = [f(a) - f(y)]^2$$

是 $a \leq y \leq b$ 上的非负连续函数, 故 $G(y) \equiv 0 (a \leq y \leq b)$. 因此, $f(y) \equiv f(a) (a \leq y \leq b)$, 即 $f(x) =$ 常数. 证毕.

3912. 下列积分有什么样的符号:

$$(a) \iint_{|x|+|y| \leq 1} \ln(x^2 + y^2) dx dy;$$

$$(b) \iint_{x^2 + y^2 \leq 4} \sqrt[3]{1 - x^2 - y^2} dx dy;$$

$$(B) \iint_{\substack{0 \leq x \leq 1 \\ -1 \leq y \leq 1-x}} \arcsin(x-y) dx dy?$$

解 (a) 由于 $0 < x^2 + y^2 \leq (|x| + |y|)^2 \leq 1$ 及 $\ln(x^2 + y^2) \leq \ln 1 = 0$, 且当 $|x| + |y| < 1$ 时 $\ln(x^2 + y^2) < 0$, 故

$$\iint_{|x|+|y| \leq 1} \ln(x^2 + y^2) dx dy < 0.$$

(b) 我们有

$$\iint_{x^2 + y^2 \leq 4} \sqrt[3]{1 - x^2 - y^2} dx dy = I_1 - I_2 - I_3,$$

其中

$$I_1 = \iint_{x^2 + y^2 \leq 1} \sqrt[3]{1 - x^2 - y^2} dx dy,$$

$$I_2 = \iint_{1 < x^2 + y^2 \leq 2} \sqrt[3]{x^2 + y^2 - 1} dx dy,$$

$$I_3 = \iint_{2 < x^2 + y^2 \leq 4} \sqrt[3]{x^2 + y^2 - 1} dx dy.$$

显然

$$0 < I_1 < \iint_{x^2+y^2 \leq 1} dx dy = \pi,$$

$$I_2 > 0,$$

$$I_3 > \iint_{2 \leq x^2+y^2 \leq 4} dx dy = 4\pi - 2\pi = 2\pi,$$

故

$$\iint_{x^2+y^2 \leq 4} \sqrt{1-x^2-y^2} dx dy < 0.$$

(B) 我们有

$$\begin{aligned} & \iint_{\substack{0 \leq x \leq 1 \\ -1 \leq y \leq 1}} \arcsin(x+y) dx dy \\ &= \iint_{\substack{0 \leq x \leq 1 \\ -1 \leq y \leq 0}} \arcsin(x+y) dx dy \\ &+ \iint_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1-x}} \arcsin(x+y) dx dy. \end{aligned}$$

上式右端第一个积分由对称性知其值为零,第二个积分因被积函数在积分域上为非负不恒为零的连续函数,因而积分值是正的.于是,原积分是正的.

3913. 求函数

$$f(x, y) = \sin^2 x \sin^2 y$$

在正方形: $0 \leq x \leq \pi, 0 \leq y \leq \pi$ 内的平均值.

解 平均值

$$\begin{aligned} I_0 &= \frac{1}{\pi^2} \iint_{\substack{0 \leq x \leq \pi \\ 0 \leq y \leq \pi}} \sin^2 x \sin^2 y dx dy \\ &= \frac{1}{\pi^2} \left(\int_0^\pi \sin^2 x dx \right)^2 = \frac{1}{\pi^2} \left[\left(\frac{x}{2} - \frac{1}{4} \sin 2x \right) \Big|_0^\pi \right]^2 \end{aligned}$$

$$= \frac{1}{4}.$$

3914. 利用中值定理, 估计积分

$$I = \iint_{|x|+|y| \leq 10} \frac{dxdy}{100 + \cos^2 x + \cos^2 y}$$

之值.

解 由于积分域的面积为 200, 故由积分中值定理知

$$\begin{aligned} I &= \frac{1}{100 + \cos^2 \xi + \cos^2 \eta} \cdot 200 \\ &= \frac{200}{100 + \cos^2 \xi + \cos^2 \eta}, \end{aligned} \quad (1)$$

其中 (ξ, η) 为域 $|x| + |y| \leq 10$ 中的某点.

显然

$$0 \leq \cos^2 \xi + \cos^2 \eta \leq 2,$$

我们证明必有

$$0 < \cos^2 \xi + \cos^2 \eta < 2. \quad (2)$$

由于函数 $\cos^2 x + \cos^2 y$ 在有界闭域 $|x| + |y| \leq 10$ 上的最大值为 2, 最小值为 0. 从而连续函数

$\frac{1}{100 + \cos^2 x + \cos^2 y}$ 在有界闭域 $|x| + |y| \leq 10$ 上的

最小值为 $\frac{1}{102}$, 最大值为 $\frac{1}{100}$. 如果 $\cos^2 \xi + \cos^2 \eta = 2$, 则

由 (1) 式知

$$\begin{aligned} &\iint_{|x|+|y| \leq 10} \left(\frac{1}{100 + \cos^2 x + \cos^2 y} - \frac{1}{102} \right) dxdy \\ &= I - I = 0. \end{aligned}$$

但 $f(x, y) = \frac{1}{100 + \cos^2 x + \cos^2 y} - \frac{1}{102}$ 是非负连续函数, 从而必有 $f(x, y) \equiv 0$ (在域 $|x| + |y| \leq 10$ 上),

即 $\cos^2 x + \cos^2 y \equiv 2$ (在域 $|x| + |y| \leq 10$ 上), 这显然是错误的. 由此可知, $\cos^2 \xi + \cos^2 \eta < 2$. 同理可证 $\cos^2 \xi + \cos^2 \eta > 0$. 于是, (2) 式成立. 从而, 得

$$\frac{200}{102} < I < \frac{200}{100}, \text{ 即 } 1.96 < I < 2.$$

3915. 求圆 $(x-a)^2 + (y-b)^2 \leq R^2$ 上的点到原点的距离之平方的平均值.

解 平均值

$$I_0 = \frac{1}{\pi R^2} \iint_{(x-a)^2 + (y-b)^2 \leq R^2} (x^2 + y^2) dx dy.$$

由于

$$\begin{aligned} & \frac{1}{\pi R^2} \iint_{(x-a)^2 + (y-b)^2 \leq R^2} y^2 dx dy \\ &= \frac{1}{\pi R^2} \int_{a-R}^{a+R} dx \int_b - \frac{\sqrt{R^2 - (x-a)^2}}{\sqrt{R^2 - (x-a)^2}} y^2 dy \\ &= \frac{1}{3\pi R^2} \left\{ 6b^2 \int_{a-R}^{a+R} \sqrt{R^2 - (x-a)^2} dx \right. \\ & \quad \left. + 2 \int_{a-R}^{a+R} [R^2 - (x-a)^2]^{\frac{3}{2}} dx \right\} \\ &= \frac{2b^2}{\pi R^2} \left[\frac{x-a}{2} \sqrt{R^2 - (x-a)^2} \right. \\ & \quad \left. + \frac{R^2}{2} \arcsin \frac{x-a}{R} \right] \Big|_{a-R}^{a+R} \\ &+ \frac{2}{3\pi R^2} \left\{ \frac{x-a}{8} [5R^2 - 2(x-a)^2] \sqrt{R^2 - (x-a)^2} \right. \\ & \quad \left. + \frac{3R^4}{8} \arcsin \frac{x-a}{R} \right\} \Big|_{a-R}^{a+R} \\ &= \frac{2b^2}{\pi R^2} \cdot \frac{\pi R^2}{2} + \frac{2}{3\pi R^2} \cdot \frac{3\pi R^4}{8} = b^2 + \frac{R^2}{4}, \end{aligned}$$

同理,有

$$\frac{1}{\pi R^2} \iint_{(x-a)^2 + (y-b)^2 \leq R^2} x^2 dx dy = a^2 + \frac{R^2}{4}.$$

于是,

$$I_0 = a^2 + b^2 + \frac{R^2}{2}.$$

在问题 3916 — 3922 中对二重积分 $\iint_{\Omega} f(x, y) dx dy$ 内按

所指示的区域 Ω 依两个不同的顺序安置积分的上下限.

3916. Ω — 以 $O(0,0), A(1,0), B(1,1)$ 为顶点的三角形.

解 为方便起见,将二重积分 $\iint_{\Omega} f(x, y) dx dy$ 记以 I .

于是, $I = \int_0^1 dx \int_0^x f(x, y) dy = \int_0^1 dy \int_y^1 f(x, y) dx$.

3917. Ω — 以 $O(0,0), A(2,1), B(-2,1)$ 为顶点的三角形.

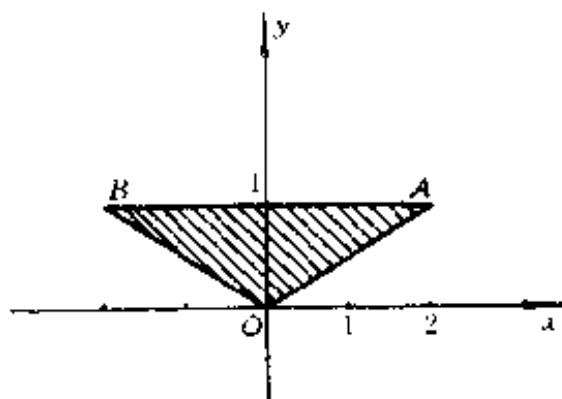


图 8.1

解 如图 8.1 所示

OA 的方程为 $y = \frac{1}{2}x$,

OB 的方程为 $y = -\frac{1}{2}x$,

AB 的方程为 $y = 1$.

于是,

$$\begin{aligned} I &= \int_0^1 dy \int_{\frac{1}{2}y}^{2y} f(x, y) dx = \int_{-2}^0 dx \int_{-\frac{1}{2}x}^1 f(x, y) dy \\ &\quad + \int_0^2 dx \int_{\frac{1}{2}x}^1 f(x, y) dy \\ &= \int_{-2}^2 dx \int_{\frac{1}{2}|x|}^1 f(x, y) dy. \end{aligned}$$

3918. Ω — 以 $O(0,0), A(1,0), B(1,2), C(0,1)$ 为顶点的梯形.

解 如图 8.2 所示, BC 的方程为 $y - 1 = x$.

于是,

$$\begin{aligned} I &= \int_0^1 dx \int_0^{1+x} f(x, y) dy \\ &= \int_0^1 dy \int_0^1 f(x, y) dx + \int_1^2 dy \int_{y-1}^1 f(x, y) dx. \end{aligned}$$

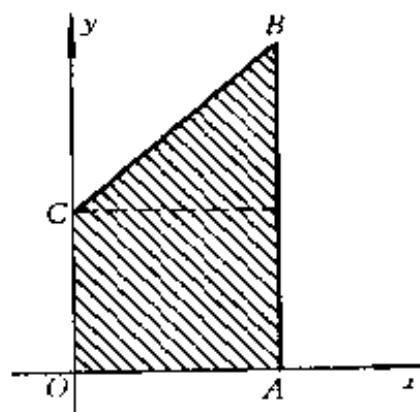


图 8.2

3919. Ω — 圆 $x^2 + y^2 \leq 1$.

$$\begin{aligned} \text{解} \quad I &= \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy \\ &= \int_{-1}^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx. \end{aligned}$$

3920. Ω — 圆 $x^2 + y^2 \leq y$.

解 如图 8.3 所示. 积分域的围线 $x^2 + y^2 = y$ 即为

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

于是,

$$\begin{aligned}
 I &= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{\frac{1}{2}-\sqrt{\frac{1}{4}-x^2}}^{\frac{1}{2}+\sqrt{\frac{1}{4}-x^2}} f(x, y) dy \\
 &= \int_0^1 dy \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} f(x, y) dx.
 \end{aligned}$$

3921. Ω —由曲线 $y = x^2$ 及 $y = 1$ 所包围的抛物线的一节.

解 曲线 $y = x^2$ 及 $y = 1$ 的交点为 $(-1, 1), (1, 1)$.

于是,

$$I = \int_{-1}^1 dx \int_{x^2}^1 f(x, y) dy = \int_0^1 dy \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx.$$

3922. Ω —圆环 $1 \leq x^2 + y^2 \leq 4$.

解 如图 8.4 所示. 若先对 y 后对 x 积分, 则

$$\begin{aligned}
 I &= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y) dy \\
 &+ \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \left\{ \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy \right. \\
 &+ \left. \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x, y) dy \right\} \\
 &+ \int_{\frac{1}{2}}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y) dy.
 \end{aligned}$$

若先对 x 后对 y 积分, 则

$$\begin{aligned}
 I &= \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y) dx \\
 &+ \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \left\{ \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx \right. \\
 &+ \left. \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x, y) dx \right\} \\
 &+ \int_{\frac{1}{2}}^2 dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y) dx.
 \end{aligned}$$

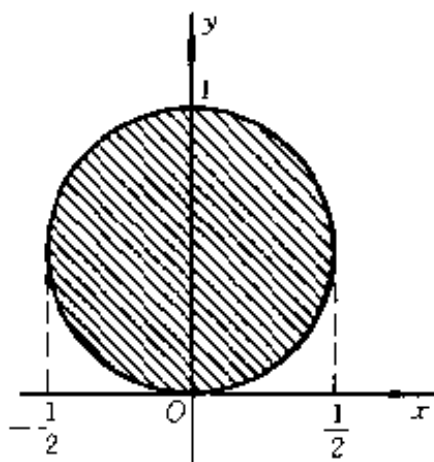


图 8.3

$$\begin{aligned}
& + \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x,y) dx \Big\} \\
& + \int_1^2 dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y) dx.
\end{aligned}$$

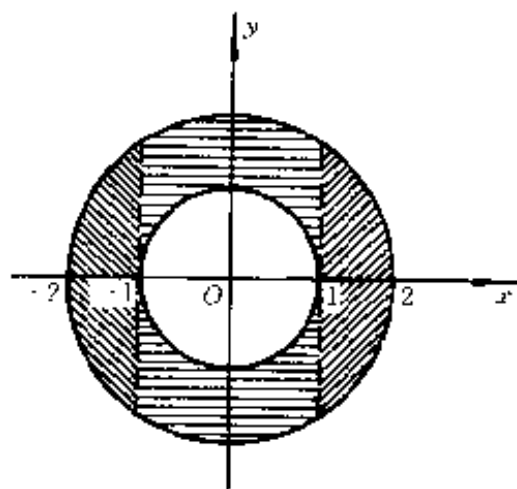


图 8.4

3923. 证明迪里黑里公式

$$\begin{aligned}
& \int_0^a dx \int_0^x f(x,y) dy \\
& = \int_0^a dy \int_y^a f(x,y) dx \quad (a > 0).
\end{aligned}$$

证 公式左端的逐次积分，
等于积分 $\iint_{\Omega} f(x,y) dx dy$ ，其
中 Ω 为三角形域 OAB (图 8.

5); $O(0,0)$, $A(a,0)$, $B(a,$

$a)$. 对于该积分, 若化为先对 x 后对 y 的逐次积分, 即为
公式的右端. 于是本题获证。

在下列积分中改变积分的顺序:

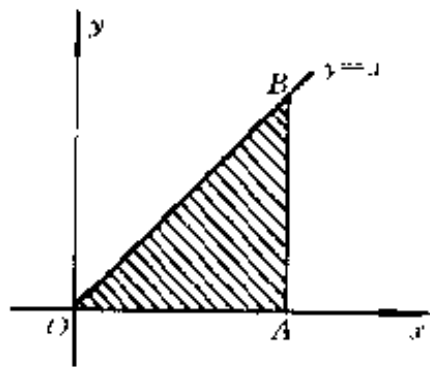


图 8.5

3924. $\int_0^2 dx \int_r^{2r} f(x, y) dy.$

解 积分域的围线为: $y = x$, $y = 2x$ 及 $x = 2$, 如图 8.6 所示. 改变积分的顺序, 即得

$$\begin{aligned} & \int_0^2 dx \int_r^{2r} f(x, y) dy \\ &= \int_0^2 dy \int_{\frac{y}{2}}^y f(x, y) dx \\ &+ \int_2^4 dy \int_{\frac{y}{2}}^2 f(x, y) dx. \end{aligned}$$

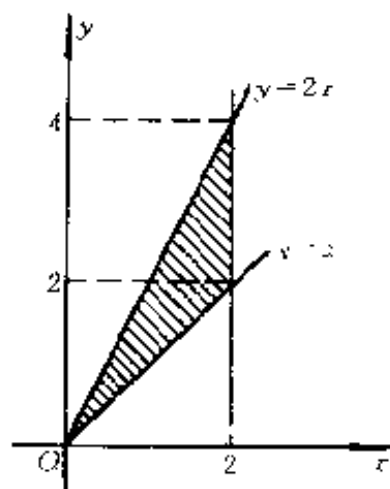


图 8.6

3925. $\int_{-6}^2 dx \int_{\frac{x^2}{4}-1}^{2-x} f(x, y) dy.$

解 积分域的围线为: $y = 2 - x$ 及 $y + 1 = \frac{x^2}{4}$, 其交点为 $(2, 0)$, $(-6, 8)$, 如图 8.7 所示. 改变积分的顺序, 即得

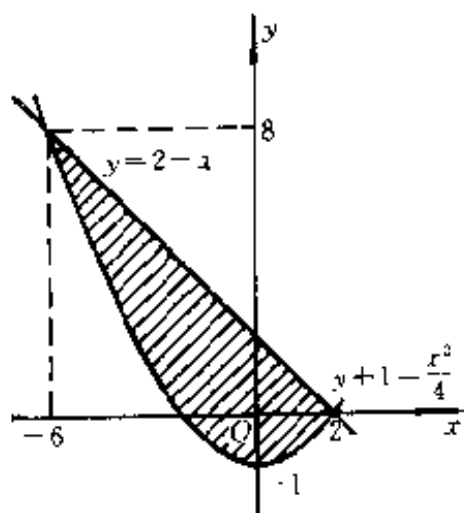


图 8.7

$$\int_{-6}^2 dx \int_{\frac{x^2}{4}-1}^{2-x} f(x, y) dy$$

$$= \int_{-1}^0 dy \int_{-2}^{\sqrt{1+y}} f(x, y) dx + \int_0^8 dy \int_{\frac{2}{\sqrt{1-y}}}^{2-y} f(x, y) dx.$$

3926. $\int_0^1 dx \int_{x^3}^{x^2} f(x, y) dy.$

解 积分域的围线为: $y = x^2$ 及 $y = x^3$, 其交点为 $(0, 0)$, $(1, 1)$, 如图 8.8 所示. 改变积分的顺序, 即得

$$\begin{aligned} & \int_0^1 dx \int_{x^3}^{x^2} f(x, y) dy \\ &= \int_0^1 dy \int_{\sqrt[3]{y}}^{\sqrt{y}} f(x, y) dx. \end{aligned}$$

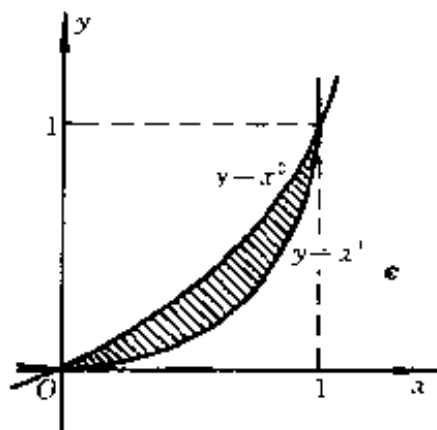


图 8.8

3927. $\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x, y) dy.$

解 积分域的围线为圆 $x^2 + y^2 = 1$ 的下半部分及抛物线 $y = 1 - x^2$, 如图 8.9 所示. 改变积分的顺序, 即得

$$\begin{aligned} & \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x, y) dy \\ &= \int_{-1}^0 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx \\ &+ \int_0^1 dy \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y) dx. \end{aligned}$$

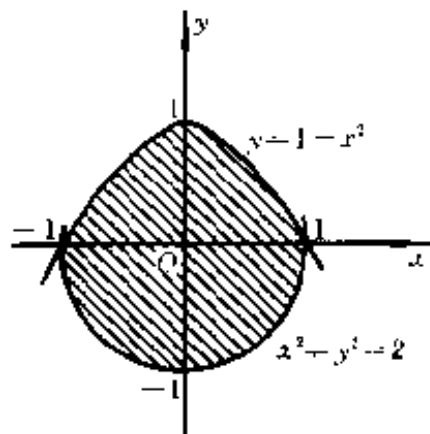


图 8.9

$$3928. \int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x, y) dy.$$

解 积分域的围线为圆 $x^2 + y^2 = 2x$ 或 $(x-1)^2 + y^2 = 1$ 及直线 $y = 2-x$, 其交点为 $(2, 0)$, $(1, 1)$, 如图 8.10 中阴影部分所示. 改变积分的顺序, 即得

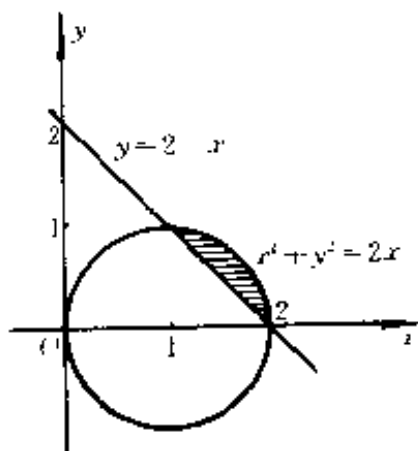


图 8.10

$$\begin{aligned} & \int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x, y) dy \\ &= \int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x, y) dx. \end{aligned}$$

$$3929. \int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y) dy \quad (a > 0).$$

解 积分域由围线 $(x-a)^2 + y^2 = a^2$ ($y \geq 0$), $y^2 = 2ax$ ($y \geq 0$) 及 $x = 2a$ 组成. 如图 8.11 中阴影部分所示. 改变积分的顺序, 即得

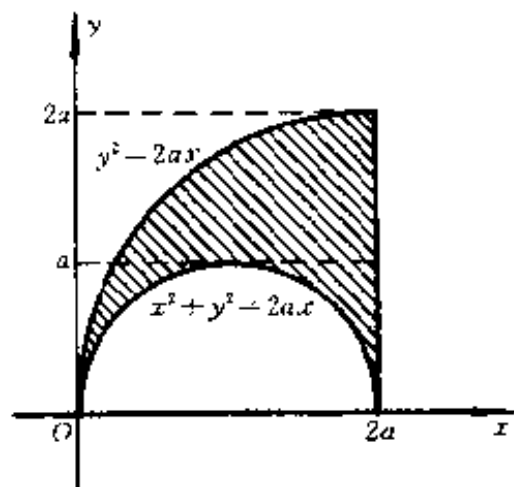


图 8.11

$$\begin{aligned} & \int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y) dy \\ &= \int_0^a dy \left\{ \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x, y) dx + \int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y) dx \right\} \\ &+ \int_a^{2a} dy \int_{\frac{y^2}{2a}}^{2a} f(x, y) dx. \end{aligned}$$

3930. $\int_1^e dx \int_0^{\ln x} f(x, y) dy.$

解 积分域如图 8.12 中阴影部分所示. 改变积分顺序, 即得

$$\begin{aligned} & \int_1^e dx \int_0^{\ln x} f(x, y) dy \\ &= \int_0^1 dy \int_{e^y}^e f(x, y) dx. \end{aligned}$$

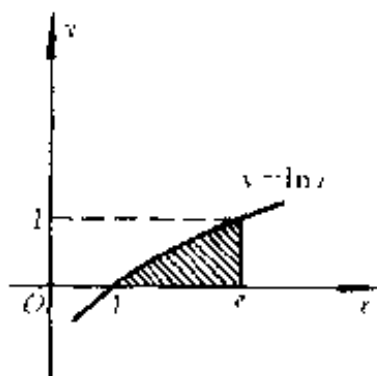


图 8.12

3931. $\int_0^{2\pi} dx \int_0^{\sin x} f(x, y) dy.$

解 积分域如图 8.13 中阴影部分所示. 由于 $y = \sin x$ 的反函数, 当 y 从 0 变到 1 时为 $x = \arcsin y$, 当 y 从 1 变到 -1 时 $x = \pi - \arcsin y$, 当 y 从 -1 变到 0 时为 $x = 2\pi + \arcsin y$, 故改变积分的顺序, 即得

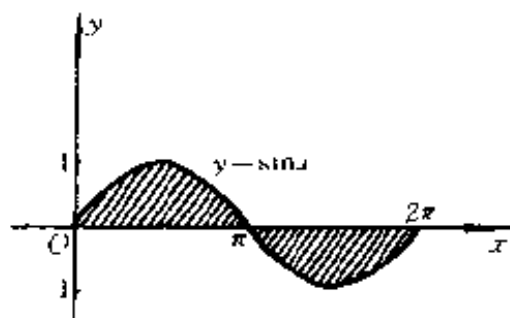


图 8.13

$$\begin{aligned} & \int_0^{2\pi} dx \int_0^{\sin x} f(x, y) dy \\ &= \int_0^1 dy \int_{\arcsin y}^{\pi - \arcsin y} f(x, y) dx \\ &+ \int_{-1}^0 dy \int_{\pi - \arcsin y}^{2\pi + \arcsin y} f(x, y) dx. \end{aligned}$$

计算下列积分:

3932. $\iint_{\Omega} xy^2 dx dy$, 设 Ω 是由抛物线

$y^2 = 2px$ 和直线 $x = \frac{p}{2}$ ($p > 0$) 所界的区域.

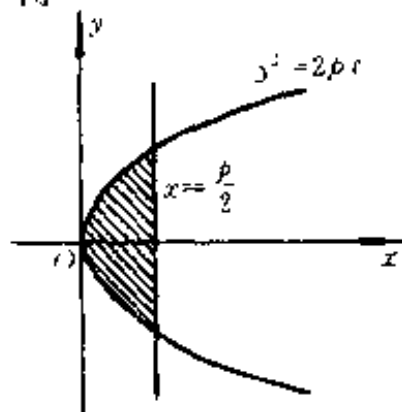


图 8.14

解 积分域如图 8.14 所示. 于是,

$$\begin{aligned} \iint_{\Omega} xy^2 dx dy &= \int_0^{\frac{p}{2}} dx \int_{\sqrt{2px}}^{\sqrt{2bx}} xy^2 dy \\ &= \int_0^{\frac{p}{2}} \frac{2}{3} x \sqrt{(2px)^3} dx = \frac{p^5}{21}. \end{aligned}$$

3933. $\iint_{\Omega} \frac{dx dy}{\sqrt{2a-x}}$ ($a > 0$), 设 Ω 是

由圆心在点 (a, a) 半径为 a 且与坐标轴相切的圆周的较短弧和坐标轴所围成的区域.

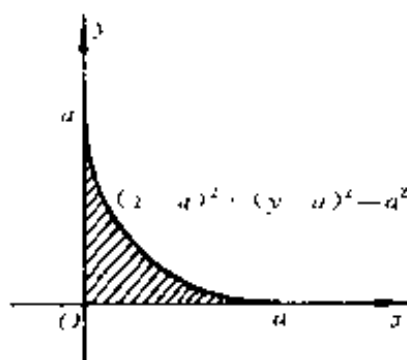


图 8.15

解 如图 8.15 所示. 当 x 从 0 变到 a 时, 对于每一固定的

x , y 从 0 变到 $a - \sqrt{2ax - x^2}$. 于是,

$$\begin{aligned} \iint_{\Omega} \frac{dx dy}{\sqrt{2a-x}} &= \int_0^a \frac{dx}{\sqrt{2a-x}} \int_0^{a-\sqrt{2ax-x^2}} dy \\ &= \int_0^a \frac{a dx}{\sqrt{2a-x}} = \int_0^a \sqrt{x} dx = \left(2\sqrt{2} - \frac{8}{3} \right) a \sqrt{a}. \end{aligned}$$

3934. $\iint_{\Omega} |xy| dx dy$, 设 Ω 是以 a 为半径, 坐标原点为圆心的圆.

解
$$\begin{aligned} \iint_{\Omega} |xy| dx dy &= \int_{-a}^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} |xy| dy \\ &= \int_{-a}^a (a^2 - x^2) |x| dx = 2 \int_0^a (a^2 - x^2) x dx = \frac{a^4}{2}. \end{aligned}$$

3935. $\iint_{\Omega} (x^2 + y^2) dx dy$, 设 Ω 是以 $y = x, y = x + a, y = a$ 和 $y = 3a$ ($a > 0$) 为边的平行四边形.

解 如图 8.16 所示. 当 y 从 a 变到 $3a$ 时, 对于每一固定的 y , x 从 $y-a$ 变到 y . 于是,

$$\begin{aligned} & \iint_{\Omega} (x^2 + y^2) dx dy \\ &= \int_a^{3a} dy \int_{y-a}^y (x^2 + y^2) dx \\ &= \int_a^{3a} \left[\frac{y^3}{3} + ay^2 - \frac{(y-a)^3}{3} \right] dy \\ &= \frac{168a^4}{12} = 14a^4. \end{aligned}$$

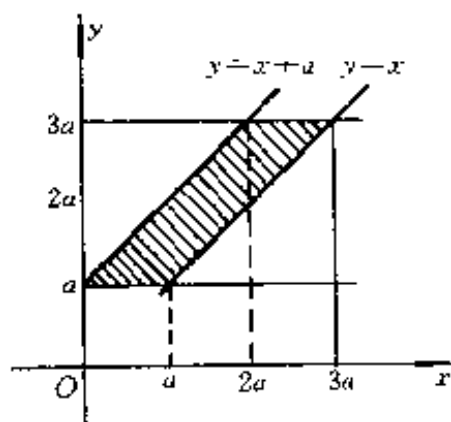


图 8.16

3936. $\iint_{\Omega} y^2 dx dy$, 设 Ω 是由横轴和摆线

$$x = a(t - \sin t), y = a(1 - \cos t) \quad (0 \leq t \leq 2\pi)$$

的第一拱所界的区域.

$$\begin{aligned} \text{解} \quad & \iint_{\Omega} y^2 dx dy = \int_0^{2\pi} dx \int_0^y y^2 dy \\ &= \frac{a^4}{3} \int_0^{2\pi} (1 - \cos t)^4 dt \\ &= \frac{2^4 a^4}{3} \int_0^{2\pi} \sin^8 \frac{t}{2} dt = \frac{2^5 a^4}{3} \int_0^{\pi} \sin^8 u du \\ &= \frac{2^5 a^4}{3} \left\{ \int_0^{\frac{\pi}{2}} \sin^8 u du + \int_{\frac{\pi}{2}}^{\pi} \sin^8 u du \right\} \\ &= \frac{2^5 a^4}{3} \left\{ \int_0^{\frac{\pi}{2}} \sin^8 u du + \int_0^{\frac{\pi}{2}} \cos^8 u du \right\} \\ &= \frac{2^5 a^4}{3} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^8 u du^{*}) \\ &= \frac{2^5 a^4}{3} \cdot 2 \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2}^{**}) = \frac{35}{12} \pi a^4. \end{aligned}$$

*) 参看 2282 题的结果.

* *) 参看 2281 题的结果。

在二重积分

$$\iint_{\Omega} f(x, y) dx dy$$

中, 假定 $x = r \cos \varphi$ 和 $y = r \sin \varphi$, 变换为极坐标 r 和 φ , 并配置积分的限, 设:

3937. Ω — 圆 $x^2 + y^2 \leq a^2$.

解 雅可比式 $I = r$, 以下各题不再写出。

φ 从 0 变到 2π , r 从 0 变到 a . 于是,

$$\iint_{\Omega} f(x, y) dx dy = \int_0^{2\pi} d\varphi \int_0^a f(r \cos \varphi, r \sin \varphi) r dr.$$

3938. Ω — 圆 $x^2 + y^2 \leq ax$ ($a > 0$).

解 圆 $x^2 + y^2 = ax$ 即 $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$, 极坐

标方程为 $r = a \cos \varphi$. 当 φ 从 $-\frac{\pi}{2}$ 变到 $\frac{\pi}{2}$ 时, 对于每一固定的 φ , r 从 0 变到 $a \cos \varphi$. 于是,

$$\iint_{\Omega} f(x, y) dx dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a \cos \varphi} f(r \cos \varphi, r \sin \varphi) r dr.$$

3939. Ω — 环 $a^2 \leq x^2 + y^2 \leq b^2$.

解 φ 从 0 变到 2π , r 从 $|a|$ 变到 $|b|$. 于是

$$\iint_{\Omega} f(x, y) dx dy = \int_0^{2\pi} d\varphi \int_a^{|b|} f(r \cos \varphi, r \sin \varphi) r dr.$$

3940. Ω — 三角形 $0 \leq x \leq 1; 0 \leq y \leq 1 - x$.

解 由于直线 $x + y = 1$ 的极坐标方程为

$$r = \frac{1}{\sin \varphi + \cos \varphi} = \frac{1}{\sqrt{2}} \csc \left(\varphi + \frac{\pi}{4} \right),$$

因而当 φ 从 0 变到 $\frac{\pi}{2}$ 时, 对于每一固定的 φ , r 从 0 变到 $\frac{1}{\sqrt{2}} \csc \left| \varphi + \frac{\pi}{4} \right|$. 于是,

$$\iint_{\Omega} f(x, y) dx dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{1}{\sqrt{2}} \csc \left(\varphi + \frac{\pi}{4} \right)} f(r \cos \varphi, r \sin \varphi) r dr.$$

3941. Ω — 抛物线节 $-a \leq x \leq a$; $\frac{x^2}{a} \leq y \leq a$.

解 如图 8.17 所示.

区域 Ω 可分为三部分:

(1) 当 φ 从 0 变到 $\frac{\pi}{4}$ 时, 对于每一固定的 φ , r 从 0 变到 $\frac{a \sin \varphi}{\cos^2 \varphi}$, 其中

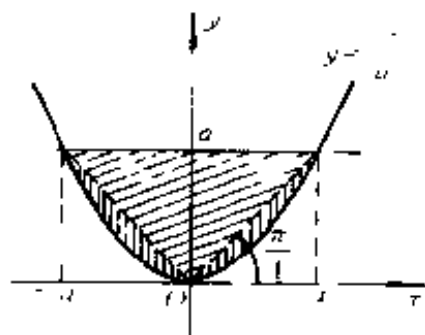


图 8.17

$r = \frac{a \sin \varphi}{\cos^2 \varphi}$ 为抛物线 $y = \frac{x^2}{a}$ 的极坐标方程;

(2) 当 φ 从 $\frac{\pi}{4}$ 变到 $\frac{3\pi}{4}$ 时, 对于每一固定的 φ , r 从 0 变到 $\frac{a}{\sin \varphi}$;

(3) 当 φ 从 $\frac{3\pi}{4}$ 变到 π 时, 对于每一固定的 φ , r 从 0 变到 $\frac{a \sin \varphi}{\cos^2 \varphi}$.

于是

$$\iint_{\Omega} f(x, y) dx dy = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\frac{a \sin \varphi}{\cos^2 \varphi}} f(r \cos \varphi, r \sin \varphi) r dr$$

$$\begin{aligned}
& + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_0^{\frac{a}{\sin\varphi}} f(r\cos\varphi, r\sin\varphi) r dr \\
& + \int_{\frac{3\pi}{4}}^{\pi} d\varphi \int_0^{\frac{a\sin\varphi}{\cos^2\varphi}} f(r\cos\varphi, r\sin\varphi) r dr.
\end{aligned}$$

3942. 在怎样的情况下, 当变换为极坐标之后, 积分的限是常数?

解 若变换为极坐标, 积分

$$\iint_{\Omega} f(x, y) dx dy = \int_{\alpha}^{\beta} d\varphi \int_a^b f(r\cos\varphi, r\sin\varphi) r dr,$$

其中 α, β, a, b 均为常数, 则表明积分域 Ω 为 $a \leq r \leq b$, $\alpha \leq \varphi \leq \beta$. 它表示圆环面 $a \leq r \leq b$ 被射线 $\varphi = \alpha, \varphi = \beta$ 截出的部分, 且只有积分域是这种情况, 变换为极坐标后积分的限才是常数. 如 3937 题及 3939 题即为其特例.

在下列积分中, 假定 $x = r\cos\varphi$ 和 $y = r\sin\varphi$, 变换为极坐标 r 和 φ , 并依两种不同的顺序配置积分的限:

3943. $\int_0^1 dx \int_0^1 f(x, y) dy.$

解 如图 8.18 所示.

若先对 r 积分, 则当 φ

从 0 变到 $\frac{\pi}{4}$ 时, 对于每一

一固定的 φ , r 从 0 变到

$\sec\varphi$; 当 φ 从 $\frac{\pi}{4}$ 变到 $\frac{\pi}{2}$

时, 对于每一固定的

φ , r 从 0 变到 $\csc\varphi$.

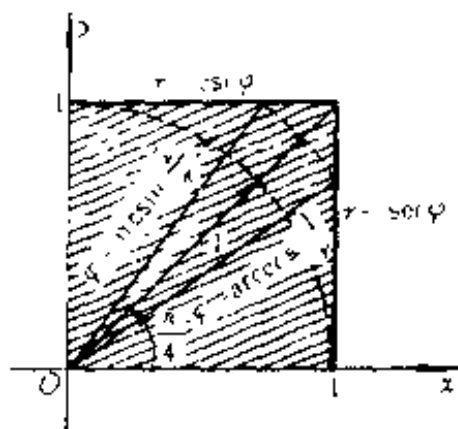


图 8.18

若先对 φ 积分, 则当 r 从 0 变到 1 时, φ 从 0 变到 $\frac{\pi}{2}$; 当 r 从 1 变到 $\sqrt{2}$ 时, 对于每一固定的 r , φ 从 $\arccos \frac{1}{r}$ 变到 $\arcsin \frac{1}{r}$. 于是,

$$\begin{aligned} & \int_0^1 dx \int_0^1 f(x, y) dy \\ &= \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sec \varphi} f(r \cos \varphi, r \sin \varphi) r dr \\ &+ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_0^{\csc \varphi} f(r \cos \varphi, r \sin \varphi) r dr \\ &= \int_0^1 r dr \int_0^{\frac{\pi}{2}} f(r \cos \varphi, r \sin \varphi) d\varphi + \int_1^{\sqrt{2}} r dr \int_{\arccos \frac{1}{r}}^{\arcsin \frac{1}{r}} \\ &f(r \cos \varphi, r \sin \varphi) d\varphi. \end{aligned}$$

3944. $\int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} f(x, y) dy.$

解 如图 8.19 所示. 若先对 r 积分, 则当 φ 从 0 变到 $\frac{\pi}{2}$ 时, 对于每一固定的 φ , r 从 $\frac{1}{\sqrt{2}} \csc\left(\varphi + \frac{\pi}{4}\right)$ 变到 1.

若先对 φ 积分, 则当 r 从 $\frac{1}{\sqrt{2}}$ 变到 1 时, 对于每一固定的

r , φ 从 $\frac{\pi}{4} - \arccos \frac{1}{r\sqrt{2}}$ 变到 $\frac{\pi}{4} + \arccos \frac{1}{r\sqrt{2}}$, 其中

直线 $x + y = 1$ 的极坐标方程为 $r \sin\left(\varphi + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$,

即 $\cos\left(\frac{\pi}{4} - \varphi\right) = \frac{1}{r\sqrt{2}}$ 或 $\frac{\pi}{4} - \varphi = \pm \arccos \frac{1}{r\sqrt{2}}$.

于是,

$$\begin{aligned}
& \int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} f(x, y) dy \\
&= \int_0^{\frac{\pi}{2}} d\varphi \int_{\frac{1}{\sqrt{2}} \csc(\varphi + \frac{\pi}{4})}^1 f(r \cos \varphi, r \sin \varphi) r dr \\
&= \int_{\frac{1}{\sqrt{2}}}^1 r dr \int_{\frac{\pi}{4} - \arccos \frac{1}{r\sqrt{2}}}^{\frac{\pi}{4} + \arccos \frac{1}{r\sqrt{2}}} f(r \cos \varphi, r \sin \varphi) d\varphi.
\end{aligned}$$

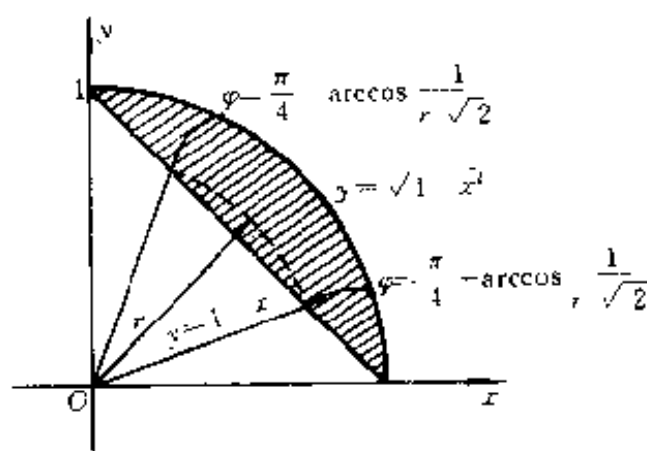


图 8.19

3945. $\int_0^2 dx \int_x^{\sqrt{3}} f(\sqrt{x^2 + y^2}) dy.$

解 如图 8.20 所示.

若先对 r 积分, 则当 φ 从 $\frac{\pi}{4}$ 变到 $\frac{\pi}{3}$ 时, 对于每一固定的 φ , r 从 0 变到 $\frac{2}{\cos \varphi}$.

若先对 φ 积分, 则当 r 从 0 变到 $2\sqrt{2}$ 时, φ 从 $\frac{\pi}{4}$ 变到 $\frac{\pi}{3}$; 当 r

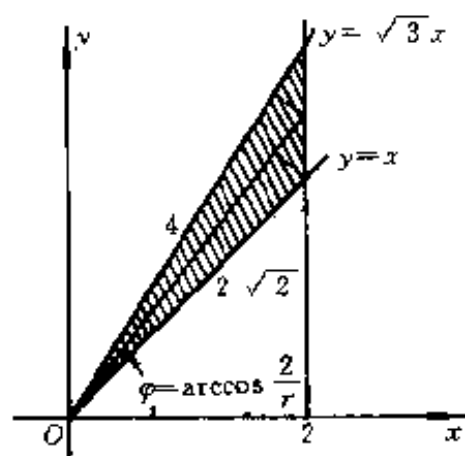


图 8.20

从 $2\sqrt{2}$ 变到 4 时, 对于每一固定的 r , φ 从 $\arccos \frac{2}{r}$ 变到 $\frac{\pi}{3}$. 于是,

$$\begin{aligned} \int_0^2 dx \int_x^{\sqrt{3}} f(\sqrt{x^2+y^2}) dy &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^{\frac{2}{\cos\varphi}} r f(r) dr \\ &= \frac{\pi}{12} \int_0^{\sqrt{3}} r f(r) + \int_2^4 \left(\frac{\pi}{3} - \arccos \frac{2}{r} \right) r f(r) dr. \end{aligned}$$

3946⁺. ① $\int_0^1 dx \int_0^{x^2} f(x, y) dy.$

解 如图 8.21 所示. 若先对 r 积分, 则当 φ 从 0 变到 $\frac{\pi}{4}$ 时, 对于每一固定的 φ , r 从 $\frac{\sin\varphi}{\cos^2\varphi}$ 变到 $\frac{1}{\cos\varphi}$, 其中 $r = \frac{\sin\varphi}{\cos^2\varphi}$ 为抛物线 $y = x^2$ 的极坐标方程.

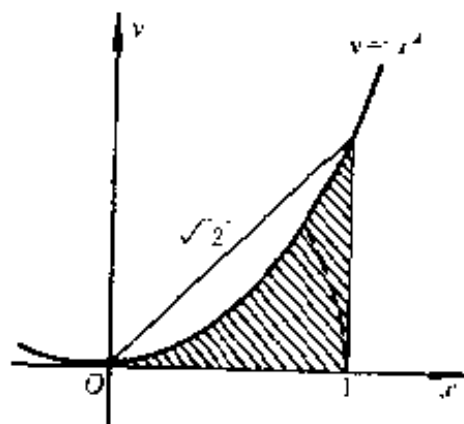


图 8.21

若先对 φ 积分, 则当 r 从 0 变到 1 时对于每一固定的 r , φ 从 0 变到 $\arcsin \frac{\sqrt{1+4r^2}-1}{2r}$ (由 $r = \frac{\sin\varphi}{\cos^2\varphi}$ 解出 φ); 当 r 从 1 变到 $\sqrt{2}$ 时, 对于每一固定的 r , φ 从 $\arccos \frac{1}{r}$ 变到 $\arcsin \frac{\sqrt{1+4r^2}-1}{2r}$. 于是,

① 题号右上角带“+”号表示题解答案与原习题集中译本所附答案不一致, 以后不再说明. 中译本基本是按俄文第二版翻译的. 俄文第二版中有一些错误已在俄文第三版中改正.

$$\begin{aligned} \int_0^1 dx \int_0^{x^2} f(x, y) dy &= \int_0^{\frac{\pi}{4}} d\varphi \int_{\frac{\sin \varphi}{\cos^2 \varphi}}^{\frac{1}{\cos \varphi}} f(r \cos \varphi, r \sin \varphi) r dr = \\ &= \int_0^1 r dr \int_0^{\arcsin \frac{\sqrt{1+4r^2}-1}{2r}} f(r \cos \varphi, r \sin \varphi) d\varphi \\ &+ \int_1^{\sqrt{2}} r dr \int_{\arccos \frac{1}{r}}^{\arcsin \frac{\sqrt{1+4r^2}-1}{2r}} f(r \cos \varphi, r \sin \varphi) d\varphi. \end{aligned}$$

3947. $\iint_{\Omega} f(x, y) dx dy$, 其中 Ω 是由曲线 $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ ($x \geq 0$) 所界的域.

解 令 $x = r \cos \varphi$, $y = r \sin \varphi$, 则曲线 $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ ($x \geq 0$) 的极坐标方程为 $r^2 = a^2 \cos 2\varphi$, 其图象是双纽线的右半部分, 如图 8.22 所示.

若先对 r 积分, 则当 φ 从 $-\frac{\pi}{4}$ 变到 $\frac{\pi}{4}$ 时, 对于每一固定的 φ , r 从 0 变到 $a \sqrt{\cos 2\varphi}$.

若先对 φ 积分, 则当 r 从 0 变到 a 时, 对于每一固定的 r , φ 从 $-\frac{1}{2} \arccos \frac{r^2}{a^2}$ 变到 $\frac{1}{2} \arccos \frac{r^2}{a^2}$. 于是, $\iint_{\Omega} f(x, y) dx dy$

$$\begin{aligned} &= \int_0^a r dr \int_{-\frac{1}{2} \arccos \frac{r^2}{a^2}}^{\frac{1}{2} \arccos \frac{r^2}{a^2}} f(r \cos \varphi, r \sin \varphi) d\varphi \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{a \sqrt{\cos 2\varphi}} f(r \cos \varphi, r \sin \varphi) r dr. \end{aligned}$$

假定 r 和 φ 为极坐标, 在下列积分中变更积分的顺序:

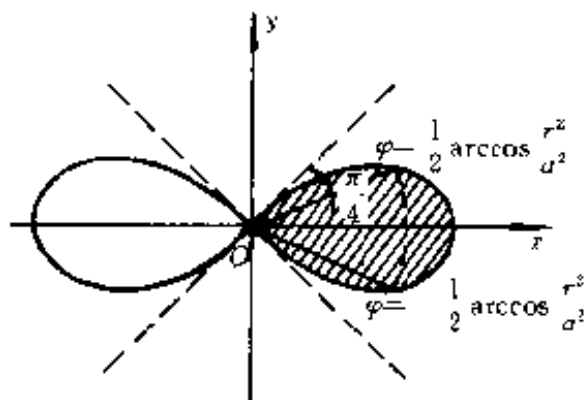


图 8.22

3948. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a \cos \varphi} f(\varphi, r) dr \quad (x > 0).$

解 积分域为由圆 $r = a \cos \varphi$ 或 $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$ 所围成的圆域.

若先对 φ 积分, 则当 r 从 0 变到 a 时, 对于每一固定的 r , φ 从 $-\arccos \frac{r}{a}$ 变到 $\arccos \frac{r}{a}$

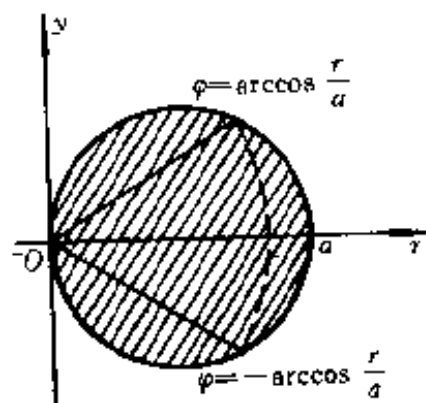


图 8.23

$\frac{r}{a}$ (图 8.23). 于是,

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a \cos \varphi} f(\varphi, r) dr \\ &= \int_0^a dr \int_{-\arccos \frac{r}{a}}^{\arccos \frac{r}{a}} f(\varphi, r) d\varphi. \end{aligned}$$

3949. $\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\sqrt{\sin 2\varphi}} f(\varphi, r) dr \quad (a > 0).$

解 积分域由双纽线 $r^2 = a^2 \sin 2\varphi$ 的右上部分围成(图

8.24).

若先对 φ 积分, 则当 r 从 0 变到 a 时, 对于每一固定的 r , φ 从 $\frac{1}{2}\arcsin \frac{r^2}{a^2}$ 变到

$\frac{\pi}{2} - \frac{1}{2}\arcsin \frac{r^2}{a^2}$. 于是,

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\sqrt{\sin 2\varphi}} f(\varphi, r) dr \\ &= \int_0^a dr \int_{\frac{1}{2}\arcsin \frac{r^2}{a^2}}^{\frac{\pi}{2} - \frac{1}{2}\arcsin \frac{r^2}{a^2}} f(\varphi, r) d\varphi. \end{aligned}$$

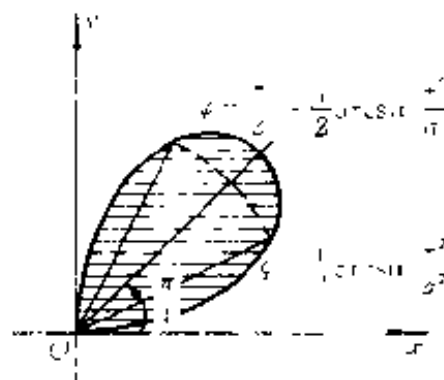


图 8.24

3950. $\int_0^a d\varphi \int_0^{\varphi} f(\varphi, r) dr$

($0 < a < 2\pi$).

解 积分域由曲线 $r = \varphi$ (阿基米德螺线) 与射线 $\varphi = a$ 围成(图 8.25).

改变积分顺序, 即得

$$\begin{aligned} & \int_0^a d\varphi \int_0^{\varphi} f(\varphi, r) dr \\ &= \int_0^a dr \int_r^a f(\varphi, r) d\varphi. \end{aligned}$$

变换成极坐标, 以一重积分来代替二重积分:

3951. $\iint_{x^2+y^2 \leq 1} f(\sqrt{x^2+y^2}) dx dy.$

$$\begin{aligned} \text{解} \quad & \iint_{x^2+y^2 \leq 1} f(\sqrt{x^2+y^2}) dx dy = \int_0^{2\pi} d\varphi \int_0^1 f(r) r dr \\ &= 2\pi \int_0^1 r f(r) dr. \end{aligned}$$

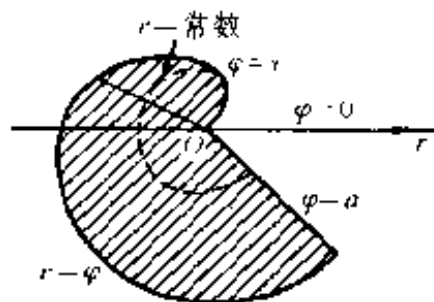


图 8.25

3952. $\iint_{\Omega} f(\sqrt{x^2+y^2})dxdy$, 其中 $\Omega = \{|y| \leq |x|; |x| \leq 1\}$.

解 域 Ω 如图 8.26 所示。

先对 φ 积分, 则当 r 从 0 变到 1 时, φ 从 $-\frac{\pi}{4}$ 变到 $\frac{\pi}{4}$;

当 r 从 1 变到 $\sqrt{2}$ 时, 对于每一固定的 r , φ 从 $\arccos \frac{1}{r}$ 变到 $\frac{\pi}{4}$. 于是,

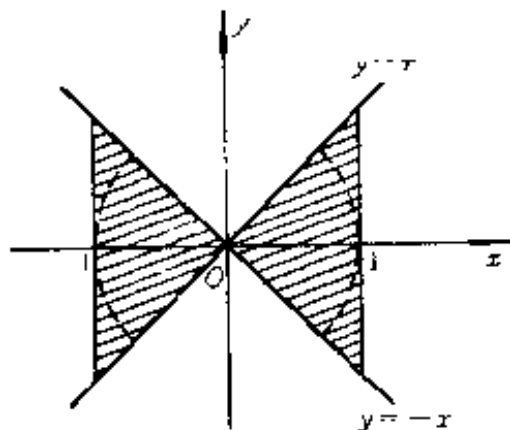


图 8.26

$$\begin{aligned} & \iint_{\Omega} f(\sqrt{x^2+y^2})dxdy \\ &= 2 \int_0^1 r f(r) \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi + 4 \int_1^{\sqrt{2}} r f(r) \int_{\arccos \frac{1}{r}}^{\frac{\pi}{4}} d\varphi \\ &= \pi \int_0^1 r f(r) dr + \int_1^{\sqrt{2}} \left(\pi - 4 \arccos \frac{1}{r} \right) r f(r) dr. \end{aligned}$$

3953. $\iint_{x^2+y^2 \leq x} f\left(\frac{y}{x}\right) dxdy.$

解 $\iint_{x^2+y^2 \leq x} f\left(\frac{y}{x}\right) dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\cos \varphi} f(\operatorname{tg} \varphi) r dr$

$$= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\operatorname{tg} \varphi) \cos^2 \varphi d\varphi.$$

变换成极坐标, 以计算下列二重积分:

3954. $\iint_{x^2+y^2 \leq a^2} \sqrt{x^2+y^2} dxdy$

$$\text{解} \quad \iint_{x^2+y^2 \leq a^2} \sqrt{x^2+y^2} dx dy = \int_0^{2\pi} d\varphi \int_0^a r \cdot r dr = \frac{2\pi a^3}{3}.$$

$$3955. \quad \iint_{x^2 \leq x^2+y^2 \leq 4\pi^2} \sin \sqrt{x^2+y^2} dx dy.$$

$$\begin{aligned} \text{解} \quad & \iint_{x^2 \leq x^2+y^2 \leq 4\pi^2} \sin \sqrt{x^2+y^2} dx dy \\ &= \int_0^{2\pi} d\varphi \int_{\pi}^{2\pi} r \sin r dr \\ &= 2\pi \int_{\pi}^{2\pi} r \sin r dr = -6\pi^2. \end{aligned}$$

3956. 利用函数组

$$u = \frac{y^2}{x}, \quad v = \sqrt{xy}$$

把矩形 $S\{a < x < a+h, b < y < b+h\}$ ($a > 0, b > 0$)

变换为域 S' . 求域 S' 的面积与 S 的面积之比.

当 $h \rightarrow 0$ 时, 此比值的极限等于什么?

解 正方形的角点 $A(a, b), B(a+h, b), C(a+h, b+h), D(a, b+h)$ 对应于 Ouv 平面上的点 $A'\left(\frac{b^2}{a}, \sqrt{ab}\right),$

$$B'\left(\frac{b^2}{(a+h)^2}, \sqrt{(a+h)b}\right),$$

$$C'\left(\frac{(b+h)^2}{a+h}, \sqrt{(a+h)(b+h)}\right),$$

$$D'\left(\frac{(b+h)^2}{a}, \sqrt{a(b+h)}\right).$$

正方形的四边 $y=b, x=a+h, y=b+h, x=a$ 对应于 Ouv 平面上的四条曲线, 即

$$A'B': u = \frac{b^3}{v^2}; \quad B'C': u = \frac{v^4}{(a+h)^3};$$

$$C'D': u = \frac{(b+h)^3}{v^2}; \quad D'A': u = \frac{v^4}{a^3}.$$

由这四条曲线围成的域即为 S' (图 8.27).

于是, 域 S' 的面积

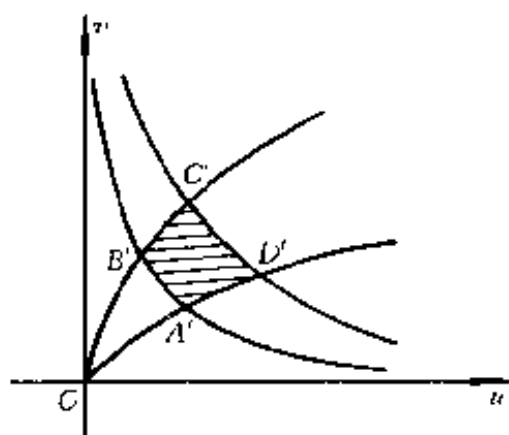


图 8.27

$$\begin{aligned} S' &= \iint_{S'} du dv = \int_{\sqrt{ab}}^{\sqrt{a(b+h)}} \frac{v^4}{a^3} dv \\ &+ \int_{\sqrt{a(b+h)}}^{\sqrt{(a+h)(b+h)}} \frac{(b+h)^3}{v^2} dv - \int_{\sqrt{ab}}^{\sqrt{(a+h)b}} \frac{b^3}{v^2} dv \\ &- \int_{\sqrt{(a+h)b}}^{\sqrt{(a+h)(b+h)}} \frac{v^4}{(a+h)^3} dv \\ &= \frac{1}{5a^3} [\sqrt{a^5(b+h)^5} - \sqrt{a^5b^5}] \\ &+ (b+h)^3 \left[\frac{1}{\sqrt{a(b+h)}} - \frac{1}{\sqrt{(a+h)(b+h)}} \right] \\ &- b^3 \left[\frac{1}{\sqrt{ab}} - \frac{1}{\sqrt{(a+h)b}} \right] \\ &- \frac{1}{5(a+h)^3} [\sqrt{(a+h)^5(b+h)^5} - \sqrt{(a+h)^5b^5}] \end{aligned}$$

$$= \frac{6}{5} [\sqrt{(b+h)^5} - \sqrt{b^5}] \left[\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{a+h}} \right].$$

从而, 域 S' 的面积与 S 的面积之比

$$\begin{aligned} \frac{S'}{S} &= \frac{6}{5h^2} [\sqrt{(b+h)^5} - \sqrt{b^5}] \left[\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{a+h}} \right] \\ &= \frac{6}{5} \cdot \frac{[\sqrt{(b+h)^5} - \sqrt{b^5}](\sqrt{a+h} - \sqrt{a})}{h^2 \sqrt{a(a+h)}} \\ &= \frac{6}{5} \cdot \frac{\sqrt{(b+h)^5} - \sqrt{b^5}}{\sqrt{a(a+h)}(\sqrt{a+h} + \sqrt{a})(\sqrt{b+h} + \sqrt{b})(\sqrt{b+h} - \sqrt{b})} \\ &= \frac{6}{5} \cdot \frac{b^2 + b(b+h) - (b+h)^2 + (2b+h)\sqrt{b(b+h)}}{\sqrt{a(a+h)}(\sqrt{a+h} + \sqrt{a})(\sqrt{b+h} + \sqrt{b})}. \end{aligned}$$

上述比式是 h 的函数, 并且在 $h=0$ 点连续. 于是,

$$\lim_{h \rightarrow 0} \frac{S'}{S} = \frac{6}{5} \cdot \frac{5b^2}{4\sqrt{a^3} \cdot \sqrt{b}} = \frac{3}{2} \left(\frac{b}{a} \right)^{\frac{3}{2}}.$$

事实上, 应用洛比塔法则求此极限更简单些, 这是因为

$$\lim_{h \rightarrow 0} \frac{\sqrt{(b+h)^5} - \sqrt{b^5}}{h} = \lim_{h \rightarrow 0} \frac{5}{2} \sqrt{(b+h)^3} = \frac{5}{2} b^{\frac{3}{2}}.$$

$$\lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{a+h}}}{h} = \lim_{h \rightarrow 0} \frac{1}{2} (a+h)^{-\frac{3}{2}} = \frac{1}{2} a^{-\frac{3}{2}}.$$

于是,

$$\lim_{h \rightarrow 0} \frac{S'}{S} = \frac{6}{5} \cdot \frac{5}{2} b^{\frac{3}{2}} \cdot \frac{1}{2} a^{-\frac{3}{2}} = \frac{3}{2} \left(\frac{b}{a} \right)^{\frac{3}{2}}.$$

注意, 若利用二重积分的变量代换, 则计算 S' 较为简

单. 容易算得 $\frac{D(u,v)}{D(x,y)} = -\frac{3}{2} \left(\frac{y}{x} \right)^{\frac{3}{2}}$, 故

$$S' = \iint_{S'} du dv = \iint_S \left| \frac{D(u,v)}{D(x,y)} \right| dx dy$$

$$\begin{aligned}
&= \frac{3}{2} \int_a^{a+h} x^{-\frac{3}{2}} dx \int_b^{b+h} y^{\frac{3}{2}} dy \\
&= \frac{6}{5} \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{a+h}} \right) (\sqrt{(b+h)^5} - \sqrt{b^5})
\end{aligned}$$

与上述结果一致. 但是, 从原习题集题目的安排来看, 似乎应从 3966 题以后才开始用一般的变量代换来计算二重积分.

引入新的变量 u, v 来代替 x, y , 并确定下列二重积分中的积分限:

3957. $\int_a^b dx \int_{\alpha x}^{\beta x} f(x, y) dy$ ($0 < a < b; 0 < \alpha < \beta$), 若 $u = x$,
 $v = \frac{y}{x}$.

解 在变换 $u = x, v = \frac{y}{x}$ 下, 区域 $\Omega = \{a \leq x \leq b, \alpha x \leq y \leq \beta x\}$ 变为 $\Omega = \{a \leq u \leq b, \alpha \leq v \leq \beta\}$. 变换的雅哥比式

$$I = \frac{D(x, y)}{D(u, v)} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u > 0.$$

于是

$$\int_a^b dx \int_{\alpha x}^{\beta x} f(x, y) dy = \int_a^b u du \int_{\alpha}^{\beta} f(u, uv) dv.$$

3958. $\int_0^2 dx \int_{1-x}^{2-x} f(x, y) dy$, 若 $u = x + y, v = x - y$.

解 在变换 $u = x + y, v = x - y$ 下, 区域 $\Omega = \{0 \leq x \leq 2, 1-x \leq y \leq 2-x\}$ 变为 $\Omega = \{1 \leq u \leq 2, -u \leq v \leq 4-u\}$. 事实上, $u + v = 2x, u - v = 2y$, 故 $0 \leq u + v \leq 4$, 即 $-u \leq v \leq 4-u$. 变换的雅哥比式 $I = -\frac{1}{2}$, 从而 $|I| = \frac{1}{2}$. 于是,

$$\int_0^2 dx \int_{1-x}^x f(x, y) dy$$

$$= \frac{1}{2} \int_1^2 du \int_{-u}^u f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) dv.$$

3959. $\iint_{\Omega} f(x, y) dx dy$, 其中 Ω 是由曲线 $\sqrt{x} + \sqrt{y} = \sqrt{a}$,

$x = 0, y = 0 (a > 0)$ 所界的区域, 若

$$x = u \cos^4 v, \quad y = u \sin^4 v.$$

解 Ω 的界线 $\sqrt{x} + \sqrt{y} = \sqrt{a}$ 的参数方程为

$$x = a \cos^4 v, y = a \sin^4 v \quad \left(0 \leq v \leq \frac{\pi}{2}\right).$$

对于变换 $x = u \cos^4 v, y = u \sin^4 v$, 有 $|I| = 4|u \cos^3 v \cdot \sin^3 v|$, 且区域 Ω 变为 $\Omega' = \{0 \leq u \leq a, 0 \leq v \leq \frac{\pi}{2}\}$.

于是,

$$\iint_{\Omega} f(x, y) dx dy$$

$$= 4 \int_0^a u du \int_0^{\frac{\pi}{2}} \cos^3 v \sin^3 v f(u \cos^4 v, v \sin^4 v) dv$$

$$= 4 \int_0^{\frac{\pi}{2}} \cos^3 v \sin^3 v dv \int_0^a u f(u \cos^4 v, u \sin^4 v) du.$$

3960. 证明: 变数代换

$$x + y = \xi, \quad y = \xi \eta$$

把三角形 $0 \leq x \leq 1, 0 \leq y \leq 1 - x$ 变为单位正方形 $0 \leq \xi \leq 1, 0 \leq \eta \leq 1$.

证 由 $0 \leq y \leq 1 - x$ 及 $0 \leq x \leq 1$ 得 $0 \leq x + y \leq 1$, 即

$$0 \leq \xi \leq 1.$$

又 $\eta = \frac{y}{\xi} \leq \frac{y}{0+y} = 1$, 且 $\eta \geq 0$, 故 $0 \leq \eta \leq 1$.

反之, 从 $0 \leq \xi \leq 1, 0 \leq \eta \leq 1$ 得 $0 \leq x+y \leq 1$,
 $y = \xi\eta, x = \xi(1-\eta)$, 故 $0 \leq x \leq 1, 0 \leq y \leq 1-x$.
 因此, 三角形域 $\{0 \leq x \leq 1, 0 \leq y \leq 1-x\}$ 变为正方形域 $\{0 \leq \xi \leq 1, 0 \leq \eta \leq 1\}$.

3961. 在什么样的变数代换下, 由曲线 $xy = 1, xy = 2, x-y = 1, x-y = -1$ ($x > 0, y > 0$) 所界的曲线四边形变换成矩形, 其边平行于坐标轴?

解 原四条曲线为 $xy = 1, xy = 2, x-y = -1, x-y = 1$ ($x > 0, y > 0$), 故显然应作变换 $xy = u, x-y = v$. 这时 u 从 1 变到 2, v 从 -1 变到 1, 故原积分域变为域:
 $1 \leq u \leq 2, -1 \leq v \leq 1$.

进行适当的变数代换, 化二重积分为一重的:

3962. $\iint_{|x|+|y| \leq 1} f(x+y) dx dy.$

解 作变换 $x+y = u, x-y = v$ 或 $x = \frac{u+v}{2}, y =$

$\frac{u-v}{2}$, 则有 $|I| = \frac{1}{2}$, 且 u 从 -1 变到 1, v 从 -1 变到

1. 于是,

$$\begin{aligned} \iint_{|x|+|y| \leq 1} f(x+y) dx dy &= \frac{1}{2} \int_{-1}^1 dv \int_{-1}^1 f(u) du \\ &= \int_{-1}^1 f(u) du. \end{aligned}$$

3963. $\iint_{x^2+y^2 \leq 1} f(ax+by+c) dx dy \quad (a^2+b^2 \neq 0).$

解 作变换 $\frac{ax+by}{\sqrt{a^2+b^2}} = u, \frac{bx-ay}{\sqrt{a^2+b^2}} = v$, 则有 $x = \frac{au+bv}{\sqrt{a^2+b^2}}, y = \frac{bu-av}{\sqrt{a^2+b^2}}$; 及 $x^2+y^2 = u^2+v^2 \leq 1$, 故域 $x^2+y^2 \leq 1$ 变为 $u^2+v^2 \leq 1$, 且有 $|I| = 1$. 于是,

$$\begin{aligned} & \iint_{x^2+y^2 \leq 1} f(ax+by+c) dx dy \\ &= \iint_{u^2+v^2 \leq 1} f(\sqrt{a^2+b^2}u+c) du dv \\ &= \int_{-1}^1 du \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f(\sqrt{a^2+b^2}u+c) dv \\ &= \int_{-1}^1 f(\sqrt{a^2+b^2}u+c) du \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} dv \\ &= 2 \int_{-1}^1 \sqrt{1-u^2} f(\sqrt{a^2+b^2}u+c) du. \end{aligned}$$

3964. $\iint_{\Omega} f(x,y) dx dy$, 其中 Ω 为由曲线 $xy=1, xy=2, y=x, y=4x (x>0, y>0)$ 所界的域.

解 作变换 $xy=u, \frac{y}{x}=v$, 则域 Ω 变为域

$\Omega' = \{1 \leq u \leq 2, 1 \leq v \leq 4\}$, 且 $|I| = \frac{1}{2v}$, 于是,

$$\iint_{\Omega} f(x,y) dx dy = \int_1^2 \frac{dv}{2v} \int_1^2 f(u) du = \ln 2 \cdot \int_1^2 f(u) du.$$

3965. $\iint_{\Omega} (x+y) dx dy$, 其中 Ω 是由曲线 $x^2+y^2=x+y$ 所包围的域.

解 域 Ω 即圆 $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \leq \left(\frac{1}{\sqrt{2}}\right)^2$. 作变

换: $x = \frac{1}{2} + r\cos\varphi, y = \frac{1}{2} + r\sin\varphi$, 则域 Ω 变为域

$\Omega' = \{0 \leq \varphi \leq 2\pi, 0 \leq r \leq \frac{1}{\sqrt{2}}\}$, 且 $|I| = r$. 于是

$$\begin{aligned} \iint_{\Omega} (x+y) dx dy &= \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{2}}} (r + r^2(\sin\varphi \\ &\quad + \cos\varphi)) dr = \frac{\pi}{2}. \end{aligned}$$

计算下列二重积分:

3966. $\iint_{|x|+|y|\leq 1} (|x|+|y|) dx dy.$

解
$$\begin{aligned} &\iint_{|x|+|y|\leq 1} (|x|+|y|) dx dy \\ &= 4 \int_0^1 dx \int_0^{1-x} (x+y) dy = \frac{4}{3}. \end{aligned}$$

3967. $\iint_{\Omega} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy$, 其积分域 Ω 是由椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 所界的域.

解 作变换: $x = a\cos\varphi, y = br\sin\varphi$, 则域 Ω 变为域 $\Omega' = \{0 \leq r \leq 1, 0 \leq \varphi \leq 2\pi\}$, 且 $|I| = abr$. 于是,

$$\begin{aligned} \iint_{\Omega} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy &= \int_0^{2\pi} d\varphi \int_0^1 ab \sqrt{1-r^2} r dr \\ &= 2\pi ab \int_0^1 \sqrt{1-r^2} r dr = \frac{2\pi ab}{3}. \end{aligned}$$

3968. $\iint_{x^4+y^4\leq 1} (x^2+y^2) dx dy.$

解 作变换: $x = r\cos\varphi, y = r\sin\varphi$, 并利用对称性, 则有

$$\begin{aligned} \iint_{x^4+y^4 \leq 1} (x^2+y^2) dx dy &= 8 \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\left(\frac{1}{\cos^4\varphi+\sin^4\varphi}\right)^{\frac{1}{4}}} r^3 dr \\ &= 2 \int_0^{\frac{\pi}{4}} \frac{d\varphi}{\cos^4\varphi+\sin^4\varphi} = 2 \int_0^{\frac{\pi}{4}} \frac{\sec^2\varphi d\tan\varphi}{1+\tan^4\varphi} = 2 \int_0^1 \frac{1+t^2}{1+t^4} dt \\ &= \frac{2}{\sqrt{2}} \operatorname{arctg} \frac{t^2-1}{t\sqrt{2}} \Big|_0^1 = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

*) 利用 1712 题的结果.

3969. $\iint_{\Omega} (x+y) dx dy$, 其积分域 Ω 是由曲线 $y^2 = 2x, x+y=4, x+y=12$ 所界的域.

解 由解方程组

$$\begin{cases} x+y=4, \\ y^2=2x \end{cases} \quad \text{及} \quad \begin{cases} x+y=12, \\ y^2=2x \end{cases}$$

求得两条直线与抛物线的交点为 $A(2,2), B(8,4), C(18,-6), D(8,-4)$ (图 8.28). 于是,

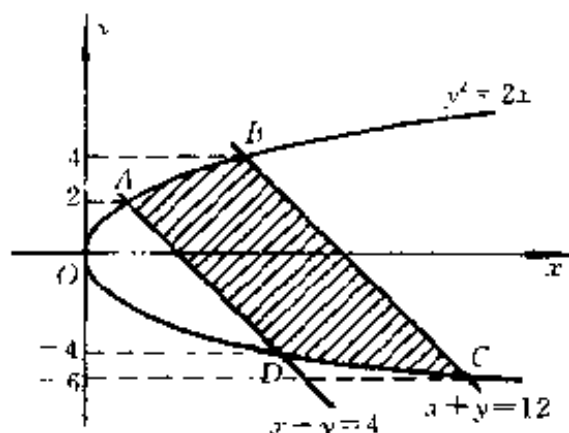


图 8.28

$$\iint_{\Omega} (x+y) dx dy = \int_{-6}^{-4} dy \int_{\frac{y^2}{2}}^{12-y} (x+y) dx$$

$$\begin{aligned}
& + \int_4^2 dy \int_{1-y}^{12-y} (x+y) dx + \int_2^4 dy \int_{\frac{y-2}{2}}^{12-y} (x+y) dx \\
& = 79 \frac{13}{15} + 384 + 79 \frac{13}{15} = 543 \frac{11}{15}.
\end{aligned}$$

3970. $\iint_{\Omega} xy dx dy$, 其中 Ω 是由曲线 $xy=1$, $x+y=\frac{5}{2}$ 所界的区域.

解 曲线 $xy=1$ 与直线 $x+y=\frac{5}{2}$ 的交点为 $(\frac{1}{2}, 2)$, $(2, \frac{1}{2})$. 于是,

$$\begin{aligned}
\iint_{\Omega} xy dx dy &= \int_{\frac{1}{2}}^2 x dx \int_{\frac{1}{x}}^{\frac{5}{2}-x} y dy \\
&= \frac{1}{2} \int_{\frac{1}{2}}^2 \left(\frac{25}{4} x - 5x^2 + x^3 - \frac{1}{x} \right) dx \\
&= 1 \frac{37}{128} - \ln 2.
\end{aligned}$$

3971. $\iint_{\substack{0 \leq x \leq \pi \\ 0 \leq y \leq \pi}} |\cos(x+y)| dx dy.$

$$\begin{aligned}
\text{解} \quad \iint_{\substack{0 \leq x \leq \pi \\ 0 \leq y \leq \pi}} |\cos(x+y)| dx dy &= \int_0^{\pi} dx \int_0^{\pi} |\cos(x+y)| dy \\
&= \int_0^{\frac{\pi}{2}} dx \int_0^{\pi} |\cos(x+y)| dy \\
&+ \int_{\frac{\pi}{2}}^{\pi} dx \int_0^{\pi} |\cos(x+y)| dy \\
&= \int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}-x} \cos(x+y) dy - \int_{\frac{\pi}{2}-x}^{\pi} \cos(x+y) dy \right) dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{\frac{\pi}{2}}^{\pi} \left(- \int_0^{\frac{3\pi}{2}} \cos(x+y) dy \right. \\
& \left. + \int_{\frac{3\pi}{2}}^{\pi} \cos(x+y) dy \right) dx \\
& = \int_0^{\frac{\pi}{2}} \left\{ \left(\sin \frac{\pi}{2} - \sin x \right) - \left[\sin(x+\pi) - \sin \frac{\pi}{2} \right] \right\} dx \\
& + \int_{\frac{\pi}{2}}^{\pi} \left\{ - \left(\sin \frac{3\pi}{2} - \sin x \right) + \left[\sin(x+\pi) \right. \right. \\
& \left. \left. - \sin \frac{3\pi}{2} \right] \right\} dx \\
& = \int_0^{\frac{\pi}{2}} 2 dx + \int_{\frac{\pi}{2}}^{\pi} 2 dx = 2\pi.
\end{aligned}$$

3972. $\iint_{x^2+y^2 \leq 1} \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx dy.$

解 积分域如图 8.29 所示, 由 $\Omega_1, \Omega_2, \Omega_3$ 和 Ω_4 所组成, 其中 Ω_1 为由圆 $\frac{x+y}{\sqrt{2}} - x^2 - y^2 = 0$, 即

$$\begin{aligned}
& \text{圆} \left(x - \frac{1}{2\sqrt{2}} \right)^2 \\
& + \left(y - \frac{1}{2\sqrt{2}} \right)^2 = \frac{1}{4}
\end{aligned}$$

围成的区域, 该圆的极坐标方程为

$$r = \sin\left(\varphi + \frac{\pi}{4}\right),$$

而圆 $x^2 + y^2 = 1$ 的极坐标方程为 $r = 1$. 于是, 各区域为

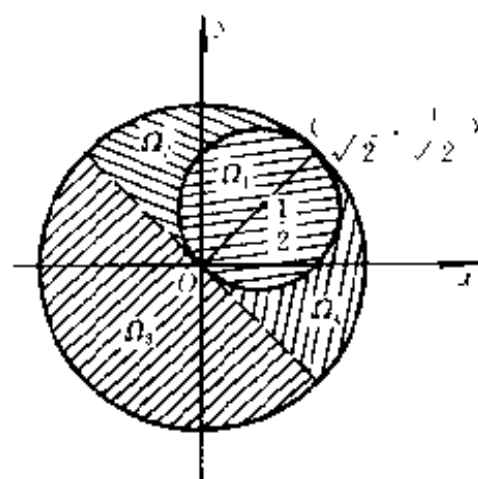


图 8.29

$$\Omega_1: -\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, 0 \leq r \leq \sin(\varphi + \frac{\pi}{4});$$

$$\Omega_2: \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \sin(\varphi + \frac{\pi}{4}) \leq r \leq 1;$$

$$\Omega_3: \frac{3\pi}{4} \leq \varphi \leq \frac{7\pi}{4}, 0 \leq r \leq 1;$$

$$\Omega_4: -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}, \sin(\varphi + \frac{\pi}{4}) \leq r \leq 1.$$

当点在 Ω_1 中时, 由于 $\left(x - \frac{1}{2\sqrt{2}}\right)^2 + \left(y - \frac{1}{2\sqrt{2}}\right)^2 \leq \frac{1}{4}$ 即 $\frac{x+y}{\sqrt{2}} - (x^2 + y^2) \geq 0$, 故

$$\left|\frac{x+y}{\sqrt{2}} - x^2 - y^2\right| = \frac{x+y}{\sqrt{2}} - x^2 - y^2 = r\sin\left(\varphi + \frac{\pi}{4}\right) - r^2; \text{ 当点在 } \Omega_2, \Omega_3, \text{ 和 } \Omega_4 \text{ 中时,}$$

$$\left|\frac{x+y}{\sqrt{2}} - x^2 - y^2\right| = x^2 + y^2 - \frac{x+y}{\sqrt{2}} = r^2 - r\sin\left(\varphi + \frac{\pi}{4}\right).$$

于是, 注意到利用对称性便得

$$\begin{aligned} & \iint_{x^2+y^2 \leq 1} \left|\frac{x+y}{\sqrt{2}} - x^2 - y^2\right| dx dy \\ &= 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{\sin(\varphi + \frac{\pi}{4})} [r \sin(\varphi + \frac{\pi}{4}) - r^2] r dr \\ &+ 2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_{\sin(\varphi + \frac{\pi}{4})}^1 [r^2 - r \sin(\varphi + \frac{\pi}{4})] r dr \\ &+ \int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} d\varphi \int_0^1 [r^2 - r \sin(\varphi + \frac{\pi}{4})] r dr \\ &= \frac{1}{6} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^4\left(\varphi + \frac{\pi}{4}\right) d\varphi + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\frac{1}{2} - \frac{2}{3} \right. \end{aligned}$$

$$\begin{aligned}
& \cdot \sin\left(\varphi + \frac{\pi}{4}\right) + \frac{1}{6}\sin^4\left(\varphi + \frac{\pi}{4}\right)\bigg]d\varphi \\
& + \int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}}\left[\frac{1}{4} - \frac{1}{3}\sin\left(\varphi + \frac{\pi}{4}\right)\right]d\varphi \\
& = \frac{1}{6}\int_0^{\frac{\pi}{2}}\sin^4 u du + \left(\frac{\pi}{4} - \frac{2}{3} + \frac{1}{6}\int_0^{\frac{\pi}{2}}\sin^4 u du\right) \\
& \quad + \left(\frac{2}{3} + \frac{\pi}{4}\right) \\
& \quad *) \\
& = \frac{\pi}{32} + \left(\frac{\pi}{4} - \frac{2}{3} + \frac{\pi}{32}\right) + \left(\frac{2}{3} + \frac{\pi}{4}\right) = \frac{9\pi}{16}.
\end{aligned}$$

*) 利用 2281 题的结果.

$$3973^+. \iint_{\substack{|x| \leq 1 \\ 0 \leq y \leq 2}} \sqrt{|y - x^2|} dx dy.$$

$$\begin{aligned}
\text{解} \quad & \iint_{\substack{|x| \leq 1 \\ 0 \leq y \leq 2}} \sqrt{|y - x^2|} dx dy = \iint_{\substack{|x| \leq 1 \\ 0 \leq y \leq x^2}} \sqrt{x^2 - y} dx dy \\
& + \iint_{\substack{|x| \leq 1 \\ x^2 \leq y \leq 2}} \sqrt{y - x^2} dx dy \\
& = \int_{-1}^1 dx \int_0^{x^2} \sqrt{x^2 - y} dy + \int_{-1}^1 dx \int_{x^2}^2 \sqrt{y - x^2} dy \\
& = \frac{4}{3} \int_0^1 x^3 dx + \frac{4}{3} \int_0^1 (2 - x^2)^{\frac{3}{2}} dx \\
& = \frac{1}{3} + \frac{16}{3} \int_0^{\frac{\pi}{4}} \cos^4 \theta d\theta \\
& = \frac{1}{3} + \frac{16}{3} \left(\frac{3\pi}{32} + \frac{1}{4} \right)^{*}) = \frac{5}{3} + \frac{\pi}{2}.
\end{aligned}$$

*) 参看 1750 题的结果.

计算不连续函数的积分:

$$3974. \iint_{x^2+y^2 \leq 4} \operatorname{sgn}(x^2 - y^2 + 2) dx dy.$$

解 当 $y^2 - x^2 < 2$ 时,

$$\operatorname{sgn}(x^2 - y^2 + 2) = 1;$$

当 $y^2 - x^2 > 2$ 时,

$$\operatorname{sgn}(x^2 - y^2 + 2) = -1;$$

当 $y^2 - x^2 = 2$ 时,

$$\operatorname{sgn}(x^2 - y^2 + 2) = 0.$$

现将域 $x^2 + y^2 \leq 4$ 分成 $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ 和 Ω_5 五部分, 其界线分别为

$x^2 + y^2 = 4, y^2 - x^2 = 2, x = \pm 1$ (图 8.30). 当点在 Ω_1 和 Ω_5 中时, $y^2 - x^2 > 2$, 故 $\operatorname{sgn}(x^2 - y^2 + 2) = -1$; 当点在 Ω_2, Ω_3 和 Ω_4 中时, $y^2 - x^2 < 2$, 故 $\operatorname{sgn}(x^2 - y^2 + 2) = 1$. 于是,

$$\begin{aligned} & \iint_{x^2+y^2 \leq 4} \operatorname{sgn}(x^2 - y^2 + 2) dx dy \\ &= - \iint_{\Omega_1} dx dy - \iint_{\Omega_5} dx dy + \iint_{\Omega_2} dx dy + \iint_{\Omega_3} dx dy \\ & \quad + \iint_{\Omega_4} dx dy \\ &= -4 \int_0^1 dx \int_{\sqrt{2+x^2}}^{\sqrt{4-x^2}} dy + 4 \int_0^1 dx \int_0^{\sqrt{2+x^2}} dy \end{aligned}$$

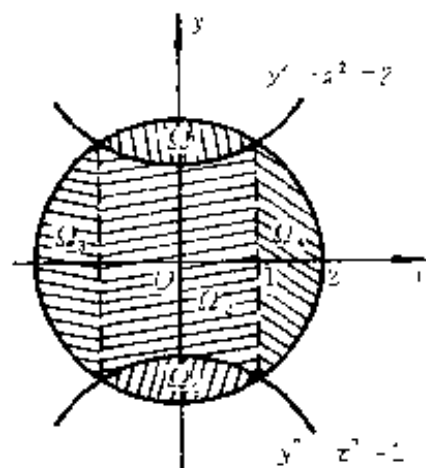


图 8.30

$$\begin{aligned}
& + 4 \int_1^2 dx \int_0^{\sqrt{4-x^2}} dy \\
& = 8 \int_0^1 \sqrt{2+x^2} dx + 4 \left(\int_1^2 \sqrt{4-x^2} dx \right. \\
& \quad \left. - \int_0^1 \sqrt{4-x^2} dx \right) \\
& = \frac{4\pi}{3} + 8 \ln \frac{1+\sqrt{3}}{\sqrt{2}}.
\end{aligned}$$

3975. $\iint_{\substack{0 \leq x \leq 2 \\ 0 \leq y \leq 2}} [x+y] dx dy.$

解 当 $0 \leq x+y < 1$ 时,
 $[x+y] = 0$;
 当 $1 \leq x+y < 2$ 时,
 $[x+y] = 1$;
 当 $2 \leq x+y < 3$ 时,
 $[x+y] = 2$;
 当 $3 \leq x+y < 4$ 时,
 $[x+y] = 3$;

当 $x+y=4$ 时, $[x+y]=4$.

如图 8.31 所示, 域 $0 \leq x \leq 2, 0 \leq y \leq 2$ 可分为下列四部分:

$$\begin{aligned}
\Omega_1: & x+y \leq 1, x \geq 0, y \geq 0; \\
\Omega_2: & 1 \leq x+y \leq 2, x=0, y=0; \\
\Omega_3: & 2 \leq x+y \leq 3, x=2, y=2; \\
\Omega_4: & x+y \geq 3, x \leq 2, y \leq 2.
\end{aligned}$$

当点属于 Ω_1 的内部时, $[x+y]=0$; 当点属于 Ω_2 的内部时, $[x+y]=1$; 当点属于 Ω_3 的内部时, $[x+y]=$

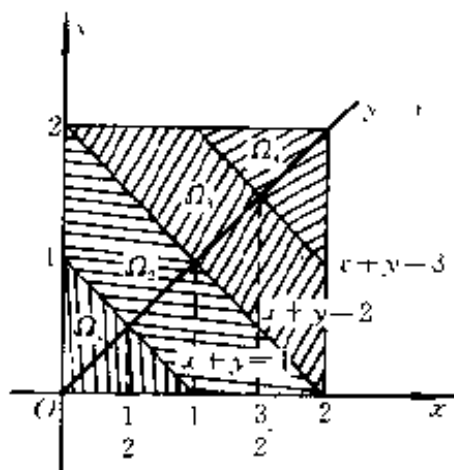


图 8.31

2; 当点属于 Ω_i 的内部时, $[x+y] = 3$. 于是,

$$\begin{aligned}
 \iint_{\substack{0 \leq x \leq 2 \\ 0 \leq y \leq 2}} [x+y] dx dy &= \iint_{\Omega_2} dx dy + 2 \iint_{\Omega_3} dx dy + 3 \iint_{\Omega_4} dx dy \\
 &= 2 \left(\int_{\frac{1}{2}}^1 dx \int_{1-x}^x dy + \int_1^2 dx \int_0^{2-x} dy \right) + 4 \left(\int_1^{\frac{3}{2}} dx \int_{2-x}^x dy \right. \\
 &\quad \left. + \int_{\frac{3}{2}}^2 dx \int_{2-x}^{3-x} dy \right) + 6 \int_{\frac{3}{2}}^2 dx \int_{3-x}^x dy \\
 &= 2 \left(\int_{\frac{1}{2}}^1 (2x-1) dx + \int_1^2 (2-x) dx \right) \\
 &\quad + 4 \left(\int_1^{\frac{3}{2}} (2x-2) dx + \int_{\frac{3}{2}}^2 dx \right) \\
 &\quad + 6 \int_{\frac{3}{2}}^2 (2x-3) dx = 6.
 \end{aligned}$$

3976. $\iint_{x^2 \leq y \leq 4} \sqrt{y-x^2} dx dy.$

解 如图 8.32 所示.

当 $x^2 \leq y < x^2 + 1$

时,

$$[y-x^2] = 0;$$

当 $1+x^2 \leq y < x^2$

+ 2 时,

$$[y-x^2] = 1;$$

当 $2+x^2 \leq y < x^2$

+ 3 时,

$$[y-x^2] = 2;$$

当 $3+x^2 \leq y < 4$ 时,

$$[y-x^2] = 3.$$

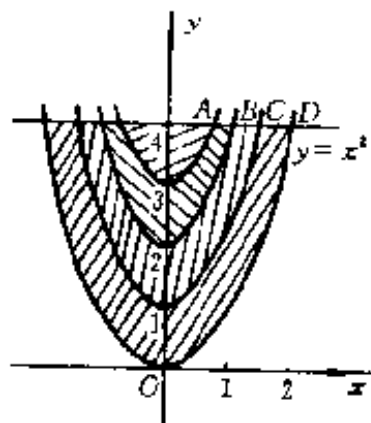


图 8.32

抛物线 $y = x^2 + 3, y = x^2 + 2, y = x^2 + 1$ 及 $y = x^2$ 与直线 $y = 4$ 在第一象限内的交点为 $A(1, 4), B(\sqrt{2}, 4), C(\sqrt{3}, 4)$ 及 $D(2, 4)$, 与 Oy 轴对称的位置还有四个交点, 于是,

$$\begin{aligned} & \iint_{x^2 \leq y \leq 4} \sqrt{y - x^2} dx dy \\ &= 2 \left[\int_0^{\sqrt{2}} dx \int_{x^2+1}^{x^2+2} dy + \int_{\sqrt{2}}^{\sqrt{3}} dx \int_{x^2+1}^4 dy \right] \\ & \quad + 2 \sqrt{2} \left[\int_0^1 dx \int_{x^2+2}^{x^2+3} dy + \int_1^{\sqrt{2}} dx \int_{x^2+2}^4 dy \right] \\ & \quad + 2 \sqrt{3} \int_0^1 dx \int_{x^2+3}^4 dy \\ &= 2 \left[\sqrt{2} + \int_{\sqrt{2}}^{\sqrt{3}} (3 - x^2) dx \right] + 2 \sqrt{2} \\ & \quad \cdot \left[1 + \int_1^{\sqrt{2}} (2 - x^2) dx \right] + 2 \sqrt{3} \int_0^1 (1 - x^2) dx \\ &= \frac{4}{3} (4 + 4 \sqrt{3} - 3 \sqrt{2}). \end{aligned}$$

3977. 设 m 及 n 为正整数且其中至少有一个是奇数, 证明

$$\iint_{x^2+y^2 \leq a^2} x^m y^n dx dy = 0.$$

证 作变换: $x = r \cos \varphi, y = r \sin \varphi$, 则得

$$\begin{aligned} \iint_{x^2+y^2 \leq a^2} x^m y^n dx dy &= \iint_{\substack{0 \leq \varphi \leq 2\pi \\ 0 \leq r \leq a}} r^{m+n+1} \cos^m \varphi \sin^n \varphi dr d\varphi \\ &= \frac{a^{m+n+2}}{m+n+2} \int_0^{2\pi} \cos^m \varphi \sin^n \varphi d\varphi \\ &= \frac{a^{m+n+2}}{m+n+2} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^m \varphi \sin^n \varphi d\varphi \end{aligned}$$

$$= \frac{a^{m+n+2}}{m+n+2} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^m \varphi \sin^n \varphi d\varphi + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^m \varphi \sin^n \varphi d\varphi \right], \quad (1)$$

若在上式右端的第二个积分中令 $\varphi = \pi + t$, 即得

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^m \varphi \sin^n \varphi d\varphi = (-1)^m \cdot (-1)^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^m t \sin^n t dt. \quad (2)$$

当 m 及 n 中有且仅有一个为奇数时, $(-1)^m \cdot (-1)^n = -1$, 因而(1)式为零, 当 m 和 n 均为奇数时, $(-1)^m \cdot (-1)^n = 1$, 因而(1)式等于

$$\frac{2a^{m+n+2}}{m+n+2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^m \varphi \sin^n \varphi d\varphi.$$

但此被积函数在对称区间 $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ 上为奇函数, 故积分仍然为零.

总之, 当 m 和 n 中至少有一个为奇数时,

$$\iint_{x^2+y^2 \leq a^2} x^m y^n dx dy = 0.$$

3978. 求:

$$\lim_{\rho \rightarrow 0} \frac{1}{\pi \rho^2} \iint_{x^2+y^2 \leq \rho^2} f(x, y) dx dy,$$

其中 $f(x, y)$ 为连续函数.

解 利用积分中值定理, 即得

$$\begin{aligned} & \iint_{x^2+y^2 \leq \rho^2} f(x, y) dx dy \\ &= f(\xi, \eta) \iint_{x^2+y^2 \leq \rho^2} dx dy = \pi \rho^2 \cdot f(\xi, \eta), \end{aligned}$$

其中点 (ξ, η) 为圆域 $x^2 + y^2 \leq \rho^2$ 内的一点显然,当 $\rho \rightarrow 0$ 时,点 $(\xi, \eta) \rightarrow O(0, 0)$. 于是,根据函数 $f(x, y)$ 的连续性知:

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{1}{\pi \rho^2} \iint_{x^2 + y^2 \leq \rho^2} f(x, y) dx dy \\ = \lim_{\rho \rightarrow 0} f(\xi, \eta) = f(0, 0). \end{aligned}$$

3979. 设

$$F(t) = \iint_{\substack{0 \leq x \leq t \\ 0 \leq y \leq t}} e^{\frac{tx}{y^2}} dx dy,$$

求 $F'(t)$.

解 令 $x = ut, y = vt$, 则

$$F(t) = \iint_{\substack{0 \leq x \leq t \\ 0 \leq y \leq t}} e^{\frac{tx}{y^2}} dx dy = \iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1}} t^2 e^{\frac{u}{v^2}} du dv. \quad (1)$$

于是,似乎应该有

$$\begin{aligned} F'(t) &= \iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1}} 2te^{\frac{u}{v^2}} du dv \\ &= \frac{2}{t} \iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1}} t^2 e^{\frac{u}{v^2}} du dv = \frac{2}{t} F(t) \quad (t > 0). \end{aligned}$$

但这是错误的. 实际上本题有问题, 因为(1)式中的二重积分都是广义二重积分. 当 $t > 0$ 时, 在 $x > 0, y = 0$ 上(即 $u > 0, v = 0$ 上)被积函数成为无穷, 而且这个广义二重积分是发散的. 这是因为, 根据被积函数的非负性, 有(参看 § 9)

$$\iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1}} e^{\frac{u}{v^2}} du dv = \int_0^1 dv \int_0^1 e^{\frac{u}{v^2}} du$$

$$= \int_0^1 v^2 (e^{\frac{1}{v^2}} - 1) dv. \quad (2)$$

对此积分, $v = 0$ 是瑕点, 由于被积函数 $v^2 (e^{\frac{1}{v^2}} - 1)$ 在 $0 \leq v \leq 1$ 上非负, 且 (令 $\frac{1}{v^2} = t$)

$$\lim_{v \rightarrow +0} v^2 [v^2 (e^{\frac{1}{v^2}} - 1)] = \lim_{t \rightarrow +\infty} \frac{e^t - 1}{t^2} = +\infty,$$

故瑕积分 $\int_0^1 v^2 (e^{\frac{1}{v^2}} - 1) dv$ 发散, 且

$$\int_0^1 v^2 (e^{\frac{1}{v^2}} - 1) dv = +\infty.$$

由此, 再根据(1)式与(2)式, 得

$$F(t) \equiv +\infty \text{ (当 } t > 0 \text{ 时)}.$$

因此, 提出求 $F'(t)$ 的问题是无意义的.

注意, 若本题换为: 设

$$F(t) = \iint_{\substack{0 \leq x \leq t \\ 0 \leq y \leq t}} e^{-\frac{t^2}{y^2}} dx dy,$$

求 $F'(t)$. 这时得 (作代换 $x = ut, y = vt$)

$$F(t) = t^2 \iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1}} e^{-\frac{u^2}{v^2}} du dv,$$

从而右端积分是收敛的, (实际上可视为常义积分). 于是,

$$F'(t) = 2t \iint_{\substack{0 \leq u \leq 1 \\ 0 \leq v \leq 1}} e^{-\frac{u^2}{v^2}} du dv = \frac{2}{t} F(t) \quad (t > 0).$$

3980⁺. 设

$$F(t) = \iint_{(x-t)^2 + (y-t)^2 \leq 1} \sqrt{x^2 + y^2} dx dy,$$

求 $F'(t)$.

解 作变量代换 $x = u + t, y = v + t$ (t 固定), 则

$$F(t) = \iint_{u^2+v^2 \leq 1} \sqrt{(u+t)^2 + (v+t)^2} du dv. \quad (1)$$

今在积分号下求导数^{*}), 得

$$\begin{aligned} F'(t) &= \iint_{u^2+v^2 \leq 1} \frac{u+t+v+t}{\sqrt{(u+t)^2 + (v+t)^2}} du dv \\ &= \iint_{(x-t)^2 + (y-t)^2 \leq 1} \frac{x+y}{\sqrt{x^2 + y^2}} dx dy \\ &\quad (-\infty < t < +\infty). \end{aligned}$$

*) 积分号下求导数的合理性, 证明如下: 令

$$f(u, v, t) = \sqrt{(u+t)^2 + (v+t)^2},$$

则

$$\begin{aligned} f'_t(u, v, t) &= \frac{u+t+v+t}{\sqrt{(u+t)^2 + (v+t)^2}} \\ &\quad ((u, v) \neq (-t, -t)). \end{aligned}$$

当 $(u, v) = (-t, -t)$ 时, 易知 $f'_t(u, v, t)$ 不存在, 但右导数存在且等于 $\sqrt{2}$, 左导数也存在且等于 $-\sqrt{2}$. 由于对任何数 a, b , 有 $a^2 + b^2 \geq 2ab$, 故 $2(a^2 + b^2) \geq (a+b)^2$, 从而 $\frac{|a+b|}{\sqrt{a^2 + b^2}} \leq \sqrt{2}$. 于是,

$$|f'_t(u, v, t)| \leq \sqrt{2} \quad ((u, v) \neq (-t, -t)). \quad (2)$$

如果 $|t| > \frac{1}{\sqrt{2}}$, 这时 $f(u, v, t), f'_t(u, v, t)$ (t 固定) 都是域 $u^2 + v^2 \leq 1$ 上的连续函数, 当然可在积分号下求导数, 得

$$F(t) = \iint_{u^2+v^2 \leq 1} f'_t(u, v, t) du dv. \quad (3)$$

但如果 $|t| \leq \frac{1}{\sqrt{2}}$, 则(3)式右端积分的被积函数 $f'_i(u, v, t)$ 在积分域 $u^2 + v^2 \leq 1$ 中的点 $(u, v) = (-t, -t)$ 不连续. 因此, 不能立即断定(3)式的正确性. 下面不论 t 为何值 $(-\infty < t < +\infty)$ 直接证明(3)式成立. 令

$$g(t) = \iint_{u^2+v^2 \leq 1} f'_i(u, v, t) du dv \quad (-\infty < t < +\infty). \quad (4)$$

由(2)式知 $f'_i(u, v, t)$ 是有界的, 且在域 $u^2 + v^2 \leq 1$ 上至多有一个不连续点(t 固定), 故(4)式右端的积分存在. 实际上, 利用(2)式以及 $f'_i(u, v, t)$ 当 $(u, v) \neq (-t, -t)$ 时的连续性, 用(必要时, 即 $|t| \leq \frac{1}{\sqrt{2}}$ 时)挖掉以点 $(-t, -t)$ 为中心的小圆域的方法, 不难证明 $g(t)$ 是 $-\infty < t < +\infty$ 上的连续函数(详细证明留给读者). 令

$$G(t) = \int_0^t g(s) ds \quad (-\infty < t < +\infty),$$

则

$$G'(t) = g(t) \quad (-\infty < t < +\infty). \quad (5)$$

但

$$\begin{aligned} G(t) &= \int_0^t ds \iint_{u^2+v^2 \leq 1} f'_i(u, v, s) du dv \\ &= \iiint_{\substack{u^2+v^2 \leq 1 \\ 0 \leq s \leq t}} f'_i(u, v, s) du dv ds \end{aligned}$$

$$= \iint_{u^2+v^2 \leq 1} dudv \int_0^t f'_{,1}(u, v, s) ds. \quad (6)$$

注意, (6) 式中的运算是合理的, 因为三维域 $u^2 + v^2 \leq 1, 0 \leq s \leq t$ (t 固定) 中, 三元函数 $f'_{,1}(u, v, s)$ 有界且只在直线 $u = v = -s$ 的一段上不连续, 从而 (6) 式中的三重积分及两个累次积分都存在, 故它们相等.

下证恒有

$$\int_0^t f'_{,1}(u, v, s) ds = f(u, v, t) - f(u, v, 0). \quad (7)$$

事实上, 若 $(u, v) \neq (-t_1, -t_1)$ ($t_1 \in [0, t]$) 则 $f'_{,1}(u, v, s)$ 是 $0 \leq s \leq t$ 上的连续函数 (u, v 固定), 从而 (7) 式成立; 若 $(u, v) = (-t_1, -t_1)$ (t_1 是属于 $[0, t]$ 的某数), 则由 $f(u, v, s)$ 对任何 u, v, s 的连续性, 有

$$\begin{aligned} \int_0^t f'_{,1}(u, v, s) ds &= \lim_{\epsilon \rightarrow +0} \int_0^{t_1-\epsilon} f'_{,1}(u, v, s) ds \\ &\quad + \lim_{\epsilon' \rightarrow 0} \int_{t_1+\epsilon'}^t f'_{,1}(u, v, s) ds \\ &= \lim_{\epsilon \rightarrow +0} [f(u, v, t_1 - \epsilon) - f(u, v, 0)] \\ &\quad + \lim_{\epsilon' \rightarrow 0} [f(u, v, t) - f(u, v, t_1 + \epsilon')] \\ &= f(u, v, t_1) - f(u, v, 0) + f(u, v, t) - f(u, v, t_1) \\ &= f(u, v, t) - f(u, v, 0), \end{aligned}$$

故 (7) 式恒成立. 代入 (6) 式, 得

$$\begin{aligned} G(t) &= \iint_{u^2+v^2 \leq 1} [f(u, v, t) - f(u, v, 0)] dudv \\ &= F(t) - F(0) \quad (-\infty < t < +\infty). \end{aligned}$$

由此, 再注意到 (5) 式, 即知 $F'(t)$ 存在, 且

$$F(t) = G(t) = g(t)$$

$$= \iint_{u^2+v^2 \leq 1} f'_{,i}(u,v,t) du dv \quad (-\infty < t < +\infty),$$

即(3)式成立.

3981. 设

$$F(t) = \iint_{x^2+y^2 \leq t^2} f(x,y) dx dy \quad (t > 0),$$

求 $F'(t)$.

解 令 $x = r \cos \varphi, y = r \sin \varphi$, 则

$$F'(t) = \int_0^t dr \int_0^{2\pi} f(r \cos \varphi, r \sin \varphi) r d\varphi,$$

故得

$$F'(t) = \int_0^{2\pi} f(t \cos \varphi, t \sin \varphi) t d\varphi.$$

注意, 此题中应假定 $f(x,y)$ 是连续函数.

3982. 设 $f(x,y)$ 是连续的, 求证函数

$$u(x,y) = \frac{1}{2} \int_0^x d\xi \int_{\xi-x+y}^{x+y-\xi} f(\xi, \eta) d\eta$$

满足方程式

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y).$$

证 利用含参变量的常义积分求导数的公式, 得

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{2} \int_0^x [f(\xi, x+y-\xi) - (-1)f(\xi, \xi-x+y)] d\xi \\ &\quad + \frac{1}{2} \int_{x-x+y}^{x+y-x} f(x, \eta) d\eta \\ &= \frac{1}{2} \int_0^x [f(\xi, x+y-\xi) + f(\xi, \xi-x+y)] d\xi, \\ \frac{\partial^2 u}{\partial x^2} &= \frac{1}{2} \int_0^x [f'_{,1}(\xi, x+y-\xi) - f'_{,1}(\xi, \xi-x+y)] d\xi \\ &\quad + \frac{1}{2} [f(x, x+y-x) + f(x, x-x+y)] \end{aligned}$$

$$= \frac{1}{2} \int_0^x [f'_y(\xi, x+y-\xi) - f'_y(\xi, \xi-x+y)] d\xi + f(x, y).$$

同理, 有

$$\frac{\partial u}{\partial y} = \frac{1}{2} \int_0^x [f(\xi, x+y-\xi) - f(\xi, \xi-x+y)] d\xi,$$

$$\frac{\partial u}{\partial y^2} = \frac{1}{2} \int_0^x [f'_y(\xi, x+y-\xi) - f'_y(\xi, \xi-x+y)] d\xi.$$

于是, 得

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y),$$

证毕.

注意, 显然本题还应假定 $f'_y(x, y)$ 存在且连续.

3983. 设函数 $f(x, y)$ 的等位线是简单封闭曲线, $S(v_1, v_2)$ 是由曲线 $f(x, y) = v_1$ 及 $f(x, y) = v_2$ 所围成的域. 证明

$$\iint_{S(v_1, v_2)} f(x, y) dx dy = \int_{v_1}^{v_2} v F'(v) dv,$$

其中 $F(v)$ 为由曲线 $f(x, y) = v_1$ 与 $f(x, y) = v_2$ 所包围的面积.

证 作 $[v_1, v_2]$ 的任一分划 T :

$$\begin{aligned} v_1 &= v'_0 < v'_1 < \\ \dots < v'_i < \dots < v'_n &= \\ v_2. \end{aligned}$$

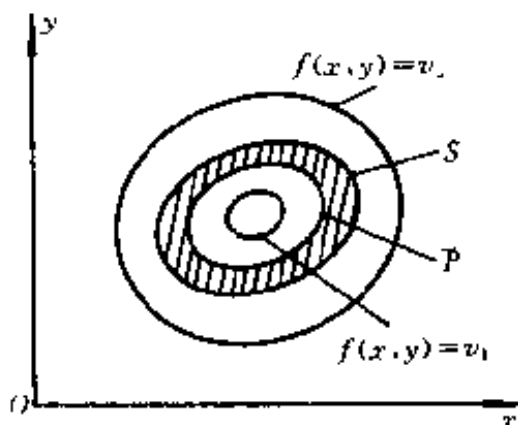


图 8.33

令 $d(T) = \max_{1 \leq i \leq n} \Delta v_i$, 这里 $\Delta v_i = v'_{i+1} - v'_{i-1} (i = 1, 2, \dots, n)$. 于是, 由积分中值定理 (这里假定了 $f(x, y)$ 在 $S(v_1, v_2)$ 上连续) 知

$$\begin{aligned} & \iint_{S(v_1, v_2)} f(x, y) dx dy \\ &= \sum_{i=1}^n \iint_{S(v'_{i-1}, v'_i)} f(x, y) dx dy = \sum_{i=1}^n f(\bar{x}_i, \bar{y}_i) \Delta S_i, \end{aligned}$$

其中 ΔS_i 表小环形域 $S(v'_{i-1}, v'_i)$ (如图 8.33 阴影部分所示) 的面积, $\bar{P}(\bar{x}_i, \bar{y}_i) \in S(v'_{i-1}, v'_i)$.

令 $v_i^* = f(\bar{x}_i, \bar{y}_i)$, 则 $v'_{i-1} \leq v_i^* \leq v'_i$. 又显然 (利用微分中值定理) 有

$$\begin{aligned} \Delta S_i &= F(v'_i) - F(v'_{i-1}) = F'(\bar{v}_i)(v'_i - v'_{i-1}) \\ &= F'(\bar{v}_i) \Delta v_i (i = 1, 2, \dots, n), \end{aligned}$$

其中 $v'_{i-1} \leq \bar{v}_i \leq v'_i$. 这里我们假定了 $F'(v)$ 在 $[v_1, v_2]$ 上存在且可积, 于是它有界, 即

$$|F'(v)| \leq M = \text{常数} (v_1 \leq v \leq v_2). \quad (1)$$

我们有

$$\iint_{S(v_1, v_2)} f(x, y) dx dy = \sum_{i=1}^n v_i^* F'(\bar{v}_i) \Delta v_i = I_1 + I_2, \quad (2)$$

其中

$$I_1 = \sum_{i=1}^n \bar{v}_i F'(\bar{v}_i) \Delta v_i, I_2 = \sum_{i=1}^n (v_i^* - \bar{v}_i) F'(\bar{v}_i) \Delta v_i.$$

由于 $F'(v)$ 在 $[v_1, v_2]$ 上可积, 故 $vF'(v)$ 也在 $[v_1, v_2]$ 上可积. 因此,

$$\begin{aligned}\lim_{d(T) \rightarrow 0} I_1 &= \lim_{d(T) \rightarrow 0} \sum_{i=1}^n \bar{v}_i F'(\bar{v}_i) \Delta v_i \\ &= \int_{v_1}^{v_2} v F'(v) dv.\end{aligned}\quad (3)$$

另一方面, 由(1) 式知

$$|I_2| \leq M d(T) \sum_{i=1}^n \Delta v_i = M(v_2 - v_1) d(T),$$

故

$$\lim_{d(T) \rightarrow 0} I_2 = 0. \quad (4)$$

现在(2) 式两端令 $d(T) \rightarrow 0$ 取极限(注意, (2) 式左端是常数), 并注意到(3) 式与(4) 式, 即得

$$\iint_{S(v_1, v_2)} f(x, y) dx dy = \int_{v_1}^{v_2} v F'(v) dv.$$

证毕.

应当指出, 正如上面所说的, 本题应假定 $f(x, y)$ 在 $S(v_1, v_2)$ 上连续, 而 $F'(v)$ 在 $[v_1, v_2]$ 上存在并且可积.

§ 2. 面积的计算法

Oxy 平面上域 S 的面积由公式

$$S = \iint_S dx dy$$

所给出.

求下列曲线所界的面积:

3984. $xy = a^2, x + y$

$$= \frac{5a}{2} (a > 0).$$

解 两曲线的交点

为 $A(\frac{a}{2}, 2a)$ 和

$B(2a, \frac{a}{2})$ (图 8.

34), 故所求面积为

$$S = \int_{\frac{a}{2}}^{2a} dx \int_{\frac{a}{x}}^{\frac{5a}{2} - x} dy =$$

$$\frac{15}{8}a^2 - 2a^2 \ln 2.$$

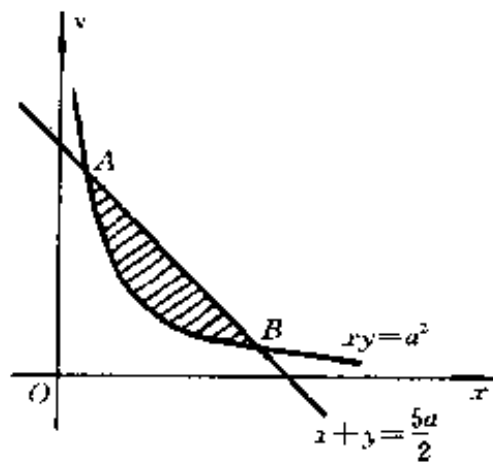


图 8.34

3985. $y^2 = 2px + p^2, y^2 = -2qx + q^2 (p > 0, q > 0).$

解 两曲线的交点为

$A(\frac{q-p}{2}, \sqrt{pq})$ 和

$B(\frac{q-p}{2}, -\sqrt{pq})$ (图

8.35), 故所求面积为

$$S = 2 \int_0^{\sqrt{pq}} dy \int_{\frac{y^2 - p^2}{2p}}^{\frac{q^2 - y^2}{2q}} dx$$

$$= \frac{2}{3}(p+q)\sqrt{pq}.$$

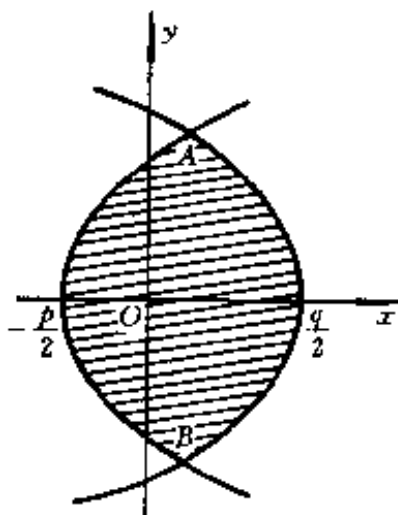


图 8.35

3986. $(x - y)^2 + x^2 = a^2$

$(a > 0).$

解 如图 8.36 所示.

所求面积的域为:

$$-a \leq x \leq a,$$

$x - \sqrt{a^2 - x^2} \leq y \leq x + \sqrt{a^2 - x^2}$. 于是,
所求的面积为

$$\begin{aligned} S &= \int_{-a}^a dx \int_x^{x+\sqrt{a^2-x^2}} \sqrt{a^2-x^2} dy \\ &= 4 \int_0^a \sqrt{a^2-x^2} dx \\ &= 4 \int_0^{\frac{\pi}{2}} a^2 \cos^2 t dt \\ &= 4 \cdot \frac{\pi a^2}{4} = \pi a^2. \end{aligned}$$

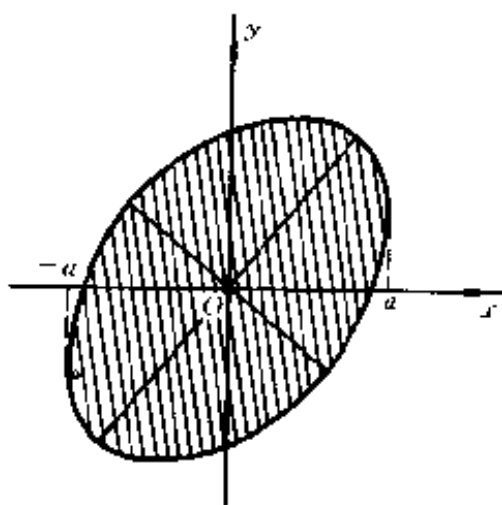


图 8.36

变换为极坐标,以计算由下列曲线所界的面积:

3987. $(x^2 + y^2)^2 = 2a^2(x^2 - y^2); x^2 + y^2 \geq a^2$.

解 曲线的极坐标方程为

$$r^2 = 2a^2 \cos 2\varphi \text{ 及 } r \geq a.$$

它们的交点(在第一象限内)为 $(a, \frac{\pi}{6})$, 如图 8.37 所示. 利用对称性,得所求面积为

$$\begin{aligned} S &= 4 \int_0^{\frac{\pi}{6}} d\varphi \int_a^{\sqrt{2a^2 \cos 2\varphi}} r dr \\ &= 2 \int_0^{\frac{\pi}{6}} (2a^2 \cos 2\varphi - a^2) d\varphi \\ &= \frac{3\sqrt{3}}{3} a^2. \end{aligned}$$

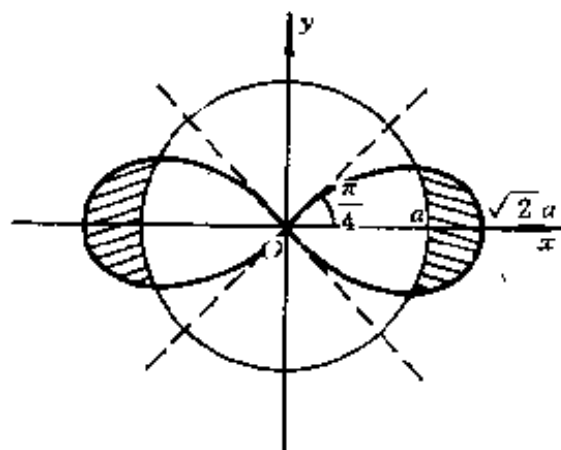


图 8.37

3988. $(x^3 + y^3)^2 = x^2 + y^2; x \geq 0, y \geq 0.$

解 将方程化为极坐标方程,得

$$(r^3 \cos^3 \theta + r^3 \sin^3 \theta)^2 = r^2,$$

即

$$r^2 = \frac{1}{\cos^3 \theta + \sin^3 \theta} (0 \leq \theta \leq \frac{\pi}{2}).$$

曲线所界的面积为

$$S = \int_0^{\frac{\pi}{2}} r dr d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos^3 \theta + \sin^3 \theta}.$$

由于

$$\frac{1}{\cos^3 \theta + \sin^3 \theta} = \frac{1}{3} \left(\frac{2}{\sin \theta + \cos \theta} + \frac{\sin \theta + \cos \theta}{1 - \sin \theta \cos \theta} \right),$$

又

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta + \cos \theta} &= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \left(\theta + \frac{\pi}{4} \right)} \\ &= \frac{1}{\sqrt{2}} \operatorname{Intg} \frac{\theta + \frac{\pi}{4}}{2} \Big|_0^{\frac{\pi}{2}} \\ &= \frac{1}{\sqrt{2}} \left(\operatorname{Intg} \frac{3\pi}{8} - \operatorname{Intg} \frac{\pi}{8} \right) \\ &= \\ &= \frac{1}{\sqrt{2}} \left[\ln \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}} - \ln \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} \right] \\ &= \sqrt{2} \ln(1 + \sqrt{2}), \end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin\theta + \cos\theta}{1 - \sin\theta\cos\theta} d\theta = 2 \int_0^{\frac{\pi}{2}} \frac{d\left(\frac{1}{2}\sin\theta - \frac{1}{2}\cos\theta\right)^{**}}{2\left(\frac{1}{2}\sin\theta - \frac{1}{2}\cos\theta\right)^2 + \frac{1}{2}}$$

$$= 2\operatorname{arctg}(\sin\theta - \cos\theta) \Big|_0^{\frac{\pi}{2}} = \pi.$$

于是,所求的面积为

$$S = \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta + \cos\theta} + \frac{1}{6} \int_0^{\frac{\pi}{2}} \frac{\sin\theta + \cos\theta}{1 - \sin\theta\cos\theta} d\theta$$

$$= \frac{\sqrt{2}}{3} \ln(1 + \sqrt{2}) + \frac{\pi}{6}.$$

*) 利用 2053 题的结果,其中

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2}, A = 2, B = 0.$$

3989. $(x^2 + y^2)^2 = a(x^3 - 3xy^2) (a > 0).$

解 显然曲线关于 Ox 轴对称,故只要求出 $y \geq 0$ 的部分. 化为极坐标,方程为

$$r = a\cos\theta(4\cos^2\theta - 3).$$

由于必须 $x^3 - 3xy^2 \geq 0$, 故 $\cos\theta(4\cos^2\theta - 3) \geq 0$. 因此,
 $\cos\theta \geq 0$ 且 $\cos\theta \geq \frac{\sqrt{3}}{2}$ 或 $\cos\theta \leq 0$ 且 $\cos\theta \geq -\frac{\sqrt{3}}{2}$,

故 $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}, \frac{\pi}{2} \leq \theta \leq \pi - \frac{\pi}{6}, -\pi + \frac{\pi}{6} \leq \theta \leq -\frac{\pi}{2}.$

于是,在 Ox 轴的上方部分 ($y \geq 0$) 为

$$0 \leq \theta \leq \frac{\pi}{6} \text{ 和 } \frac{\pi}{2} \leq \theta \leq \pi - \frac{\pi}{6}.$$

由此可知

$$S = \iint_S r dr d\theta = 2 \left[\frac{1}{2} \int_0^{\frac{\pi}{6}} r^2 d\theta + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi - \frac{\pi}{6}} r^2 d\theta \right]$$

$$= \int_0^{\frac{\pi}{6}} a^2 \cos^2 \theta (4 \cos^2 \theta - 3)^2 d\theta \\ + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} a^2 \cos^2 \theta (4 \cos^2 \theta - 3)^2 d\theta.$$

在上式右端第二个积分中作代换 $\theta = \pi - \varphi$, 则

$$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} a^2 \cos^2 \theta (4 \cos^2 \theta - 3)^2 d\theta \\ = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} a^2 \cos^2 \theta (4 \cos^2 \theta - 3)^2 d\theta,$$

故

$$S = \int_0^{\frac{\pi}{2}} a^2 \cos^2 \theta (4 \cos^2 \theta - 3)^2 d\theta \\ = a^2 \int_0^{\frac{\pi}{2}} (16 \cos^6 \theta - 24 \cos^4 \theta + 9 \cos^2 \theta) d\theta \\ = a^2 \left(16 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2} - 24 \cdot \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2} \right. \\ \left. + 9 \cdot \frac{1}{2} \frac{\pi}{2} \right) \\ = \frac{\pi a^2}{4}.$$

3990. $(x^2 + y^2)^2 = 8a^2 xy; (x-a)^2 + (y-a)^2 \leq a^2 (a > 0).$

解 将方程化为极坐标方程, 得(双纽线)

$$r^4 = 8a^2 r^2 \cos \theta \sin \theta,$$

即

$$r = 2a \sqrt{\sin 2\theta};$$

与圆周

$$(r \cos \theta - a)^2 + (r \sin \theta - a)^2 = a^2,$$

即

$$r = a(\cos\theta + \sin\theta) \pm a \sqrt{\sin 2\theta}.$$

显然, 两条曲线关于射线 $\theta = \frac{\pi}{4}$ 是对称的. 令

$$2a \sqrt{\sin 2\theta} = a(\cos\theta + \sin\theta) - a \sqrt{\sin 2\theta},$$

解得交点的极角

$$\theta = \frac{1}{2} \arcsin \frac{1}{8}.$$

于是, 所求的面积为

$$\begin{aligned} S &= \iint_S r dr d\theta \\ &= \int_{\frac{1}{2} \arcsin \frac{1}{8}}^{\frac{\pi}{4}} \{ (2a \sqrt{\sin 2\theta})^2 - [a(\cos\theta + \sin\theta) \\ &\quad - a \sqrt{\sin 2\theta}]^2 \} d\theta \\ &= \int_{\frac{1}{2} \arcsin \frac{1}{8}}^{\frac{\pi}{4}} (2a^2 \sin 2\theta + 2a^2 (\sin\theta + \cos\theta) \sqrt{\sin 2\theta} \\ &\quad - a^2) d\theta. \end{aligned}$$

注意到

$$\begin{aligned} &\int (\sin\theta + \cos\theta) \sqrt{\sin 2\theta} d\theta \\ &= \frac{1}{2} (\sin\theta - \cos\theta) \sqrt{\sin 2\theta} \\ &\quad + \frac{1}{2} \arcsin(\sin\theta - \cos\theta) + C'', \end{aligned}$$

即得

$$\begin{aligned} S &= a^2 \left[-\cos 2\theta + (\sin\theta - \cos\theta) \sqrt{\sin 2\theta} \right. \\ &\quad \left. + \arcsin(\sin\theta - \cos\theta) - \theta \right] \Big|_{\frac{1}{2} \arcsin \frac{1}{8}}^{\frac{\pi}{4}} \\ &= a^2 \left\{ -\frac{\pi}{4} + \frac{3\sqrt{7}}{8} + \frac{\sqrt{14}}{4} \sqrt{\frac{1}{8}} \right\} \end{aligned}$$

$$\begin{aligned}
& + \arcsin \frac{\sqrt{14}}{4} + \frac{1}{2} \arcsin \frac{1}{8} \Big) \\
& = a^2 \left(\frac{\sqrt{7}}{2} + \arcsin \frac{\sqrt{14}}{4} - \frac{1}{2} \arccos \frac{1}{8} \right) \\
& = a^2 \left(\frac{\sqrt{7}}{2} + \arcsin \frac{\sqrt{14}}{8} \right)^{**}.
\end{aligned}$$

*) 利用三角恒等式

$$\begin{aligned}
\sqrt{\sin 2x} \sin x &= \frac{1}{\sqrt{2}} \left(\frac{2 \operatorname{tg} x}{1 + \operatorname{tg}^2 x} \right) \sqrt{\operatorname{tg} x}, \\
\sqrt{\sin 2x} \cos x &= \frac{1}{\sqrt{2}} \left(\frac{2 \operatorname{tg} x}{1 + \operatorname{tg}^2 x} \right) \sqrt{\operatorname{ctg} x}
\end{aligned}$$

化为二项型微分的积分, 参看 А. Ф. Тимофеев
《ИНТЕГРИРОВАНИЕ ФУНКЦИЙ》第五章 § 15.

**) 容易证明:

$$\arcsin \frac{\sqrt{14}}{4} - \frac{1}{2} \arccos \frac{1}{8} = \arcsin \frac{\sqrt{14}}{8}.$$

事实上, 我们有

$$\begin{aligned}
& \sin \left(\arcsin \frac{\sqrt{14}}{8} + \frac{1}{2} \arccos \frac{1}{8} \right) \\
&= \frac{3 \sqrt{14}}{32} + \frac{5 \sqrt{14}}{32} = \frac{\sqrt{14}}{4}.
\end{aligned}$$

根据公式

$$x = a \cos^a \varphi, y = b r \sin^a \varphi (r \geq 0)$$

引入普遍的极坐标(其中 a, b 和 α 为以适当的方法选出的常数, 且 $\frac{D(x, y)}{D(r, \varphi)} = a a b r \cos^{a-1} \varphi \sin^{a-1} \varphi$), 以求由下列曲线所界的面积(假定参数是正的):

$$3991. \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{h} + \frac{y}{k}.$$

解 不失一般性, 设 $k > 0, h > 0$. 令

$$x = a \cos \varphi, y = b r \sin \varphi,$$

则方程化为

$$r = \frac{a}{h} \cos \varphi + \frac{b}{k} \sin \varphi.$$

由于 $r \geq 0$, 故有

$$\frac{a}{h} \cos \varphi + \frac{b}{k} \sin \varphi \geq 0,$$

因此, 首先必须 $-\frac{\pi}{2} \leq \varphi \leq \pi$. 同时, 应有 $\cos \varphi \geq 0$ 且

$$\operatorname{tg} \varphi \geq -\frac{ak}{bh} \text{ 或者 } \cos \varphi < 0 \text{ 且 } \operatorname{tg} \varphi \leq -\frac{ak}{bh}.$$

从而, 极角 φ 应满足不等式

$$-\arctg \frac{ak}{bh} \leq \varphi \leq \pi - \arctg \frac{ak}{bh}.$$

于是, 曲线所界的面积为

$$\begin{aligned} S &= \int_S ab r dr d\varphi \\ &= \frac{ab}{2} \int_{-\arctg \frac{ak}{bh}}^{\pi - \arctg \frac{ak}{bh}} \left(\frac{a}{h} \cos \varphi + \frac{b}{k} \sin \varphi \right)^2 d\varphi \\ &= \frac{ab}{2} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right) \int_{-\arctg \frac{ak}{bh}}^{\pi - \arctg \frac{ak}{bh}} \sin^2(\varphi + \alpha_0) d\varphi, \end{aligned}$$

其中 $\alpha_0 = \arctg \frac{ak}{bh}$. 从而, 我们有

$$\begin{aligned} S &= \frac{ab}{2} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right) \left[\frac{\varphi + \alpha_0}{2} - \frac{1}{4} \sin 2(\varphi + \alpha_0) \right] \Big|_{-\arctg \frac{ak}{bh}}^{\pi - \arctg \frac{ak}{bh}} \\ &= \frac{ab}{2} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right) \frac{\pi}{2} = \frac{\pi ab}{4} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right). \end{aligned}$$

$$3992. \frac{x^3}{a^3} + \frac{y^3}{b^3} = \frac{x^2}{h^2} + \frac{y^2}{k^2}; x = 0 \text{ } y = 0.$$

解 令

$$x = a \cos \varphi, y = b r \sin \varphi,$$

则方程化为

$$r = \frac{\left(\frac{a}{h}\right)^2 \cos^2 \varphi + \left(\frac{b}{k}\right)^2 \sin^2 \varphi}{\cos^3 \varphi + \sin^3 \varphi}.$$

于是,曲线所界的面积为

$$\begin{aligned} S &= \iint_S a b r d r d \theta = \frac{a b}{2} \int_0^{\frac{\pi}{2}} r^2 d \varphi \\ &= \frac{a b}{2} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{a}{h}\right)^4 \cos^4 \varphi + \left(\frac{b}{k}\right)^4 \sin^4 \varphi + 2 \left(\frac{a}{h}\right)^2 \left(\frac{b}{k}\right)^2 \cos^2 \varphi \sin^2 \varphi}{(\cos^3 \varphi + \sin^3 \varphi)^2} d \varphi. \end{aligned}$$

根据 И. М. 雷日克、И. С. 格拉德什坦编著的《函数表与积分表》2.125、2.126 知:

$$\begin{aligned} \int \frac{\cos^4 \varphi d \varphi}{(\cos^3 \varphi + \sin^3 \varphi)^2} &= \int \frac{1}{(1 + \operatorname{tg}^3 \varphi)} d(\operatorname{tg} \varphi) \\ &= \frac{\operatorname{tg} \varphi}{3(1 + \operatorname{tg}^3 \varphi)} + \frac{2}{9} \left\{ \frac{1}{2} \ln \frac{(\operatorname{tg} \varphi + 1)^2}{\operatorname{tg}^2 \varphi - \operatorname{tg} \varphi + 1} \right. \\ &\quad \left. + \sqrt{3} \operatorname{arctg} \frac{2 \operatorname{tg} \varphi - 1}{\sqrt{3}} \right\} + C. \end{aligned}$$

从而

$$\begin{aligned} &\frac{a b}{2} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{a}{h}\right)^4 \cos^4 \varphi}{(\cos^3 \varphi + \sin^3 \varphi)^2} d \varphi \\ &= \frac{a b}{2} \left(\frac{a}{h}\right)^4 \left\{ \frac{\operatorname{tg} \varphi}{3(1 + \operatorname{tg}^3 \varphi)} + \frac{2}{9} \left\{ \frac{1}{2} \ln \frac{(\operatorname{tg} \varphi + 1)^2}{\operatorname{tg}^2 \varphi - \operatorname{tg} \varphi + 1} \right. \right. \\ &\quad \left. \left. + \sqrt{3} \operatorname{arctg} \frac{2 \operatorname{tg} \varphi - 1}{\sqrt{3}} \right\} \right\} \Big|_0^{\frac{\pi}{2}-0} \end{aligned}$$

$$= \frac{ab}{2} \left(\frac{a}{h} \right)^4 \cdot \frac{2\sqrt{3}}{9} \left(\frac{\pi}{2} + \frac{\pi}{6} \right) = \frac{2\pi ab}{9\sqrt{3}} \left(\frac{a}{h} \right)^4;$$

又

$$\begin{aligned} \int \frac{\sin^4 \varphi d\varphi}{(\cos^3 \varphi + \sin^3 \varphi)^2} &= \int \frac{\operatorname{tg}^4 \varphi}{(1 + \operatorname{tg}^3 \varphi)^2} d(\operatorname{tg} \varphi) \\ &= \frac{\operatorname{tg}^5 \varphi}{3(1 + \operatorname{tg}^3 \varphi)} - \frac{2}{3} \int \frac{\operatorname{tg}^4 \varphi}{1 + \operatorname{tg}^3 \varphi} d(\operatorname{tg} \varphi) \\ &= \frac{\operatorname{tg}^5 \varphi}{3(1 + \operatorname{tg}^3 \varphi)} - \frac{2}{3} \left\{ \frac{\operatorname{tg}^2 \varphi}{2} + \frac{1}{3} \left[\frac{1}{2} \ln \frac{(\operatorname{tg} \varphi + 1)^2}{\operatorname{tg}^2 \varphi - \operatorname{tg} \varphi + 1} \right. \right. \\ &\quad \left. \left. - \sqrt{3} \operatorname{arctg} \frac{2\operatorname{tg} \varphi - 1}{\sqrt{3}} \right] \right\} + C, \end{aligned}$$

从而

$$\begin{aligned} \frac{ab}{2} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{b}{k} \right)^4 \sin^4 \varphi}{(\cos^3 \varphi + \sin^3 \varphi)^2} d\varphi \\ &= \frac{ab}{2} \left(\frac{b}{k} \right)^4 \left\{ \frac{\operatorname{tg}^5 \varphi}{3(1 + \operatorname{tg}^3 \varphi)} - \frac{\operatorname{tg}^2 \varphi}{3} \right. \\ &\quad \left. - \frac{2}{9} \left[\frac{1}{2} \ln \frac{(\operatorname{tg} \varphi + 1)^2}{\operatorname{tg}^2 \varphi - \operatorname{tg} \varphi + 1} \right. \right. \\ &\quad \left. \left. - \sqrt{3} \operatorname{arctg} \frac{2\operatorname{tg} \varphi - 1}{\sqrt{3}} \right] \right\} \Big|_0^{\frac{\pi}{2}-0} \\ &= \frac{ab}{2} \left(\frac{b}{k} \right)^4 \cdot \frac{2\sqrt{3}}{9} \left(\frac{\pi}{2} + \frac{\pi}{6} \right) = \frac{2\pi ab}{9\sqrt{3}} \left(\frac{b}{k} \right)^4; \end{aligned}$$

此外, 还有

$$\begin{aligned} \int \frac{\cos^2 \varphi \sin^2 \varphi d\varphi}{(\cos^3 \varphi + \sin^3 \varphi)^2} &= \int \frac{\operatorname{tg}^2 \varphi}{(1 + \operatorname{tg}^3 \varphi)^2} d(\operatorname{tg} \varphi) \\ &= -\frac{1}{3(1 + \operatorname{tg}^3 \varphi)} + C, \end{aligned}$$

从而

$$\begin{aligned}
& \frac{ab}{2} \int_0^{\frac{\pi}{2}} \frac{2 \left(\frac{a}{h} \right)^2 \left(\frac{b}{k} \right)^2 \cos^2 \varphi \sin^2 \varphi}{(\cos^3 \varphi + \sin^3 \varphi)^2} d\varphi \\
&= ab \left(\frac{a}{h} \right)^2 \left(\frac{b}{k} \right)^2 \left[-\frac{1}{3(1 + \operatorname{tg}^3 \varphi)} \right] \Big|_0^{\frac{\pi}{2}} \\
&= \frac{ab}{3} \left(\frac{a}{h} \right)^2 \left(\frac{b}{k} \right)^2.
\end{aligned}$$

于是, 曲线所界的面积为

$$\begin{aligned}
S &= \frac{2\pi ab}{9\sqrt{3}} \left(\frac{a}{h} \right)^4 + \frac{2\pi ab}{9\sqrt{3}} \left(\frac{b}{k} \right)^4 + \frac{ab}{3} \left(\frac{a}{h} \right)^2 \left(\frac{b}{k} \right)^2 \\
&= \frac{ab}{3} \left[\frac{2\pi}{9\sqrt{3}} \left(\frac{a^4}{h^4} + \frac{b^4}{k^4} \right) + \frac{a^2 b^2}{h^2 k^2} \right].
\end{aligned}$$

3993. $\left(\frac{x}{a} + \frac{y}{b} \right)^4 = \frac{x^2}{h^2} + \frac{y^2}{k^2} (x > 0, y > 0).$

解 方法一

令

$$x = a \cos \varphi, y = b \sin \varphi,$$

则方程化为

$$r^2 = \frac{\left(\frac{a}{h} \right)^2 \cos^2 \varphi + \left(\frac{b}{k} \right)^2 \sin^2 \varphi}{(\cos \varphi + \sin \varphi)^4} \quad (0 \leq \varphi \leq \frac{\pi}{2}).$$

于是, 曲线所界的面积为

$$S = \iint_S ab r dr d\varphi = \frac{ab}{2} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{a}{h} \right)^2 \cos^2 \varphi + \left(\frac{b}{k} \right)^2 \sin^2 \varphi}{(\cos \varphi + \sin \varphi)^4} d\varphi.$$

注意到

$$\begin{aligned}
\int \frac{\cos^2 \varphi}{(\cos \varphi + \sin \varphi)^4} d\varphi &= \int \frac{1}{(1 + \operatorname{tg} \varphi)^4} d(\operatorname{tg} \varphi) \\
&= -\frac{1}{3(1 + \operatorname{tg} \varphi)^3} + C.
\end{aligned}$$

$$\begin{aligned}
\int \frac{\sin^2 \varphi}{(\cos \varphi + \sin \varphi)^4} d\varphi &= \int \frac{\operatorname{tg}^2 \varphi}{(1 + \operatorname{tg} \varphi)^4} d(\operatorname{tg} \varphi) \\
&= \int \frac{(\operatorname{tg} \varphi - 1)(\operatorname{tg} \varphi + 1) + 1}{(1 + \operatorname{tg} \varphi)^4} d(\operatorname{tg} \varphi) \\
&= \int \frac{1}{(1 + \operatorname{tg} \varphi)^2} d(\operatorname{tg} \varphi) - 2 \int \frac{1}{(1 + \operatorname{tg} \varphi)^3} d(\operatorname{tg} \varphi) \\
&\quad + \int \frac{1}{(1 + \operatorname{tg} \varphi)^4} d(\operatorname{tg} \varphi) \\
&= -\frac{1}{1 + \operatorname{tg} \varphi} + \frac{1}{(1 + \operatorname{tg} \varphi)^2} - \frac{1}{3} \frac{1}{(1 + \operatorname{tg} \varphi)^3} + C,
\end{aligned}$$

于是, 所求的面积为

$$\begin{aligned}
S &= \frac{ab}{2} \left(\frac{a}{h} \right)^2 \left[-\frac{1}{3(1 + \operatorname{tg} \varphi)^3} \right] \Big|_0^{\frac{\pi}{2}-0} \\
&\quad + \frac{ab}{2} \left(\frac{b}{k} \right)^2 \left[-\frac{1}{1 + \operatorname{tg} \varphi} + \frac{1}{(1 + \operatorname{tg} \varphi)^2} \right. \\
&\quad \left. - \frac{1}{3(1 + \operatorname{tg} \varphi)^3} \right] \Big|_c^{\frac{\pi}{2}-0} \\
&= \frac{ab}{6} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right).
\end{aligned}$$

方法二

令

$$x = hr \cos \varphi, \quad y = kr \sin \varphi,$$

则方程化为

$$\begin{aligned}
r^2 &= \frac{1}{\left(\frac{h}{a} \cos \varphi + \frac{k}{b} \sin \varphi \right)^2} \\
&= \left[\frac{a^2 b^2}{(hb)^2 + (ka)^2} \right]^2 \frac{1}{\sin^4(\varphi + \alpha)} \quad (0 \leq \varphi \leq \frac{\pi}{2}),
\end{aligned}$$

其中 $\operatorname{tg} \alpha = \frac{hb}{ka}$. 于是, 曲线所界的面积为

$$\begin{aligned}
S &= \iint_S hkrdrd\varphi = \frac{hka^4b^4}{[(hb)^2 + (ka)^2]^2} \\
&\quad \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sin^4(\varphi + \alpha)} \\
&= \frac{hka^4b^4}{[(hb)^2 + (ka)^2]^2} \left[-\frac{1}{6} \frac{\cos(\varphi + \alpha)}{\sin^3(\varphi + \alpha)} \right. \\
&\quad \left. - \frac{1}{3} \frac{\cos(\varphi + \alpha)}{\sin(\varphi + \alpha)} \right] \Big|_0^{\frac{\pi}{2}} \\
&= \frac{hka^4b^4}{[(hb)^2 + (ka)^2]^2} \left[\frac{1}{6} \left(\frac{\sin\alpha}{\cos^3\alpha} + \frac{\cos\alpha}{\sin^3\alpha} \right) \right. \\
&\quad \left. + \frac{1}{3} (\operatorname{tg}\alpha + \operatorname{ctg}\alpha) \right] \\
&= \frac{hka^4b^4}{[(hb)^2 + (ka)^2]^2} \cdot \frac{1}{6} \frac{[(hb)^2 + (ka)^2]^{3**})}{(hbka)^3} \\
&= \frac{ab}{6} \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right).
\end{aligned}$$

*) 参看 2012 题的结果.

**) 由 $\operatorname{tg}\alpha = \frac{hb}{ka}$ 知:

$$\operatorname{ctg}\alpha = \frac{ka}{hb}, \quad \sin\alpha = \frac{hb}{\sqrt{(hb)^2 + (ka)^2}},$$

$$\cos\alpha = \frac{ka}{\sqrt{(hb)^2 + (ka)^2}}.$$

$$3994. \left(\frac{x}{a} + \frac{y}{b} \right)^4 = \frac{x^2}{h^2} - \frac{y^2}{k^2} (x > 0, y > 0).$$

解 令

$$x = \operatorname{arccos}\varphi, \quad y = br\sin\varphi,$$

则方程化为

$$r^2 = \frac{\left(\frac{a}{h} \right)^2 \cos^2\varphi - \left(\frac{b}{k} \right)^2 \sin^2\varphi}{(\cos\varphi + \sin\varphi)^4}.$$

由于

$$\begin{aligned}\left(\frac{a}{h}\right)^2 \cos^2 \varphi - \left(\frac{b}{k}\right)^2 \sin^2 \varphi &\geq 0, \\ \left(\frac{ak}{bh}\right)^2 &\geq \operatorname{tg}^2 \varphi,\end{aligned}$$

注意到 $0 \leq \varphi \leq \frac{\pi}{2}$, 可知极角的变化区间为

$$0 \leq \varphi \leq \operatorname{arctg} \frac{ak}{bh}.$$

于是, 注意利用上题中两个不定积分, 便得到曲线所界的面积为

$$\begin{aligned}S &= \iint_S ab r dr d\varphi = \frac{ab}{2} \int_0^{\operatorname{arctg} \frac{ak}{bh}} r^2 d\varphi \\ &= \frac{ab}{2} \left(\frac{a}{h}\right)^2 \int_0^{\operatorname{arctg} \frac{ak}{bh}} \frac{\cos^2 \varphi}{(\cos \varphi + \sin \varphi)^4} d\varphi \\ &\quad - \frac{ab}{2} \left(\frac{b}{k}\right)^2 \int_0^{\operatorname{arctg} \frac{ak}{bh}} \frac{\sin^2 \varphi}{(\cos \varphi + \sin \varphi)^4} d\varphi \\ &= \frac{ab}{2} \left(\frac{a}{h}\right)^2 \left[-\frac{1}{3(1 + \operatorname{tg} \varphi)^3} \right] \Big|_0^{\operatorname{arctg} \frac{ak}{bh}} \\ &\quad - \frac{ab}{2} \left(\frac{b}{k}\right)^2 \left[-\frac{3\operatorname{tg}^2 \varphi + 3\operatorname{tg} \varphi + 1}{(1 + \operatorname{tg} \varphi)^3} \right] \Big|_0^{\operatorname{arctg} \frac{ak}{bh}} \\ &= \frac{ab}{6} \left(\frac{a}{h}\right)^2 \left[\frac{-1}{\left(1 + \frac{ak}{bh}\right)^3} + 1 \right] \\ &\quad + \frac{ab}{6} \left(\frac{b}{k}\right)^2 \left[\frac{3\left(\frac{ak}{bh}\right)^2 + 3\left(\frac{ak}{bh}\right) + 1}{\left(1 + \frac{ak}{bh}\right)^3} - 1 \right] \\ &= \frac{ab}{6} \left(\frac{a}{h}\right)^2 \frac{(ak)^3 + 3(ak)^2 bh + 3ak(bh)^2}{(ak + bh)^3}\end{aligned}$$

$$\begin{aligned}
& + \frac{ab}{5} \left(\frac{b}{k} \right)^2 \frac{-(ak)^3}{(ak+bh)^3} \\
& = \frac{a^4bk}{6h^2(ak+bh)^3} (a^2k^2 + 3akbh + 2b^2h^2) \\
& = \frac{a^4bk(ak+2bh)}{6h^2(ak+bh)^2}.
\end{aligned}$$

3995. $\sqrt[4]{\frac{x}{a}} + \sqrt[4]{\frac{y}{b}} = 1; x=0, y=0.$

解 令

$$x = ar \cos^8 \varphi, y = br \sin^8 \varphi,$$

则方程化为

$$r = 1 (0 \leq \varphi \leq \frac{\pi}{2}).$$

于是, 曲线所界的面积为

$$\begin{aligned}
S &= \iint_S 8abr \cos^7 \varphi \sin^7 \varphi d\varphi \\
&= 4ab \int_0^{\frac{\pi}{2}} \cos^7 \varphi \sin^7 \varphi d\varphi \\
&= 4ab \int_0^1 u^7 (1-u^2)^3 du \\
&= 4ab \int_0^1 (u^7 - 3u^9 + 3u^{11} - u^{13}) du \\
&= 4ab \left(\frac{1}{8} - \frac{3}{10} + \frac{1}{4} - \frac{1}{14} \right) \\
&= \frac{ab}{70}.
\end{aligned}$$

进行适当的变量代换, 求由下列曲线所界的面积:

3996. $x+y=a, x+y=b, y=ax, y=\beta x$
 $(0 < a < b; 0 < \alpha < \beta).$

解 作变换: $x+y=u, \frac{y}{x}=\nu$, 则 $a \leq u \leq b, \alpha \leq \nu \leq$

β , 且有

$$|I| = \frac{u}{(1+v)^2}.$$

于是, 所求的面积为

$$S = \int_a^b u du \int_a^\beta \frac{dv}{(1+v)^2} = \frac{1}{2} \cdot \frac{(\beta - \alpha)(b^2 - a^2)}{(1 + \alpha)(1 + \beta)}.$$

3997. $xy = a^2, xy = 2a^2, y = x, y = 2x (x > 0; y > 0)$.

解 作变换: $xy = u, \frac{y}{x} = v$, 则 $a^2 \leq u \leq 2a^2, 1 \leq v \leq 2$, 且有

$$|I| = \frac{1}{2v}.$$

于是, 所求的面积为

$$S = \frac{1}{2} \int_{a^2}^{2a^2} du \int_1^2 \frac{dv}{v} = \frac{1}{2} a^2 \ln 2.$$

3998. $y^2 = 2px, y^2 = 2qx, x^2 = 2ry, x^2 = 2sy (0 < p < q; 0 < r < s)$.

解 作变换: $\frac{y^2}{x} = u, \frac{x^2}{y} = v$, 则 $2p \leq u \leq 2q, 2r \leq v \leq 2s$, 且有

$$|I| = \frac{1}{3}.$$

于是, 所求的面积为

$$S = \frac{1}{3} \int_{2p}^{2q} du \int_{2r}^{2s} dv = \frac{4}{3} (q - p)(s - r).$$

3999. $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1, \sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 2, \frac{x}{a} = \frac{y}{b}, \frac{4x}{a} = \frac{y}{b}$
($a > 0, b > 0$).

解 作变换: $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = u, \frac{x}{y} = v$, 即

$$x = \frac{u^2 v}{\left[\sqrt{\frac{v}{a}} + \frac{1}{\sqrt{b}} \right]^2}, \quad y = \frac{u^2}{\left[\sqrt{\frac{v}{a}} + \frac{1}{\sqrt{b}} \right]^2},$$

则 $1 \leq u \leq 2$, $\frac{a}{4b} \leq v \leq \frac{a}{b}$, 且有

$$|I| = \frac{2u^3}{\left[\sqrt{\frac{v}{a}} + \frac{1}{\sqrt{b}} \right]^4}.$$

于是, 所求的面积为

$$\begin{aligned} S &= \int_1^2 2u^3 du \int_{\frac{a}{4b}}^{\frac{a}{b}} \frac{dv}{\left[\sqrt{\frac{v}{a}} + \frac{1}{\sqrt{b}} \right]^4} \\ &= \frac{15}{2} \cdot \int_{\frac{1}{2\sqrt{b}}}^{\frac{1}{\sqrt{b}}} \frac{2atdt}{\left(t + \frac{1}{\sqrt{b}} \right)^4} \\ &= 15a \int_{\frac{1}{2\sqrt{b}}}^{\frac{1}{\sqrt{b}}} \left[\frac{1}{\left(t + \frac{1}{\sqrt{b}} \right)^3} - \frac{1}{\sqrt{b}} \cdot \frac{1}{\left[\sqrt{\frac{1}{b}} + t \right]^4} \right] dt \\ &= 15a \cdot \left(\frac{7b}{72} - \frac{37b}{648} \right) = \frac{65ab}{108}. \end{aligned}$$

*) 作代换 $v = at^2$.

4000. $\frac{x^2}{\lambda} + \frac{y^2}{\lambda - c^2} = 1$, 其中 λ 取下列各值: $\frac{1}{3}c^2, \frac{2}{3}c^2, \frac{4}{3}c^2, \frac{5}{3}c^2 (x > 0, y > 0)$.

解 方程 $\frac{x^2}{\lambda} + \frac{y^2}{\lambda - c^2} = 1$ 可变为

$$\lambda^2 - (x^2 + y^2 + c^2)\lambda + c^2x^2 = 0.$$

将 λ 作为未知量解方程, 不妨记方程的两个解为 λ 及 μ , 则

$$\lambda = \frac{x^2 + y^2 + c^2 + \sqrt{(x^2 + y^2 + c^2)^2 - 4c^2x^2}}{2},$$

$$\mu = \frac{x^2 + y^2 + c^2 - \sqrt{(x^2 + y^2 + c^2)^2 - 4c^2x^2}}{2},$$

今设按上式作变量代换, 将 (x, y) 变为 (λ, μ) . 易知

$$\begin{aligned} \left| \frac{D(\lambda, \mu)}{D(x, y)} \right| &= \frac{4c^2xy}{\sqrt{(x^2 + y^2 + c^2)^2 - 4c^2x^2}} \\ &= \frac{4\sqrt{\lambda\mu(c^2 - \mu)(\lambda - c^2)}}{\lambda - \mu}, \end{aligned}$$

从而

$$\begin{aligned} \frac{D(x, y)}{D(\lambda, \mu)} &= \frac{1}{\frac{D(\lambda, \mu)}{D(x, y)}} \\ &= \frac{\lambda - \mu}{4\sqrt{\lambda\mu(c^2 - \mu)(\lambda - c^2)}}. \end{aligned}$$

于是, 所求的面积为

$$\begin{aligned} S &= \iint_D dx dy = \iint \frac{\lambda - \mu}{4\sqrt{\lambda\mu(c^2 - \mu)(\lambda - c^2)}} d\lambda d\mu \\ &\quad \frac{4c^2}{3} \leq \lambda \leq \frac{5c^2}{3} \\ &\quad \frac{c^2}{3} \leq \mu \leq \frac{2c^2}{3} \\ &= \frac{c^2}{4} \int_{\frac{4}{3}}^{\frac{5}{3}} \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{u - v}{\sqrt{uv(1 - v)(u - 1)}} du dv \\ &= \frac{c^2}{4} \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{\sqrt{u} du}{\sqrt{u - 1}} \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{dv}{\sqrt{v(1 - v)}} \end{aligned}$$

$$- \frac{c^2}{4} \int_{\frac{4}{3}}^{\frac{5}{3}} \frac{du}{\sqrt{u(u-1)}} \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{\sqrt{v} dv}{\sqrt{1-v}}.$$

由于

$$\int_{\frac{4}{3}}^{\frac{5}{3}} \frac{\sqrt{u}}{\sqrt{u-1}} du = \frac{\sqrt{10}}{3} - \frac{2}{3} + \lg \frac{\sqrt{5} - \sqrt{2}}{2 - \sqrt{2}},$$

$$\int_{\frac{4}{3}}^{\frac{5}{3}} \frac{du}{\sqrt{u(u-1)}} = 2 \lg \frac{\sqrt{5} - \sqrt{2}}{2 - \sqrt{2}},$$

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \frac{dv}{\sqrt{v(1-v)}} = 2 \arcsin \sqrt{\frac{2}{3}} - 2 \arcsin \sqrt{\frac{1}{3}},$$

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \frac{\sqrt{v}}{\sqrt{1-v}} dv = \arcsin \sqrt{\frac{2}{3}} - \arcsin \sqrt{\frac{1}{3}},$$

故最后得

$$\begin{aligned} S &= \frac{c^2}{4} \left[\left(\frac{\sqrt{10}}{3} - \frac{2}{3} + \lg \frac{\sqrt{5} - \sqrt{2}}{2 - \sqrt{2}} \right) \right. \\ &\quad \cdot \left. \left[2 \arcsin \sqrt{\frac{2}{3}} - 2 \arcsin \sqrt{\frac{1}{3}} \right] \right] \\ &\quad - \frac{c^2}{4} \left[\left(2 \lg \frac{\sqrt{5} - \sqrt{2}}{2 - \sqrt{2}} \right) \left(\arcsin \sqrt{\frac{2}{3}} \right. \right. \\ &\quad \left. \left. - \arcsin \sqrt{\frac{1}{3}} \right) \right] \\ &= \frac{c^2}{6} (\sqrt{10} - 2) \arcsin \frac{1}{3}. \end{aligned}$$

4001. 求由椭圆

$$(a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2 = 1$$

(其中 $\delta = a_1b_2 - a_2b_1 \neq 0$) 所界的面积,

解 作变换: $a_1x + b_1y + c_1 = u, a_2x + b_2y + c_2 = v,$

则椭圆所围成的域变为 $u_2 + v_2 \leq 1$, 且有

$$|I| = \frac{1}{|\delta|}.$$

于是, 所求的面积为

$$S = \frac{1}{|\delta|} \iint_{u^2+v^2 \leq 1} du dv = \frac{\pi}{|\delta|}.$$

4002. 求由椭圆

$$\frac{x^2}{\operatorname{ch}^2 u} - \frac{y^2}{\operatorname{sh}^2 u} = c^2 (u = u_1, u_2)$$

和双曲线

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = c^2 (v = v_1, v_2)$$

$$(0 < u_1 < u_2; 0 < v_1 < v_2; x > 0, y > 0)$$

所界的面积.

解 作变换: $x = c \operatorname{ch} u \cos v, y = c \operatorname{sh} u \sin v$,

则有

$$|I| = |c^2 \operatorname{ch}^2 u - c^2 \cos^2 v|.$$

因为 $\operatorname{ch}^2 u \geq 1 \geq \cos^2 v$, 故所求的面积为

$$\begin{aligned} S &= c^2 \int_{u_1}^{u_2} \int_{v_1}^{v_2} (\operatorname{ch} 2u - \cos^2 v) du dv \\ &= c^2 [(v_2 - v_1) \int_{u_1}^{u_2} \frac{1 + \operatorname{ch}^2 u}{2} du - (u_2 - u_1) \\ &\quad \cdot \int_{v_1}^{v_2} \cos^2 v dv] \\ &= \frac{c^2}{4} [(v_2 - v_1)(\operatorname{sh} 2u_2 - \operatorname{sh} 2u_1) - (u_2 - u_1) \\ &\quad \cdot (\sin 2v_2 - \sin 2v_1)]. \end{aligned}$$

4003. 求用平面 $x + y + z = b$ 与曲面 $x^2 + y^2 + z^2 - xy - xz - yz = a^2$ 相截所得截断面之面积.

解 为简化平面和曲面的方程,作变量代换:

$$x' = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}z,$$

$$y' = \frac{1}{\sqrt{6}}x - \frac{2}{\sqrt{6}}y + \frac{1}{\sqrt{6}}z,$$

$$z' = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z,$$

这是一个正交变换,故 $Ox'y'z'$ 成为一新的直角坐标系. 在新的坐标系下,平面方程为

$$z' = \frac{1}{\sqrt{3}}(x + y + z) = \frac{b}{\sqrt{3}}.$$

由于

$$x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z',$$

$$y = -\frac{\sqrt{6}}{3}y' + \frac{1}{\sqrt{3}}z',$$

$$z = -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z',$$

故有

$$\begin{aligned} x^2 + y^2 + z^2 - xy - xz - yz &= \frac{1}{2}[(x - y)^2 \\ &\quad + (y - z)^2 + (z - x)^2] \\ &= \frac{3}{2}(x'^2 + y'^2). \end{aligned}$$

从而,曲面方程变为

$$x'^2 + y'^2 = \frac{2}{3}a^2.$$

于是,所求的面积为

$$S = \iint_{x'^2 + y'^2 \leq \frac{2}{3}a^2} dx' dy' = \frac{2}{3}\pi a^2.$$

4004. 求用平面 $z = 1 - 2(x + y)$ 与曲面 $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ 相截所得截断面之面积.

解 平面被曲面所截部分记为 S , 它在 Oxy 平面上的投影记为 D . 由于平面 $z = 1 - 2(x + y)$ 的法线之方向余弦为 $\cos\alpha = \cos\beta = \frac{2}{3}$, $\cos\gamma = \frac{1}{3}$, 故 $D = S\cos\gamma = \frac{1}{3}S$, 从而 $S = 3D$, 显然 D 为 Oxy 平面上由曲线 $\frac{1}{x} + \frac{1}{y} + \frac{1}{1-2(x+y)} = 0$ (也即 $2x^2 + 2y^2 + 3xy - x - y = 0$) 所界的区域. 作变量代换

$$x = u + v + \frac{1}{7}, \quad y = u - v + \frac{1}{7}.$$

于是, $\frac{D(x,y)}{D(u,v)} = -2$, 且曲线 $2x^2 + 2y^2 + 3xy - x - y = 0$ 变为 $7u^2 + v^2 - \frac{1}{7} = 0$, 这是一个椭圆 (在 uv 平面上). 从而, 即得

$$\begin{aligned} D &= \iint_D dx dy = 2 \iint_{49u^2 + 7v^2 \leq 1} du dv \\ &= 2 \cdot \pi \left(\frac{1}{7} \right) \left(\frac{1}{\sqrt{7}} \right) = \frac{2\pi}{7\sqrt{7}}. \end{aligned}$$

由此, 最后得

$$S = 3D = \frac{6\pi}{7\sqrt{7}}.$$

§ 3. 体积的计算法

设柱体上顶是连续的曲面 $z = f(x, y)$, 下底是平面 $z = 0$, 侧面为从平面 Oxy 中的可求面积的区域 Ω (图 8.38) 竖起的垂直柱面所界定.

柱体的体积等于

$$V = \iint_{\Omega} f(x, y) dx dy,$$

4005. 试绘出一物体, 其体积等于积分

$$V = \int_0^1 dx \int_0^{1-x} (x^2 + y^2) dy.$$

解 积分域为三角形

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$$

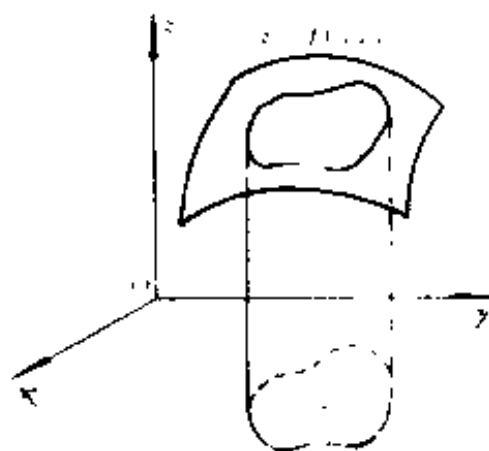


图 8.38

柱体上顶为旋转抛物面 $z = x^2 + y^2$, 物体的形状如图 8.39 所示.

4006. 描出下列二重积分所表示的体积:

$$(a) \iint_{\substack{0 \leq x+y \leq 1 \\ x \geq 0, y \geq 0}} (x+y) dx dy;$$

$$(6) \iint_{\frac{x^2}{4} + \frac{y^2}{9} \leq 1} \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} dx dy;$$

$$(B) \iint_{|x| + |y| \leq 1} (x^2 - y^2) dx dy;$$

$$(\Gamma) \iint_{x^2 + y^2 \leq 4} \sqrt{x^2 + y^2} dx dy;$$

$$(\Lambda) \iint_{\substack{1 \leq x \leq 2 \\ x \leq y \leq 2x}} \sqrt{xy} dx dy;$$

$$(e) \iint_{x^2 + y^2 \leq 1} \sin \pi \sqrt{x^2 + y^2} dx dy.$$

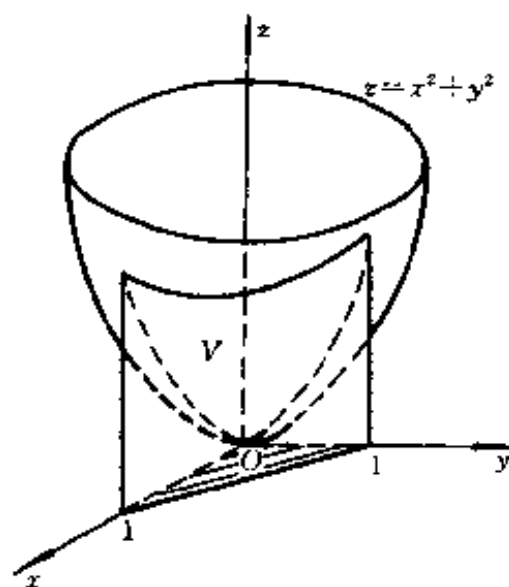


图 8.39

解 (a) 积分域为三角形

$$0 \leq x + y \leq 1, x \geq 0, y \geq 0.$$

柱体的上顶为平面 $z = x + y$ (图 8.40).

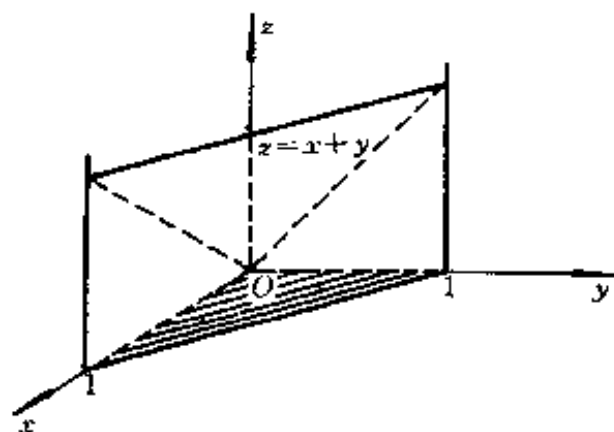


图 8.40

(6) 积分域为椭圆

$$\frac{x^2}{4} + \frac{y^2}{9} \leq 1,$$

即立体的底面, 顶面为椭球面 $z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$ (图 8.41).

(b) 积分域为由直线

$$x + y = 1, x + y = -1, x - y = 1, y - x = 1$$

围成的正方形, 柱体的顶面为旋转抛物面 $z = x^2 + y^2$.

图 8.42 中仅画了第一卦限部分的体积.

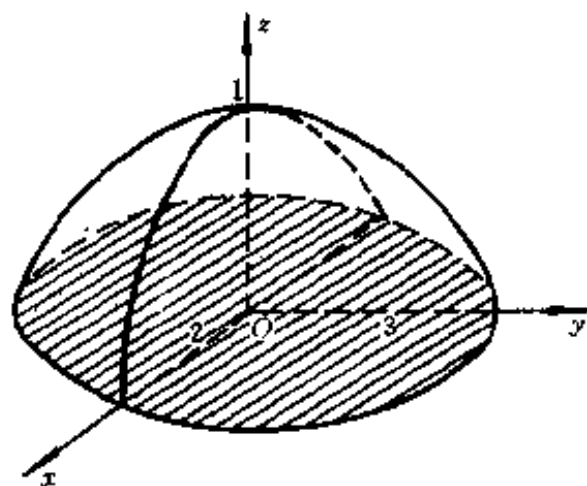


图 8.41

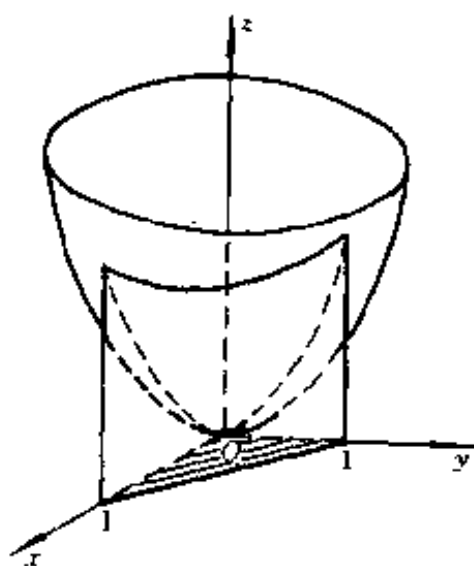


图 8.42

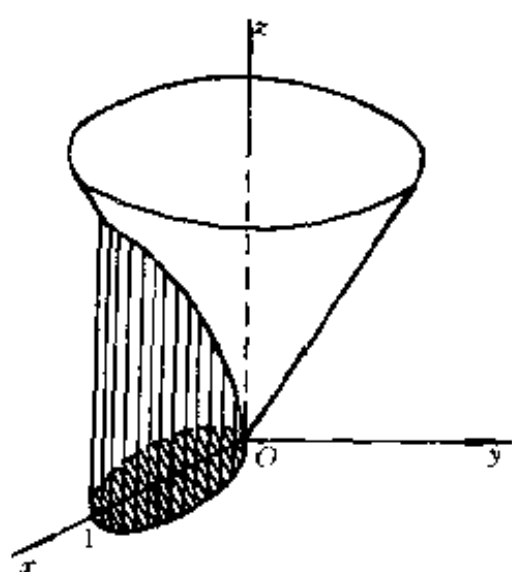


图 8.43

(r) 积分域为圆

$$\left(x - \frac{1}{2}\right)^2 + y^2 \leq \frac{1}{4}.$$

柱体的顶面为圆锥面 $z = \sqrt{x^2 + y^2}$ (图 8.43).

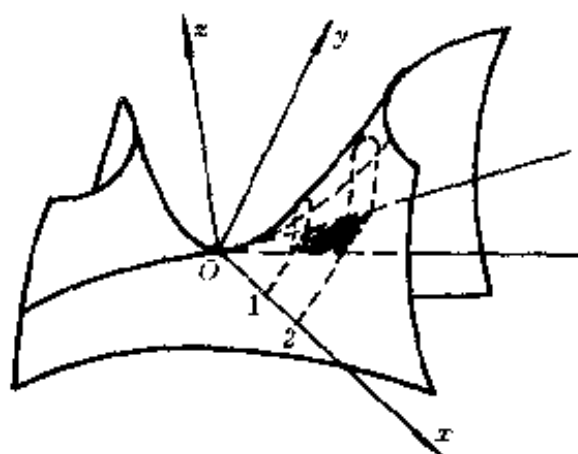


图 8.44

(n) 积分域为梯形 $1 \leq x \leq 2, x \leq y \leq 2x$. 柱体的顶面为双曲抛物面 $z = \sqrt{xy}$ (图 8.44).

(e) 积分域为圆

$$x^2 + y^2 \leq 1.$$

即立体的底面, 顶面是由正弦曲线 $z = \sin \pi x$ 绕 Oz 轴旋转一周而得的旋转曲面 (图 8.45).

求由下列曲面所界的体积:

4007. $z = 1 + x + y, z = 0, x + y = 1, x = 0, y = 0$.

解
$$V = \int_0^1 dx \int_0^{1-x} (1 + x + y) dy$$

$$= \int_0^1 \left(\frac{3}{2} - x - \frac{x^2}{2} \right) dx = \frac{5}{6}.$$

4008. $x + y + z = a, x^2 + y^2 = R^2, x = 0, y = 0, z = 0 (a \geq R\sqrt{2})$.

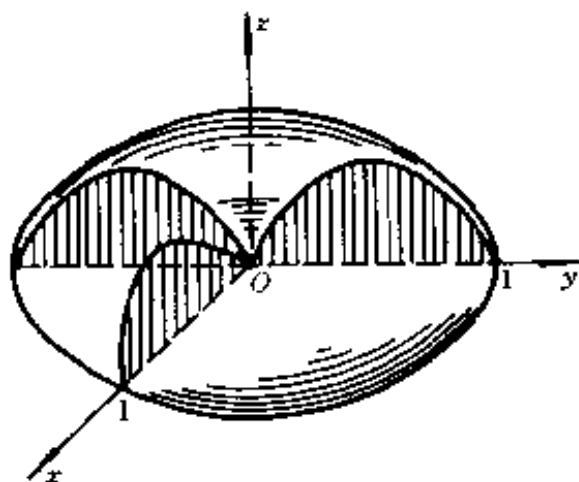


图 8.45

解
$$V = \int_0^R dx \int_0^{\sqrt{R^2-x^2}} (a - x - y) dy$$

$$= \int_0^R \left[(a - x) \sqrt{R^2 - x^2} - \frac{R^2 - x^2}{2} \right] dx$$

$$= \int_0^R a \sqrt{R^2 - x^2} dx - \int_0^R \left(x \sqrt{R^2 - x^2} + \frac{R^2 - x^2}{2} \right) dx = \frac{\pi a R^2}{4} - \frac{2R^3}{3}.$$

4009. $z = x^2 + y^2, y = x^2, y = 1, z = 0$.

解
$$V = \int_{-1}^1 dx \int_{x^2}^1 (x^2 + y^2) dy = \frac{88}{105}.$$

4010. $z = \cos x \cos y, z = 0, |x + y| \leq \frac{\pi}{2}, |x - y| \leq \frac{\pi}{2}$.

解 因函数 $z = \cos x \cdot \cos y$ 的图形关于 Oyz 平面对称, 而积分域 (图 8.46). 关于 Oy 轴对称, 故所求的体积为

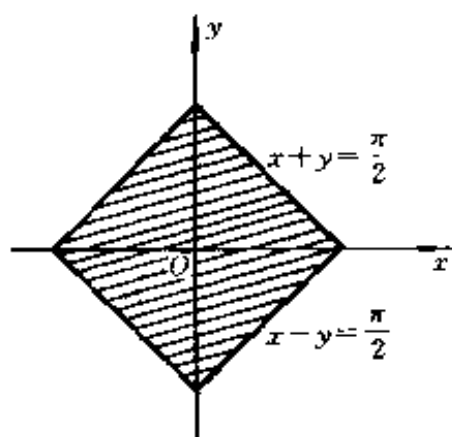


图 8.46

$$\begin{aligned} V &= 2 \int_0^{\frac{\pi}{2}} dx \int_x^{\frac{\pi}{2}-x} \cos x \\ &\quad \cdot \cos y dy \\ &= 4 \int_0^{\frac{\pi}{2}} \cos^2 x dx \\ &= 4 \left(\frac{x}{2} + \frac{\sin 2x}{4} \right) \Big|_0^{\frac{\pi}{2}} = \pi. \end{aligned}$$

4011. $z = \sin \frac{\pi y}{2x}, z = 0, y = x, y = 0, x = \pi.$

解 $V = \int_0^{\pi} dx \int_0^x \sin \frac{\pi y}{2x} dy = \frac{2}{\pi} \int_0^{\pi} x dx = \pi.$

4012. $z = xy, x + y + z = 1, z = 0.$

解 体积 V 由两部分组成:

$$\begin{aligned} V_1: & 0 \leq x \leq 1, 0 \leq y \\ & \leq \frac{1-x}{1+x}, z = xy. \end{aligned}$$

$$\begin{aligned} V_2: & 0 \leq x \leq 1, \frac{1-x}{1+x} \leq \\ & y \leq 1-x, z = 1-x-y. \end{aligned}$$

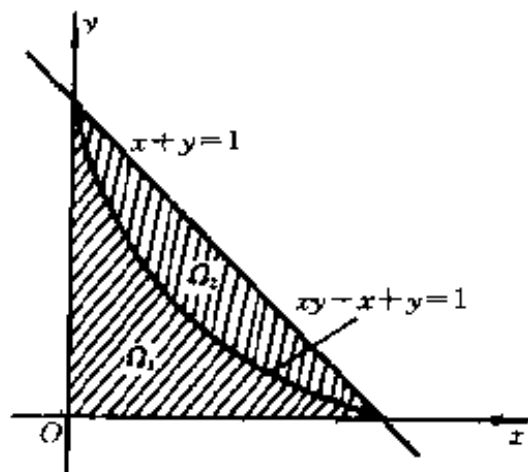


图 8.47

它们在 Oxy 平面上的射影域 Ω_1 及 Ω_2 如图 8.47 所示. 于是, 所求的体积为

$$\begin{aligned} V &= V_1 + V_2 \\ &= \int_0^1 x dx \int_0^{\frac{1-x}{1+x}} y dy \\ &\quad + \int_0^1 dx \int_{\frac{1-x}{1+x}}^{1-x} (1-x-y) dy \\ &= \left(-\frac{11}{4} + 4\ln 2 \right) \\ &\quad + \left(\frac{25}{6} - 6\ln 2 \right) \\ &= \frac{17}{12} - 2\ln 2. \end{aligned}$$

变换成极坐标, 以求由下列曲面所界的体积:

4013. $z^2 = xy, x^2 + y^2 = a^2$.

解 因为 $z = \sqrt{xy}$, 故所求的体积为

$$\begin{aligned} V &= 4 \iint_{\substack{x^2+y^2 \leq a^2 \\ x \geq 0, y \geq 0}} \sqrt{xy} dx dy \\ &= 4 \int_0^a dr \int_0^{\frac{\pi}{2}} \sqrt{\cos \varphi \sin \varphi} \cdot r^2 d\varphi \\ &= \frac{4}{3} a^3 \int_0^{\frac{\pi}{2}} \cos^{\frac{1}{2}} \varphi \sin^{\frac{1}{2}} \varphi d\varphi \\ &= \frac{4}{3} a^3 \cdot \frac{1}{2} B\left(\frac{3}{4}, \frac{3}{4}\right)^{*)} = \frac{2}{3} a^3 \cdot \frac{\Gamma^2\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{2}\right)} \\ &= \frac{2}{3} a^3 \cdot \frac{\Gamma^2\left(\frac{3}{4}\right)}{\frac{1}{2} \Gamma\left(\frac{1}{2}\right)} = \frac{4}{3} a^3 \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}}. \end{aligned}$$

*) 利用 3856 题的结果.

$$4014. z = x + y, (x^2 + y^2)^2 = 2xy, z = 0 (x > 0, y > 0).$$

解 令 $x = r\cos\varphi, y = r\sin\varphi$, 则方程

$$(x^2 + y^2)^2 = 2xy \text{ 及 } z = x + y$$

变为

$$r^2 = 2\sin\varphi\cos\varphi = \sin 2\varphi \text{ 及 } z = r(\cos\varphi + \sin\varphi).$$

于是, 所求的体积为

$$\begin{aligned} V &= \iint_{\Omega} (x + y) dx dy \\ &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\sqrt{\sin 2\varphi}} r^2 (\cos\varphi + \sin\varphi) dr \\ &= \frac{2}{3} \frac{\sqrt{2}}{3} \int_0^{\frac{\pi}{2}} (\sin^{\frac{5}{2}}\varphi \cos^{\frac{3}{2}}\varphi + \cos^{\frac{5}{2}}\varphi \sin^{\frac{3}{2}}\varphi) d\varphi \\ &= \frac{2}{3} \frac{\sqrt{2}}{3} \cdot B\left(\frac{5}{4}, \frac{7}{4}\right) \\ &= \frac{2}{3} \frac{\sqrt{2}}{3} \cdot \frac{\Gamma\left(\frac{5}{4}\right) \Gamma\left(\frac{7}{4}\right)}{\Gamma(3)} \\ &= \frac{2}{3} \frac{\sqrt{2}}{3} \cdot \frac{\frac{1}{4} \cdot \frac{3}{4} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{2!} \\ &= \frac{\sqrt{2}}{16} \cdot \frac{\pi}{\sin \frac{\pi}{4}} = \frac{\pi}{8}. \end{aligned}$$

*) 利用 3856 题的结果.

$$4015. z = x^2 + y^2, x^2 + y^2 = x, x^2 + y^2 = 2x, z = 0.$$

解 令 $x = r\cos\varphi, y = r\sin\varphi$, 则方程

$$x^2 + y^2 = x, x^2 + y^2 = 2x \text{ 及 } z = x^2 + y^2$$

变为

$$r = \cos\varphi, r = 2\cos\varphi \text{ 及 } z = r^2.$$

于是,所求的体积为

$$\begin{aligned} V &= \iint_{\Omega} (x^2 + y^2) dx dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{\cos\varphi}^{2\cos\varphi} r^3 dr \\ &= \frac{2}{4} \int_0^{\frac{\pi}{2}} (16\cos^4\varphi - \cos^4\varphi) d\varphi \\ &= \frac{15}{2} \int_0^{\frac{\pi}{2}} \cos^4\varphi d\varphi = \frac{15}{2} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{45}{32}\pi. \end{aligned}$$

4016. $x^2 + y^2 + z^2 = a^2, x^2 + y^2 \geq a|x| (a > 0).$

解 只须计算由下列曲面所围成的体积:

$$x^2 + y^2 + z^2 = a^2, x^2 + y^2 \leq a|x|.$$

若引用极坐标,则

$$r^2 + z^2 = a^2, r^2 \leq a|r\cos\varphi|,$$

其体积为

$$\begin{aligned} V_1 &= 8 \iint_{\substack{x^2+y^2 \leq a|x| \\ x \geq 0, y \geq 0}} \sqrt{a^2 - (x^2 + y^2)} dx dy \\ &= 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{a\cos\varphi} r \cdot \sqrt{a^2 - r^2} dr \\ &= -\frac{8}{3} \int_0^{\frac{\pi}{2}} (a^2 - r^2)^{\frac{3}{2}} \Big|_0^{a\cos\varphi} d\varphi \\ &= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^2\varphi) d\varphi \\ &= \frac{4\pi a^3}{3} - \frac{16a^3}{9}. \end{aligned}$$

于是,所求的体积为

$$V = \frac{4\pi a^3}{5} - \left(\frac{4\pi a^3}{3} - \frac{16a^3}{9} \right) = \frac{16a^3}{9}.$$

4017. $x^2 + y^2 - az = 0, (x^2 + y^2)^2 = a^2(x^2 - y^2), z = 0 (a > 0).$

解 若引用极坐标, 则有

$$z = \frac{r^2}{a}, r^2 = a^2 \cos 2\varphi (a > 0).$$

于是, 利用对称性知, 所求的体积为

$$\begin{aligned} V &= 4 \iint_D \frac{1}{a} (x^2 + y^2) dx dy \\ &= 4 \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sqrt{\cos 2\varphi}} \frac{r^2}{a} \cdot r dr \\ &= a^3 \int_0^{\frac{\pi}{4}} \cos^2 2\varphi d\varphi = \frac{\pi a^3}{8}. \end{aligned}$$

4018. $z = e^{-(x^2 + y^2)}, z = 0, x^2 + y^2 = R^2.$

解 利用对称性, 得所求的体积为

$$\begin{aligned} V &= 4 \iint_{\substack{x^2 + y^2 \leq R^2 \\ x \geq 0, y \geq 0}} e^{-(x^2 + y^2)} dx dy \\ &= 4 \int_0^{\frac{\pi}{4}} d\varphi \int_0^R e^{-r^2} r dr = \pi(1 - e^{-R^2}). \end{aligned}$$

4019. $z = c \cos \frac{\pi \sqrt{x^2 + y^2}}{2a}, x^2 + y^2 = a^2, y = x \tan \alpha, y = x \tan \beta (a > 0, c > 0, 0 \leq \alpha < \beta \leq 2\pi).$

解 所求的体积为

$$\begin{aligned} V &= \iint_D c \cos \frac{\pi \sqrt{x^2 + y^2}}{2a} dx dy \\ &= \int_\alpha^\beta d\varphi \int_0^a c r \cos \frac{\pi r}{2a} dr \end{aligned}$$

$$\begin{aligned}
&= c(\beta - \alpha) \left[\frac{2ar}{\pi} \sin \frac{\pi r}{2a} + \frac{4a^2}{\pi^2} \cos \frac{\pi r}{2a} \right] \Big|_0^a \\
&= 2a^2 c(\beta - \alpha) \left(\frac{1}{\pi} - \frac{2}{\pi^2} \right) \\
&= \frac{2a^2 c(\beta - \alpha)(\pi - 2)}{\pi^2}.
\end{aligned}$$

4020. $z = x^2 + y^2, z = x + y$.

解 立体的射影域的围线为 $x^2 + y^2 = x + y$ 或

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}. \text{ 若引用代换 } x = \frac{1}{2} + r\cos\varphi, y = \frac{1}{2} + r\sin\varphi, \text{ 则有}$$

$$z = r^2 + \frac{1}{2} + r(\cos\varphi + \sin\varphi), z = 1 + r(\cos\varphi + \sin\varphi)$$

$$(0 \leq \varphi \leq 2\pi, 0 \leq r \leq \frac{1}{\sqrt{2}}).$$

于是,所求的体积为

$$\begin{aligned}
V &= \iint_{\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \leq \frac{1}{2}} [(x+y) - (x^2 + y^2)] dx dy \\
&= \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{2}}} \left\{ [1 + r(\cos\varphi + \sin\varphi)] \right. \\
&\quad \left. - \left[r^2 + \frac{1}{2} + r(\cos\varphi + \sin\varphi) \right] \right\} r dr \\
&= \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{2} - r^2 \right) r dr = \frac{\pi}{8}.
\end{aligned}$$

求由下列曲面所界的体积(假定参数是正的):

4021. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} (z > 0).$

解 曲面的交线在 Oxy 平面上的射影为椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2}$

$= \frac{1}{2}$, 令 $x = \arccos \varphi, y = br \sin \varphi$, 则方程化为

$$z = c \sqrt{1 - r^2} \text{ 及 } z = cr (0 \leq \varphi \leq 2\pi, 0 \leq r \leq \frac{1}{\sqrt{2}}).$$

于是, 曲面所界的体积为

$$\begin{aligned} V &= \iint_D \left[c \sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)} - c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} \right] dx dy \\ &= \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{2}}} abr (c \sqrt{1 - r^2} - cr) dr \\ &= abc \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{2}}} (r \sqrt{1 - r^2} - r^2) dr \\ &= -\frac{1}{3} abc \int_0^{2\pi} \left[r^3 - (1 - r^2)^{\frac{3}{2}} \right] \Big|_0^{\frac{1}{\sqrt{2}}} d\varphi \\ &= \frac{1}{3} \pi abc (2 - \sqrt{2}). \end{aligned}$$

4022. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

解 若令 $x = \arccos \varphi, y = br \sin \varphi$, 则曲面方程化为

$$z = \pm c \sqrt{1 + r^2} (0 \leq \varphi \leq 2\pi, 0 \leq r \leq 1).$$

于是, 曲面所界的体积为

$$\begin{aligned} V &= \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} 2c \sqrt{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}} dx dy \\ &= \int_0^{2\pi} d\varphi \int_0^1 2abcr \sqrt{1 + r^2} dr \\ &= 2abc \int_0^{2\pi} d\varphi \int_0^1 r \sqrt{1 + r^2} dr \\ &= \frac{4\pi}{3} abc (2\sqrt{2} - 1). \end{aligned}$$

$$4023. \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}, \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}, z = 0.$$

解 立体在 Oxy 平面上的射影域的界线为椭圆 $\frac{x^2}{a^2} +$

$\frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$, 即 $\left(\frac{x}{a} - \frac{1}{2}\right)^2 + \left(\frac{y}{b} - \frac{1}{2}\right)^2 = \frac{1}{2}$. 若令

$\frac{x}{a} = \frac{1}{2} + r\cos\varphi, \frac{y}{b} = \frac{1}{2} + r\sin\varphi$, 则曲面方程化为

$$z = c \left[\frac{1}{2} + r(\cos\varphi + \sin\varphi) + r^2 \right] \\ \left(0 \leq \varphi \leq 2\pi, 0 \leq r \leq \frac{1}{\sqrt{2}} \right).$$

于是, 曲面所界的体积为

$$V = \iint_{\left(\frac{x}{a} - \frac{1}{2}\right)^2 + \left(\frac{y}{b} - \frac{1}{2}\right)^2 \leq \frac{1}{2}} c \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) dx dy \\ = abc \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{2}}} r \left[\frac{1}{2} + r(\cos\varphi + \sin\varphi) + r^2 \right] dr \\ = abc \int_0^{2\pi} \left[\frac{1}{8} + \frac{1}{6\sqrt{2}} (\cos\varphi + \sin\varphi) + \frac{1}{16} \right] d\varphi \\ = abc \cdot \frac{3 \cdot 2\pi}{16} = \frac{3}{8} \pi abc.$$

$$4024. \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2 + \frac{z}{c} = 1, z = 0.$$

解 若令 $x = a\cos\varphi, y = br\sin\varphi$, 则曲面方程化为

$$z = c(1 - r^4) (0 \leq \varphi \leq 2\pi, 0 \leq r \leq 1).$$

于是, 曲面所界的体积为

$$V = \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} c \left[1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \right] dx dy$$

$$= \int_0^{2\pi} d\varphi \int_0^1 abcr(1-r^2)dr = \frac{2}{3}\pi abc.$$

4025. $\left(\frac{x}{a} + \frac{y}{b}\right)^2 + \frac{z^2}{c^2} = 1, x=0, y=0, z=0.$

解 下面计算位于第一卦限部分的体积.

令 $x = ar\cos^2\varphi, y = br\sin^2\varphi$, 则方程化为

$$z = c\sqrt{1-r^2} \quad (0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq r \leq 1).$$

于是, 曲面所界的体积为

$$\begin{aligned} V &= \iint_D c\sqrt{1-\left(\frac{x}{a} + \frac{y}{b}\right)^2} dxdy \\ &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 abcsin2\varphi \cdot \sqrt{1-r^2} r dr \\ &= abc \left(\int_0^{\frac{\pi}{2}} \sin2\varphi d\varphi \right) \left(\int_0^1 r\sqrt{1-r^2} dr \right) \\ &= \frac{abc}{3}. \end{aligned}$$

4026. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$

解 令

$$x = ar\cos\varphi, y = br\sin\varphi,$$

则方程化为

$$z = \pm c\sqrt{1-r^2},$$

$$r^2 = \cos^2\varphi - \sin^2\varphi$$

$$= \cos2\varphi \quad (\text{因 } r^2 = \cos2\varphi \geq 0,$$

$$\text{故 } -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}, \frac{3\pi}{4} \leq \varphi \leq \frac{5\pi}{4}).$$

于是, 利用对称性知, 曲面所界的体积为

$$\begin{aligned}
V &= 8c \iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy \\
&= 8abc \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\varphi}} \sqrt{1 - r^2} r dr d\varphi \\
&= 8abc \int_0^{\frac{\pi}{4}} \frac{1}{3} (1 - \sqrt{8} \sin^3 \varphi) d\varphi \\
&= \frac{8abc}{3} \left(\varphi + \sqrt{8} \cos \varphi - \frac{\sqrt{8}}{3} \cos^3 \varphi \right) \Big|_0^{\frac{\pi}{4}} \\
&= \frac{8abc}{3} \left(\frac{\pi}{4} + \frac{5}{3} - \frac{4\sqrt{2}}{3} \right) \\
&= \frac{2abc}{9} (3\pi + 20 - 16\sqrt{2}).
\end{aligned}$$

4027. $z^2 = xy, x + y = a, x + y = b (0 < a < b)$.

解 由于 $z = \pm \sqrt{xy}$, 又所界立体在 Oxy 平面上的射影域 Ω 由直线 $x + y = a, x + y = b, x = 0$ 及 $y = 0$ 围成. 于是, 利用对称性知, 曲面所界的体积为

$$\begin{aligned}
V &= 2 \iint_{\Omega} \sqrt{xy} dx dy \\
&= 2 \left(\int_0^a dx \int_{a-x}^{b-x} \sqrt{xy} dy + \int_a^b dx \int_0^{b-x} \sqrt{xy} dy \right) \\
&= \frac{4}{3} \int_0^a [\sqrt{x(b-x)^3} - \sqrt{x(a-x)^3}] dx \\
&\quad + \frac{4}{3} \int_a^b \sqrt{x(b-x)^3} dx \\
&= \frac{4}{3} \int_0^b (b-x) \sqrt{x(b-x)} dx \\
&\quad - \frac{4}{3} \int_0^a (a-x) \sqrt{x(a-x)} dx.
\end{aligned}$$

令 $x = b \sin^2 t$, 可得

$$\begin{aligned}
& \int_0^b (b-x) \sqrt{x(b-x)} dx \\
&= 2b^3 \int_0^{\frac{\pi}{2}} \cos^4 t \sin^2 t dt \\
&= 2b^3 \left(\int_0^{\frac{\pi}{2}} \cos^4 t dt - \int_0^{\frac{\pi}{2}} \cos^6 t dt \right) \\
&= 2b^3 \left(\frac{3 \cdot 1}{4 \cdot 2} - \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \right) \frac{\pi}{2} = \frac{1}{16} \pi b^3;
\end{aligned}$$

同理,有

$$\int_0^a (a-x) \sqrt{x(a-x)} dx = \frac{1}{16} \pi a^3.$$

于是,所求的体积为

$$V = \frac{4}{3} \left(\frac{\pi b^3}{16} - \frac{\pi a^3}{16} \right) = \frac{\pi}{12} (b^3 - a^3).$$

4028. $z = x^2 + y^2, xy = a^2, xy = 2a^2, y = \frac{x}{2}, y = 2x, z = 0.$

解 曲面所界的立体在 Oxy 平面上的射影域 Ω 由曲线 $xy = a^2, xy = 2a^2$ 和直线 $y = \frac{x}{2}, y = 2x$ 围成. 利用对称性, 曲面所界体积可表示为

$$V = 2 \iint_{\Omega} z dx dy = 2 \iint_{\Omega} (x^2 + y^2) dx dy.$$

作变量代换

$$xy = ua^2, y = vx,$$

则积分域 Ω 变为长方形域

$$1 \leq u \leq 2, \frac{1}{2} \leq v \leq 2,$$

且 $|I| = \frac{a^2}{2v}, z = x^2 + y^2 = a^2 \left(\frac{u}{v} + uv \right).$

于是,所求的体积为

$$\begin{aligned}
V &= 2 \iint_{\Omega} (x^2 + y^2) dx dy \\
&= 2 \iint_{\substack{1 \leq u \leq 2 \\ \frac{1}{2} \leq v \leq 2}} a^2 \left(\frac{u}{v} + uv \right) \frac{a^2}{2v} du dv \\
&= a^4 \int_1^2 u du \int_{\frac{1}{2}}^2 \left(1 + \frac{1}{v^2} \right) dv = \frac{9}{2} a^4.
\end{aligned}$$

4029. $z = xy, x^2 = y, x^2 = 2y, y^2 = x, y^2 = 2x, z = 0$.

解 曲面所界立体 V 在 Oxy 平面上的射影域 Ω 由曲线 $x^2 = y, x^2 = 2y, y^2 = x$ 及 $y^2 = 2x$ 围成.

我们有

$$V = \iint_{\Omega} z dx dy = \iint_{\Omega} xy dx dy.$$

作变量代换

$$x = uy^2, y = vx^2,$$

或

$$x = u^{-\frac{1}{3}} v^{-\frac{2}{3}}, y = u^{-\frac{2}{3}} v^{-\frac{1}{3}},$$

则积分域 Ω 变为正方形域

$$\frac{1}{2} \leq u \leq 1, \frac{1}{2} \leq v \leq 1,$$

且 $|I| = \frac{1}{3} u^{-2} v^{-2}$. 于是, 曲面所界的体积为

$$\begin{aligned}
V &= \iint_{\Omega} xy dx dy = \frac{1}{3} \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 u^{-3} v^{-3} du dv \\
&= \frac{1}{3} \left(\int_{\frac{1}{2}}^1 u^{-3} du \right)^2 \\
&= \frac{1}{3} \cdot \frac{9}{4} = \frac{3}{4}.
\end{aligned}$$

4030. $z = c \sin \frac{\pi xy}{a^2}, z = 0, xy = a^2, y = ax, y = \beta x (0 < \alpha < \beta; x > 0)$.

解 曲面所界的立体在 Oxy 平面上的投影域 Ω 由曲线 $xy = a^2$ 和直线 $y = ax, y = \beta x (x > 0)$ 围成. 于是, 曲面所界的体积为

$$V = \iint_{\Omega} z dx dy = c \iint_{\Omega} \sin \frac{\pi xy}{a^2} dx dy.$$

作变量代换 $x = a \cos \varphi, y = a \sin \varphi$, 则 $|I| = a^2 r$. 于是,

$$\begin{aligned} V &= \iint_{\Omega} z dx dy = c \iint_{\Omega} \sin \frac{\pi xy}{a^2} dx dy \\ &= a^2 c \int_{\arctg \alpha}^{\arctg \beta} \int_0^{\frac{1}{\sqrt{-\ln \varphi \cos \varphi}}} \sin(\pi r^2 \sin \varphi \cos \varphi) r dr d\varphi \\ &= \frac{a^2 c}{\pi} \int_{\arctg \alpha}^{\arctg \beta} \frac{1}{\sin \varphi \cos \varphi} d\varphi \\ &= \frac{a^2 c}{\pi} \ln \varphi \bigg|_{\arctg \alpha}^{\arctg \beta} = \frac{a^2 c}{\pi} \ln \frac{\beta}{\alpha}. \end{aligned}$$

4031. $z = x^{\frac{3}{2}} + y^{\frac{3}{2}}, z = 0, x + y = 1, x = 0, y = 0$.

解 曲面所界的立体在 Oxy 平面上的投影域 Ω 由直线 $x + y = 1, x = 0, y = 0$ 围成. 于是, 曲面所界的体积为

$$\begin{aligned} V &= \iint_{\Omega} z dx dy = \iint_{\Omega} (x^{\frac{3}{2}} + y^{\frac{3}{2}}) dx dy \\ &= \int_0^1 \int_0^{1-x} (x^{\frac{3}{2}} + y^{\frac{3}{2}}) dx dy \\ &= \int_0^1 \left[x^{\frac{3}{2}}(1-x) + \frac{2}{5}(1-x)^{\frac{5}{2}} \right] dx \\ &= \frac{2}{5} x^{\frac{5}{2}} \bigg|_0^1 - \frac{2}{7} x^{\frac{7}{2}} \bigg|_0^1 + \frac{1}{35} (1-x)^{\frac{7}{2}} \bigg|_0^1 \end{aligned}$$

$$= \frac{2}{5} - \frac{2}{7} + \frac{4}{35} = \frac{8}{35}.$$

4032. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z}{c} = 1, \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1, z = 0.$

解 令

$$x = ar\cos^3\varphi, y = br\sin^3\varphi,$$

则方程化为

$$z = c[1 - r^2(\cos^6\varphi + \sin^6\varphi)],$$

$$r = 1 \quad (0 \leq \varphi \leq 2\pi).$$

于是,利用对称性知,曲面所界的体积为

$$\begin{aligned} V &= 4 \iint_{\Omega} z dx dy \\ &= 12abc \int_0^{\frac{\pi}{2}} \int_0^1 [1 - r^2(\cos^6\varphi + \sin^6\varphi)] \\ &\quad \cdot r \cos^2\varphi \sin^2\varphi dr d\varphi \\ &= 12abc \left[\int_0^{\frac{\pi}{2}} \frac{1}{2} \cos^2\varphi \sin^2\varphi d\varphi \right. \\ &\quad \left. - \frac{1}{4} \int_0^{\frac{\pi}{2}} (\cos^6\varphi + \sin^6\varphi) \cos^2\varphi \sin^2\varphi d\varphi \right] \\ &= 6abc \left(\int_0^{\frac{\pi}{2}} \cos^2\varphi \sin^2\varphi d\varphi - \int_0^{\frac{\pi}{2}} \cos^6\varphi \cos^2\varphi \sin^2\varphi d\varphi \right) \\ &= 6abc \left[\frac{\pi}{4} \left(1 - \frac{3}{4}\right) - \frac{1}{10} \cdot \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \right] \\ &= \frac{3\pi abc}{2} \left(\frac{1}{4} - \frac{105}{1920} \right) = \frac{75}{256} \pi abc. \end{aligned}$$

4033. $z = \operatorname{arctg} \frac{y}{x}, z = 0, \sqrt{x^2 + y^2} = a \operatorname{arctg} \frac{y}{x} (y \geq 0).$

解 令

$$x = r \cos \varphi, y = r \sin \varphi,$$

则方程化为

$$\begin{aligned} z &= c\varphi, \\ r &= a\varphi \left(0 \leq \varphi \leq \frac{\pi}{2} \right) \end{aligned}$$

于是, 曲面所界的体积为

$$\begin{aligned} V &= \iint_{\Omega} z dx dy = \int_0^{\frac{\pi}{2}} \int_0^{a\varphi} c\varphi r dr d\varphi \\ &= \frac{a^2 c}{2} \int_0^{\frac{\pi}{2}} \varphi^2 d\varphi = \frac{a^2 c}{6} \varphi^3 \Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi^3 a^2 c}{128}. \end{aligned}$$

4034. $\frac{x^n}{a^n} + \frac{y^n}{b^n} + \frac{z^n}{c^n} = 1, x = 0, y = 0, z = 0 (n > 0).$

解 曲面方程可表示为

$$z = c \sqrt[n]{1 - \left(\frac{x^n}{a^n} + \frac{y^n}{b^n} \right)}.$$

若令

$$x = a \cos^{\frac{2}{n}} \varphi, y = b r \sin^{\frac{2}{n}} \varphi \left(0 \leq \varphi \leq \frac{\pi}{2} \right),$$

则曲面所界的体积为

$$\begin{aligned} V &= c \iint_{\Omega} \sqrt[n]{1 - \left(\frac{x^n}{a^n} + \frac{y^n}{b^n} \right)} dx dy \\ &= \frac{2abc}{n} \int_0^1 \sqrt[n]{1 - r^n} r dr \int_0^{\frac{\pi}{2}} \cos^{\frac{2-n}{n}} \varphi \sin^{\frac{2-n}{n}} \varphi d\varphi. \end{aligned}$$

若令 $r^n = t$ 可得

$$\int_0^1 \sqrt[n]{1 - r^n} r dr = \int_0^1 (1 - t)^{\frac{1}{n}} t^{\frac{2}{n} - 1} dt$$

$$\begin{aligned}
&= B\left(\frac{1}{n} + 1, \frac{2}{n}\right) = \frac{\Gamma\left(\frac{1}{n} + 1\right) \Gamma\left(\frac{2}{n}\right)}{\Gamma\left(1 + \frac{3}{n}\right)} \\
&= \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right)}{3\Gamma\left(\frac{3}{n}\right)};
\end{aligned}$$

令 $\sin^2 \varphi = t$ 可得

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^{\frac{2-n}{n}} \varphi \sin^{\frac{2-n}{n}} \varphi d\varphi &= \frac{1}{2n} \int_0^1 (1-t)^{\frac{1}{n}-1} t^{\frac{1}{n}-1} dt \\
&= \frac{1}{2n} \cdot B\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2n} \cdot \frac{\Gamma^2\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}.
\end{aligned}$$

于是, 所求的体积为

$$\begin{aligned}
V &= \frac{abc}{n^2} \cdot \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right)}{3\Gamma\left(\frac{3}{n}\right)} \cdot \frac{\Gamma^2\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)} \\
&= \frac{abc}{3n^2} \cdot \frac{\Gamma^3\left(\frac{1}{n}\right)}{\Gamma\left(\frac{3}{n}\right)}.
\end{aligned}$$

4035. $\left(\frac{x}{a} + \frac{y}{b}\right)^n + \left(\frac{z}{c}\right)^m = 1, x=0, y=0, z=0 (n>0, m>0).$

解 令

$$x = a \cos^2 \varphi, y = b \sin^2 \varphi (0 \leq \varphi \leq \frac{\pi}{2}),$$

则曲面所界的体积为

$$\begin{aligned}
V &= c \iint_{\Omega} \sqrt{1 - \left(\frac{x}{a} + \frac{y}{b} \right)^n} dx dy \\
&= 2abc \int_0^1 \sqrt[n]{1 - r^n} r dr \int_0^{\frac{\pi}{2}} \cos \varphi \sin \varphi d\varphi \\
&= abc \int_0^1 \sqrt[n]{1 - r^n} r dr \\
&= \frac{abc}{n} \int_0^1 (1 - t)^{\frac{1}{n}} t^{\frac{2}{n}-1} dt \\
&= \frac{abc}{n} \cdot B\left(\frac{1}{n} + 1, \frac{2}{n}\right) \\
&= \frac{abc}{n} \cdot \frac{\Gamma\left(\frac{1}{n} + 1\right) \Gamma\left(\frac{2}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{2}{n} + 1\right)} \\
&= \frac{abc}{n + 2m} \cdot \frac{\Gamma\left(\frac{1}{m}\right) \Gamma\left(\frac{2}{n}\right)}{\Gamma\left(\frac{1}{m} + \frac{2}{n}\right)}.
\end{aligned}$$

§ 4. 曲面面积算法

1° 曲面由显函数给出的情形 平滑曲面 $z = z(x, y)$ 的面积由积分

$$S = \iint_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dx dy$$

(其中 Ω 为已知曲面在 Oxy 平面上的射影) 所表出.

2° 曲面由参数方程给出的情形 若曲面的方程是用参数给出

$$x = x(u, v), y = y(u, v), z = z(u, v),$$

其中 $(u, v) \in \Omega$, Ω 为封闭可求积的有界区域, 假定函数 x, y 和 z 为在域 Ω 内连续可微分的函数, 则对于曲面的面积有公式

$$S = \iint_{\sigma} \sqrt{EG - F^2} du dv,$$

其中

$$E = \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2,$$

$$G = \left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2,$$

$$F = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v}.$$

4036. 求曲面 $az = xy$ 包含在圆柱 $x^2 + y^2 = a^2$ 内那部分的面积.

解 所求的面积为

$$\begin{aligned} S &= \iint_{x^2 + y^2 \leq a^2} \sqrt{1 + \left(\frac{y}{a} \right)^2 + \left(\frac{x}{a} \right)^2} dx dy \\ &= \iint_{x^2 + y^2 \leq a^2} \sqrt{\frac{a^2 + (x^2 + y^2)}{a^2}} dx dy \\ &= \frac{1}{a} \iint_{x^2 + y^2 \leq a^2} \sqrt{a^2 + (x^2 + y^2)} dx dy \\ &= \frac{1}{a} \int_0^{\frac{\pi}{2}} d\varphi \int_0^a r \sqrt{a^2 + r^2} dr \\ &= \frac{2\pi a^2}{3} (2\sqrt{2} - 1). \end{aligned}$$

4037. 求由曲面 $x^2 + z^2 = a^2$, $y^2 + z^2 = a^2$ 所界物体的面积.

解 如图 8.48 所示. 两曲面的交线在 Oyz 平面上的射影为圆

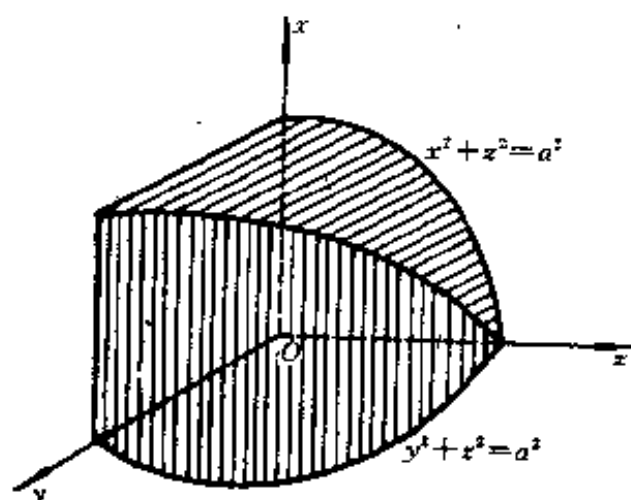


图 8.48

$$y^2 + z^2 = a^2, x = 0.$$

于是,利用对称性知,所求的面积为

$$\begin{aligned}
 S &= 4 \iint_{y^2+z^2 \leq a^2} \sqrt{1 + \left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} dydz \\
 &= 4 \cdot 4 \int_0^a dz \int_0^{\sqrt{a^2-z^2}} \sqrt{1 + 0^2 + \left(-\frac{z}{x}\right)^2} dy \\
 &= 16 \int_0^a dz \int_0^{\sqrt{a^2-z^2}} \sqrt{\frac{x^2 + z^2}{x^2}} dy \\
 &= 16a \int_0^a dz \int_0^{\sqrt{a^2-z^2}} \frac{dy}{\sqrt{a^2-z^2}} \\
 &= 16a \int_0^a \frac{1}{\sqrt{a^2-z^2}} \cdot \sqrt{a^2-z^2} dz = 16a^2.
 \end{aligned}$$

4038. 求球面 $x^2 + y^2 + z^2 = a^2$ 包含在柱面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (b \leq$

a) 内那部分的面积.

解 因为

$$\begin{aligned}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \\ &= \sqrt{\frac{x^2 + y^2 + z^2}{z^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}\end{aligned}$$

又积分域 $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ 位于第一象限部分为

$$0 \leq x \leq a, 0 \leq y \leq \frac{b}{a} \sqrt{a^2 - x^2}.$$

于是, 利用对称性知, 所求的面积为

$$\begin{aligned}S &= 2 \cdot 4 \int_0^a dx \int_0^{\frac{b}{a} \sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} dy \\ &= 8a \int_0^a \arcsin \frac{y}{\sqrt{a^2 - x^2}} \Big|_0^{\frac{b}{a} \sqrt{a^2 - x^2}} dx \\ &= 8a^2 \arcsin \frac{b}{a}.\end{aligned}$$

4039. 求曲面 $z^2 = 2xy$ 被平面 $x + y = 1, x = 0, y = 0$ 所截下那部分的面积.

解 因为

$$\begin{aligned}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1 + \frac{y^2}{z^2} + \frac{x^2}{z^2}} \\ &= \sqrt{\frac{x^2 + y^2 + z^2}{z^2}} = \frac{1}{\sqrt{2}} \sqrt{\frac{x^2 + y^2 + 2xy}{xy}} \\ &= \frac{1}{\sqrt{2}} \frac{x + y}{\sqrt{xy}},\end{aligned}$$

于是, 所求的面积为

$$\begin{aligned}
S &= \frac{2}{\sqrt{2}} \int_0^1 dx \int_0^{1-x} \frac{x+y}{\sqrt{xy}} dy \\
&= \frac{2}{\sqrt{2}} \int_0^1 [2\sqrt{x(1-x)} \\
&\quad + \frac{2}{3\sqrt{x}}(1-x)\sqrt{1-x}] dx \\
&= \sqrt{2} \int_0^1 \frac{2\sqrt{1-x}(1+2x)}{3\sqrt{x}} dx \\
&= \frac{4\sqrt{2}}{3} \int_0^1 \sqrt{1-x}(1+2x) d(\sqrt{x}) \\
&= \frac{4\sqrt{2}}{3} \int_0^1 \sqrt{1-t^2}(1+2t^2) dt \\
&= \frac{4\sqrt{2}}{3} \left(\frac{\pi}{4} + \frac{\pi}{8} \right) = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

4040. 求曲面 $x^2 + y^2 + z^2 = a^2$ 在圆柱 $x^2 + y^2 = \pm ax$ 外那部分的面积(维维安尼问题).

解 只须求出球面被圆柱面割出部分的面积. 因为

$$\begin{aligned}
\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \\
&= \frac{a}{\sqrt{a^2 - x^2 - y^2}},
\end{aligned}$$

于是, 利用对称性知, 割出部分的面积为

$$\begin{aligned}
S &= 8 \iint_D \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy \\
&= 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{a \cos \varphi} \frac{ar}{\sqrt{a^2 - r^2}} dr = 8a^2 \left(\frac{\pi}{2} - 1 \right).
\end{aligned}$$

因而, 所求的面积为

$$A = 4\pi a^2 - S = 4\pi a^2 - 8a^2\left(\frac{\pi}{2} - 1\right) = 8a^2.$$

4041. 求曲面 $z = x^2 + y^2$ 包含在圆柱 $x^2 + y^2 = 2x$ 内那部分的面积.

解 因为

$$\begin{aligned} & \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \\ &= \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} = \sqrt{2}, \end{aligned}$$

又积分域为: $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}, 0 \leq r \leq 2\cos\varphi$, 于是,

所求的面积为

$$\begin{aligned} S &= \iint_D \sqrt{2} \, dx dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \sqrt{2} \, r dr \\ &= 2\sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi \, d\varphi = 2\pi. \end{aligned}$$

4042. 求曲面 $z = x^2 - y^2$ 包含在柱面 $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ 内那部分的面积.

解 因为

$$\begin{aligned} & \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \\ &= \sqrt{1 + \left(\frac{x}{\sqrt{x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{x^2 - y^2}}\right)^2} \\ &= \frac{\sqrt{2}x}{\sqrt{x^2 - y^2}}, \end{aligned}$$

又积分域由双纽线 $r^2 = a^2 \cos 2\varphi$ 所围成, 于是, 利用对称性知, 所求的面积为

$$\begin{aligned}
 S &= \iint_{\Omega} \frac{\sqrt{2}x}{\sqrt{x^2 - y^2}} dx dy \\
 &= 4 \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sqrt{\cos 2\varphi}} \sqrt{2}r \cdot \frac{r \cos \varphi}{r \sqrt{\cos 2\varphi}} dr \\
 &= 2 \sqrt{2} \int_0^{\frac{\pi}{4}} a^2 \cos \varphi \sqrt{\cos 2\varphi} d\varphi \\
 &= 2a^2 \int_0^{\frac{\pi}{4}} \sqrt{1 - 2\sin^2 \varphi} d(\sqrt{2} \sin \varphi) \\
 &= 2a^2 \int_0^{\frac{\pi}{4}} \sqrt{1 - 2\sin^2 \varphi} d(\sqrt{2} \sin \varphi) \\
 &= 2a^2 \left[\frac{\sqrt{2} \sin \varphi}{2} \sqrt{1 - 2\sin^2 \varphi} \right. \\
 &\quad \left. + \frac{1}{2} \arcsin(\sqrt{2} \sin \varphi) \right] \Big|_0^{\frac{\pi}{4}} = \frac{\pi a^2}{2}.
 \end{aligned}$$

4043. 求曲面 $z = \frac{1}{2}(x^2 - y^2)$ 被平面 $x - y = \pm 1, x + y = \pm 1$ 所截那部分的面积.

解 因为

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + x^2 + y^2}$$

故所求的面积为

$$S = \iint_{\Omega} \sqrt{1 + x^2 + y^2} dx dy$$

, 其中 Ω 为由直线 $x - y = \pm 1, x + y = \pm 1$ 围成的正方形域. 为便于计算, 作变换

$$x = \frac{\sqrt{2}}{2}u - \frac{\sqrt{2}}{2}v, y = \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v,$$

从而积分域变为由方程 $u = \pm \frac{\sqrt{2}}{2}, v = \pm \frac{\sqrt{2}}{2}$ 围成的正方形, 且 $I = 1$. 于是, 注意利用对称性, 即得所求的面积为

$$\begin{aligned} S &= \iint_D \sqrt{1+x^2+y^2} dx dy \\ &= 4 \int_0^{\frac{\sqrt{2}}{2}} du \int_{-u}^u \sqrt{1+u^2+v^2} dv \\ &= \\ &4 \int_0^{\frac{\sqrt{2}}{2}} \left\{ u \sqrt{1+2u^2} + \frac{1+u^2}{2} [\ln(\sqrt{1+2u^2}+u) \right. \\ &\quad \left. - \ln(\sqrt{1+2u^2}-u)] \right\} du \\ &= \frac{2}{3} (1+2u^2)^{\frac{3}{2}} \Big|_0^{\frac{\sqrt{2}}{2}} + 2 \int_0^{\frac{\sqrt{2}}{2}} [\ln(\sqrt{1+2u^2}+u) \\ &\quad - \ln(\sqrt{1+2u^2}-u)] d\left(u + \frac{u^3}{3}\right) \\ &= \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{7\sqrt{2}}{6} \ln 3 - \int_0^{\frac{\sqrt{2}}{2}} 4 \left(u + \frac{u^3}{3}\right) \\ &\quad \cdot \frac{du}{\sqrt{1+2u^2}(1+u^2)} \\ &= \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{7\sqrt{2}}{6} \ln 3 - \int_0^{\frac{\sqrt{2}}{2}} \frac{1 + \frac{u^2}{3}}{1+u^2} \\ &\quad \cdot \frac{d(1+2u^2)}{\sqrt{1+2u^2}} \\ &= \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{7\sqrt{2}}{6} \ln 3 - \frac{2}{3} \int_1^{\sqrt{2}} \frac{t^2+5}{t^2+1} dt^{**}) \end{aligned}$$

$$\begin{aligned}
&= \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{7\sqrt{2}}{6}\ln 3 - \frac{2}{3}(\sqrt{2} - 1) \\
&\quad - \frac{8}{3} \int_1^{\sqrt{2}} \frac{dt}{t^2 + 1} \\
&= \frac{2\sqrt{2}}{3} (1 + \frac{7}{4}\ln 3) + \frac{2\pi}{3} - \frac{8}{3} \operatorname{arctg} \sqrt{2} \\
&= -\frac{2\pi}{3} + \frac{2\sqrt{2}}{3} (1 + \frac{7\ln 3}{4}) + \frac{8}{3} \operatorname{arctg} \frac{1}{\sqrt{2}}.
\end{aligned}$$

*) 作代换 $1 + 2u^2 = t^2$.

4044. 求曲面 $x^2 + y^2 = 2az$ 包含在柱面 $(x^2 + y^2)^2 = 2a^2xy$ 内那部分的面积.

解 因为

$$\begin{aligned}
\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1 + \left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2} \\
&= \frac{1}{a} \sqrt{a^2 + (x^2 + y^2)},
\end{aligned}$$

又积分域由双纽线 $r^2 = a^2 \sin 2\varphi$ 围成, 于是, 利用对称性, 即得所求的面积为

$$\begin{aligned}
S &= \iint_D \frac{1}{a} \sqrt{a^2 + (x^2 + y^2)} \, dx dy \\
&= 4 \int_0^{\frac{\pi}{4}} d\varphi \int_0^{a\sqrt{\sin 2\varphi}} \frac{1}{a} \sqrt{a^2 + r^2} r dr \\
&= \frac{4}{3a} \int_0^{\frac{\pi}{4}} [a^3 (1 + \sin 2\varphi)^{\frac{3}{2}} - a^3] d\varphi \\
&= \frac{4a^2}{3} \int_0^{\frac{\pi}{4}} (1 + \sin 2\varphi)^{\frac{3}{2}} d\varphi - \frac{\pi a^2}{3}.
\end{aligned}$$

于是

$$\int_0^{\frac{\pi}{4}} (1 + \sin 2\varphi)^{\frac{3}{2}} d\varphi = \int_0^{\frac{\pi}{4}} [1 + \cos 2(\frac{\pi}{4} - \varphi)]^{\frac{3}{2}} d\varphi$$

$$\begin{aligned}
&= 2\sqrt{2} \int_0^{\frac{\pi}{4}} \cos^3\left(\frac{\pi}{4} - \varphi\right) d\varphi \\
&= 2\sqrt{2} \int_0^{\frac{\pi}{4}} \cos^3 t dt = 2\sqrt{2} \left(\sin t - \frac{\sin^3 t}{3} \right) \bigg|_0^{\frac{\pi}{4}} \\
&= \frac{5}{3},
\end{aligned}$$

故最后得

$$S = \frac{4a^2}{3} \cdot \frac{5}{3} - \frac{\pi a^2}{3} = \frac{a^2}{9}(20 - 3\pi).$$

4045. 求曲面 $x^2 + y^2 = a^2$ 被平面 $x + z = 0, x - z = 0$ ($x > 0, y > 0$) 所截那部分的面积.

解 因为

$$\begin{aligned}
\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} &= \sqrt{1 + \left(\frac{x}{y}\right)^2} \\
&= \frac{a}{\sqrt{a^2 - x^2}},
\end{aligned}$$

于是, 所求的面积为

$$\begin{aligned}
S &= \iint_D \frac{a}{\sqrt{a^2 - x^2}} dx dz \\
&= \int_0^a dx \int_{-x}^x \frac{a}{\sqrt{a^2 - x^2}} dz \\
&= \int_0^a \frac{2ax}{\sqrt{a^2 - x^2}} dx = 2a^2.
\end{aligned}$$

4046. 求由曲面 $x^2 + y^2 = \frac{1}{3}z^2, x + y + z = 2a$ ($a > 0$) 所界物体的表面积和体积.

解 曲面的交线在 Oxy 平面上的射影为

$$3x^2 + 3y^2 = (2a - x - y)^2,$$

即

$$x^2 + y^2 - xy + 2a(x + y) = 2a^2.$$

令

$$x = \frac{x' - y'}{\sqrt{2}}, y = \frac{x' + y'}{\sqrt{2}},$$

则方程变为

$$\frac{\left(x' + \frac{4a}{\sqrt{2}}\right)^2}{(2\sqrt{3}a)^2} + \frac{y'^2}{(2a)^2} = 1.$$

由此可见,曲面所界的物体在 Oxy 平面上的射影域为以 $2a$ 为短半轴、 $2\sqrt{3}a$ 为长半轴的椭圆.

物体的表面积由截面和截出的锥面两部分组成.

对于 $z = 2a - x - y, z = \sqrt{3x^2 - 3y^2}$ 分别有

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3},$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = 2.$$

于是,物体的表面积

$$\begin{aligned} S &= \iint_D \sqrt{3} dx dy + \iint_D 2 dx dy \\ &= (\sqrt{3} + 2) \cdot \pi \cdot 2a \cdot 2\sqrt{3}a \\ &= 4\pi(3 + 2\sqrt{3})a^2. \end{aligned}$$

又所截圆锥之高为

$$H = \left| \frac{-2a}{\sqrt{1^2 + 1^2 + 1^2}} \right| = \frac{2a}{\sqrt{3}}$$

(即坐标原点到平面 $x + y + z = 2a$ 的距离). 于是,物体的体积为

$$V = \frac{1}{3} \cdot A \cdot \frac{2a}{\sqrt{3}},$$

其中 A 为截圆锥的底面积:

$$\begin{aligned} A &= \iint_D \sqrt{3} \, dx dy = \sqrt{3} \cdot \pi \cdot 2a \cdot 2\sqrt{3}a \\ &= 12\pi a^2. \end{aligned}$$

因此, 所求物体的体积为

$$V = \frac{1}{3} \cdot 12\pi a^2 \cdot \frac{2a}{\sqrt{3}} = \frac{8}{\sqrt{3}}\pi a^3.$$

4047. 求球壳被包含在两条纬线和两条经线间那部分的面积.

解 球壳的参数方程为

$$x = R\cos\varphi\cos\psi, y = R\sin\varphi\cos\psi, z = R\sin\psi$$

其中 R 为球的半径. 因为

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 \\ &= R^2\sin^2\varphi\cos^2\psi + R^2\cos^2\varphi\cos^2\psi = R^2\cos^2\psi, \end{aligned}$$

$$\begin{aligned} G &= \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2 \\ &= R^2\cos^2\varphi\sin^2\psi + R^2\sin^2\varphi\sin^2\psi + R^2\cos^2\psi \\ &= R^2, \end{aligned}$$

$$\begin{aligned} F &= \frac{\partial x}{\partial \varphi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \varphi} \frac{\partial y}{\partial \psi} + \frac{\partial z}{\partial \varphi} \frac{\partial z}{\partial \psi} \\ &= R^2\sin\varphi\cos\psi\cos\varphi\sin\psi - R^2\sin\varphi\cos\varphi\sin\psi\cos\psi \\ &\quad + 0 = 0, \end{aligned}$$

故

$$\sqrt{EG - F^2} = R^2\cos\psi.$$

于是, 所求的面积为

$$\begin{aligned} S &= \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\psi_1}^{\psi_2} R^2\cos\psi d\psi \\ &= (\varphi_2 - \varphi_1)(\sin\psi_2 - \sin\psi_1)R^2, \end{aligned}$$

其中 φ_1 及 φ_2 为经线的经度, ψ_1 及 ψ_2 为纬线的纬度.

4048. 求螺旋面

$$x = r \cos \varphi, y = r \sin \varphi, z = h \varphi,$$

其中 $0 < r < a, 0 < \varphi < 2\pi$ 那部分的面积.

解 因为

$$E = \left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2 = 1,$$

$$G = \left(\frac{\partial x}{\partial \varphi} \right)^2 + \left(\frac{\partial y}{\partial \varphi} \right)^2 + \left(\frac{\partial z}{\partial \varphi} \right)^2 = r^2 + h^2,$$

$$F = \frac{\partial x}{\partial r} \frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial \varphi} = 0,$$

故

$$\sqrt{EG - F^2} = \sqrt{r^2 + h^2}.$$

于是, 所求的面积为

$$\begin{aligned} S &= \int_0^{2\pi} d\varphi \int_0^a \sqrt{r^2 + h^2} dr \\ &= 2\pi \left[\frac{r}{2} \sqrt{r^2 + h^2} + \frac{h^2}{2} \ln(r + \sqrt{r^2 + h^2}) \right] \Big|_0^a \\ &= \pi a \sqrt{a^2 + h^2} + \pi h^2 \ln \frac{a + \sqrt{a^2 + h^2}}{h}. \end{aligned}$$

4049. 求环面

$$x = (b + a \cos \psi) \cos \varphi, y = (b + a \cos \psi) \sin \varphi,$$

$$z = a \sin \psi (a < a \leq b)$$

被两条经线 $\varphi = \varphi_1, \varphi = \varphi_2$ 和两条纬线 $\psi = \psi_1, \psi = \psi_2$ 所界那部分的面积. 整个环的表面积等于什么?

解 因为

$$E = \left(\frac{\partial x}{\partial \varphi} \right)^2 + \left(\frac{\partial y}{\partial \varphi} \right)^2 + \left(\frac{\partial z}{\partial \varphi} \right)^2 = (b + a \cos \psi)^2$$

$$G = \left(\frac{\partial x}{\partial \psi} \right)^2 + \left(\frac{\partial y}{\partial \psi} \right)^2 + \left(\frac{\partial z}{\partial \psi} \right)^2 = a^2,$$

$$F = \frac{\partial x}{\partial \varphi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \varphi} \frac{\partial y}{\partial \psi} + \frac{\partial z}{\partial \varphi} \frac{\partial z}{\partial \psi}$$

故

$$\sqrt{EG - F^2} = a(b + a \cos \psi)$$

于是,所求的面积为

$$\begin{aligned} S &= \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\psi_1}^{\psi_2} a(b + a \cos \psi) d\psi \\ &= a(\varphi_2 - \varphi_1)[b(\psi_2 - \psi_1) + a(\sin \psi_2 - \sin \psi_1)]. \end{aligned}$$

整个环的表面积

$$A = \int_0^{2\pi} d\varphi \int_{-\pi}^{\pi} a(b + a \cos \psi) d\psi = 4\pi^2 ab.$$

4050. 求立体角 ω , 在这个角里从坐标原点看得见矩形

$$x = a > 0, 0 \leq y \leq b, 0 \leq z \leq c.$$

若 a 很大, 对于 ω 推出近似公式.

解 以原点为球心作单位球, 则 ω 即为该球面含于四面体 $O-ABCD$ 内的面积, 其中 $ABCD$ 是以 b, c 为边长的矩形(图 8.49).

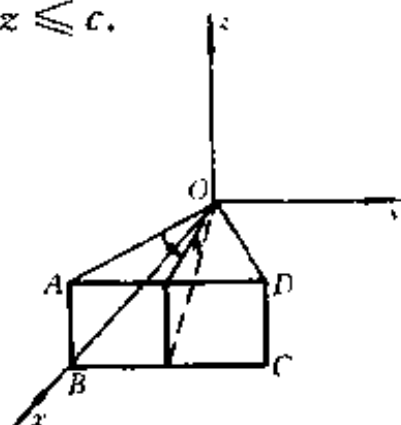


图 8.49

取球坐标系, 由 4047

题知:

$$\sqrt{EG - F^2} = \cos \psi$$

又 φ 和 ψ 的变化域为

$$0 \leq \varphi \leq \arcsin \frac{b}{\sqrt{a^2 + b^2}},$$

$$0 \leq \psi \leq \arcsin \frac{c \cos \varphi}{\sqrt{a^2 + c^2 \cos^2 \varphi}}.$$

于是,立体角

$$\begin{aligned}
 \omega &= \int_0^{\arcsin \frac{b}{\sqrt{a^2+b^2}}} \int_0^{\arcsin \frac{c \cos \varphi}{\sqrt{a^2+c^2 \cos^2 \varphi}}} \cos \psi d\psi d\varphi \\
 &= \int_0^{\arcsin \frac{b}{\sqrt{a^2+b^2}}} \left[\frac{c \cos \varphi}{\sqrt{a^2+c^2 \cos^2 \varphi}} \right] d\varphi \\
 &= \int_0^{\arcsin \frac{b}{\sqrt{a^2+b^2}}} \frac{d \left(\frac{c}{\sqrt{a^2+c^2}} \sin \varphi \right)}{\sqrt{1 - \left(\frac{c}{\sqrt{a^2+c^2}} \sin \varphi \right)^2}} \\
 &= \arcsin \left(\frac{c}{\sqrt{a^2+c^2}} \sin \arcsin \frac{b}{\sqrt{a^2+b^2}} \right) \\
 &= \arcsin \frac{bc}{\sqrt{(a^2+c^2)(a^2+b^2)}}.
 \end{aligned}$$

当 a 很大时,有

$$\begin{aligned}
 \frac{bc}{\sqrt{(a^2+c^2)(a^2+b^2)}} &= \frac{bc}{a^2 \sqrt{\left(1 + \frac{c^2}{a^2}\right) \left(1 + \frac{b^2}{a^2}\right)}} \\
 &\doteq \frac{bc}{a^2},
 \end{aligned}$$

于是,得 ω 的近似公式

$$\omega \doteq \frac{bc}{a^2}$$

§ 5. 二重积分在力学上的应用

1° 重心 若 x_0 和 y_0 为平面 Oxy 内薄板 Ω 的重心坐标, $\rho = \rho(x, y)$ 为薄板的密度,则

$$x_0 = \frac{1}{M} \iint_{\Omega} \rho x dx dy, y_0 = \frac{1}{M} \iint_{\Omega} \rho y dx dy, \quad (1)$$

其中 $M = \iint_D \rho dx dy$ 为薄板的质量.

若薄板是均匀的, 则于公式(1)中应令 $\rho = 1$.

2° 转动惯量 I_x 和 I_y 分别为平面 Oxy 内薄板 Ω 对于坐标轴 Ox 和 Oy 的转动惯量 —— 用公式来表示

$$I_x = \iint_D \rho y^2 dx dy, I_y = \iint_D \rho x^2 dx dy, \quad (2)$$

其中 $\rho = \rho(x, y)$ 为薄板的密度.

于公式(2)中假定 $\rho = 1$, 我们得到平面图形的几何转动惯量.

4051. 求边长为 a 的正方形薄板的质量, 设薄板上每一点的密度与该点距正方形顶点之一的距离成比例且在正方形的中点等于 ρ_0 .

解 取坐标系如图 8.50

所示, 则密度

$$\rho = k \sqrt{x^2 + y^2}.$$

由于 $\rho_0 = k$

$$\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2},$$

故 $k = \frac{\rho_0}{a} \sqrt{2}$, 从而

$$\rho = \frac{\rho_0 \sqrt{2}}{a} \sqrt{x^2 + y^2}.$$

若引用极坐标, 即得质量

$$\begin{aligned} M &= \iint_D \frac{\rho_0 \sqrt{2}}{a} \sqrt{x^2 + y^2} dx dy \\ &= \frac{\rho_0}{a} \sqrt{2} \left(\int_0^{\frac{\pi}{4}} d\varphi \int_0^{\frac{a}{\cos \varphi}} r^2 dr + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{a}{\sin \varphi}} r^2 dr \right) \end{aligned}$$

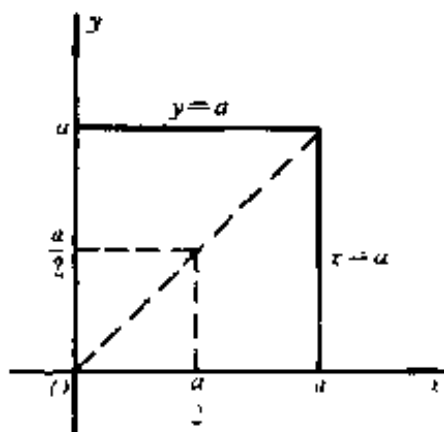


图 8.50

$$\begin{aligned}
&= \frac{\rho_0 a^2}{3} \sqrt{2} \left(\int_0^{\frac{\pi}{4}} \frac{d\varphi}{\cos^3 \varphi} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\varphi}{\sin^3 \varphi} \right) \\
&= \frac{\rho_0 a^2}{3} 2 \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{d\varphi}{\cos^3 \varphi} \\
&= \frac{\rho_0 a^2}{3} 2 \sqrt{2} \int_0^{\frac{\pi}{4}} \sqrt{1 + \operatorname{tg}^2 \varphi} d(\operatorname{tg} \varphi) \\
&= \frac{\rho_0 a^2}{3} 2 \sqrt{2} \left[\frac{\operatorname{tg} \varphi}{2} \sqrt{1 + \operatorname{tg}^2 \varphi} \right. \\
&\quad \left. + \frac{1}{2} \ln |\operatorname{tg} \varphi + \sqrt{1 + \operatorname{tg}^2 \varphi}| \right] \Big|_0^{\frac{\pi}{4}} \\
&= \frac{\rho_0 a^2}{3} [2 + \sqrt{2} \ln(1 + \sqrt{2})].
\end{aligned}$$

求由下列曲线所界均
匀薄板的重心坐标:

4052. $ay = x^2, x + y = 2a$
($a > 0$).

解 密度 ρ 为常数.
积分域如图 8.51 所
示. 质量

$$\begin{aligned}
M &= \rho \int_{-2a}^a dx \int_{\frac{x^2}{a}}^{2a-x} dy \\
&= \frac{9}{2} \rho a^2.
\end{aligned}$$

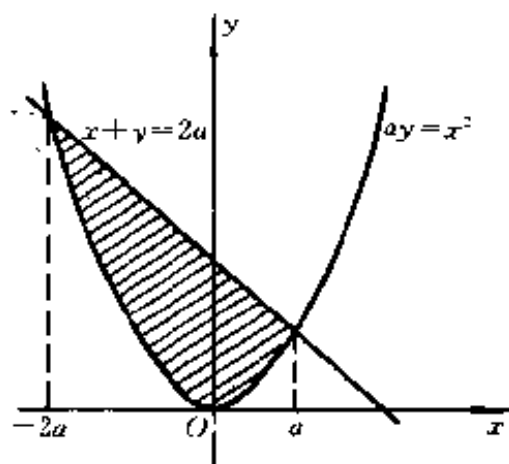


图 8.51

对于坐标轴的一次矩为

$$\begin{aligned}
M_y &= \rho \int_{-2a}^a x dx \int_{\frac{x^2}{a}}^{2a-x} dy = -\frac{9}{4} \rho a^3, \\
M_x &= \rho \int_{-2a}^a dx \int_{\frac{x^2}{a}}^{2a-x} y dy = \frac{36}{5} \rho a^3.
\end{aligned}$$

于是,重心 (x_0, y_0) 为

$$x_0 = \frac{M_y}{M} = -\frac{a}{2}, y_0 = \frac{M_x}{M} = \frac{8}{5}a.$$

4053. $\sqrt{x} + \sqrt{y} = \sqrt{a}, x = 0, y = 0.$

解 质量和对 Oy 轴的一次矩分别为

$$M = \rho \int_0^a dx \int_0^{(\sqrt{a}-\sqrt{x})^2} dy = \frac{1}{6} \rho a^2,$$

$$M_y = \rho \int_0^a x dx \int_0^{(\sqrt{a}-\sqrt{x})^2} dy = \frac{1}{30} \rho a^3.$$

于是,重心的横坐标为

$$x_0 = \frac{M_y}{M} = \frac{a}{5}.$$

由关于直线 $y = x$ 的对称性知

$$x_0 = y_0 = \frac{a}{5}.$$

4054. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} (x > 0, y > 0).$

解 质量和对 Oy 轴的一次矩分别为

$$M = \rho \int_0^a dx \int_0^{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}} dy = \rho \int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} dx$$

$$= 3a^2 \rho \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t dt^{**})$$

$$= 3a^2 \rho \int_0^{\frac{\pi}{2}} (\sin^4 t - \sin^6 t) dt$$

$$= 3a^2 \rho \left(\frac{3}{4} \cdot \frac{1}{2} - \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \right) \frac{\pi}{2} = \frac{3\pi a^2 \rho}{32},$$

$$M_y = \rho \int_0^a x dx \int_0^{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}} dy = \rho \int_0^a x (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} dx$$

$$= 3a^3 \rho \int_0^{\frac{\pi}{2}} \sin^4 t \cos^3 t dt = \frac{8a^3 \rho}{105}.$$

于是,重心的横坐标为

$$x_0 = \frac{M_y}{M} = \frac{256a}{315\pi}.$$

由关于直线 $y = x$ 的对称性知

$$x_0 = y_0 = \frac{256a}{315\pi}.$$

*) 代作换 $x = a\cos^3 t$.

$$4055. \left(\frac{x}{a} + \frac{y}{b} \right)^3 = \frac{xy}{c^2} \text{ (线圈)}.$$

解 此曲线在第一象限部分是一封闭曲线, 围成一图形 Ω . 作变量代

$$\begin{aligned} x &= \frac{a^2 b}{c^2} r \cos^4 \theta \sin^2 \theta, \\ y &= \frac{a b^2}{c^2} r \cos^2 \theta \sin^4 \theta, \end{aligned} \quad (0 \leq \theta \leq \frac{\pi}{2})$$

则原曲线方程变为 $r = 1$. 又容易算得

$$\frac{D(x, y)}{D(r, \theta)} = \frac{2a^3 b^3}{c^4} r (\sin^5 \theta \cos^7 \theta + \sin^7 \theta \cos^5 \theta),$$

故(利用 3856 题的结果)

$$\begin{aligned} M &= \iint_{\Omega} \rho dx dy \\ &= \frac{2a^3 b^3}{c^4} \rho \int_0^1 r dr \int_0^{\frac{\pi}{2}} (\sin^5 \theta \cos^7 \theta \\ &\quad + \sin^7 \theta \cos^5 \theta) d\theta \\ &= \frac{a^3 b^3}{c^4} \rho \left[\frac{1}{2} B(3, 4) + \frac{1}{2} B(4, 3) \right] \\ &= \frac{a^3 b^3}{c^4} \rho B(3, 4), \end{aligned}$$

$$\begin{aligned} M_y &= \iint_{\Omega} \rho x dx dy \\ &= \frac{2a^5 b^4}{c^6} \rho \int_0^1 r^2 dr \int_0^{\frac{\pi}{2}} \cos^4 \theta \sin^2 \theta (\sin^5 \theta \cos^7 \theta \end{aligned}$$

$$\begin{aligned}
& + \sin^7 \theta \cos^5 \theta) d\theta \\
& = \frac{2}{3} \frac{a^5 b^4}{c^6} \rho \left(\int_0^{\frac{\pi}{2}} \sin^7 \theta \cos^{11} \theta d\theta \right. \\
& \quad \left. + \int_0^{\frac{\pi}{2}} \sin^9 \theta \cos^9 \theta d\theta \right) \\
& = \frac{1}{3} \cdot \frac{a^5 b^4}{c^6} \rho [B(4, 6) + B(5, 5)].
\end{aligned}$$

于是,

$$x_0 = \frac{M_y}{M} = \frac{a^2 b}{3c^2} \cdot \frac{B(4, 6) + B(5, 5)}{B(3, 4)}.$$

由于,

$$B(4, 6) = \frac{\Gamma(4)\Gamma(6)}{\Gamma(10)} = \frac{3!5!}{9!},$$

$$B(5, 5) = \frac{[\Gamma(5)]^2}{\Gamma(10)} = \frac{(4!)^2}{9!},$$

$$B(3, 4) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2!3!}{6!},$$

代入, 化简得

$$x_0 = \frac{a^2 b}{3c^2} \cdot \frac{6! [3!5! + (4!)^2]}{2!3!9!} = \frac{a^2 b}{14c^2}.$$

同理, 可求得重心的纵坐标为

$$y_0 = \frac{M_x}{M} = \frac{\int_D \rho y dx dy}{\int_D \rho dx dy} = \frac{ab^2}{14c^2}.$$

4056. $(x^2 + y^2)^2 = 2a^2 xy (x > 0, y > 0).$

解 曲线的极坐标方程为

$$r^2 = a^2 \sin 2\varphi$$

质量和对 Oy 轴的一次矩为

$$\begin{aligned}
M &= \iint_D \rho dx dy \\
&= \rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^a \sqrt{\sin 2\varphi} r dr \\
&= \frac{\rho a^2}{2} \int_0^{\frac{\pi}{2}} \sin 2\varphi d\varphi = \frac{\rho a^2}{2}, \\
M_y &= \iint_D \rho x dx dy \\
&= \rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^a \sqrt{\sin 2\varphi} r \cdot r \cos \varphi dr \\
&= \frac{\rho a^3}{3} \int_0^{\frac{\pi}{2}} \cos \varphi \sin^{\frac{3}{2}} 2\varphi d\varphi \\
&= \frac{2\sqrt{2}\rho a^3}{3} \int_0^{\frac{\pi}{2}} \cos^{\frac{5}{2}} \varphi \sin^{\frac{3}{2}} \varphi d\varphi \\
&= \frac{2\sqrt{2}}{3} \rho a^3 \cdot \frac{1}{2} B\left(\frac{7}{4}, \frac{5}{4}\right) \\
&= \frac{2\sqrt{2}}{3} \rho a^3 \frac{\Gamma\left(\frac{7}{4}\right) \Gamma\left(\frac{5}{4}\right)}{2\Gamma(3)} \\
&= \frac{2\sqrt{2}}{3} \rho a^3 \frac{\frac{3}{4} \Gamma\left(\frac{3}{4}\right) \cdot \frac{1}{4} \Gamma\left(\frac{1}{4}\right)}{2 \cdot 2} \\
&= \frac{1}{8\sqrt{2}} \rho a^3 \frac{\pi}{2 \sin \frac{\pi}{4}} = \frac{1}{16} \pi \rho a^3.
\end{aligned}$$

于是,重心的横坐标为

$$x_0 = \frac{M_y}{M} = \frac{\pi a}{8}.$$

由关于直线 $y = x$ 的对称性知

$$x_0 = y_0 = \frac{\pi a}{8}.$$

*) 利用 3856 题的结果.

4057. $r = a(1 + \cos\varphi)$, $\varphi = 0$.

解 质量和一次矩分别为

$$\begin{aligned} M &= \rho \int_0^\pi d\varphi \int_0^{a(1+\cos\varphi)} r dr \\ &= \frac{1}{2} \rho a^2 \int_0^\pi (1 + \cos\varphi)^2 d\varphi = \frac{3}{4} \pi \rho a^2, \\ M_y &= \rho \int_0^\pi d\varphi \int_0^{a(1+\cos\varphi)} r \cdot r \cos\varphi dr \\ &= \frac{\rho a^3}{3} \int_0^\pi (1 + \cos\varphi)^3 \cos\varphi d\varphi \\ &= \frac{\rho a^3}{3} \left[\int_0^\pi (1 + \cos\varphi)^4 d\varphi \right. \\ &\quad \left. - \int_0^\pi (1 + \cos\varphi)^3 d\varphi \right] \\ &= \frac{\rho a^3}{3} \left(32 \int_0^{\frac{\pi}{2}} \cos^8 t dt - 16 \int_0^{\frac{\pi}{2}} \cos^6 t dt \right) \\ &= \frac{\rho a^3}{3} \left(32 \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right. \\ &\quad \left. - 16 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) = \frac{5\pi\rho a^3}{8}, \\ M_x &= \rho \int_0^\pi d\varphi \int_0^{a(1+\cos\varphi)} r \cdot r \sin\varphi dr \\ &= \frac{\rho a^3}{3} \int_0^\pi (1 + \cos\varphi)^3 \sin\varphi d\varphi \\ &= -\frac{\rho a^3}{3} \cdot \frac{(1 + \cos\varphi)^4}{4} \Big|_0^\pi = \frac{4\rho a^3}{3}. \end{aligned}$$

于是,重心的坐标为

$$x_0 = \frac{M_y}{M} = \frac{5}{6}a, \quad y_0 = \frac{M_x}{M} = \frac{16}{9\pi}a.$$

4058. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$), $y =$

解 质量和对 Ox 轴的一次矩为

$$M = \rho \int_0^{2\pi a} dx \int_0^y dy = \rho \int_0^{2\pi} a^2 (1 - \cos t)^2 dt \\ = 3\pi \rho a^2.$$

$$M_x = \rho \int_0^{2\pi a} dx \int_0^y y dy = \frac{1}{2} \rho \int_0^{2\pi} y^2 dx \\ = \frac{1}{2} \rho a^3 \int_0^{2\pi} (1 - \cos t)^3 dt = \frac{5}{2} \pi \rho a^3.$$

于是,

$$y_0 = \frac{M_x}{M} = \frac{5}{6}a.$$

由对称性知: $x_0 = \pi a$.

4059. 求圆形薄板 $x^2 + y^2 \leq a^2$ 的重心坐标, 设它在点 $M(x, y)$ 的密度与 M 点到 $A(a, 0)$ 点的距离成比例.

解 按题设, 密度

$$\rho = k \sqrt{(x-a)^2 + y^2} \quad (k \text{ 为常数}).$$

于是, 质量为

$$M = \int_{-a}^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} k \sqrt{(x-a)^2 + y^2} dy \\ = k \int_{-a}^a \left[y \sqrt{(x-a)^2 + y^2} + (x-a)^2 \right. \\ \left. \cdot \ln(y + \sqrt{(x-a)^2 + y^2}) \right] \Big|_0^{\sqrt{a^2-x^2}} dx \\ = k \int_{-a}^a \sqrt{2a}(a-x) \sqrt{a+x} dx \\ - k \int_{-a}^a \left[\frac{1}{2} \ln(a-x) \right] (a-x)^2 dx \\ + k \int_{-a}^a (a-x)^2 \ln(\sqrt{a+x} + \sqrt{2a}) dx$$

$$= I_1 - I_2 + I_3.$$

由于

$$\begin{aligned} I_1 &= k \int_{-a}^a \sqrt{2a} [- (a+x)^{\frac{3}{2}} + 2a(x+a)^{\frac{1}{2}}] dx \\ &= \sqrt{2a} k \cdot \left[-\frac{2}{5}(a+x)^{\frac{5}{2}} \right. \\ &\quad \left. + \frac{4a}{3}(a+x)^{\frac{3}{2}} \right] \Big|_{-a}^a \\ &= \frac{32}{15} ka^3, \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{k}{2} \int_0^{2a} t^2 \ln t dt = \frac{k}{6} t^3 \ln t \Big|_0^{2a} - \frac{k}{6} \int_0^{2a} t^3 \cdot \frac{1}{t} dt \\ &= \frac{4}{3} ka^3 \cdot \ln 2a - \frac{4}{9} ka^3, \end{aligned}$$

$$\begin{aligned} I_3 &= k \cdot 2 \int_0^{\sqrt{2a}} t(2a - t^2) \ln(t + \sqrt{2a}) dt \\ &= 8a^2 k \int_0^{\sqrt{2a}} t \ln(t + \sqrt{2a}) dt \\ &\quad - 8ka \int_0^{\sqrt{2a}} t^3 \ln(t + \sqrt{2a}) dt \\ &\quad + 2k \int_0^{\sqrt{2a}} t^5 \ln(t + \sqrt{2a}) dt \\ &= 8ka^2 \left(\frac{a}{2} + a \ln \sqrt{2a} \right) \\ &\quad - 8ka \left(\frac{7}{12} a^2 + a^2 \ln \sqrt{2a} \right) \\ &\quad + 2k \left(\frac{37}{45} a^3 + \frac{4}{3} a^3 \ln \sqrt{2a} \right) \\ &= \frac{44}{45} ka^3 + \frac{8}{3} ka^3 \ln \sqrt{2a} = \frac{44}{45} ka^3 + \frac{4}{3} ka^3 \ln 2a. \end{aligned}$$

因而最后得

$$\begin{aligned}
 M &= \frac{32}{15}ka^3 - \left(\frac{4}{3}ka^3\ln 2a - \frac{4}{9}ka^3 \right) \\
 &\quad + \left(\frac{44}{45}ka^3 + \frac{4}{3}ka^3\ln 2a \right) \\
 &= \frac{32}{9}ka^3.
 \end{aligned}$$

仿照上述方法可求得一次矩

$$\begin{aligned}
 M_y &= \int_{-a}^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} kx \sqrt{(x-a)^2 + y^2} dy \\
 &= -\frac{32}{45}ka^4.
 \end{aligned}$$

而由对称性知: $M_x = 0$.

于是,重心的坐标为

$$x_0 = \frac{M_y}{M} = -\frac{a}{5}, \quad y_0 = \frac{M_x}{M} = 0.$$

4060. 求由变动的面积的重心所描写出来的曲线,所指的变动面积是被曲线

$$y = \sqrt{2px}, \quad y = 0, \quad x = X$$

所界的.

解 变动面积的质量

$$M = \rho \int_0^X dx \int_0^{\sqrt{2px}} dy = \rho \frac{2\sqrt{2p}}{3} X^{\frac{3}{2}},$$

而一次矩

$$M_y = \rho \int_0^X x dx \int_0^{\sqrt{2px}} dy = \rho \frac{2\sqrt{2p}}{5} X^{\frac{5}{2}},$$

$$M_x = \rho \int_0^X dx \int_0^{\sqrt{2px}} y dy = \rho \frac{1}{2} p X^2.$$

于是,变动面积的重心为

$$x_0 = \frac{M_y}{M} = \frac{3}{5}X, \quad y_0 = \frac{M_x}{M} = \frac{3\sqrt{pX}}{4\sqrt{2}}.$$

因此,重心的轨迹方程为

$$y_0 = \frac{3}{4\sqrt{2}} \sqrt{p \cdot \frac{5}{3}x_0} = \frac{1}{8} \sqrt{30px_0},$$

此即所求的曲线方程,其图形是抛物线的一半.

求由下列曲线所界的面只($\rho=1$)对于坐标轴 Ox 和 Oy 的转动惯量 I_x 和 I_y :

4061. $\frac{x}{b_1} + \frac{y}{h} = 1, \quad \frac{x}{b_2} + \frac{y}{h} = 1, \quad y=0 (b_1 > 0, b_2 > 0, h > 0).$

解 若设 $b_2 > b_1$, 则

$$\begin{aligned} I_x &= \int_0^h y^2 dy \int_{\left(1-\frac{y}{h}\right)b_1}^{\left(1-\frac{y}{h}\right)b_2} dx = (b_2 - b_1) \\ &\quad \cdot \int_0^h y^2 \left(1 - \frac{y}{h}\right) dy = \frac{(b_2 - b_1)h^3}{12}, \\ I_y &= \int_0^h dy \int_{\left(1-\frac{y}{h}\right)b_1}^{\left(1-\frac{y}{h}\right)b_2} x^2 dx = \frac{b_2^3 - b_1^3}{3} \int_0^h \left(1 - \frac{y}{h}\right)^3 dy \\ &= \frac{h(b_2^3 - b_1^3)}{12}; \end{aligned}$$

若设 $b_1 > b_2$, 则

$$I_x = \frac{(b_1 - b_2)h^3}{12}, \quad I_y = \frac{h(b_1^3 - b_2^3)}{12}.$$

4062. $(x-a)^2 + (y-a)^2 = a^2, \quad x=0, \quad y=0 (0 \leq x \leq a).$

解
$$\begin{aligned} I_x &= \int_0^a dx \int_0^{\sqrt{2ax-x^2}} y^2 dy \\ &= \frac{1}{3} \int_0^a [a^3 - 3a^2 \sqrt{2ax-x^2} + 3a(2ax-x^2) \\ &\quad - (2ax-x^2)^{\frac{3}{2}}] dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \left[a^3 x - 3a^2 \left(\frac{x-a}{2} \sqrt{2ax-x^2} \right. \right. \\
&\quad \left. \left. + \frac{a^2}{2} \arcsin \frac{x-a}{2} \right) + 3a^2 x^2 - ax^3 \right] \Big|_0^a \\
&\quad - \frac{1}{3} \int_0^a (2ax-x^2)^{\frac{3}{2}} dx \\
&= a^4 \left(1 - \frac{\pi}{4} \right) - \frac{1}{3} \int_{-\frac{\pi}{2}}^0 a^4 \cos^4 t dt \\
&= a^4 \left(1 - \frac{\pi}{4} \right) - \frac{1}{3} \int_0^{\frac{\pi}{2}} a^4 \cos^4 t dt \\
&= a^4 \left(1 - \frac{\pi}{4} \right) - \frac{a^4}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{a^4}{16} (16 - 5\pi).
\end{aligned}$$

5π).

利用图形的对称性, 即得 $I_y = I_x = \frac{a^4}{16} (16 - 5\pi)$.

*) 作代换 $x - a = a \sin t$.

4063. $r = a(1 + \cos \varphi)$.

解 曲线所界的平面域可表示为

$$-\pi \leq \varphi \leq \pi, \quad 0 \leq r \leq a(1 + \cos \varphi).$$

于是,

$$\begin{aligned}
I_x &= \int_{-\pi}^{\pi} \int_0^{a(1+\cos\varphi)} r^2 \sin^2 \varphi \cdot r dr d\varphi \\
&= \int_{-\pi}^{\pi} \frac{1}{4} a^4 (1 + \cos \varphi)^4 \sin^2 \varphi d\varphi \\
&= 2 \cdot \frac{1}{4} a^4 \int_0^{\pi} (1 + 4\cos \varphi + 6\cos^2 \varphi + 4\cos^3 \varphi \\
&\quad + \cos^4 \varphi) \sin^2 \varphi d\varphi \\
&= \frac{1}{2} \pi a^4 \cdot \frac{21}{16} = \frac{21}{32} \pi a^4. \\
I_y &= \int_{-\pi}^{\pi} \int_0^{a(1+\cos\varphi)} r^2 \cos^2 \varphi \cdot r dr d\varphi
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}a^4 \int_0^\pi (1 + \cos\varphi)^4 \cos^2\varphi d\varphi \\
&= \frac{1}{2}a^4 \int_0^\pi (\cos^2\varphi + 4\cos^3\varphi + 6\cos^4\varphi \\
&\quad + 4\cos^5\varphi - \cos^6\varphi) d\varphi \\
&= \frac{49}{32}\pi a^4.
\end{aligned}$$

*) 对于任意自然数 n , 有

$$\int_0^\pi \cos^n \varphi d\varphi = \begin{cases} 2 \int_0^{\frac{\pi}{2}} \cos^n \varphi d\varphi, & \text{当 } n \text{ 为偶数;} \\ 0, & \text{当 } n \text{ 为奇数.} \end{cases}$$

为算出 I_x, I_y 的值, 也可变换被积函数的形式, 直接用换元法计算, 这样较简单.

事实上, 我们有

$$\begin{aligned}
I_x &= \frac{a^4}{2} \int_0^\pi (1 + \cos\varphi)^4 \sin^2\varphi d\varphi \\
&= 2^6 a^4 \int_0^\pi \cos^{10} \frac{\varphi}{2} \sin^2 \frac{\varphi}{2} d\left(\frac{\varphi}{2}\right) \\
&= 2^6 a^4 \int_0^{\frac{\pi}{2}} \cos^{10} x (1 - \cos^2 x) dx \\
&= 2^6 a^4 \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \left(1 - \frac{11}{12}\right) \frac{\pi}{2} \\
&= \frac{21}{32} \pi a^4.
\end{aligned}$$

$$\begin{aligned}
I_y &= \frac{a^4}{2} \int_0^\pi (1 + \cos\varphi)^4 \cos^2\varphi d\varphi \\
&= \frac{a^4}{2} \int_0^\pi (1 + \cos\varphi)^4 d\varphi - \frac{21}{32} \pi a^4 \\
&= 2^4 a^4 \int_0^{\frac{\pi}{2}} \cos^3 x dx - \frac{21}{32} \pi a^4 \\
&= \frac{70}{32} \pi a^4 - \frac{21}{32} \pi a^4
\end{aligned}$$

$$= \frac{49}{32}\pi a^4.$$

$$4064. x^4 + y^4 = a^2(x^2 + y^2).$$

解 曲线的图形关于两坐标轴和直线 $y = x$ 是对称的, 参看 1542 题的图形. 曲线的极坐标方程为

$$r^2 = \frac{a^2}{\cos^4 \varphi + \sin^4 \varphi} \quad (0 \leq \varphi \leq 2\pi).$$

根据对称性, 只要算出从 $\varphi = 0$ 到 $\varphi = \frac{\pi}{4}$ 部分面积的转动惯量再八倍起来即得结果, 并且显然有 $I_x = I_y$. 于是, 我们有

$$\begin{aligned} I_x = I_y &= 4 \iint_D (x^2 + y^2) dx dy \\ &= 4 \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sqrt{\frac{a^2}{\cos^4 \varphi + \sin^4 \varphi}}} r^3 dr \\ &= \int_0^{\frac{\pi}{4}} \frac{a^4 d\varphi}{(\cos^4 \varphi + \sin^4 \varphi)^2} \\ &= \int_0^{\frac{\pi}{4}} \frac{a^4 d\varphi}{(1 - 2\sin^2 \varphi \cos^2 \varphi)^2} \\ &= \int_0^{\frac{\pi}{4}} \frac{a^4 d\varphi}{\left(\frac{3}{4} + \frac{1}{4}\cos 4\varphi\right)^2} \\ &= 16a^4 \int_0^{\frac{\pi}{4}} \frac{d\varphi}{(3 + \cos 4\varphi)^2} \\ &= \frac{4a^4}{9} \int_0^{\pi} \frac{dx}{\left(1 + \frac{1}{3}\cos x\right)^2} \quad *) \end{aligned}$$

$$\begin{aligned}
&= \frac{4a^4}{9} \left[-\frac{\frac{1}{3}\sin x}{\left(1 - \frac{1}{9}\right)\left(1 + \frac{1}{3}\cos x\right)} \right. \\
&\quad \left. + \frac{2}{\left(1 - \frac{1}{9}\right)^{\frac{3}{2}}} \operatorname{arctg} \left(\sqrt{\frac{1 - \frac{1}{3}}{1 + \frac{1}{3}}} \operatorname{tg} \frac{x}{2} \right) \right] \Big|_0^{\pi} \\
&= \frac{4a^4}{9} \cdot 2 \left(\frac{9}{8} \right)^{\frac{3}{2}} \frac{\pi}{2} = \frac{3\pi a^4}{4\sqrt{2}}.
\end{aligned}$$

*) 作代换 $x = 4\varphi$.

* *) 利用 2063 题的结果.

4065. $xy = a^2, xy = 2a^2, x = 2y, 2x = y \quad (x > 0, y > 0)$.

解 作代换 $xy = u, \frac{y}{x} = v$, 则 $x = \sqrt{\frac{u}{v}}, y = \sqrt{uv}$

且雅哥比式的绝对值 $|I| = \frac{1}{2v}$, 曲线所界的面积即积分域变为

$$a^2 \leq u \leq 2a^2, \quad \frac{1}{2} \leq v \leq 2.$$

于是,

$$I_x = \iint_{\Omega} y^2 dx dy = \int_{\frac{1}{2}}^2 dv \int_{a^2}^{2a^2} \frac{uv}{2v} du = \frac{9a^4}{8},$$

$$I_y = \iint_{\Omega} x^2 dx dy = \int_{\frac{1}{2}}^2 dv \int_{a^2}^{2a^2} \frac{u}{2v^2} du = \frac{9a^4}{8}.$$

4066. 求面积 S 的极转动惯量

$$I_0 = \iint_S (x^2 + y^2) dx dy,$$

面积 S 是由曲线

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

所界的.

解 引用极坐标, 则面积 S 的界线的极坐标方程为

$$r^2 = a^2 \cos 2\varphi,$$

这是双纽线. 利用对称性, 得

$$\begin{aligned} I_0 &= \iint_S (x^2 + y^2) dx dy = 4 \int_0^{\frac{\pi}{4}} d\varphi \int_0^a \sqrt{\cos 2\varphi} r^3 dr \\ &= \int_0^{\frac{\pi}{4}} a^4 \cos^2 2\varphi d\varphi = \frac{\pi a^4}{8}. \end{aligned}$$

4067. 证明公式

$$I_l = I_{l_0} + Sd^2,$$

其中 I_l, I_{l_0} 是面积 S 对于二平行轴 l 和 l_0 的转动惯量, 其中 l_0 是通过面积的重心, 而 d 为两轴间的距离.

证 取 l_0 轴为 Ox 轴, 面积的重心为坐标原点, 则

$$\begin{aligned} I_l &= \iint_S (y - d)^2 dx dy = \iint_S y^2 dx dy \\ &\quad - 2d \iint_S y dx dy + d^2 \iint_S dx dy. \end{aligned}$$

因为 l_0 通过面积 S 的重心, 故

$$y_0 = \frac{1}{S} \iint_S y dx dy = 0, \text{ 即 } \iint_S y dx dy = 0.$$

又

$$\iint_S y^2 dx dy = I_{l_0}, \quad \iint_S dx dy = S,$$

于是,

$$I_l = I_{l_0} + Sd^2.$$

4068. 证明面积 S 对于通过重心 $O(0,0)$ 并与 Ox 轴组成 α 角的直线的转动惯量等于

$$I = I_x \cos^2 \alpha - 2I_{xy} \sin \alpha \cos \alpha + I_y \sin^2 \alpha,$$

其中 I_x 和 I_y 为面积 S 对于 Ox 轴和 Oy 轴的转动惯量及 I_{xy} 为离心惯量:

$$I_{xy} = \iint_S \rho xy dx dy.$$

证 今取直角坐标系 $Ox'y'$, 使 Ox' 轴与 Ox 轴的夹角为 α , 则有

$$x' = x \cos \alpha + y \sin \alpha, y' = -x \sin \alpha + y \cos \alpha.$$

这就是旋转变换, 雅哥比式的绝对值

$$|I| = \left| \frac{D(x', y')}{D(x, y)} \right| = 1.$$

于是,

$$\begin{aligned} I &= \iint_S y'^2 \rho dx' dy' = \iint_S (-x \sin \alpha + y \cos \alpha)^2 \rho dx dy \\ &= \cos^2 \alpha \iint_S y^2 \rho dx dy - 2 \sin \alpha \cos \alpha \\ &\quad \cdot \iint_S \rho xy dx dy + \sin^2 \alpha \iint_S \rho x^2 dx dy \\ &= I_x \cos^2 \alpha - 2I_{xy} \sin \alpha \cos \alpha + I_y \sin^2 \alpha. \end{aligned}$$

4069. 求以 a 为边的正三角形的面积对于通过三角形重心并与它的高成 α 角的直线的转动惯量.

解 利用上题的结果. 取重心为坐标原点. 不妨取 Ox 轴平行于三角形的一条边, 则过重心与高成 α 角的直线, 即为过坐标原点与 Ox 轴成 α 角的直线. 于是, 要求的转动惯量为

$$I_o = I_x \cos^2 \alpha - 2I_{xy} \sin \alpha \cos \alpha + I_y \sin^2 \alpha.$$

由于三角形三边所在的直线方程为

$$y = -\frac{a}{2\sqrt{3}}, y = -\sqrt{3}x + \frac{a}{\sqrt{3}},$$

$$y = \sqrt{3}x + \frac{a}{\sqrt{3}},$$

所以,根据对称性知:

$$\begin{aligned} I_x &= 2 \int_0^{\frac{a}{2}} dx \int_{-\frac{a}{2\sqrt{3}}}^{-\sqrt{3}x + \frac{a}{\sqrt{3}}} y^2 dy \\ &= 2 \int_0^{\frac{a}{2}} \frac{1}{3} \left[\left(-\sqrt{3}x + \frac{a}{\sqrt{3}} \right)^3 \right. \\ &\quad \left. - \left(-\frac{a}{2\sqrt{3}} \right)^3 \right] dy \\ &= 2 \int_0^{\frac{a}{2}} \left(-\sqrt{3}x^3 + \sqrt{3}ax^2 - \frac{\sqrt{3}}{3}a^2x \right. \\ &\quad \left. + \frac{\sqrt{3}}{24} \right) dx \end{aligned}$$

$$= 2\sqrt{3}a^4 \left(\frac{1}{48} - \frac{1}{64} \right) = \frac{a^4}{32\sqrt{3}};$$

$$I_{xy} = \iint_S xy dx dy = 0;$$

$$\begin{aligned} I_y &= 2 \int_0^{\frac{a}{2}} dx \int_{-\frac{a}{2\sqrt{3}}}^{-\sqrt{3}x + \frac{a}{\sqrt{3}}} x^2 dy \\ &= 2 \int_0^{\frac{a}{2}} x^2 \left[\left(-\sqrt{3}x + \frac{a}{\sqrt{3}} \right) + \frac{a}{2\sqrt{3}} \right] dx \\ &= 2 \int_0^{\frac{a}{2}} \left(-\sqrt{3}x^3 + \frac{\sqrt{3}a}{2}x^2 \right) dx \end{aligned}$$

$$= \sqrt{3} a^4 \left(\frac{1}{24} - \frac{1}{32} \right) = \frac{a^4}{32 \sqrt{3}}.$$

于是,

$$I_a = \frac{a^4}{32 \sqrt{3}} \cos^2 \alpha + \frac{a^4}{32 \sqrt{3}} \sin^2 \alpha = \frac{a^4}{32 \sqrt{3}}.$$

4070. 设有水平面为 $z = h$ 的圆柱形容器 $x^2 + y^2 = a^2, z = 0$, 求它侧壁上 ($x \geq 0$) 水的压力.

解 用 X 与 Y 分别表示压力在 Ox 轴与 Oy 轴上的投影. 由对称性, 显然有 $Y = 0$. 下面求 X . 由于 $dS = ad\theta dz$ ($-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$), 而在面元 dS 上的压力在 Ox 轴上的投影 dX 为 $(zdS)\cos\theta$. 于是,

$$\begin{aligned} X &= \iint_S z \cos\theta dS = \iint_S az \cos\theta d\theta dz \\ &= a \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta \right] \cdot \left(\int_0^h z dz \right) = ah^2. \end{aligned}$$

4071. 半径为 a 的球体沉入密度为 δ 的液体中的深度为 h (由球心量起), 这里 $h \geq a$. 求在球表面的上部和下部的液体压力.

解 设球面方程为 $x^2 + y^2 + z^2 = a^2$, 则在球面上的点 (x, y, z) 处沉入液体的深度 d 为

$$d = h - z \quad (-a \leq z \leq h).$$

于是, 上半球面 S_1 的点和下半球面 S_2 的点的深度分别为:

$$\begin{aligned} d &= h - \sqrt{a^2 - (x^2 + y^2)}, \\ d &= h + \sqrt{a^2 - (x^2 + y^2)}. \end{aligned}$$

根据对称性知, 压力在 Ox 轴上和 Oy 轴的射影均为零,

故只要计算压力在 Oz 轴上的射影. 液体作用于球面上部和下部的压力分别记以 p_1 和 p_2 , 并设 γ 为球上各点处压力的方向(即内法线方向)与 Oz 轴正向的夹角, 则

$$\begin{aligned}
 p_1 &= \iint_{S_1} d\delta \cos \gamma ds \\
 &= - \iint_{x^2+y^2 \leq a^2} \delta [h - \sqrt{a^2 - (x^2 + y^2)}] dxdy \\
 &= -h\pi a^2 \delta + \int_0^{2\pi} d\theta \int_0^a \sqrt{1-r^2} r dr \\
 &= -h\pi a^2 \delta + \left[-\frac{2\pi\delta}{3} \sqrt{(a^2 - r^2)^3} \right] \Big|_0^a \\
 &= -\pi a^2 \delta \left(h - \frac{2a}{3} \right) \quad (p_1 < 0 \text{ 表示压力向下}).
 \end{aligned}$$

同理, 我们有

$$\begin{aligned}
 p_2 &= \iint_{S_2} d\delta \cos \gamma dS \\
 &= \iint_{x^2+y^2 \leq a^2} \delta [h + \sqrt{a^2 - (x^2 + y^2)}] dxdy \\
 &= \pi a^2 \delta \left(h + \frac{2a}{3} \right) \quad (p_2 > 0 \text{ 表示压力向上}).
 \end{aligned}$$

4072. 底半径为 a 高为 b 的直圆柱完全沉入密度为 δ 的液体中, 其中心在液面下的深度为 h , 而圆柱的轴与铅垂线成 α 角, 求在圆柱上底和下底的液体压力.

解 取圆柱的中心为坐标原点, 取 Oxy 平面是水平的, 再取圆柱的轴(朝上的方向)在 Oxy 平面上的投影所在的方向为 Ox 轴, 取 Oz 轴垂直朝上, 最后取 Oy 轴使 Ox 轴 Oy 轴和 Oz 轴构成右手系.

于是, 液面方程为 $z = h$. 设圆柱上底为 S_1 , 下底为

S_2 , 则 S_1 所在平面的方程为

$$x\sin\alpha + z\cos\alpha = \frac{b}{2}, \quad (1)$$

S_2 所在平面的方程为

$$x\sin\alpha + z\cos\alpha = -\frac{b}{2}. \quad (2)$$

在点 (x, y, z) 处 ($z \leq h$) 液体的深度为 $h - z$. 用 X_1, Y_1 和 Z_1 分别表示液体在圆柱上底 S_1 上压力在 Ox 轴, Oy 轴和 Oz 轴上的投影. 同样, 用 X_2, Y_2 和 Z_2 分别表示在 S_2 上压力在 Ox 轴, Oy 轴和 Oz 轴上的投影. 显然, $Y_1 = Y_2 = 0$. 我们有

$$X_1 = - \iint_{S_1} \delta(h - z)\sin\alpha dS = -\delta\sin\alpha \iint_{S_1} (h - z)dS, \quad (3)$$

$$Z_1 = - \iint_{S_1} \delta(h - z)\cos\alpha dS = -\delta\cos\alpha \iint_{S_1} (h - z)dS. \quad (4)$$

由(1)式知, 在 S_1 上有

$$z = \frac{1}{\cos\alpha} \left(\frac{b}{2} - x\sin\alpha \right).$$

于是, 注意到 S_1 的面积为 πa^2 , 可知

$$\begin{aligned} \iint_{S_1} (h - z)dS &= \iint_{S_1} \left[h - \frac{1}{\cos\alpha} \left(\frac{b}{2} - x\sin\alpha \right) \right] dS \\ &= \left(h - \frac{b}{2} \cdot \frac{1}{\cos\alpha} \right) \iint_{S_1} dS + \frac{\sin\alpha}{\cos\alpha} \iint_{S_1} x dS \\ &= \left(h - \frac{b}{2} \cdot \frac{1}{\cos\alpha} \right) \pi a^2 + \frac{\sin\alpha}{\cos\alpha} \iint_{S_1} x dS. \end{aligned}$$

由于 $\frac{1}{\pi a^2} \int_{S_1} x dS$ 是 S_1 的重心的 x 坐标, 也即 $\frac{b}{2} \sin \alpha$, 故

$$\int_{S_1} x dS = \frac{1}{2} \pi a^2 b \sin \alpha, \text{ 代入即得}$$

$$\begin{aligned} \int_{S_1} (h - z) dS &= \left(h - \frac{b}{2 \cos \alpha} \right) \pi a^2 + \frac{1}{2} \pi a^2 b \frac{\sin^2 \alpha}{\cos \alpha} \\ &= \left(h - \frac{b}{2} \cos \alpha \right) \pi a^2. \end{aligned}$$

以此代入(3)式与(4)式, 得

$$X_1 = -\pi a^2 \delta \left(h - \frac{b}{2} \cos \alpha \right) \sin \alpha,$$

$$Z_1 = -\pi a^2 \delta \left(h - \frac{b}{2} \cos \alpha \right) \cos \alpha.$$

同理, 我们有

$$X_2 = \iint_{S_2} \delta (h - z) \sin \alpha dS = \delta \sin \alpha \iint_{S_2} (h - z) dS,$$

$$Z_2 = \iint_{S_2} \delta (h - z) \cos \alpha dS = \delta \cos \alpha \iint_{S_2} (h - z) dS.$$

再注意到(2)式, 类似地可计算得

$$\begin{aligned} \iint_{S_2} (h - z) dS &= \iint_{S_2} \left(h + \frac{1}{\cos \alpha} \left(\frac{b}{2} + x \sin \alpha \right) \right) dS \\ &= \left(h + \frac{b}{2} \cos \alpha \right) \pi a^2. \end{aligned}$$

于是,

$$X_2 = \pi a^2 \delta \left(h + \frac{b}{2} \cos \alpha \right) \sin \alpha,$$

$$Z_2 = \pi a^2 \delta \left(h + \frac{b}{2} \cos \alpha \right) \cos \alpha.$$

4073. 求均匀的圆柱 $x^2 + y^2 \leq a^2, 0 \leq z \leq h$ 对质点 $P(0, 0,$

b) 的引力, 设圆柱的质量等于 M , 而点的质量等于 m .

解 根据对称性知, 引力在 Ox 轴和 Oy 轴上的射影等于零, 故只要计算引力在 Oz 轴上的射影 F_z . 今取圆环, 其体积为

$$dV = 2\pi r dr dz,$$

则相应的质量为

$$dM = \frac{2\pi r M dr dz}{\pi a^2 h} = \frac{2Mr}{a^2 h} dr dz,$$

吸引质点 P 的引力

$$dF_z = - \frac{2krmM(b-z)}{a^2 h \sqrt{[r^2 + (b-z)^2]^3}} dr dz.$$

于是, 所求的引力

$$\begin{aligned} F_z &= - \frac{2kmM}{a^2 h} \int_0^h \int_0^a \frac{r(b-z)}{\sqrt{[r^2 + (b-z)^2]^3}} dr dz \\ &= - \frac{2kmM}{a^2 h} \left[\int_0^h \operatorname{sgn}(b-z) dz \right. \\ &\quad \left. - \int_0^h \frac{b-z}{\sqrt{a^2 + (b-z)^2}} dz \right] \\ &= - \frac{2kmM}{a^2 h} [|b| - |b-h| + \sqrt{a^2 + (b-h)^2} \\ &\quad - \sqrt{a^2 + b^2}], \end{aligned}$$

其中 k 为引力常数.

4074. 物体在椭圆面

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

上压力的分布由公式

$$p = p_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)$$

所给出. 求物体在此面上的平均压力.

解 物体在椭圆面上的平均压力

$$\begin{aligned}
 p_{cp} &= \frac{1}{\pi ab} \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} p_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) dx dy \\
 &= \frac{4}{\pi ab} \int_0^{\frac{\pi}{2}} d\theta \int_0^1 p_0 (1 - r^2) ab r dr \\
 &= \frac{4}{\pi ab} \cdot \frac{\pi}{2} \cdot \frac{p_0 ab}{4} = \frac{p_0}{2}.
 \end{aligned}$$

4075. 草地的形状为以 a 和 b 为边的矩形, 均匀地盖上密度为 p 千克/平方米的砍倒的草. 假设运送 P 千克重到距离为 r 远的地方所化的功为 kPr ($0 < k < 1$). 要把所有的干草聚集在草地的中心, 最少必须化多少功?

解 不妨将坐标原点取在矩形的中心, Ox 轴平行于 a 边, Oy 轴平行于 b 边. 由于将面积 $dx dy$ 上的草移到中心要化的功为

$$dW = kp \sqrt{x^2 + y^2} dx dy,$$

并利用对称性, 便知所要求的功为

$$\begin{aligned}
 W &= 4kp \int_0^{\frac{b}{2}} \int_0^{\frac{a}{2}} \sqrt{x^2 + y^2} dx dy \\
 &= 4kp \left(\int_0^{\operatorname{arctg} \frac{b}{a}} \int_0^{\frac{a}{2\cos\varphi}} r^2 dr d\varphi \right. \\
 &\quad \left. + \int_{\operatorname{arctg} \frac{b}{a}}^{\frac{\pi}{2}} \int_0^{\frac{b}{2\sin\varphi}} r^2 dr d\varphi \right) \\
 &= \frac{kp}{6} \left(a^3 \int_0^{\operatorname{arctg} \frac{b}{a}} \frac{1}{\cos^3 \varphi} d\varphi \right. \\
 &\quad \left. + b^3 \int_{\operatorname{arctg} \frac{b}{a}}^{\frac{\pi}{2}} \frac{1}{\sin^3 \varphi} d\varphi \right).
 \end{aligned}$$

但是,

$$\begin{aligned}
\int_0^{\operatorname{arctg} \frac{b}{a}} \frac{1}{\cos^3 \varphi} d\varphi &= \left[\frac{\sin \varphi}{2 \cos^2 \varphi} \right. \\
&\quad \left. + \frac{1}{2} \ln \left| \operatorname{tg} \left(\frac{\varphi}{2} + \frac{\pi}{4} \right) \right| \right] \Big|_0^{\operatorname{arctg} \frac{b}{a}} \\
&= \frac{b \sqrt{a^2 + b^2}}{2a^2} + \frac{1}{2} \ln \frac{b + \sqrt{a^2 + b^2}}{a}, \\
\int_{\operatorname{arctg} \frac{b}{a}}^{\frac{\pi}{2}} \frac{1}{\sin^3 \varphi} d\varphi &= \left[-\frac{\cos \varphi}{2 \sin^2 \varphi} \right. \\
&\quad \left. + \frac{1}{2} \ln \left| \operatorname{tg} \frac{\varphi}{2} \right| \right] \Big|_{\operatorname{arctg} \frac{b}{a}}^{\frac{\pi}{2}} \\
&= \frac{a \sqrt{a^2 + b^2}}{2b^2} + \frac{1}{2} \ln \frac{a + \sqrt{a^2 + b^2}}{b},
\end{aligned}$$

于是,我们有

$$\begin{aligned}
W = \frac{kp}{12} &\left(2ab \sqrt{a^2 + b^2} + a^3 \ln \frac{b + \sqrt{a^2 + b^2}}{a} \right. \\
&\quad \left. + b^3 \ln \frac{a + \sqrt{a^2 + b^2}}{b} \right).
\end{aligned}$$

*) 利用 2000 题的结果.

* *) 利用 1999 题的结果.

§ 6. 三 重 积 分

1° 三重积分的直接算法 设函数 $f(x, y, z)$ 是连续的, 且有界域 V 由下列不等式确定出来:

$$\begin{aligned}
x_1 &\leq x \leq x_2, y_1(x) \leq y \leq y_2(x), \\
z_1(x, y) &\leq z \leq z_2(x, y),
\end{aligned}$$

其中 $y_1(x), y_2(x), z_1(x, y), z_2(x, y)$ 皆为连续函数. 则函数 $f(x, y, z)$

展布于域 V 内的三重积分可按公式

$$\begin{aligned} \iiint_V f(x, y, z) dx dy dz \\ = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \end{aligned}$$

来计算. 有时采用下面的公

式也很方便

$$\begin{aligned} \iiint_V f(x, y, z) dx dy dz \\ = \int_{x_1}^{x_2} dx \iint_{S(x)} f(x, y, z) dy dz, \end{aligned}$$

其中 $S(x)$ 是用平面 $X = x$ 截域 V 所得的截断面.

2° 三重积分中的变量代换 若 $Oxyz$ 空间的有界三维闭域 V 借助于下列连续可微分的函数双方单值地反应到 $O'uvw$ 空间的域 V'

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w),$$

并且当 $(u, v, w) \in V'$ 时,

$$I = \frac{D(x, y, z)}{D(u, v, w)} \neq 0,$$

则下面的公式成立

$$\begin{aligned} \iiint_V f(x, y, z) dx dy dz \\ = \iiint_{V'} f[x(u, v, w), y(u, v, w), z(u, v, w)] |I| du dv dw. \end{aligned}$$

在特殊情况下, 有: 1) 圆柱坐标系 φ, r, h , 其中

$$x = r \cos \varphi, y = r \sin \varphi, z = h,$$

及

$$\frac{D(x, y, z)}{D(\varphi, r, h)} = r,$$

2) 球坐标系 φ, ψ, r , 其中

$$x = r \cos \varphi \cos \psi, y = r \sin \varphi \cos \psi, z = r \sin \psi,$$

及

$$\frac{D(x, y, z)}{D(\varphi, \psi, r)} = r^2 \cos \psi.$$

计算下列三重积分：

4076. $\iiint_V xy^2z^3 dx dy dz$, 此处 V 是由曲面 $z = xy, y = x, x = 1, z = 0$ 所界的区域.

解
$$\begin{aligned}\iiint_V xy^2z^3 dx dy dz &= \int_0^1 x dx \int_0^x y^2 dy \int_0^{xy} z^3 dz \\ &= \frac{1}{364}.\end{aligned}$$

4077. $\iiint_V \frac{dx dy dz}{(1+x+y+z)^3}$, 此处 V 是由曲面 $x+y+z=1, x=0, y=0, z=0$ 所界的区域.

解
$$\begin{aligned}\iiint_V \frac{dx dy dz}{(1+x+y+z)^3} &= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} \\ &= \int_0^1 dx \int_0^{1-x} \left(-\frac{1}{2(1+x+y+z)^2} \right) \Big|_0^{1-x-y} dy \\ &= \int_0^1 dx \int_0^{1-x} \left(-\frac{1}{8} + \frac{1}{2(1+x+y)^2} \right) dy \\ &= \int_0^1 \left(-\frac{1}{8}y + \frac{1}{2(1+x+y)} \right) \Big|_0^{1-x} dx \\ &= \int_0^1 \left(-\frac{3}{8} + \frac{x}{8} + \frac{1}{2(1+x)} \right) dx \\ &= \left(-\frac{3}{8}x + \frac{1}{16}x^2 + \frac{1}{2}\ln(1+x) \right) \Big|_0^1 = \frac{1}{2}\ln 2 - \frac{5}{16}.\end{aligned}$$

4078. $\iiint_V xyz dx dy dz$, 此处 V 是由曲面 $x^2 + y^2 + z^2 = 1, x=0, y=0, z=0$ 所界的区域.

$$\begin{aligned}
 \text{解} \quad & \iiint_V xyz dx dy dz \\
 &= \int_0^1 x dx \int_0^{\sqrt{1-x^2}} y dy \int_0^{\sqrt{1-x^2-y^2}} z dz \\
 &= \frac{1}{2} \int_0^1 x dx \int_0^{\sqrt{1-x^2}} y(1-x^2-y^2) dy \\
 &= \frac{1}{8} \int_0^1 x(1-x^2)^2 dx = \frac{1}{48}.
 \end{aligned}$$

4079. $\iiint_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx dy dz$, 此处 V 是由曲面

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

所界的区域.

解 设 P_x, Q_y, R_z 分别表示立体 V 与平面 $x = \text{常数}$, $y = \text{常数}$, $z = \text{常数}$ 所截部分在 O_{yx}, O_{xz}, O_{xy} 平面上的射影, 则有

$$\begin{aligned}
 & \iiint_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx dy dz \\
 &= \int_{-a}^a \frac{x^2}{a^2} dx \int_{P_x} dy dz + \int_{-b}^b \frac{y^2}{b^2} dy \int_{Q_y} dz dx \\
 &+ \int_{-c}^c \frac{z^2}{c^2} dz \int_{R_z} dx dy \\
 &= \frac{\pi bc}{a^2} \int_{-a}^a x^2 \left(1 - \frac{x^2}{a^2} \right) dx + \frac{\pi ac}{b^2} \int_{-b}^b y^2 \left(1 - \frac{y^2}{b^2} \right) dy \\
 &+ \frac{\pi ab}{c^2} \int_{-c}^c z^2 \left(1 - \frac{z^2}{c^2} \right) dz \\
 &= 3 \cdot \frac{4\pi abc}{15} = \frac{4\pi abc}{5}.
 \end{aligned}$$

*) P_x 在平面 $X = x$ 上的方程为

$$\frac{y^2}{b^2 \left(1 - \frac{x^2}{a^2}\right)} + \frac{z^2}{c^2 \left(1 - \frac{x^2}{a^2}\right)} = 1,$$

故其面积为

$$\pi b \sqrt{1 - \frac{x^2}{a^2}} \cdot c \sqrt{1 - \frac{x^2}{a^2}} = \pi bc \left(1 - \frac{x^2}{a^2}\right).$$

Q_y 及 R_z 的面积类推.

4080. $\iiint_V \sqrt{x^2 + y^2} dx dy dz$, 其中 V 是由曲面

$$x^2 + y^2 = z^2, z = 1$$

所界的区域.

解 曲面在 Oxy 平面上的射影 Q 为圆盘 $x^2 + y^2 \leq 1$.

于是,

$$\begin{aligned} & \iiint_V \sqrt{x^2 + y^2} dx dy dz \\ &= \iint_Q dx dy \int_{\sqrt{x^2 + y^2}}^1 \sqrt{x^2 + y^2} dz \\ &= \iint_{x^2 + y^2 \leq 1} [\sqrt{x^2 + y^2} - (x^2 + y^2)] dx dy \\ &= \int_0^{2\pi} d\varphi \int_0^1 (r - r^2) r dr = \frac{\pi}{6}. \end{aligned}$$

于下列三重积分内用各种方法来配置积分的限:

4081. $\int_0^1 dx \int_0^{1-x} dy \int_0^{x+y} f(x, y, z) dz$.

解 有界域 V 如图 8.52 所示.

如果先对 y 积分, 再对 z, x 积分, 如图 8.53 所示, 则积分域在 Oyz 平面上的射影域由诸直线

$$z = 0, z = x + y,$$

$$y = 0, y = 1 - x (x \text{ 固定})$$

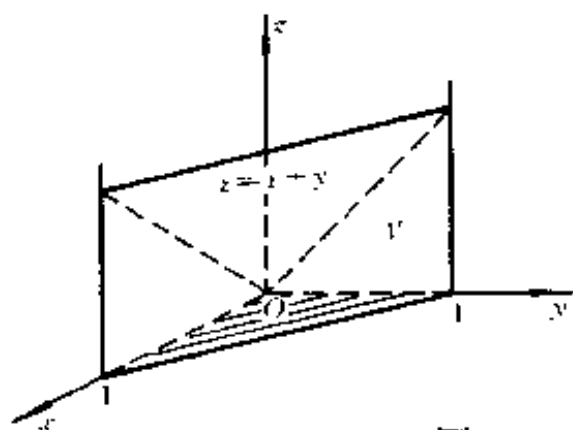


图 8.52

围成. 于是, 我们有

$$\begin{aligned} & \int_0^1 dx \int_0^{1-x} dy \int_0^{x+y} f(x, y, z) dz \\ &= \int_0^1 dx \left\{ \int_0^x dz \int_0^{z-x} f(x, y, z) dy \right. \\ & \quad \left. + \int_x^1 dz \int_{z-x}^{1-x} f(x, y, z) dy \right\}^{**} \end{aligned}$$

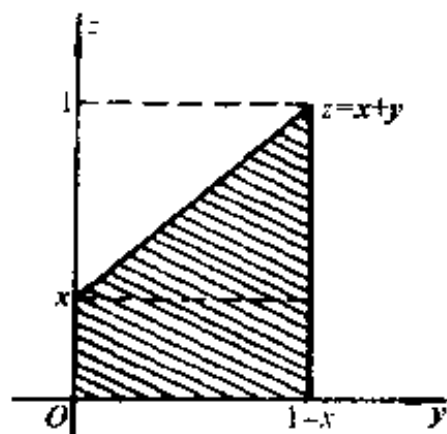


图 8.53

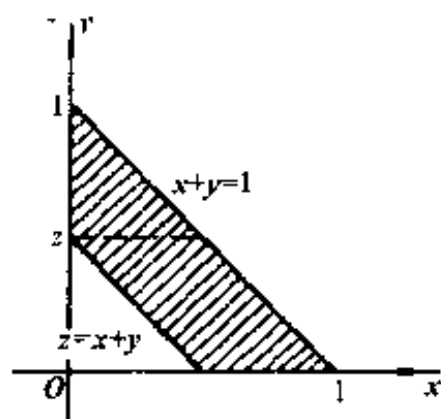


图 8.54

如果先对 x 积分, 再对 y, z 积分, 如图 8.54 所示, 则有

$$\int_0^1 dx \int_0^{1-x} dy \int_0^{x+y} f(x, y, z) dz$$

$$= \int_0^1 dz \left\{ \int_0^z dy \int_{z-y}^{1-y} f(x, y, z) dx \right. \\ \left. + \int_z^1 dy \int_0^{1-y} f(x, y, z) dx \right\}.$$

*) 这里用的公式为 $\iiint_V f(x, y, z) dx dy dz$

$$= \int_a^b dx \iint_{S(x)} f(x, y, z) dy dz.$$

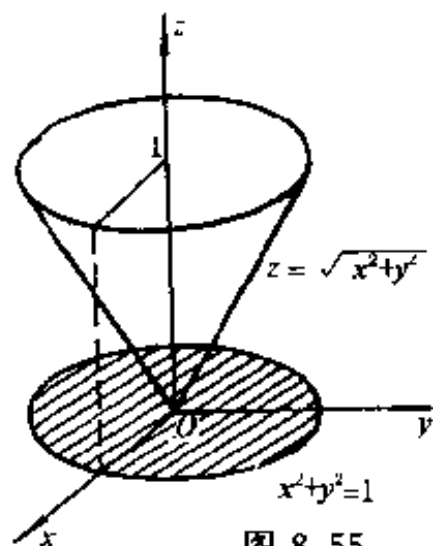


图 8.55

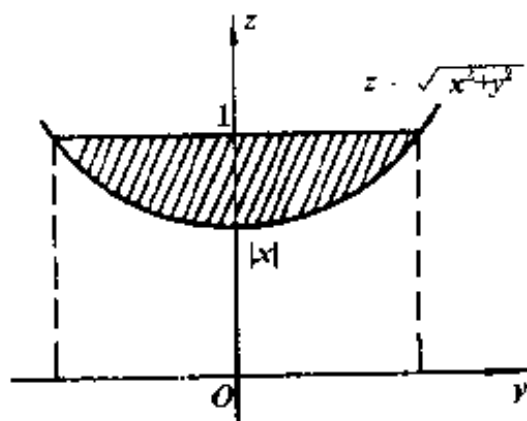


图 8.56

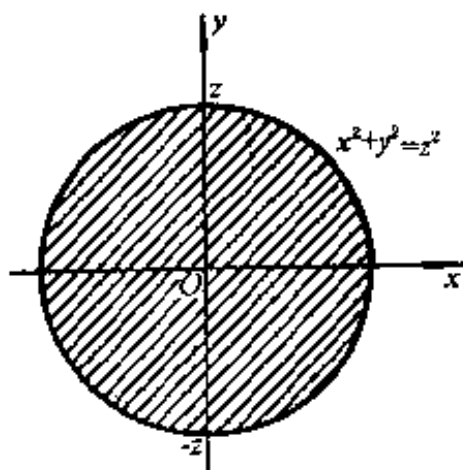


图 8.57

4082. $\int_{-1}^1 dx \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^1 f(x, y, z) dz.$

解 有界域 V 如图 8.55 所示.

如果先对 y 积分, 再对 z, x 积分, 如图 8.56 所示, 则积分域在 Oyz 平面上的射影域由不等式

$|x| \leq z \leq 1, \sqrt{z^2 - x^2} \leq y \leq \sqrt{z^2 - x^2}$ (x 固定) 给出. 于是, 我们有

$$\begin{aligned} & \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^1 f(x, y, z) dz \\ &= \int_{-1}^1 dx \int_{|x|}^1 dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x, y, z) dy. \end{aligned}$$

如果先对 x 积分, 再对 y, z 积分, 如图 8.57 所示, 则有

$$\begin{aligned} & \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^1 f(x, y, z) dz \\ &= \int_0^1 dz \int_{-z}^z dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x, y, z) dx. \end{aligned}$$

4083. $\int_0^1 dx \int_0^1 dy \int_0^{x^2+y^2} f(x, y, z) dz.$

解 如果先对 y 积分, 再对 z, x 积分, 则积分域在 Oxy 平面上的射影域 *) 由方程

$$x = 1, z = 0, z = x^2$$

及

$$x = 0, x = 1, z = x^2, z = x^2 + 1$$

所表示的线围成. 于是, 我们有

$$\begin{aligned} & \int_0^1 dx \int_0^1 dy \int_0^{x^2+y^2} f(x, y, z) dz \\ &= \int_0^1 dx \left(\int_0^{x^2} dz \int_0^1 f(x, y, z) dy \right. \end{aligned}$$

$$+ \int_{x^2}^{x^2+1} dz \int_{\sqrt{z-x^2}}^1 f(x, y, z) dy \Big].$$

如果先对 x 积分, 再对 z, y 积分, 不难由轮换对称关系得出结果。

如果先对 x 积分, 再对 y, z 积分, 则积分域在 Oyz 平面上的射影域由方程

$$y = 1, z = 0, y = \sqrt{z}$$

及

$$y = 0, y = 1, y = \sqrt{z}, y = \sqrt{z-1}$$

所表示的线围成. 于是, 我们有

$$\begin{aligned} & \int_0^1 dx \int_0^1 dy \int_0^{x^2+y^2} f(x, y, z) dz \\ &= \int_0^1 dz \left\{ \int_0^{\sqrt{z}} dy \int_{\sqrt{z-y^2}}^1 f(x, y, z) dx \right. \\ &+ \left. \int_{\sqrt{z}}^1 dy \int_0^1 f(x, y, z) dx \right\} \\ &+ \int_1^2 dz \int_{\sqrt{z-1}}^1 dy \int_{\sqrt{z-y^2}}^1 f(x, y, z) dx. \end{aligned}$$

*) 这里采用的投影方式与前两题不同, 系用结果

$$\iiint_V f(x, y, z) dx dy dz = \iint_D dx dz \int_{y_1}^{y_2} f(x, y, z) dy.$$

用一重积分以代替三重积分:

$$4084. \int_0^x d\xi \int_0^\xi d\eta \int_0^\eta f(\zeta) d\zeta.$$

$$\begin{aligned} \text{解} \quad & \int_0^x d\xi \int_0^\xi d\eta \int_0^\eta f(\xi) d\xi = \int_0^x d\xi \int_0^\xi d\zeta \int_\zeta^\xi f(\zeta) d\eta \\ &= \int_0^x d\xi \int_0^\xi f(\zeta) (\xi - \zeta) d\zeta \\ &= \int_0^x d\zeta \int_\zeta^x f(\zeta) (\xi - \zeta) d\xi \end{aligned}$$

$$= \frac{1}{2} \int_0^x f(\xi)(x-\xi)^2 d\xi.$$

4085. $\int_0^1 dx \int_0^1 dy \int_0^{x+y} f(z) dz.$

解 化为先对 y 积分, 再对 x, z 积分, 可将原积分表示成如下两部分:

$$\begin{aligned} & \int_0^1 dz \left[\int_z^1 dx \int_0^1 f(z) dy - \int_0^z dx \int_{z-x}^1 f(z) dy \right] \\ &= \int_0^1 dz \int_x^1 f(z) dx + \int_0^1 dz \int_0^z f(z)(1-z+x) dx \\ &= \int_0^1 f(z)(1-z) dz + \int_0^1 f(z)(1-z)z dz + \frac{1}{2} \int_0^1 f(z)z^2 dz \\ &= \int_0^1 f(z) \left(1 - \frac{z^2}{2} \right) dz = \frac{1}{2} \int_0^1 f(z)(2-z^2) dz; \end{aligned}$$

$$\begin{aligned} & \int_1^2 dz \int_{z-1}^1 dx \int_z^1 f(z) dy \\ &= \int_1^2 dz \int_{z-1}^1 f(z)(1-z+x) dx \\ &= \int_1^2 \left[f(z)(1-z)x + \frac{1}{2} f(z)x^2 \right] \Big|_{z-1}^1 dz \\ &= \int_1^2 f(z) \left[1-z + (z-1)^2 + \frac{1}{2} - \frac{1}{2}(z-1)^2 \right] dz \\ &= \frac{1}{2} \int_1^2 f(z)(z-2)^2 dz, \end{aligned}$$

于是,

$$\begin{aligned} \int_0^1 dx \int_0^1 dy \int_0^{x+y} f(z) dz &= \frac{1}{2} \int_0^1 f(z)(2-z^2) dz \\ &+ \frac{1}{2} \int_1^2 f(z)(2-z)^2 dz. \end{aligned}$$

4086. 设 $f(x, y, z) = F'_{xyz}(x, y, z)$ 及 a, b, c, A, B, C 为常数, 求:

$$\int_a^A dx \int_b^B dy \int_c^C f(x, y, z) dz.$$

$$\begin{aligned} \text{解} \quad & \int_a^A dx \int_b^B dy \int_c^C f(x, y, z) dz \\ &= \int_a^A dx \int_b^B [F'_{xy}(x, y, C) - F'_{xy}(x, y, c)] dy \\ &= \int_a^A [F'_x(x, B, C) - F'_x(x, b, C) - F'_x(x, B, c) \\ &\quad + F'_x(x, b, c)] dx \\ &= F(A, B, C) - F(a, B, C) - F(A, b, C) \\ &\quad + F(a, b, C) - F(A, B, c) + F(a, B, c) \\ &\quad + F(A, b, c) - F(a, b, c). \end{aligned}$$

变换为球坐标以计算积分:

$$4087. \iiint_V \sqrt{x^2 + y^2 + z^2} dx dy dz, \text{ 此处 } V \text{ 是由曲面}$$

$$x^2 + y^2 + z^2 = z$$

所界的区域.

解 令 $x = r \cos \varphi \cos \psi$, $y = r \sin \varphi \cos \psi$, $z = r \sin \psi$, 则曲面 $x^2 + y^2 + z^2 = z$ 化为 $r = \sin \psi$. 从而

$$V: 0 \leq \varphi \leq 2\pi, 0 \leq \psi \leq \frac{\pi}{2}, 0 \leq r \leq \sin \psi.$$

$$|I| = r^2 \cos \psi.$$

于是,

$$\begin{aligned} & \iiint_V \sqrt{x^2 + y^2 + z^2} dx dy dz \\ &= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\sin \psi} r \cdot r^2 \cos \psi dr \\ &= \frac{1}{4} \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin^4 \psi \cos \psi d\psi = \frac{\pi}{10}. \end{aligned}$$

$$4088. \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz.$$

解 变换为球坐标, 积分域 V 为

$$0 \leq \varphi \leq \frac{\pi}{2}, \frac{\pi}{4} \leq \psi \leq \frac{\pi}{2}, 0 \leq r \leq \sqrt{2}.$$

于是,

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\psi \int_0^{\sqrt{2}} r^2 \cos\psi \cdot$$

$$r^2 \sin^2\psi dr$$

$$= \frac{1}{5} 4 \sqrt{2} \cdot \frac{\pi}{2} \cdot$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\psi \sin^2\psi d\psi$$

$$= \frac{\pi}{5} 2 \sqrt{2} \cdot \frac{1}{3} \sin^3\psi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{15} (2\sqrt{2} - 1).$$

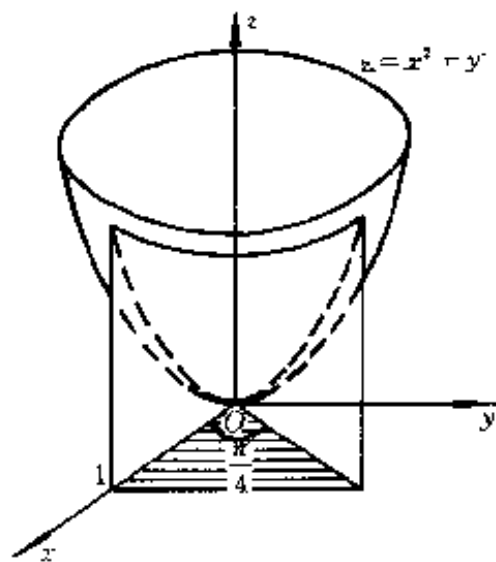


图 8.58

4089. 于积分中变换为球坐标

$$\iiint_V f(\sqrt{x^2 + y^2 + z^2}) dx dy dz.$$

此处 V 是由曲面 $z = x^2 + y^2, x = y, x = 1, y = 0, z = 0$ 所界的区域.

解 引用球坐标, 由 $x = y, x = 1, y = 0$ 知: $0 \leq \varphi \leq$

$\frac{\pi}{4}$ (图 8.58).

又从原点引半射线,由曲面 $z = x^2 + y^2$ 穿进,平面 $x = 1$ 穿出,于是,得 r 的下限为 $r = \frac{\sin\phi}{\cos^2\phi}$, r 的上限为 $r = \frac{1}{\cos\varphi\cos\phi}$. 而 ϕ 的变化域由 $z = 0$ 到 $z = x^2 + y^2$, $x = 1$ 所决定,即

$$0 \leq \phi \leq \operatorname{arctg} \frac{1}{\cos\varphi} \quad *)$$

于是,

$$\begin{aligned} & \iiint_V f(\sqrt{x^2 + y^2 + z^2}) dx dy dz \\ &= \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\operatorname{arctg} \frac{1}{\cos\varphi}} \cos\phi d\phi \int_{\frac{\sin\phi}{\cos^2\phi}}^{\frac{1}{\cos\varphi\cos\phi}} r^2 f(r) dr. \end{aligned}$$

*) 因为 $x = 1$ 对应 $r = \frac{1}{\cos\varphi\cos\phi}$, $z = x^2 + y^2$ 对应

$$r = \frac{\sin\phi}{\cos^2\phi}, \text{ 故 } \frac{1}{\cos\varphi\cos\phi} = \frac{\sin\phi}{\cos^2\phi}, \text{ 即 } \phi = \operatorname{arctg} \frac{1}{\cos\varphi}.$$

4090. 进行适当的变量代换,以计算三重积分

$$\iiint_V \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}} dx dy dz,$$

此处 V 为椭球 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 的内部.

解 作变量代换

$x = a \cos\varphi \cos\phi$, $y = b r \sin\varphi \cos\phi$, $z = c r \sin\phi$, 则有 $|I| = abc r^2 \cos\phi$, 且对于 V 的 $\frac{1}{8}$ 部分有

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq r \leq 1.$$

于是,

$$\begin{aligned}
& \iiint_V \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}} dx dy dz \\
&= 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^1 abc r^2 \cos\psi \sqrt{1-r^2} dr \\
&= 4\pi \int_0^1 abc r^2 \sqrt{1-r^2} dr = 4\pi abc \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt \\
&= \frac{\pi abc}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 4t) dt = \frac{\pi^2 abc}{4}.
\end{aligned}$$

4091. 变换为圆柱坐标, 以计算积分

$$\iiint_V (x^2 + y^2) dx dy dz,$$

此处 V 是由曲面 $x^2 + y^2 = 2z, z = 2$ 所界的区域.

解 令 $x = r \cos \varphi, y = r \sin \varphi, z = z$, 则 $x^2 + y^2 = 2z$ 化为 $r^2 = 2z$. 积分域

$$V: 0 \leq \varphi \leq 2\pi, 0 \leq r \leq 2, \frac{r^2}{2} \leq z \leq 2.$$

$$|I| = r.$$

于是,

$$\begin{aligned}
\iiint_V (x^2 + y^2) dx dy dz &= \int_0^{2\pi} d\varphi \int_0^2 r^2 \cdot r dr \int_{\frac{r^2}{2}}^2 dz \\
&= \frac{16\pi}{3}.
\end{aligned}$$

4092. 计算积分

$$\iiint_V x^2 dx dy dz$$

此处 V 是由曲面 $z = ay^2, z = by^2, y > 0 (0 < a < b)$, $z = \alpha x, z = \beta x (0 < \alpha < \beta), z = h (h > 0)$ 所界的区域.

解 作变换 $\frac{z}{y^2} = u, \frac{z}{x} = v, z = \omega$, 则 $x = \frac{\omega}{v}, y =$

$\sqrt{\frac{\omega}{u}}, z = \omega$, 从而积分域变为

$$V: a \leq u \leq b, \alpha \leq v \leq \beta, 0 \leq \omega \leq h,$$

且雅哥比行列式

$$I = \begin{vmatrix} 0 & -\frac{\omega}{v^2} & \frac{1}{v} \\ -\frac{\sqrt{\omega}}{2u\sqrt{u}} & 0 & \frac{1}{2\sqrt{u\omega}} \\ 0 & 0 & 1 \end{vmatrix} = \frac{-\omega\sqrt{\omega}}{2u\sqrt{u}v^2}.$$

于是,

$$\begin{aligned} \iiint_V x^2 dx dy dz &= \int_0^h \omega^{\frac{7}{2}} d\omega \int_{\alpha}^{\beta} \frac{1}{v^4} dv \int_a^b \frac{1}{2u\sqrt{u}} du \\ &= \frac{2}{27} \left(\frac{1}{\alpha^3} - \frac{1}{\beta^3} \right) \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right) h^4 \sqrt{h}. \end{aligned}$$

4093. 求积分

$$\iiint_V xyz dx dy dz.$$

其中 V 位于 $x > 0, y > 0, z > 0$ 这一卦限内且由下列曲面所界:

$$\begin{aligned} z &= \frac{x^2 + y^2}{m}, z = \frac{x^2 + y^2}{n}, xy = a^2, xy = b^2, y = \alpha x, y \\ &= \beta x (0 < a < b; 0 < \alpha < \beta; 0 < m < n), \end{aligned}$$

解 作变换 $\frac{z}{x^2 + y^2} = u, xy = v, \frac{y}{x} = \omega$, 则 $x = \sqrt{\frac{v}{\omega}},$
 $y = \sqrt{v\omega}, z = uv \left(\omega + \frac{1}{\omega} \right)$, 且

$$I = \begin{vmatrix} 0 & \frac{1}{2\sqrt{v\omega}} & -\frac{\sqrt{v}}{2\omega\sqrt{\omega}} \\ 0 & \frac{\sqrt{\omega}}{2\sqrt{v}} & \frac{\sqrt{v}}{2\sqrt{\omega}} \\ v\left(\omega + \frac{1}{\omega}\right) & u\left(\omega + \frac{1}{\omega}\right) & uv\left(1 - \frac{1}{\omega^2}\right) \end{vmatrix}$$

$$= \frac{v}{2\omega}\left(\omega + \frac{1}{\omega}\right),$$

$$V: \frac{1}{n} \leq u \leq \frac{1}{m}, a^2 \leq v \leq b^2, \alpha \leq \omega \leq \beta.$$

于是,

$$\begin{aligned} & \iiint_V xyz dx dy dz \\ &= \int_{\frac{1}{n}}^{\frac{1}{m}} \frac{u}{2} du \int_{a^2}^{b^2} v^3 dv \int_{\alpha}^{\beta} \left(\omega + \frac{1}{\omega^3} + \frac{2}{\omega} \right) d\omega \\ &= \frac{1}{32} \left(\frac{1}{m^2} - \frac{1}{n^2} \right) (b^8 - a^8) \left[(\beta^2 - \alpha^2) \left(1 + \frac{1}{\alpha^2 \beta^2} \right) + 4 \ln \frac{\beta}{\alpha} \right]. \end{aligned}$$

4094. 求函数

$$f(x, y, z) = x^2 + y^2 + z^2$$

在域 $x^2 + y^2 + z^2 \leq x + y + z$ 内的平均值.

解 域 $x^2 + y^2 + z^2 \leq x + y + z$ 即

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 \leq \frac{3}{4},$$

$$\text{其体积 } V = \frac{4}{3} \pi \left(\frac{\sqrt{3}}{2} \right)^3 = \frac{\sqrt{3}}{2} \pi.$$

作变换: $x = r \cos \varphi \cos \psi + \frac{1}{2}, y = r \sin \varphi \cos \psi + \frac{1}{2}, z =$

$\frac{1}{2} + r \sin \psi$, 则有

$$\begin{aligned}
f_{\overline{V}} &= \frac{1}{V} \iiint_V (x^2 + y^2 + z^2) dx dy dz, \\
&= \frac{1}{V} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{\frac{\sqrt{3}}{2}} r^2 \cos\psi \cdot \left(\frac{3}{4} + r^2 \right. \\
&\quad \left. + r\sin\psi + r\cos\varphi\cos\psi + r\sin\varphi\cos\psi \right) dr \\
&= \frac{1}{V} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{\frac{\sqrt{3}}{2}} r^2 \cos\psi \cdot \left(\frac{3}{4} + r^2 \right) dr \\
&= \frac{1}{V} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3\sqrt{3}}{20} \cos\psi d\psi \\
&= \frac{1}{V} \int_0^{2\pi} \frac{3\sqrt{3}}{10} d\varphi \\
&= \frac{1}{V} \cdot \frac{3\sqrt{3}}{5} \pi = \frac{2}{\sqrt{3}\pi} \cdot \frac{3\sqrt{3}\pi}{5} = \frac{6}{5}.
\end{aligned}$$

4095. 求函数

$$\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}$$

$$f(x, y, z) = e$$

在域 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ 内的平均值.

解 由于域 $V: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ 为椭球, 其体积等于 $\frac{4}{3}\pi abc$, 故平均值

$$f_{\overline{V}} = \frac{3}{4\pi abc} \iiint_V e \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} dx dy dz.$$

若作变换: $x = a \cos\varphi \cos\psi, y = b \sin\varphi \cos\psi, z = c \sin\psi$, 并利用对称性, 则

$$\begin{aligned}
 f_{(V,E)} &= \frac{3}{4\pi abc} \cdot 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^1 abce^r r^2 \cos\psi dr \\
 &= 3 \left(\int_0^{\frac{\pi}{2}} \cos\psi d\psi \right) \left(\int_0^1 r^2 e^r dr \right) \\
 &= 3(e-2).
 \end{aligned}$$

4096. 利用中值定理, 估计积分

$$u = \iiint_{x^2+y^2+z^2 \leq R^2} \frac{dx dy dz}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

之值, 其中 $a^2 + b^2 + c^2 > R^2$.

解 由积分中值定理, 有

$$\begin{aligned}
 u &= \iiint_{x^2+y^2+z^2 \leq R^2} \frac{dx dy dz}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} \\
 &= \frac{1}{\sqrt{(\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2}} \cdot \frac{4}{3}\pi R^3, \quad (1)
 \end{aligned}$$

其中 $\xi^2 + \eta^2 + \zeta^2 \leq R^2$. 由于函数

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

代表点 (x, y, z) 与点 (a, b, c) 之间的距离, 显然在域 $x^2 + y^2 + z^2 \leq R^2$ 中此距离的最小值是 $\sqrt{a^2 + b^2 + c^2} - R$, 最大值是 $\sqrt{a^2 + b^2 + c^2} + R$, 并且只在一个点达到最小值, 也只在一点达到最大值. 因此, 函数

$$\frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$$

在域 $x^2 + y^2 + z^2 \leq R^2$ 中的最大值是

$\frac{1}{\sqrt{a^2 + b^2 + c^2} - R}$, 最小值是 $\frac{1}{\sqrt{a^2 + b^2 + c^2} + R}$, 并且

只在一个点达到最大值, 也只在一点达到最小值. 我们证明(1)式中的中值不可能是函数的最大值, 也不可

能是函数的最小值,事实上,例如,若是最大值,即

$$\frac{1}{\sqrt{(\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2}} \\ = \frac{1}{\sqrt{a^2 + b^2 + c^2} - R},$$

则由(1)式知

$$\iiint_{x^2+y^2+z^2 \leq R^2} f(x,y,z) dx dy dz = 0, \quad (2)$$

$$\text{其中 } f(x,y,z) = \frac{1}{\sqrt{a^2 + b^2 + c^2} - R} \\ - \frac{1}{\sqrt{(\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2}}.$$

显然,在域 $x^2 + y^2 + z^2 \leq R^2$ 上 $f(x,y,z) \geq 0$ 且 $f(x,y,z)$ 为连续函数. 于是,由(2)式知在域 $x^2 + y^2 + z^2 \leq R^2$ 上必有 $f(x,y,z) \equiv 0$,这显然是不可能的. 因此,

$$\frac{1}{\sqrt{a^2 + b^2 + c^2} + R} < \frac{1}{\sqrt{(\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2}} \\ < \frac{1}{\sqrt{a^2 + b^2 + c^2} - R},$$

即

$$\sqrt{a^2 + b^2 + c^2} - R < \sqrt{(\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2} \\ < \sqrt{a^2 + b^2 + c^2} + R,$$

故

$$\sqrt{(\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2} \\ = \sqrt{a^2 + b^2 + c^2} + \theta R,$$

其中 $|\theta| < 1$. 于是,由(1)式得

$$u = \frac{4\pi}{3} \cdot \frac{R^3}{\sqrt{a^2 + b^2 + c^2} + \theta R}.$$

4097. 证明: 若函数 $f(x, y, z)$ 于域 V 内是连续的且对于任何的域 $\omega \subset V$

$$\iiint_{\omega} f(x, y, z) dx dy dz = 0,$$

则当 $(x, y, z) \in V$ 时, $f(x, y, z) \equiv 0$.

证 用反证法. 若当 $(x, y, z) \in V$ 时, $f(x, y, z) \not\equiv 0$. 不失一般性, 设对于 V 的某内点 (x_0, y_0, z_0) , 有 $f(x_0, y_0, z_0) > 0$, 则由于 $f(x, y, z)$ 的连续性, 故存在点 (x_0, y_0, z_0) 的某个闭邻域 $\omega' \subset V$, 使当 $(x, y, z) \in \omega'$ 时,

$$f(x, y, z) > 0.$$

这样一来, 利用中值定理, 即有

$$\iiint_{\omega'} f(x, y, z) dx dy dz = f(\xi, \eta, \zeta) \cdot V_{\omega'} > 0,$$

其中 $(\xi, \eta, \zeta) \in \omega' \subset V$. 这与假设

$$\iiint_{\omega} f(x, y, z) dx dy dz \equiv 0$$

矛盾. 因此, 当 $(x, y, z) \in V$ 时, $f(x, y, z) \equiv 0$.

4098. 求 $F(t)$, 设:

$$(a) F(t) = \iiint_{x^2+y^2+z^2 \leq t^2} f(x^2+y^2+z^2) dx dy dz, \text{ 其中 } f \text{ 为}$$

可微分函数;

$$(b) F(t) = \iiint_{\substack{0 \leq x \leq t \\ 0 \leq y \leq t \\ 0 \leq z \leq t}} f(xyz) dx dy dz, \text{ 其中 } f \text{ 为可微分函数.}$$

解 (a) 作球坐标变换得

$$\begin{aligned}
F(t) &= \iiint_{x^2+y^2+z^2 \leq t^2} f(x^2+y^2+z^2) dx dy dz \\
&= 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi \int_0^t f(r^2) r^2 dr \\
&= 4\pi \int_0^t (r^2) r^2 dr,
\end{aligned}$$

于是,

$$F'(t) = 4\pi t^2 f(t^2).$$

(σ) 作变换 $x = t\xi, y = t\eta, z = t\zeta$ 得

$$\begin{aligned}
F(t) &= \iiint_{\substack{0 \leq x \leq t \\ 0 \leq y \leq t \\ 0 \leq z \leq t}} (xyz) dx dy dz \\
&= \iiint_{\substack{0 \leq \xi \leq 1 \\ 0 \leq \eta \leq 1 \\ 0 \leq \zeta \leq 1}} f(t^3\xi\eta\zeta) t^3 d\xi d\eta d\zeta,
\end{aligned}$$

于是,

$$\begin{aligned}
F'(t) &= 3 \iiint_{\substack{0 \leq \xi \leq 1 \\ 0 \leq \eta \leq 1 \\ 0 \leq \zeta \leq 1}} t^2 f(t^3\xi\eta\zeta) d\xi d\eta d\zeta \\
&+ 3 \iiint_{\substack{0 \leq \xi \leq 1 \\ 0 \leq \eta \leq 1 \\ 0 \leq \zeta \leq 1}} f(t^3\xi\eta\zeta) t^5 \xi\eta\zeta d\xi d\eta d\zeta \\
&= \frac{3}{t} \left[F(t) + \iiint_{\substack{0 \leq x \leq t \\ 0 \leq y \leq t \\ 0 \leq z \leq t}} f'(xyz) xyz dx dy dz \right]
\end{aligned}$$

($t > 0$).

4099. 求

$$\iiint_{x^2+y^2+z^2 \leq 1} x^m y^n z^p dx dy dz,$$

其中 m, n, p 为非负整数.

解 分两种情况:

i) 设 m, n, p 中至少有一个是奇数. 例如, 设 p 为奇数. 于是,

$$\begin{aligned} I &= \iiint_{x^2+y^2+z^2 \leq 1} x^m y^n z^p dx dy dz \\ &= \iiint_{\substack{x^2+y^2+z^2 \leq 1 \\ z \geq 0}} x^m y^n z^p dx dy dz \\ &\quad + \iiint_{\substack{x^2+y^2+z^2 \leq 1 \\ z \leq 0}} x^m y^n z^p dx dy dz = I_1 + I_2. \end{aligned}$$

今在积分 I_2 中作变量代换 $x = u, y = v, z = -\omega$, 则

$$\frac{D(x, y, z)}{D(u, v, \omega)} = -1, \text{ 从而, 注意到 } p \text{ 为奇数, 可知}$$

$$I_2 = - \iiint_{\substack{x^2+y^2+z^2 \leq 1 \\ \omega \geq 0}} u^m v^n \omega^p du dv d\omega = -I_1$$

于是, $I = I_1 - I_1 = 0$.

ii) 设 m, n, p 均为偶数. 此时被积函数 $x^m y^n z^p$ 关于三个坐标平面皆对称. 于是,

$$\begin{aligned} I &= \iiint_{x^2+y^2+z^2 \leq 1} x^m y^n z^p dx dy dz \\ &= 8 \iiint_{\substack{x^2+y^2+z^2 \leq 1 \\ x \geq 0, y \geq 0, z \geq 0}} x^m y^n z^p dx dy dz. \end{aligned}$$

引用球坐标, $x = r \cos \varphi \cos \psi, y = r \sin \varphi \cos \psi, z = r \sin \psi$, 得

$$\begin{aligned}
& \iiint_{\substack{x^2+y^2+z^2 \leq 1 \\ x \geq 0, y \geq 0, z \geq 0}} x^m y^n z^p dx dy dz \\
&= \int_0^{\frac{\pi}{2}} \cos^m \varphi \sin^2 \varphi d\varphi \int_0^{\frac{\pi}{2}} \cos^{m+n+1} \phi \sin^p \phi d\phi \\
&\quad \cdot \int_0^1 r^{m+n+p+2} dr \\
&= \frac{1}{m+n+p+3} \cdot \frac{1}{2} \cdot \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)} \\
&\quad \cdot \frac{1}{2} \cdot \frac{\Gamma\left(\frac{m+n+2}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+n+p+3}{2}\right)} \\
&= \frac{1}{4(m+n+p+3)} \\
&\quad \cdot \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+n+p+3}{2}\right)} \\
&= \frac{1}{4(m+n+p+3)} \\
&\quad \cdot \frac{\frac{(m-1)!!}{2^{\frac{m}{2}}} \cdot \frac{(n-1)!!}{2^{\frac{n}{2}}} \cdot \frac{(p-1)!!}{2^{\frac{p}{2}}} \cdot \pi \sqrt{\pi}}{\frac{(m+n+p+1)!!}{2^{\frac{m+n+p+2}{2}}} \cdot \sqrt{\pi}} \\
&= \frac{\pi}{2(m+n+p+3)} \\
&\quad \cdot \frac{(m-1)!! \cdot (n-1)!! \cdot (p-1)!!}{(m+n+p+1)!!},
\end{aligned}$$

故

$$I = \iiint_{x^2+y^2+z^2 \leq 1} x^m y^n z^p dx dy dz$$

$$= \frac{4\pi}{m+n+p+3} \cdot \frac{(m-1)!! \cdot (n-1)!! \cdot (p-1)!!}{(m+n+p+1)!!}.$$

*) 利用 3856 题的结果.

4100. 计算迪里黑里积分

$$\iiint_V x^p y^q z^r (1-x-y-z)^s dx dy dz$$

($p > 0, q > 0, r > 0, s > 0$),

此处 V 是由平面 $x+y+z=1, x=0, y=0, z=0$ 所界的区域, 假定

$$x+y+z=\xi, y+z=\xi\eta, z=\xi\eta\zeta.$$

解 由假设知

$$x=\xi(1-\eta), y=\xi\eta(1-\zeta), z=\xi\eta\zeta.$$

在此变换下可求得 $|I| = \xi^2\eta$, 并且积分域 V 变为:

$$0 \leq \xi \leq 1, 0 \leq \eta \leq 1, 0 \leq \zeta \leq 1.$$

于是,

$$\begin{aligned} & \iiint_V x^p y^q z^r (1-x-y-z)^s dx dy dz \\ &= \int_0^1 \xi^{p+q+r+2} (1-\xi)^s d\xi \int_0^1 \eta^{q+r+1} (1-\eta)^p d\eta \\ & \quad \cdot \int_0^1 \zeta^r (1-\zeta)^s d\zeta \\ &= B(p+q+r+3, s+1) \cdot B(q+r+2, p+1) \\ & \quad \cdot B(r+1, s+1) \\ &= \frac{\Gamma(p+q+r+3) \cdot \Gamma(s+1) \cdot \Gamma(q+r+2)}{\Gamma(p+q+r+s+4) \cdot \Gamma(p+q+r+3)} \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{\Gamma(p+1) \cdot \Gamma(r+1) \cdot \Gamma(q+1)}{\Gamma(q+r+2)} \\
&= \frac{\Gamma(p+1)\Gamma(q+1)\Gamma(s+1)\Gamma(r+1)}{\Gamma(p+q+r+s+4)}.
\end{aligned}$$

§ 7. 利用三重积分计算体积法

域的体积 V 由下公式来表示

$$V = \iiint_V dx dy dz.$$

求由下列曲面所界的体积:

4101. $z = x^2 + y^2, z = 2x^2 + 2y^2, y = x, y = x^2.$

解 域 V 为

$$0 \leq x \leq 1, x^2 \leq y \leq x, x^2 + y^2 \leq z \leq 2x^2 + 2y^2,$$

$$\begin{aligned}
\text{故体积为 } V &= \int_0^1 dx \int_{x^2}^x dy \int_{x^2+y^2}^{2x^2+2y^2} dz \\
&= \int_0^1 dx \int_{x^2}^x (x^2 + y^2) dy \\
&= \int_0^1 \left(\frac{4}{3}x^3 - x^4 - \frac{1}{3}x^6 \right) dx \\
&= \left(\frac{1}{3}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7 \right) \Big|_0^1 = \frac{3}{35}.
\end{aligned}$$

4102. $z = x + y, z = xy, x + y = 1, x = 0, y = 0.$

解 域 V 为

$$0 \leq x \leq 1, 0 \leq y \leq 1-x, xy \leq z \leq x+y,$$

故体积为

$$\begin{aligned}
V &= \int_0^1 dx \int_0^{1-x} dy \int_{xy}^{x+y} dz \\
&= \int_0^1 dx \int_0^{1-x} (x + y - xy) dy
\end{aligned}$$

$$= \int_0^1 \left[x(1-x) + \frac{(1-x)^3}{2} \right] dx = \frac{7}{24}.$$

*) 因为 $0 \leq y \leq 1$, 故有 $xy \leq z \leq x+y$.

4103. $x^2 + z^2 = a^2, x+y = \pm a, x-y = \pm a$.

解
$$\begin{aligned} V &= 8 \int_0^a dx \int_0^{a-x} dy \int_0^{\sqrt{a^2-x^2}} dz \\ &= 8 \int_0^a (a-x) \sqrt{a^2-x^2} dx \\ &= 8a \left[\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} \right] \Big|_0^a \\ &\quad + \frac{8}{3} (a^2-x^2)^{\frac{3}{2}} \Big|_0^a \\ &= \frac{2a^3}{3} (3\pi-4). \end{aligned}$$

4104. $az = x^2 + y^2, z = \sqrt{x^2 + y^2} (a > 0)$.

解 对立体 V 在 Oxy 平面上的射影作极坐标变换

$$x = r \cos \varphi, \quad y = r \sin \varphi,$$

则域 V 为

$$0 \leq \varphi \leq 2\pi, 0 \leq r \leq a, \frac{r^2}{a} \leq z \leq r,$$

且有 $|I| = r$. 于是, 体积为

$$\begin{aligned} V &= \iint_{x^2+y^2 \leq a^2} \int_{\frac{r^2}{a}}^{\sqrt{r^2+y^2}} dz \\ &= \int_0^{2\pi} d\varphi \int_0^a r dr \int_{\frac{r^2}{a}}^r dz \\ &= 2\pi \int_0^a \left(r^2 - \frac{r^3}{a} \right) dr = \frac{\pi a^3}{6}. \end{aligned}$$

4105. $az = a^2 - x^2 - y^2, z = a - x - y, x = 0, y = 0, z = 0 (a > 0)$.

解 由 $az = a^2 - x^2 - y^2, x = 0, y = 0, z = 0$ 所界的体积

为

$$V_1 = \iint_{\substack{x^2+y^2 \leq a^2 \\ x \geq 0, y \geq 0}} \left(\int_0^{\frac{a^2-x^2-y^2}{a}} dz \right) dx dy$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^a \frac{a^2-r^2}{a} r dr = \frac{\pi a^3}{8}.$$

由 $z = a-x-y, x=0, y=0, z=0$ 所界的体积为

$$V_2 = \iiint_{\substack{x+y+z \leq a \\ x \geq 0, y \geq 0, z \geq 0}} dx dy dz = \int_0^a dx \int_0^{a-x} dy \int_0^{a-x-y} dz = \frac{a^3}{6}.$$

于是,所求的体积为

$$V = V_1 - V_2 = \frac{a^3}{24}(3\pi-4).$$

4106. $z = 6-x^2-y^2, z = \sqrt{x^2+y^2}.$

解 引用圆柱坐标,则域 V 为

$$0 \leq \varphi \leq 2\pi, 0 \leq r \leq 2, r \leq z \leq 6-r^2.$$

于是,体积为

$$V = \int_0^{2\pi} d\varphi \int_0^2 r dr \int_r^{6-r^2} dz$$

$$= 2\pi \int_0^2 (6r-r^3-r^2) dr = \frac{32\pi}{3}.$$

变换为球坐标或圆柱坐标,以计算曲面所界的体积:

4107. $x^2 + y^2 + z^2 = 2az, x^2 + y^2 \leq z^2.$

解 变换为圆柱坐标,则有

$$r^2 + z^2 = 2az \text{ 及 } r^2 \leq z^2.$$

因而域 V 为

$$0 \leq \varphi \leq 2\pi, 0 \leq r \leq a, r \leq z \leq a + \sqrt{a^2-r^2}.$$

于是,体积为

$$\begin{aligned}
 V &= \int_0^{2\pi} d\varphi \int_0^a r dr \int_r^{a+\sqrt{a^2-r^2}} dz \\
 &= 2\pi \int_0^a r(a + \sqrt{a^2-r^2}-r) dr \\
 &= 2\pi \left[\frac{ar^2}{2} - \frac{1}{3}(a^2-r^2)^{\frac{3}{2}} - \frac{r^3}{3} \right] \Big|_0^a = \pi a^3.
 \end{aligned}$$

*) 球面的方程应该是 $z = a \pm \sqrt{a^2-r^2}$, 但因体积 V 的一部分为球 $x^2 + y^2 + z^2 = 2az$ 的上半部, 故取

$$z = a + \sqrt{a^2-r^2}.$$

$$4108. (x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2 + z^2).$$

解 变换为球坐标, 则域 V 的 $\frac{1}{8}$ 部分为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{4}, 0 \leq r \leq a \sqrt{\cos 2\psi}.$$

于是, 体积为

$$\begin{aligned}
 V &= 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{4}} d\psi \int_0^{a\sqrt{\cos 2\psi}} r^2 \cos \psi dr \\
 &= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}} \cos \psi \cdot (\cos 2\psi)^{\frac{3}{2}} d\psi \\
 &= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{4}} (1-2\sin^2 \psi)^{\frac{3}{2}} d(\sin \psi) \\
 &= \frac{4\pi a^3}{3} \int_0^{\frac{1}{\sqrt{2}}} (1-2x^2)^{\frac{3}{2}} dx \\
 &= \frac{4\pi a^3}{3} \cdot \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \cos^4 t dt^{*}) \\
 &= \frac{4\pi a^3}{3\sqrt{2}} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^3}{4\sqrt{2}}.
 \end{aligned}$$

*) 作代换 $\sqrt{2}x = \sin t$.

$$4109. (x^2 + y^2 + z^2)^3 = 3xyz.$$

解 立体在第一,第三,第六及第八卦限内,对于这些卦限分别有:

$$x \geq 0, y \geq 0, z \geq 0; x \leq 0, y \leq 0, z \geq 0;$$

$$x \leq 0, y \geq 0, z \leq 0; x \geq 0, y \leq 0, z \leq 0.$$

立体在这四个卦限内的各部分,一对一对地对称于坐标轴之一.这是因为左端及右端当 x, y, z 中的任何两个同时变号时等式不变.

变换为球坐标,计算得体积

$$\begin{aligned} V &= 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\sqrt{\frac{1}{3\cos^2\varphi \cos\psi \sin\varphi \sin\psi}}} r^2 \cos\psi dr \\ &= 4 \int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi d\varphi \int_0^{\frac{\pi}{2}} \cos^4\psi \sin\psi d\psi \\ &= 4 \left(\frac{\sin^2\varphi}{2} \Big|_0^{\frac{\pi}{2}} \right) \cdot \left(-\frac{1}{4} \cos^4\psi \Big|_0^{\frac{\pi}{2}} \right) = \frac{1}{2}. \end{aligned}$$

4110. $x^2 + y^2 + z^2 = a^2, x^2 + y^2 + z^2 = b^2, x^2 + y^2 = z^2 (z \geq 0) (0 < a < b).$

解 变换为球坐标,得域 V 为

$$0 \leq \varphi \leq 2\pi, \quad \frac{\pi}{4} \leq \psi \leq \frac{\pi}{2}, \quad a \leq r \leq b.$$

于是,体积为

$$\begin{aligned} V &= \int_0^{2\pi} d\varphi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\psi \int_a^b r^2 \cos\psi dr \\ &= 2\pi \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\psi d\psi \right) \left(\int_a^b r^2 dr \right) \\ &= \frac{\pi(2 - \sqrt{2})(b^3 - a^3)}{3}. \end{aligned}$$

在下列各例中最好利用普遍的极坐标

r, φ 及 ψ ($r \geq 0; 0 \leq \varphi \leq 2\pi; -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$),

根据下列各式来引入它们

$$x = a r \cos^{\alpha} \varphi \cos^{\beta} \psi,$$

$$y = b r \sin^{\alpha} \varphi \cos^{\beta} \psi,$$

$$z = c r \sin^{\beta} \psi$$

(a, b, c, α, β 为常数), 并且

$$\frac{D(x, y, z)}{D(r, \varphi, \psi)} = \alpha \beta a b c r^2 \cos^{\alpha-1} \varphi \sin^{\alpha-1} \varphi \cos^{2\beta-1} \psi \sin^{\beta-1} \psi.$$

计算下列曲面所界的体积:

$$4111. \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)^2 = \frac{x}{h}.$$

解 令 $x = a r \cos \varphi \cos \psi, y = b r \sin \varphi \cos \psi, z = c r \sin \psi$,

则域的 $\frac{1}{4}$ 部分 (第一卦限内) 为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{2}, 0 \leq r \leq \sqrt[3]{\frac{a}{h} \cos \varphi \cos \psi}.$$

于是, 体积为

$$\begin{aligned} V &= 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\sqrt[3]{\frac{a}{h} \cos \varphi \cos \psi}} a b c r^2 \cos \psi dr \\ &= \frac{4a^2bc}{3h} \left(\int_0^{\frac{\pi}{2}} \cos \varphi d\varphi \right) \left(\int_0^{\frac{\pi}{2}} \cos^2 \psi d\psi \right) \\ &= \frac{\pi a^2bc}{3h}. \end{aligned}$$

$$4112. \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

解 令 $x = a r \cos \varphi \cos \psi, y = b r \sin \varphi \cos \psi, z = c r \sin \psi$, 并利用对称性, 即得体积

$$V = 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\cos \psi} a b c r^2 \cos \psi dr$$

$$\begin{aligned}
&= 8 \cdot \frac{\pi}{2} \cdot \frac{abc}{3} \int_0^{\frac{\pi}{2}} \cos^4 \psi d\psi \\
&= \frac{4\pi abc}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4} abc.
\end{aligned}$$

4113. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}.$

解 令 $x = a \cos \varphi, y = br \sin \varphi, z = z$, 则 r 满足方程 $r^4 + r^2 - 1 = 0$.

解得 $r = \sqrt{\frac{\sqrt{5}-1}{2}}$. 于是, 体积为

$$\begin{aligned}
V &= \int_0^{2\pi} d\varphi \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} abrd r \int_{\frac{c}{2}}^{\sqrt{1-r^2}} dz \\
&= 2\pi abc \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} r(\sqrt{1-r^2} - r^2) dr \\
&= 2\pi abc \left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}} - \frac{1}{4}r^4 \right] \Big|_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} \\
&= \frac{5\pi abc(3-\sqrt{5})}{12}.
\end{aligned}$$

4114. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^4}{c^4} = 1.$

解 令 $x = a \cos \varphi, y = br \sin \varphi, z = z$, 则得体积

$$\begin{aligned}
V &= \int_0^{2\pi} d\varphi \int_0^1 abrd r \int_{c(1-r^2)^{\frac{1}{4}}}^{c(1-r^2)^{\frac{1}{4}}} dz \\
&= 4\pi abc \int_0^1 r(1-r^2)^{\frac{1}{4}} dr \\
&= 4\pi abc \left[-\frac{2}{5}(1-r^2)^{\frac{5}{4}} \right] \Big|_0^1 = \frac{8}{5}\pi abc.
\end{aligned}$$

4115. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2 + \frac{z^4}{c^4} = 1.$

解 令 $x = a \cos \varphi \cos^{\frac{1}{2}} \psi, y = b r \sin \varphi \cos^{\frac{1}{2}} \psi,$
 $z = c r \sin^{\frac{1}{2}} \psi,$

则有 $|I| = \frac{1}{2} abcr^2 \sin^{-\frac{1}{2}} \psi$ 且 $\frac{1}{8}$ 域 V (第一卦限内) 为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{2}, 0 \leq r \leq 1.$$

于是, 体积为

$$\begin{aligned} V &= 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^1 \frac{1}{2} abcr^2 \sin^{-\frac{1}{2}} \psi dr \\ &= \frac{2}{3} \pi abc \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}} \psi d\psi \\ &= \frac{2}{3} \pi abc \cdot \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{4}\right)^{*}) \\ &= \frac{2}{3} \pi abc \cdot \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \\ &= \frac{1}{3} \pi abc \cdot \frac{\sqrt{\pi} \cdot \Gamma^2\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)} \\ &= \frac{1}{3} \pi abc \cdot \frac{\sqrt{\pi} \Gamma^2\left(\frac{1}{4}\right)^{**})}{\sqrt{2} \pi} \\ &= \frac{1}{3} abc \sqrt{\frac{\pi}{2}} \Gamma^2\left(\frac{1}{4}\right). \end{aligned}$$

*) 利用 3856 题的结果.

**) 利用余元公式: $\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin \pi x}.$

利用适当的变量代换, 以计算由曲面所界的体积 (假定参数是正的):

$$4116. \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \right)^2 = \frac{x}{h} + \frac{y}{k} \quad (x > 0, y > 0, z > 0).$$

解 令 $x = ar \cos^2 \varphi \cos^2 \psi$, $y = br \sin^2 \varphi \cos^2 \psi$,

$z = cr \sin^2 \psi$, 则有 $|I| = 4abc r^2 \cos \varphi \sin \varphi \cos^3 \psi \sin \psi$, 且域 V 为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{2}.$$

$$0 \leq r \leq \left(\frac{a}{h} \cos^2 \varphi + \frac{b}{k} \sin^2 \varphi \right) \cos^2 \psi.$$

于是, 体积为

$$\begin{aligned} V &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\left(\frac{a}{h} \cos^2 \varphi + \frac{b}{k} \sin^2 \varphi \right) \cos^2 \psi} 4abc r^2 \cos \varphi \sin \varphi \cos^3 \psi \sin \psi dr \\ &= \frac{4}{3} abc \int_0^{\frac{\pi}{2}} \cos \varphi \sin \varphi \left(\frac{a}{h} \cos^2 \varphi + \frac{b}{k} \sin^2 \varphi \right)^3 d\varphi \\ &\quad \cdot \int_0^{\frac{\pi}{2}} \cos^9 \psi \sin \psi d\psi \\ &= \frac{2}{15} abc \left(\int_0^{\frac{\pi}{2}} \frac{a^3}{h^3} \cos^7 \varphi \sin \varphi d\varphi \right. \\ &\quad + \int_0^{\frac{\pi}{2}} \frac{b^3}{k^3} \cos \varphi \sin^7 \varphi d\varphi \\ &\quad + 3 \cdot \frac{a^2 b}{h^2 k} \int_0^{\frac{\pi}{2}} \cos^5 \varphi \sin^3 \varphi d\varphi \\ &\quad \left. + 3 \cdot \frac{ab^2}{hk^2} \int_0^{\frac{\pi}{2}} \cos^3 \varphi \sin^5 \varphi d\varphi \right) \\ &= \frac{2}{15} abc \left(\frac{a^3}{8h^3} + \frac{b^3}{8k^3} + 3 \cdot \frac{a^2 b}{h^2 k} \cdot \frac{1}{24} + 3 \cdot \frac{ab^2}{hk^2} \cdot \frac{1}{24} \right)^{*)} \\ &= \frac{1}{60} abc \left(\frac{a}{h} + \frac{b}{k} \right) \left(\frac{a^2}{h^2} + \frac{b^2}{k^2} \right). \end{aligned}$$

*) 利用 3856 题的结果.

$$4117. \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \right)^4 = \frac{xyz}{abc} \quad (x > 0, y > 0, z > 0).$$

解 令 $x = \arccos^2 \varphi \cos^2 \psi$, $y = br \sin^2 \varphi \cos^2 \psi$.

$z = cr \sin^2 \psi$, 则有 $|I| = 4abc r^2 \cos \varphi \sin \varphi \cos^3 \psi \sin \psi$, 且域 V 为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{2}, 0 \leq r \leq \cos^2 \varphi \sin^2 \varphi \cos^4 \psi \sin^2 \psi.$$

于是, 体积为

$$\begin{aligned} V &= 4abc \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\cos^2 \varphi \sin^2 \varphi \cos^4 \psi \sin^2 \psi} r^2 \cos \varphi \sin \varphi \cos^3 \psi \sin \psi dr \\ &= \frac{4}{3} abc \int_0^{\frac{\pi}{2}} \cos^7 \varphi \sin^7 \varphi d\varphi \int_0^{\frac{\pi}{2}} \cos^{15} \psi \sin^7 \psi d\psi \\ &= \frac{4}{3} abc \cdot \frac{1}{2} \frac{\Gamma(4)\Gamma(4)}{\Gamma(8)} \cdot \frac{1}{2} \frac{\Gamma(8)\Gamma(4)^*}{\Gamma(12)} \\ &= \frac{1}{3} abc \cdot \frac{3!3!}{7!} \cdot \frac{7! \cdot 3!}{11!} = \frac{abc}{554400}. \end{aligned}$$

*) 利用 3856 题的结果.

$$4118. \left(\frac{x}{a} + \frac{y}{b} \right)^2 + \left(\frac{z}{c} \right)^2 = 1 \quad (x > 0, y > 0, z > 0).$$

解 令 $x = \arccos^2 \varphi \cos \psi$, $y = br \sin^2 \varphi \cos \psi$, $z = cr \sin \psi$, 则有 $|I| = 2abc r^2 \cos \varphi \sin \varphi \cos \psi$, 且域 V 为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{2}, 0 \leq r \leq 1.$$

于是, 体积为

$$\begin{aligned} V &= 2abc \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^1 r^2 \cos \varphi \sin \varphi \cos \psi dr \\ &= \frac{2}{3} abc \int_0^{\frac{\pi}{2}} \cos \varphi \sin \varphi d\varphi \int_0^{\frac{\pi}{2}} \cos \psi d\psi = \frac{1}{3} abc. \end{aligned}$$

$$4119. z = x^2 + y^2, z = 2(x^2 + y^2), xy = a^2, xy = 2a^2, x = 2y, 2x = y (x > 0, y > 0).$$

解 令 $z = u(x^2 + y^2)$, $xy = v$, $x = yw$, 则

$$x = \sqrt{vw}, y = \sqrt{\frac{v}{w}}, z = u \left(vw + \frac{v}{w} \right).$$

变换的雅哥比式为

$$I = \begin{vmatrix} 0 & \frac{\sqrt{w}}{2\sqrt{v}} & \frac{\sqrt{v}}{2\sqrt{w}} \\ 0 & \frac{1}{2\sqrt{vw}} & -\frac{\sqrt{v}}{2\sqrt{w^3}} \\ vw + \frac{v}{w} & u \left(w + \frac{1}{w} \right) & u \left(v - \frac{v}{w^2} \right) \end{vmatrix}$$

$$= - \left(\frac{v}{2} + \frac{v}{2w^2} \right),$$

而域 V 为

$$1 \leq u \leq 2, a^2 \leq v \leq 2a^2, \frac{1}{2} \leq w \leq 2.$$

于是, 体积为

$$\begin{aligned} V &= \int_1^2 du \int_{a^2}^{2a^2} dv \int_{\frac{1}{2}}^2 \left(\frac{v}{2} + \frac{v}{2w^2} \right) dw \\ &= \frac{3a^4}{4} \int_{\frac{1}{2}}^2 \left(1 + \frac{1}{w^2} \right) dw \\ &= \frac{9a^4}{4}. \end{aligned}$$

4120. $x^2 + z^2 = a^2, x^2 + z^2 = b^2, x^2 - y^2 - z^2 = 0 (x > 0).$

解 令 $x = r \cos \varphi, y = y, z = r \sin \varphi$, 则域 V 为

$$a \leq r \leq b, -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4},$$

$$-r \sqrt{\cos 2\varphi} \leq y \leq r \sqrt{\cos 2\varphi}$$

于是, 体积为

$$V = \int_a^b r dr \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_{-r\sqrt{\cos 2\varphi}}^{r\sqrt{\cos 2\varphi}} dy$$

$$\begin{aligned}
&= \frac{4}{3}(b^3 - a^3) \int_0^{\frac{\pi}{4}} \sqrt{\cos 2\varphi} d\varphi \\
&= \frac{2}{3}(b^3 - a^3) \int_0^{\frac{\pi}{2}} \sqrt{\cos \varphi} d\varphi \\
&= \frac{1}{3}(b^3 - a^3) \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})^{*})}{\Gamma(\frac{5}{4})} \\
&= \frac{2}{3}(b^3 - a^3) \sqrt{\frac{2}{\pi}} \Gamma^2\left(\frac{3}{4}\right) \cdot **)
\end{aligned}$$

*) 利用 3856 题的结果.

**) 利用余元公式有

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi.$$

$$4121. (x^2 + y^2 + z^2)^3 = \frac{a^6 z^2}{x^2 + y^2}.$$

解 采用球坐标: $x = r \cos \varphi \cos \psi$, $y = r \sin \varphi \cos \psi$, $z = r \sin \psi$, 则域 V 的 $\frac{1}{8}$ 部分 (第一卦限内) 为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{2}, 0 \leq r \leq a \operatorname{tg}^{\frac{1}{3}} \psi.$$

于是, 体积为

$$\begin{aligned}
V &= 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{a \operatorname{tg}^{\frac{1}{3}} \psi} r^2 \cos \psi dr \\
&= \frac{4\pi a^3}{3} \int_0^{\frac{\pi}{2}} \sin \psi d\psi = \frac{4\pi a^3}{3}.
\end{aligned}$$

$$4122. \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = \frac{z}{h} \cdot e^{-\frac{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}{\frac{z}{h}}}.$$

解 令 $x = a \cos \varphi \cos \psi$, $y = b \sin \varphi \cos \psi$, $z = c \sin \psi$,

则域 V 的 $\frac{1}{4}$ 部分(第一卦限内)为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{2}, 0 \leq r \leq \left(\frac{c}{h} \sin \psi e^{-\sin^2 \psi} \right)^{\frac{1}{3}}.$$

这是由于 $z \geq 0$, 故域 V 在 Oxy 平面的上方.

于是, 体积为

$$\begin{aligned} V &= 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\left(\frac{c}{h} \sin \psi e^{-\sin^2 \psi}\right)^{\frac{1}{3}}} abc r^2 \cos \psi dr \\ &= \frac{4c^2 ab}{3h} \cdot \frac{\pi}{2} \cdot \int_0^{\frac{\pi}{2}} \sin \psi \cos \psi e^{-\sin^2 \psi} d\psi \\ &= -\frac{\pi abc^2}{3h} e^{-\sin^2 \psi} \Big|_0^{\frac{\pi}{2}} = \frac{\pi abc^2}{3h} (1 - e^{-1}). \end{aligned}$$

$$4123. \frac{\frac{x}{a} + \frac{y}{b}}{\frac{x}{a} + \frac{y}{b} + \frac{z}{c}} = \frac{2}{\pi} \arcsin \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \right), \frac{x}{a} + \frac{y}{b} = 1, x=0, x=a.$$

解 令 $u = \frac{x}{a}, v = \frac{x}{a} + \frac{y}{b}, w = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$,
则

$$\frac{D(u, v, w)}{D(x, y, z)} = \begin{vmatrix} \frac{1}{a} & 0 & 0 \\ \frac{1}{a} & \frac{1}{b} & 0 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix} = \frac{1}{abc},$$

且域 V 变为

$$0 \leq u \leq 1, \frac{2}{\pi} \arcsin w \leq v \leq 1, -1 \leq w \leq 1.$$

于是, $\frac{D(x, y, z)}{D(u, v, w)} = abc$, 且体积为

$$\begin{aligned}
V &= abc \int_0^1 du \int_{-1}^1 d\omega \int_{\frac{2}{\pi} \arcsin \omega}^1 dv \\
&= 2abc \int_0^1 \left[1 - \frac{2}{\pi} \omega \arcsin \omega \right] d\omega \\
&= 2abc - \frac{2abc}{\pi} \int_0^1 \arcsin \omega d(\omega^2) \\
&= abc + \frac{2abc}{\pi} \int_0^1 \omega^2 (1 - \omega^2)^{-\frac{1}{2}} d\omega \\
&= abc + \frac{abc}{\pi} \int_0^1 t^{\frac{1}{2}} (1 - t)^{-\frac{1}{2}} dt \\
&= abc + \frac{abc}{\pi} B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{3}{2} abc.
\end{aligned}$$

$$4124. \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \ln \frac{\frac{x}{a} + \frac{y}{b} + \frac{z}{c}}{\frac{x}{a} + \frac{y}{b}}, x = 0, z = 0,$$

$$\frac{y}{b} + \frac{z}{c} = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

解 令 $u = \frac{x}{a}, v = \frac{x}{a} + \frac{y}{b}, \omega = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$, 则

$$\frac{D(u, v, \omega)}{D(x, y, z)} = \frac{1}{abc},$$

且域 V 变为

$$0 \leq u \leq \omega, \omega e^{-\omega} \leq v \leq \omega, 0 \leq \omega \leq 1.$$

于是, $\frac{D(x, y, z)}{D(u, v, \omega)} = abc$, 且体积为

$$\begin{aligned}
V &= abc \int_0^1 d\omega \int_0^\omega du \int_{\omega e^{-\omega}}^\omega dv \\
&= abc \int_0^1 (\omega^2 - \omega^2 e^{-\omega}) d\omega \\
&= abc \left(\frac{1}{3} - 2 + 5e^{-1} \right)
\end{aligned}$$

$$= 5abc \left(\frac{1}{e} - \frac{1}{3} \right).$$

4125. 求曲面 $x^2 + y^2 + az = 4a^2$ 将球 $x^2 + y^2 + z^2 = 4az$ 分成两部分的体积的比.

解 曲面 $x^2 + y^2 + az = 4a^2$ 与球面 $x^2 + y^2 + (z - 2a)^2 = 4a^2$ 的交线为圆周

$$\begin{cases} x^2 + y^2 = 3a^2 \\ z = a, \end{cases}$$

且有公共的顶点 $(0, 0, 4a)$. 球内位于曲面 $x^2 + y^2 + az = 4a^2$ 下方部分的体积为

$$\begin{aligned} V_1 &= \int_0^a dz \iint_{x^2 + y^2 \leq 4az - z^2} dx dy \\ &\quad + \int_a^{4a} dz \iint_{x^2 + y^2 \leq 4a^2 - az} dx dy \\ &= \int_0^a \pi(4az - z^2) dz + \int_a^{4a} \pi(4a^2 - az) dz \\ &= 2\pi a^3 - \frac{1}{3}\pi a^3 + 12\pi a^3 - \frac{15}{2}\pi a^3 \\ &= \frac{37}{6}\pi a^3. \end{aligned}$$

从而, 另一部分的体积

$$V_2 = \frac{4}{3}\pi(2a)^3 - \frac{37}{6}\pi a^3 = \frac{27}{6}\pi a^3.$$

于是, 球被曲面所分的两部分体积之比为

$$\frac{V_1}{V_2} = \frac{37}{27}.$$

4126. 求由曲面

$$x^2 + y^2 = az, z = 2a - \sqrt{x^2 + y^2} \quad (a > 0)$$

所界的体积和表面积.

解 两曲面的交线为圆周

$$\begin{cases} x^2 + y^2 = a^2, \\ z = a. \end{cases}$$

又曲面 $z = 2a - \sqrt{x^2 + y^2}$ 的顶点为 $(0, 0, 2a)$. 于是, 体积为

$$\begin{aligned} V &= \int_0^a dz \iint_{x^2+y^2 \leq az} dxdy + \int_a^{2a} dz \iint_{x^2+y^2 \leq (2a-z)^2} dxdy \\ &= \int_0^a \pi az dz + \int_a^{2a} \pi (2a-z)^2 dz \\ &= \frac{\pi a^3}{2} + \frac{\pi a^3}{3} = \frac{5\pi a^3}{6}. \end{aligned}$$

由两曲面方程分别可得

$$\frac{\partial z}{\partial x} = \frac{2x}{a}, \frac{\partial z}{\partial y} = \frac{2y}{a},$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \frac{1}{a} \sqrt{a^2 + 4x^2 + 4y^2};$$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{x^2 + y^2}}, \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2 + y^2}},$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{2}.$$

于是, 曲面的表面积为

$$\begin{aligned} S &= \iint_{x^2+y^2 \leq a^2} \frac{1}{a} \sqrt{a^2 + 4x^2 + 4y^2} dxdy \\ &\quad + \iint_{x^2+y^2 \leq a^2} \sqrt{2} dxdy \\ &= \int_0^{2\pi} d\varphi \int_0^1 \sqrt{1 + 4r^2} a^2 r dr + \sqrt{2} \pi a^2 \end{aligned}$$

$$= \frac{\pi a^2}{6} (6\sqrt{2} + 5\sqrt{5} - 1).$$

4127. 求由平面

$$a_1x + b_1y + c_1z = \pm h_1,$$

$$a_2x + b_2y + c_2z = \pm h_2,$$

$$a_3x + b_3y + c_3z = \pm h_3.$$

所界平行六面体的体积, 设

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

解 令

$$a_1x + b_1y + c_1z = u,$$

$$a_2x + b_2y + c_2z = v,$$

$$a_3x + b_3y + c_3z = w,$$

则有 $\frac{D(u, v, w)}{D(x, y, z)} = \Delta$. 于是, $\frac{D(x, y, z)}{D(u, v, w)} = \frac{1}{\Delta}$, 且体积为

$$V = \int_{-h_1}^{h_1} du \int_{-h_2}^{h_2} dv \int_{-h_3}^{h_3} \frac{1}{|\Delta|} dw = \frac{8h_1h_2h_3}{|\Delta|}.$$

4128. 求由曲面

$$(a_1x + b_1y + c_1z)^2 + (a_2x + b_2y + c_2z)^2 + (a_3x + b_3y + c_3z)^2 = h^2$$

所界的体积, 设

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

解 令

$$a_1x + b_1y + c_1z = u,$$

$$a_2x + b_2y + c_2z = v,$$

$$a_3x + b_3y + c_3z = w.$$

则有 $\frac{D(u, v, w)}{D(x, y, z)} = \Delta$. 于是, $\frac{D(x, y, z)}{D(u, v, w)} = \frac{1}{\Delta}$, 且体积为

$$V = \frac{1}{|\Delta|} \iiint_{x^2 + y^2 + z^2 \leq h^2} dudvdw = \frac{4\pi h^3}{3|\Delta|}.$$

4129. 求曲面

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^n + \frac{z^{2n}}{c^{2n}} = \frac{z}{h} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{n-2} \quad (n > 1)$$

所界的体积.

解 令 $x = a \cos \varphi \cos \psi$, $y = b r \sin \varphi \cos \psi$, $z = c r \sin \psi$, 则

有 $|I| = abcr^2 \cos \psi$, 且域 V 的 $\frac{1}{4}$ 为

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \psi \leq \frac{\pi}{2},$$

$$0 \leq r \leq \sqrt[n]{\frac{c}{h} \cdot \frac{\sin \psi \cos^{2n-4} \psi}{\cos^{2n} \psi + \sin^{2n} \psi}}.$$

于是, 体积为

$$\begin{aligned} V &= 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\sqrt[n]{\frac{c}{h} \cdot \frac{\sin \psi \cos^{2n-4} \psi}{\cos^{2n} \psi + \sin^{2n} \psi}}} abcr^2 \cos \psi dr \\ &= \frac{2}{3h} \pi abc^2 \int_0^{\frac{\pi}{2}} \frac{\sin \psi \cos^{2n-3} \psi d\psi}{\cos^{2n} \psi + \sin^{2n} \psi} \\ &= \frac{2}{3h} \pi abc^2 \int_0^1 \frac{t^{2n-3} dt}{t^{2n} + (1-t^2)^n} \\ &= -\frac{1}{3h} \pi abc^2 \int_0^1 \frac{t^{2n-4} d(1-t^2)}{t^{2n} + (1-t^2)^n} \\ &= \frac{1}{3h} \pi abc^2 \int_0^1 \frac{(1-x)^{n-2} dx}{(1-x)^n + x^n} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3h} \pi abc^2 \int_0^1 \frac{\frac{1}{(1-x)^2} dx}{1 + \left(\frac{x}{1-x}\right)^n} \\
&= \frac{1}{3h} \pi abc^2 \int_0^{+\infty} \frac{dt}{1+t^n} \quad *) \\
&= \frac{1}{3h} \pi abc^2 \cdot \frac{\pi}{n \sin \frac{\pi}{n}} \quad **) \\
&= \frac{\pi^2}{3n \sin \frac{\pi}{n}} \cdot \frac{abc^2}{h}.
\end{aligned}$$

*) 作代换 $t = \frac{x}{1-x}$.

**) 利用 3851 题的结果.

4130. 求在正卦限 $Oxyz (x > 0, y > 0, z > 0)$ 内由曲面

$$\frac{x^m}{a^m} + \frac{y^n}{b^n} + \frac{z^p}{c^p} = 1 \quad (m > 0, n > 0, p > 0)$$

$$x = 0, y = 0, z = 0$$

所界的体积.

解 令

$$x = ar^{\frac{2}{m}} \cos^{\frac{2}{m}} \varphi \cos^{\frac{2}{m}} \psi,$$

$$y = br^{\frac{2}{n}} \sin^{\frac{2}{n}} \varphi \cos^{\frac{2}{n}} \psi,$$

$$z = cr^{\frac{2}{p}} \sin^{\frac{2}{p}} \psi,$$

则

$$\begin{aligned}
\frac{D(x, y, z)}{D(r, \varphi, \psi)} &= \frac{8abc}{mnp} r^{\frac{2}{m} + \frac{2}{n} + \frac{2}{p} - 1} \cos^{\frac{2}{m} - 1} \varphi \\
&\cdot \sin^{\frac{2}{n} - 1} \varphi \cos^{\frac{2}{m} + \frac{2}{n} - 1} \psi \sin^{\frac{2}{p} - 1} \psi.
\end{aligned}$$

于是, 体积为

$$\begin{aligned}
V &= \frac{8abc}{mnp} \int_0^{\frac{\pi}{2}} \cos^{\frac{2}{m}-1} \varphi \sin^{\frac{2}{n}-1} \varphi d\varphi \\
&\quad \cdot \int_0^{\frac{\pi}{2}} \cos^{\frac{2}{m}+\frac{2}{n}-1} \psi \sin^{\frac{2}{p}-1} \psi d\psi \int_0^{r^{\frac{2}{m}+\frac{2}{n}+\frac{2}{p}-1}} r^{\frac{2}{m}+\frac{2}{n}+\frac{2}{p}-1} dr \\
&= \frac{8abc}{mnp} \cdot \frac{1}{2} B\left(\frac{1}{m}, \frac{1}{n}\right) \cdot \frac{1}{2} B\left(\frac{1}{m} + \frac{1}{n}, \frac{1}{p}\right) \\
&\quad \cdot \frac{1}{\frac{2}{m} + \frac{2}{n} + \frac{2}{p}}^{*)} \\
&= \frac{8abc}{mnp} \cdot \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{m}\right) \Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{m} + \frac{1}{n}\right)} \cdot \frac{1}{2} \\
&\quad \cdot \frac{\Gamma\left(\frac{1}{m} + \frac{1}{n}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{m} + \frac{1}{n} + \frac{1}{p}\right)} \\
&\quad \cdot \frac{mnp}{2(mn + np + mp)} \\
&= \frac{abc}{mn + np + mp} \cdot \frac{\Gamma\left(\frac{1}{m}\right) \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{1}{m} + \frac{1}{n} + \frac{1}{p}\right)}.
\end{aligned}$$

*) 利用 3856 题的结果.

§ 8. 三重积分在力学上的应用

1° 物体的质量 若一物体占有体积 V , $\rho = \rho(x, y, z)$ 为在点 (x, y, z) 的密度, 则该物体的质量等于

$$M = \iiint_V \rho dx dy dz.$$

2° 物体的重心 物体的重心坐标 (x_0, y_0, z_0) 按下列公式来计算

$$\left. \begin{aligned} x_0 &= \frac{1}{M} \iiint_V \rho x dx dy dz, \\ y_0 &= \frac{1}{M} \iiint_V \rho y dx dy dz, \\ z_0 &= \frac{1}{M} \iiint_V \rho z dx dy dz, \end{aligned} \right\} \quad (1)$$

若物体是均匀的,则在公式(1)中可令 $\rho = 1$.

3° 转动惯量 积分

$$\begin{aligned} I_{xy} &= \iiint_V \rho z^2 dx dy dz, \\ I_{yz} &= \iiint_V \rho x^2 dx dy dz, \\ I_{zx} &= \iiint_V \rho y^2 dx dy dz. \end{aligned}$$

分别称为物体对于坐标平面的转动惯量.

积分

$$I_l = \iiint_V \rho r^2 dx dy dz$$

(其中 r 为物体的动点 (x, y, z) 与轴 l 的距离) 称为物体对于某轴 l 的转动惯量. 特别是, 对于坐标轴 Ox, Oy, Oz 分别有

$$I_x = I_{xy} + I_{xz}, I_y = I_{yz} + I_{xy}, I_z = I_{xz} + I_{yz}.$$

积分

$$I_0 = \iiint_V \rho (x^2 + y^2 + z^2) dx dy dz$$

称为物体对于坐标原点的转动惯量.

显而易见, 有

$$I_0 = I_{xy} + I_{yz} + I_{zx}.$$

4° 引力场的位 积分

$$u(x, y, z) = \iiint_V \rho(\xi, \eta, \zeta) \frac{d\xi d\eta d\zeta}{r}$$

(其中 V 为物体的体积, $\rho = \rho(\xi, \eta, \zeta)$ 为物体的密度及

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}$$

称为物体在点 $P(x, y, z)$ 的牛顿位.

质量为 m 的质点被物体吸引的力在坐标轴 Ox, Oy, Oz 上的射影 X, Y, Z 等于

$$X = km \frac{\partial u}{\partial x} = km \iiint_V \rho \frac{\xi - x}{r^3} d\xi d\eta d\zeta,$$

$$Y = km \frac{\partial u}{\partial y} = km \iiint_V \rho \frac{\eta - y}{r^3} d\xi d\eta d\zeta,$$

$$Z = km \frac{\partial u}{\partial z} = km \iiint_V \rho \frac{\zeta - z}{r^3} d\xi d\eta d\zeta.$$

其中 k 为引力定律常数.

4131. 设物体在点 $M(x, y, z)$ 的密度由公式 $\rho = x + y + z$ 所给出, 求占有单位体积 $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ 之物体的质量.

解 质量以 M 表示, 则按题设有

$$M = \int_0^1 dx \int_0^1 dy \int_0^1 (x + y + z) dz = \frac{3}{2}.$$

4132. 若物体的密度按规律 $\rho = \rho_0 e^{-k\sqrt{x^2+y^2+z^2}}$ (其中 $\rho_0 > 0$ 及 $k > 0$ 为常数) 而变更, 求占有无限域 $x^2 + y^2 + z^2 \geq 1$ 的物体的质量.

解 若令 $x = r \cos \varphi \cos \psi, y = r \sin \varphi \cos \psi, z = r \sin \psi$, 则质量为

$$\begin{aligned} M &= \iiint_{x^2+y^2+z^2 \geq 1} \rho_0 e^{-k\sqrt{x^2+y^2+z^2}} dx dy dz \\ &= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_1^{\infty} r^2 \rho_0 e^{-kr} \cos \psi dr \\ &= 4\pi \rho_0 \int_1^{\infty} r^2 e^{-kr} dr \end{aligned}$$

$$\begin{aligned}
&= -\frac{4\pi\rho_0}{k}\int_1^{+\infty} r^2 e^{-kr} \\
&= -\frac{4\pi\rho_0}{k} r^2 e^{-kr} \Big|_1^{+\infty} \\
&\quad + \frac{4\pi\rho_0}{k} \int_1^{+\infty} 2re^{-kr} dr \\
&= \frac{4\pi\rho_0}{k} e^{-k} - \frac{8\pi\rho_0}{k^2} \Big|_1^{+\infty} r de^{-kr} \\
&= \frac{4\pi\rho_0}{k} e^{-k} - \frac{8\pi\rho_0}{k^2} \gamma e^{-kr} \Big|_1^{+\infty} \\
&\quad + \frac{8\pi\rho_0}{k^2} \int_1^{+\infty} e^{-kr} dr \\
&= 4\pi\rho_0 e^{-k} \left(\frac{1}{k} + \frac{2}{k^2} + \frac{2}{k^3} \right).
\end{aligned}$$

求由下列曲面所界的均匀物体的重心坐标:

4133. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, z = c.$

解 若令 $x = a \cos \varphi, y = b \sin \varphi, z = z$, 则质量为

$$M = ab \int_c^c dz \int_0^{2\pi} d\varphi \int_0^{\frac{z}{c}} r dr = \frac{\pi abc}{3}.$$

设重心坐标为 x_0, y_0, z_0 , 由对称性知 $x_0 = y_0 = 0$,

而

$$\begin{aligned}
z_0 &= \frac{ab}{M} \int_0^c z dz \int_0^{2\pi} d\varphi \int_0^{\frac{z}{c}} r dr \\
&= \frac{3}{\pi abc} \cdot \frac{\pi abc^2}{4} = \frac{3c}{4}.
\end{aligned}$$

于是, 重心为点 $\left(0, 0, \frac{3c}{4}\right)$.

4134. $z = x^2 + y^2, x + y = a, x = 0, y = 0, z = 0.$

解 物体的质量为

$$M = \int_0^a dx \int_0^{a-x} dy \int_0^{x^2+y^2} dz = \frac{1}{6}a^4.$$

重心的横坐标为

$$\begin{aligned} x_0 &= \frac{1}{M} \int_0^a x dx \int_0^{a-x} dy \int_0^{x^2+y^2} dz \\ &= \frac{6}{a^4} \cdot \frac{a^5}{15} = \frac{2a}{5}. \end{aligned}$$

同理可求得 $y_0 = \frac{2a}{5}$, 而

$$\begin{aligned} z_0 &= \frac{1}{M} \int_0^a dx \int_0^{a-x} dy \int_0^{x^2+y^2} z dz \\ &= \frac{1}{M} \int_0^a \left(\frac{a^5}{10} - \frac{1}{2}a^4x + \frac{4}{3}a^3x^2 \right. \\ &\quad \left. - 2a^2x^3 + 2ax^4 - \frac{14}{15}x^5 \right) dx \\ &= \frac{6}{a^4} \cdot \frac{7}{180}a^6 = \frac{7}{30}a^2. \end{aligned}$$

于是, 重心的坐标为 $x_0 = y_0 = \frac{2}{5}a, z_0 = \frac{7}{30}a^2$.

4135. $x^2 = 2pz, y^2 = 2px, x = \frac{p}{2}, z = 0$.

解 物体的质量为

$$\begin{aligned} M &= \int_0^{\frac{p}{2}} dx \int_{-\sqrt{2px}}^{\sqrt{2px}} dy \int_0^{\frac{x^2}{2p}} dz \\ &= \sqrt{\frac{2}{p}} \int_0^{\frac{p}{2}} x^{\frac{5}{2}} dx = \frac{p^3}{28}. \end{aligned}$$

重心的坐标为

$$\begin{aligned} x_0 &= \frac{1}{M} \int_0^{\frac{p}{2}} x dx \int_{-\sqrt{2px}}^{\sqrt{2px}} dy \int_0^{\frac{x^2}{2p}} dz \\ &= \frac{p^4}{72} \cdot \frac{28}{p^3} = \frac{7}{18}p. \end{aligned}$$

$$y_0 = \frac{1}{M} \int_0^{\frac{p}{2}} dx \int_{-\frac{\sqrt{2px}}{\sqrt{2px}}}^{\frac{\sqrt{2px}}{\sqrt{2px}}} y dy \int_0^{\frac{x^2}{2p}} dz = 0.$$

$$\begin{aligned} z_0 &= \frac{1}{M} \int_0^{\frac{p}{2}} dx \int_{-\frac{\sqrt{2px}}{\sqrt{2px}}}^{\frac{\sqrt{2px}}{\sqrt{2px}}} dy \int_0^{\frac{x^2}{2p}} z dz \\ &= \frac{p^4}{704} \cdot \frac{28}{p^3} = \frac{7}{176} p. \end{aligned}$$

4136. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, x \geq 0, y \geq 0, z \geq 0.$

解 若令

$$x = a r \cos \varphi \cos \psi, y = b r \sin \varphi \cos \psi, z = c r \sin \psi,$$

则质量为

$$\begin{aligned} M &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^1 abc r^2 \cos \psi dr \\ &= \frac{1}{6} \pi abc. \end{aligned}$$

于是,

$$\begin{aligned} x_0 &= \frac{1}{M} \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^1 abc r^2 \cos \psi \\ &\quad \cdot a r \cos \varphi \cos \psi dr \\ &= \frac{1}{M} \int_0^{\frac{\pi}{2}} \cos \varphi d\varphi \int_0^{\frac{\pi}{2}} \cos^2 \psi d\psi \int_0^1 a^2 b c r^3 dr \\ &= \frac{1}{16} \pi a^2 b c \cdot \frac{6}{\pi abc} = \frac{3}{8} a. \end{aligned}$$

利用对称性知重心的坐标为 $x_0 = \frac{3}{8}a, y_0 = \frac{3}{8}b, z_0 = \frac{3}{8}c.$

4137. $x^2 + z^2 = a^2, y^2 + z^2 = a^2 (z > 0).$

解 物体的质量为

$$\begin{aligned}
 M &= \int_0^a dz \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dy \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dx \\
 &= 4 \int_0^a (a^2 - z^2) dz = \frac{8a^3}{3}.
 \end{aligned}$$

于是,

$$x_0 = \frac{1}{M} \int_0^a dz \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dy \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} x dx = 0.$$

同理可得 $y_0 = 0$, 而

$$\begin{aligned}
 z_0 &= \frac{1}{M} \int_0^a z dz \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dy \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} dx \\
 &= a^4 \cdot \frac{3}{8a^3} = \frac{3}{8}a.
 \end{aligned}$$

于是, 重心的坐标为 $x_0 = y_0 = 0, z_0 = \frac{3}{8}a$.

4138. $x^2 + y^2 = 2z, x + y = z$.

解 由 $x^2 + y^2 = 2z, x + y = z$ 所围成的立体在平面 $z = 0$ 上的投影为圆 $(x - 1)^2 + (y - 1)^2 = 2$.

若引用代换

$$x = 1 + r \cos \theta, y = 1 + r \sin \theta,$$

则质量为

$$\begin{aligned}
 M &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r dr \int_{1+r(\cos\theta+\sin\theta)+\frac{r^2}{2}}^{2+1+r(\cos\theta+\sin\theta)} dz \\
 &= 2\pi \int_0^{\sqrt{2}} (1 - \frac{r^2}{2}) r dr = \pi.
 \end{aligned}$$

于是,

$$\begin{aligned}
 x_0 &= \frac{1}{M} \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r dr \\
 &\quad \int_{1+r(\cos\theta+\sin\theta)+\frac{r^2}{2}}^{2+1+r(\cos\theta+\sin\theta)} (1 + r \cos \theta) dz
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{M} \left[\pi + \int_0^{2\pi} \cos \theta d\theta \right] \int_0^{\sqrt{2}} r^2 \left(1 - \frac{r^2}{2} \right) dr \\
&= \frac{\pi}{M} = 1.
\end{aligned}$$

同理可得 $y_0 = 1$, 而

$$\begin{aligned}
z_0 &= \frac{1}{M} \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r dr \int_{1-r(\cos\theta+\sin\theta)+\frac{r^2}{2}}^{2+r(\cos\theta+\sin\theta)} z dz \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \left(3 + (\sin\theta + \cos\theta)(2r - r^3) \right. \\
&\quad \left. - \frac{1}{4}r^4 - r^2 \right) r dr \\
&= \frac{1}{2\pi} \cdot \frac{10\pi}{3} = \frac{5}{3}.
\end{aligned}$$

于是, 重心坐标为 $x_0 = y_0 = 1, z_0 = \frac{5}{3}$.

4139. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)^2 = \frac{xyz}{abc} (x > 0, y > 0, z > 0).$

解 作代换: $x = ar \cos \varphi \cos \psi, y = br \sin \varphi \cos \psi,$

$z = cr \sin \psi$, 则物体的质量为

$$\begin{aligned}
M &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\cos \varphi \sin \varphi \cos^2 \psi \sin \psi} abc r^2 \cos \psi dr \\
&= \frac{1}{3} abc \int_0^{\frac{\pi}{2}} \cos^3 \varphi \sin^3 \varphi d\varphi \int_0^{\frac{\pi}{2}} \cos^7 \psi \sin^3 \psi d\psi \\
&= \frac{1}{3} abc \cdot \frac{1}{2} B(2, 2) \cdot \frac{1}{2} B(4, 2) \\
&= \frac{1}{12} abc \cdot \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} \cdot \frac{\Gamma(4)\Gamma(2)}{\Gamma(6)} \\
&= \frac{abc}{1440}.
\end{aligned}$$

于是,

$$\begin{aligned}
x_0 &= \frac{1}{M} a^2 bc \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \\
&\quad \int_c^{\cos\varphi \sin\psi \cos^2\psi \sin\psi} r^3 \cos^2\psi \cos\varphi dr \\
&= \frac{a^2 bc}{4M} \int_0^{\frac{\pi}{2}} \cos^5\varphi \sin^4\varphi d\varphi \int_0^{\frac{\pi}{2}} \cos^{10}\psi \sin^4\psi d\psi \\
&= \frac{a^2 bc}{4M} \cdot \frac{1}{4} B\left(3, \frac{5}{2}\right) B\left(\frac{11}{2}, \frac{5}{2}\right) \\
&= \frac{a^2 bc}{4M} \cdot \frac{1}{4} \cdot \frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} \cdot \frac{\Gamma\left(\frac{11}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(8)} \\
&= \frac{18a^2 bc\pi}{16 \cdot 16 \cdot 7!} \cdot \frac{1440}{abc} = \frac{9\pi}{448} a.
\end{aligned}$$

由对称性知,重心坐标为 $x_0 = \frac{9\pi}{448}a$,

$$y_0 = \frac{9\pi}{448}b, \quad z_0 = \frac{9\pi}{448}c.$$

4140. $z = x^2 + y^2, \quad z = \frac{1}{2}(x^2 + y^2), \quad x + y = \pm 1,$

$$x - y = \pm 1.$$

解 作代换: $x - y = u, \quad x + y = v$, 则有

$$x = \frac{u+v}{2}, \quad y = \frac{v-u}{2},$$

$$z = \frac{u^2+v^2}{4} \text{ 及 } z = \frac{u^2+v^2}{2},$$

且 $|I| = \frac{1}{2}$ 及域 V 为: $-1 \leq u \leq 1, -1 \leq v \leq 1$,

$$\frac{u^2+v^2}{4} \leq z \leq \frac{u^2+v^2}{2}. \text{ 于是,}$$

$$M = \frac{1}{2} \int_{-1}^1 du \int_{-1}^1 dv \int_{\frac{u^2+v^2}{4}}^{\frac{u^2+v^2}{2}} dz = \frac{1}{3}.$$

$$\begin{aligned}
x_0 &= \frac{1}{4M} \int_{-1}^1 du \int_{-1}^1 dv \int_{\frac{u^2-v^2}{4}}^{\frac{u^2+v^2}{4}} (u+v) dz = 0, \\
y_0 &= \frac{1}{4M} \int_{-1}^1 du \int_{-1}^1 dv \int_{\frac{u^2-v^2}{4}}^{\frac{u^2+v^2}{4}} (v-u) dz = 0, \\
z_0 &= \frac{1}{2M} \int_{-1}^1 du \int_{-1}^1 dv \int_{\frac{u^2-v^2}{4}}^{\frac{u^2+v^2}{4}} z dz \\
&\quad - \frac{3}{64} \cdot \frac{1}{M} \int_{-1}^1 du \int_{-1}^1 (u^4 + 2u^2v^2 + v^4) dv \\
&= \frac{3}{64M} \int_{-1}^1 \left(2u^4 + \frac{4u^2}{3} + \frac{2}{5} \right) du \\
&= \frac{7}{20},
\end{aligned}$$

即重心坐标为 $x_0 = y_0 = 0$, $z_0 = \frac{7}{20}$.

4141. $\frac{x^n}{a^n} + \frac{y^n}{b^n} + \frac{z^n}{c^n} = 1$, $x = 0$, $y = 0$, $z = 0$
 $(n > 0, x > 0, y > 0, z > 0)$.

解 作代换:

$$\begin{aligned}
x &= a \cos^{\frac{2}{n}} \varphi \cos^{\frac{2}{n}} \psi, & y &= b r \sin^{\frac{2}{n}} \varphi \cos^{\frac{2}{n}} \psi, \\
z &= c r \sin^{\frac{2}{n}} \psi,
\end{aligned}$$

则有 $|I| = \frac{4}{n^2} abc r^2 \sin^{\frac{2}{n}-1} \varphi \cos^{\frac{2}{n}-1} \varphi \cos^{\frac{4}{n}-1} \psi \cdot \sin^{\frac{2}{n}-1} \psi$. 于是,

$$\begin{aligned}
M &= \frac{4}{n^2} abc \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^1 r^2 \sin^{\frac{2}{n}-1} \varphi \cos^{\frac{2}{n}-1} \varphi \\
&\quad \cos^{\frac{4}{n}-1} \psi \sin^{\frac{2}{n}-1} \psi dr \\
&= \frac{4}{n^2} abc \cdot \frac{1}{3} \cdot \frac{1}{2} B\left(\frac{1}{n}, \frac{1}{n}\right) \cdot \frac{1}{2} B\left(\frac{2}{n}, \frac{1}{n}\right)
\end{aligned}$$

$$= \frac{abc}{3n^2} \cdot \frac{\Gamma^3\left(\frac{1}{n}\right)}{\Gamma\left(\frac{3}{n}\right)}.$$

重心坐标为

$$\begin{aligned} x_0 &= \frac{1}{M} \cdot \frac{4}{n^2} a^2 bc \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \\ &\quad \cdot \int_0^1 r \cos^{\frac{2}{n}} \varphi \cos^{\frac{2}{n}} \psi r^2 \sin^{\frac{2}{n}-1} \varphi \cos^{\frac{2}{n}-1} \psi \\ &\quad \cdot \cos^{\frac{4}{n}-1} \psi \sin^{\frac{2}{n}-1} \psi dr \\ &= \frac{1}{M} \cdot \frac{a^2 bc}{n^2} \int_0^{\frac{\pi}{2}} \sin^{\frac{2}{n}} \varphi \cos^{\frac{4}{n}-1} \varphi d\varphi \\ &\quad \cdot \int_0^{\frac{\pi}{2}} \cos^{\frac{6}{n}-1} \psi \sin^{\frac{2}{n}-1} \psi d\psi \\ &= \frac{1}{M} \cdot \frac{a^2 bc}{n^2} \cdot \frac{1}{2} B\left(\frac{1}{n}, \frac{2}{n}\right) \\ &\quad \cdot \frac{1}{2} B\left(\frac{3}{n}, \frac{1}{n}\right) \\ &= \frac{1}{M} \cdot \frac{a^2 bc}{4n^2} \cdot \frac{\Gamma^2\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right)}{\Gamma\left(\frac{4}{n}\right)} \\ &= \frac{3n^2}{abc} \cdot \frac{\Gamma\left(\frac{3}{n}\right)}{\Gamma^3\left(\frac{1}{n}\right)} \cdot \frac{a^2 bc}{4n^2} \\ &\quad \cdot \frac{\Gamma^2\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right)}{\Gamma\left(\frac{4}{n}\right)} \end{aligned}$$

$$= \frac{3}{4} \cdot \frac{\Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{3}{n}\right)}{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{4}{n}\right)} a,$$

同理可求得

$$y_0 = \frac{3}{4} \cdot \frac{\Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{3}{n}\right)}{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{4}{n}\right)} b,$$

$$z_0 = \frac{3}{4} \cdot \frac{\Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{3}{n}\right)}{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{4}{n}\right)} c.$$

4142. 求形状为立方体:

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1$$

的物体的重心坐标, 设此物体在点 (x, y, z) 的密度等于

$$\rho = x^{\frac{2\alpha-1}{1-\alpha}} y^{\frac{2\beta-1}{1-\beta}} z^{\frac{2\gamma-1}{1-\gamma}},$$

其中 $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$.

解 物体的质量为

$$\begin{aligned} M &= \int_0^1 x^{\frac{2\alpha-1}{1-\alpha}} dx \int_0^1 y^{\frac{2\beta-1}{1-\beta}} dy \int_0^1 z^{\frac{2\gamma-1}{1-\gamma}} dz \\ &= \frac{1-\alpha}{\alpha} x^{\frac{\alpha}{1-\alpha}} \Big|_0^1 \cdot \frac{1-\beta}{\beta} y^{\frac{\beta}{1-\beta}} \Big|_0^1 \\ &\quad \cdot \frac{1-\gamma}{\gamma} z^{\frac{\gamma}{1-\gamma}} \Big|_0^1 \\ &= \frac{(1-\alpha)(1-\beta)(1-\gamma)}{\alpha\beta\gamma}. \end{aligned}$$

于是, 重心的坐标为

$$\begin{aligned} x_0 &= \frac{1}{M} \int_0^1 x^{\frac{2\alpha-1}{1-\alpha}+1} dx \int_0^1 y^{\frac{2\beta-1}{1-\beta}} dy \\ &\quad \cdot \int_0^1 z^{\frac{2\gamma-1}{1-\gamma}} dz \end{aligned}$$

$$= \frac{\alpha\beta\gamma}{(1-\alpha)(1-\beta)(1-\gamma)} \\ \cdot (1-\alpha) \frac{(1-\beta)(1-\gamma)}{\beta\gamma} = \alpha,$$

同理可求得 $y_0 = \beta$, $z_0 = \gamma$.

求由下列曲面(参变量是正的)所界均匀物体对于坐标平面的转动惯量:

4143. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \quad x=0, \quad y=0, \quad z=0.$

解
$$I_{xy} = \int_0^a dx \int_0^{b(1-\frac{x}{a})} dy \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} z^2 dz$$

$$= \frac{c^3}{3} \int_0^a dx \int_0^{b(1-\frac{x}{a})} \left(1 - \frac{x}{a} - \frac{y}{b}\right)^3 dy$$

$$= -\frac{bc^3}{12} \int_0^a \left(1 - \frac{x}{a} - \frac{y}{b}\right)^4 \Big|_0^{b(1-\frac{x}{a})} dx$$

$$= \frac{bc^3}{12} \int_0^a \left(1 - \frac{x}{a}\right)^4 dx$$

$$= \frac{abc^3}{60}.$$

利用对称性可得

$$I_{yx} = \frac{a^3bc}{60}, \quad I_{zx} = \frac{ab^3c}{60}.$$

4144. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

解 若令 $x = a \cos\varphi \cos\psi$, $y = b \sin\varphi \cos\psi$,
 $z = c \sin\psi$, 则有

$$I_{xy} = abc^3 \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^1 r^4 \cos\psi \sin^2\psi dr$$

$$= \frac{abc^3}{5} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\psi \sin^2\psi d\psi$$

$$\begin{aligned}
&= \frac{abc^3}{15} \cdot 2\pi \cdot \sin^3\psi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{4}{15}\pi abc^3.
\end{aligned}$$

利用对称性可得

$$I_{yz} = \frac{4}{15}\pi a^3bc, \quad I_{xz} = \frac{4}{15}\pi ab^3c.$$

4145. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}, \quad z = c.$

解 若令 $x = \arccos\varphi, y = br\sin\varphi$, 则有

$$\begin{aligned}
I_{xy} &= \int_0^{2\pi} d\varphi \int_0^1 abrd r \int_{-c}^c z^2 dz \\
&= \frac{1}{5}\pi abc^3, \\
I_{yz} &= \int_0^{2\pi} d\varphi \int_0^1 abrd r \int_{-c}^c (\arccos\varphi)^2 dz \\
&= a^3bc \int_0^{2\pi} \cos^2\varphi d\varphi \int_0^1 (1-r)r^3 dr \\
&= \frac{1}{20}\pi a^3bc.
\end{aligned}$$

利用对称性可得

$$I_{xx} = \frac{1}{20}\pi ab^3c.$$

4146. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a}.$

解 若令 $x = \arccos\varphi, y = br\sin\varphi$, 则得域 V 为

$$\begin{aligned}
-\frac{\pi}{2} &\leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq r \leq \cos\varphi, \\
-c\sqrt{1-r^2} &\leq z \leq c\sqrt{1-r^2}.
\end{aligned}$$

于是,

$$\begin{aligned}
I_{xy} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} abrd r \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} z^2 dz \\
&= \frac{2}{3} abc^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} (1-r^2)^{\frac{3}{2}} r dr \\
&= \frac{2}{15} abc^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 - (\sin^2\varphi)^{\frac{5}{2}}] d\varphi \\
&= \frac{4}{15} abc^3 \int_0^{\frac{\pi}{2}} (1 - \sin^5\varphi) d\varphi \\
&= \frac{4}{15} abc^3 \left(\varphi + \cos\varphi - \frac{2}{3} \cos^3\varphi \right. \\
&\quad \left. + \frac{1}{5} \cos^5\varphi \right) \Big|_0^{\frac{\pi}{2}} \\
&= \frac{2abc^3}{225} (15\pi - 16).
\end{aligned}$$

$$\begin{aligned}
I_{yz} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} abrd r \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} (\arccos\varphi)^2 dz \\
&= 2a^3bc \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi d\varphi \int_0^{\cos\varphi} \sqrt{1-r^2} r^3 dr \\
&= 2a^3bc \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi d\varphi \int_{\varphi}^{\frac{\pi}{2}} |\sin t| \sin t \cos^3 t dt \\
&= 2a^3bc \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi \left\{ \int_{\varphi}^0 |\sin t| \sin t \cos^3 t dt \right. \\
&\quad \left. + \int_0^{\frac{\pi}{2}} |\sin t| \sin t \cos^3 t dt \right\} d\varphi \\
&= 2a^3bc \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \frac{2}{15} \right. \\
&\quad \left. + \int_{\varphi}^0 |\sin t| \sin t \cos^3 t dt \right\} \cos^2\varphi d\varphi
\end{aligned}$$

$$\begin{aligned}
&= 2a^3bc \left\{ \frac{\pi}{15} \right. \\
&\quad + \int_{-\frac{\pi}{2}}^0 \left(- \int_{\varphi}^0 \sin^2 t \cos^3 t dt \right) \cos^2 \varphi d\varphi \\
&\quad \left. + \int_0^{\frac{\pi}{2}} \left(\int_{\varphi}^0 \sin^2 t \cos^2 t dt \right) \cos^2 \varphi d\varphi \right\} \\
&= 2a^3bc \left\{ \frac{\pi}{15} + \int_0^{\frac{\pi}{2}} \left(\frac{1}{5} \sin^5 \varphi \right. \right. \\
&\quad \left. \left. - \frac{1}{3} \sin^3 \varphi \right) \cos^2 \varphi d\varphi \right. \\
&\quad \left. + \int_0^{\frac{\pi}{2}} \left(\frac{1}{5} \sin^5 \varphi - \frac{1}{3} \sin^3 \varphi \right) \cos^2 \varphi d\varphi \right\} \\
&= 2a^3bc \left(\frac{\pi}{15} - \frac{92}{1575} \right) \\
&= \frac{2a^3bc}{1575} (105\pi - 92),
\end{aligned}$$

$$\begin{aligned}
I_{zx} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\cos \varphi} abrd r \int_{-\frac{\sqrt{1-r^2}}{\sqrt{1-r^2}}}^{\frac{\sqrt{1-r^2}}{\sqrt{1-r^2}}} (br \sin \varphi)^2 dz \\
&= 2ab^3c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\cos \varphi} \sqrt{1-r^2} r^3 \sin^2 \varphi dr \\
&= 2ab^3c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \varphi d\varphi \int_0^{\frac{\pi}{2}} |\sin t| \\
&\quad \cdot \sin t \cos^3 t dt \\
&= 2ab^3c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \frac{2}{15} \right. \\
&\quad \left. + \int_{\varphi}^0 |\sin t| \cdot \sin t \cos^3 t \cdot dt \right\} \sin^2 \varphi d\varphi \\
&= 2ab^3c \left\{ \frac{\pi}{15} + \int_0^{\frac{\pi}{2}} \left(\frac{1}{5} \sin^5 \varphi \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{3} \sin^3 \varphi \Big) \sin^2 \varphi d\varphi \\
& + \int_0^{\frac{\pi}{2}} \left(\frac{1}{5} \sin^5 \varphi - \frac{1}{3} \sin^3 \varphi \right) \sin^2 \varphi d\varphi \Big\} \\
& = 2ab^3c \left(\frac{\pi}{15} - \frac{272}{1575} \right) \\
& = \frac{2ab^3c}{1575} (105\pi - 272).
\end{aligned}$$

4147. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \frac{z}{c}, \quad \frac{x}{a} + \frac{y}{b} = \frac{z}{c}.$

解 两曲面在 Oxy 平面上的投影为 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \frac{x}{a} +$

$2 \frac{y}{b} = 0$, 即 $\left(\frac{x}{a} - 1 \right)^2 + \left(\frac{y}{b} - 1 \right)^2 = 2$. 若令

$$\frac{x}{a} = 1 + r \cos \varphi, \quad \frac{y}{b} = 1 + r \sin \varphi,$$

则得域 V 为

$$0 \leq \varphi \leq 2\pi, \quad 0 \leq r \leq \sqrt{2},$$

$$\begin{aligned}
& c \left[1 + \frac{r^2}{2} + r(\cos \varphi + \sin \varphi) \right] \\
& \leq z \leq c[2 + r(\cos \varphi + \sin \varphi)].
\end{aligned}$$

于是,

$$\begin{aligned}
I_{xy} &= \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} abrd r \\
&\quad \int_{c[1 + \frac{r^2}{2} + r(\cos \varphi + \sin \varphi)]}^{c[2 + r(\cos \varphi + \sin \varphi)]} z^2 dz \\
&= \frac{1}{3} abc^3 \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} r[8 + 12r(\cos \varphi + \sin \varphi) \\
&\quad + 6r^2(\cos \varphi + \sin \varphi)^2 - \left(1 + \frac{r^2}{2} \right)^3]
\end{aligned}$$

$$\begin{aligned}
& - 3 \left(1 + \frac{r^2}{2} \right)^2 r (\cos \varphi - \sin \varphi) \\
& - 3 \left(1 + \frac{r^2}{2} \right) r^2 (\cos \varphi + \sin \varphi)^2 \Big] dr \\
& = \frac{7}{2} \pi abc^3.
\end{aligned}$$

$$\begin{aligned}
I_{yz} &= \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} a^3 b r (1 + r \cos \varphi)^2 dr \\
&\quad \int_{c(1 + \frac{r^2}{2} + r(\cos \varphi + \sin \varphi))}^{c(2 + r(\cos \varphi + \sin \varphi))} dz \\
&= a^3 b c \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} r (1 + 2r \cos \varphi \\
&\quad + r^2 \cos^2 \varphi) \left(1 + \frac{r^2}{2} \right) dr \\
&= \frac{4}{3} \pi a^3 b c.
\end{aligned}$$

利用对称性得 $I_{xz} = \frac{4}{3} \pi a b^3 c$.

求由下列曲面所界均匀物体对于 Oz 轴的转动惯量:

4148. $z = x^2 + y^2, x + y = \pm 1, x - y = \pm 1, z = 0$.

解 曲面所界的均匀物体对于 Oz 轴的转动惯量记以 I_z , 则

$$I_z = I_{xz} + I_{zy}.$$

若令 $x + y = u, x - y = v$, 则有

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}, \quad z = \frac{u^2+v^2}{2},$$

且 $|I| = \frac{1}{2}$. 于是,

$$I_z = \int_{-1}^1 du \int_{-1}^1 dv \int_0^{\frac{u^2+v^2}{2}} \frac{1}{2} \left(\frac{u-v}{2} \right)^2$$

$$+ \left(\frac{u+v}{2} \right)^2 \Big\} dz \\ = \int_0^1 du \int_0^1 \frac{(u^2+v^2)^2}{8} dv = \frac{14}{45}.$$

4149. $x^2 + y^2 + z^2 = 2, x^2 + y^2 = z^2 (z > 0).$

解 若令 $x = r \cos \varphi, y = r \sin \varphi$, 则有

$$0 \leq \varphi \leq 2\pi, \quad 0 \leq r \leq 1, \quad r \leq z \leq \sqrt{2-r^2}.$$

于是,

$$\begin{aligned} I_z &= \int_0^{2\pi} d\varphi \int_0^1 r dr \int_r^{\sqrt{2-r^2}} r^2 dz \\ &= \int_0^{2\pi} d\varphi \int_0^1 (r^3 \sqrt{2-r^2} - r^4) dr \\ &= \int_0^{2\pi} \left(\frac{8}{15} \sqrt{2} - \frac{7}{5} - \frac{1}{5} \right) d\varphi^*) \\ &= \frac{4\pi}{15} (4\sqrt{2} - 5). \end{aligned}$$

*) 作代换 $r = \sqrt{2} \sin t$.

4150. 设球在动点 $P(x, y, z)$ 的密度与该点至球心距离成比例, 求质量为 M 的非均匀球体 $x^2 + y^2 + z^2 \leq R^2$ 对于其直径的转动惯量.

解 不失一般性, 取 Oz 轴在球内的一段作为直径.

若令

$$x = r \cos \varphi \cos \psi, \quad y = r \sin \varphi \cos \psi, \quad z = r \sin \psi,$$

则质量为

$$M = \int_0^{2\pi} d\varphi \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^R r^2 \cos \psi \cdot k r dr = k\pi R^4,$$

由此得 $k = \frac{M}{\pi R^4}$. 从而密度 $\rho = \frac{Mr}{\pi R^4}$. 于是, 所求的转动

惯量为

$$\begin{aligned} I_z &= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^R r^2 \cos^2 \psi \cdot \frac{Mr^3}{\pi R^4} \cos \psi dr \\ &= \frac{2M}{R^4} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 \psi d\psi \right) \left(\int_0^R r^5 dr \right) = \frac{4MR^2}{9}. \end{aligned}$$

4151. 证明等式

$$I_l = I_{l_0} + Md^2.$$

其中 I_l 为物体对于某轴 l 的转动惯量, I_{l_0} 为对于平行于 l 并通过物体重心的轴 l_0 的转动惯量, d 为轴

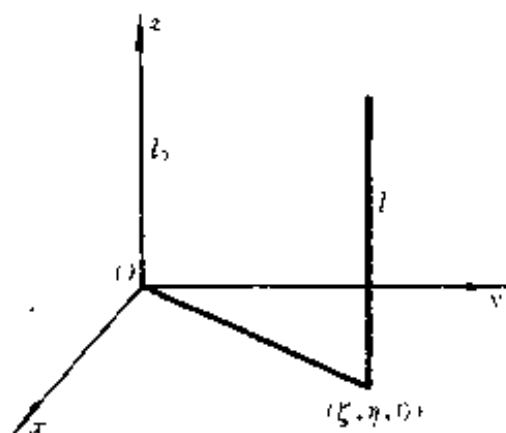


图 8.59

与轴之间的距离及 M 为物体的质量.

证 以重心为坐标原点 O , z 轴与 l_0 重合, l 与 Oxy 平面的交点为 $(\xi, \eta, 0)$, 如图 8.59 所示, 则

$$\begin{aligned} I_l &= \iiint_V [(x - \xi)^2 + (y - \eta)^2] \rho dv \\ &= \iiint_V (x^2 + y^2) \rho dv + (\xi^2 + \eta^2) \\ &\quad \cdot \iiint_V \rho dv - 2\xi \iiint_V x \rho dv - 2\eta \iiint_V y \rho dv \end{aligned} \quad (1)$$

由于重心在原点, 故 $x_0 = y_0 = 0$, 即

$$x_0 = \frac{1}{M} \iiint_V x \rho dv = 0$$

及

$$y_0 = \frac{1}{M} \iiint_V y \rho dv = 0,$$

并且 $M = \iiint_V \rho dv, d^2 = \zeta^2 + \eta^2$, 代入(1)式, 最后得

$$I_l = I_{l_0} + Md^2.$$

4152. 证明: 体积为 V 的物体对于过其重心 $O(0,0,0)$ 并与坐标轴成角 α, β, γ 的轴 l 的转动惯量等于

$$\begin{aligned} I_l = & I_x \cos^2 \alpha + I_y \cos^2 \beta \\ & + I_z \cos^2 \gamma - 2K_{xy} \cos \alpha \cos \beta \\ & - 2K_{yz} \cos \beta \cos \gamma - 2K_{xz} \cos \alpha \cos \gamma. \end{aligned}$$

其中 I_x, I_y, I_z 为物体对于坐标轴的转动惯量及

$$K_{xy} = \iiint_V \rho xy dx dy dz,$$

$$K_{xz} = \iiint_V \rho xz dx dy dz,$$

$$K_{yz} = \iiint_V \rho yz dx dy dz$$

为离心距.

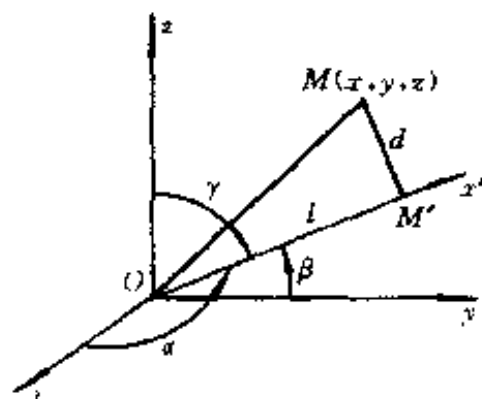


图 8.60

证 如图 8.60 所示, 距离

$$d = \frac{|\vec{OM} \times \vec{OM'}|}{|\vec{OM}|}$$

$$= \frac{1}{r} \sqrt{\begin{vmatrix} y & z \\ r\cos\beta & r\cos\gamma \end{vmatrix}^2 + \begin{vmatrix} z & x \\ r\cos\gamma & r\cos\alpha \end{vmatrix}^2 + \begin{vmatrix} x & y \\ r\cos\alpha & r\cos\beta \end{vmatrix}^2}$$

其中 $r = |\vec{OM}|$. 由于 $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, 故有

$$d^2 = (x^2 + y^2)\cos^2\gamma + (y^2 + z^2)\cos^2\alpha$$

$$+ (z^2 + x^2)\cos^2\beta - 2xy\cos\alpha\cos\beta$$

$$- 2yz\cos\beta\cos\gamma - 2xz\cos\alpha\cos\gamma.$$

于是,

$$I_t = \iiint_V \rho d^2 \cdot dx dy dz$$

$$= \cos^2\gamma \iiint_V \rho \cdot (x^2 + y^2) dx dy dz$$

$$+ \cos^2\alpha \iiint_V \rho \cdot (y^2 + z^2) dx dy dz$$

$$+ \cos^2\beta \iiint_V \rho \cdot (x^2 + z^2) dx dy dz$$

$$- 2\cos\alpha\cos\beta \iiint_V \rho xy dx dy dz$$

$$- 2\cos\beta\cos\gamma \iiint_V \rho yz dx dy dz$$

$$- 2\cos\gamma\cos\alpha \iiint_V \rho xz dx dy dz$$

$$= I_x \cos^2\alpha + I_y \cos^2\beta + I_z \cos^2\gamma$$

$$- 2K_{xy} \cos\alpha\cos\beta - 2K_{yz} \cos\beta\cos\gamma$$

$$- 2K_{xz} \cos\gamma\cos\alpha.$$

证毕.

4153. 求密度为 ρ_0 的均匀圆柱 $x^2 + y^2 \leq a^2, z = \pm h$ 对于直线 $x = y = z$ 的转动惯量.

解 直线 $x = y = z$ 通过圆柱的重心 $O(0, 0, 0)$ 且具有方向余弦 $\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$. 若取极坐标, 则有

$$\begin{aligned} I_x &= \int_0^{2\pi} d\varphi \int_0^a r dr \int_{-h}^h \rho_0 (r^2 \sin^2 \varphi + z^2) dz \\ &= \left(\frac{1}{2} \pi a^4 h + \frac{2}{3} \pi a^2 h^3 \right) \rho_0, \end{aligned}$$

$$\begin{aligned} I_y &= \int_0^{2\pi} d\varphi \int_0^a r dr \int_{-h}^h \rho_0 (r^2 \cos^2 \varphi + z^2) dz \\ &= \left(\frac{1}{2} \pi a^4 h + \frac{2}{3} \pi a^2 h^3 \right) \rho_0, \end{aligned}$$

$$I_z = \int_0^{2\pi} d\varphi \int_0^a r dr \int_{-h}^h \rho_0 r^2 dz = \pi h a^4 \rho_0,$$

$$K_{xy} = \int_0^{2\pi} d\varphi \int_0^a r dr \int_{-h}^h \rho_0 r^2 \cos\varphi \sin\varphi dz = 0,$$

$$K_{yz} = \int_0^{2\pi} d\varphi \int_0^a r dr \int_{-h}^h \rho_0 r \sin\varphi \cdot z dz = 0,$$

$$K_{zx} = \int_0^{2\pi} d\varphi \int_0^a r dr \int_{-h}^h \rho_0 \cdot r \cos\varphi \cdot z dz = 0,$$

于是, 根据 4152 题结果即得

$$\begin{aligned} I_t &= I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma \\ &\quad - 2K_{xy} \cos\alpha \cos\beta - 2K_{yz} \cos\beta \cos\gamma \\ &\quad - 2K_{zx} \cos\alpha \cos\gamma \\ &= \frac{\rho_0}{3} \left(\frac{1}{2} \pi a^4 h + \frac{2}{3} \pi a^2 h^3 + \frac{1}{2} \pi a^4 h \right. \\ &\quad \left. + \frac{2}{3} \pi a^2 h^3 + \pi a^4 h \right) = \frac{2}{3} \pi \rho_0 a^2 h \left(a^2 + \frac{2}{3} h^2 \right) \end{aligned}$$

$$= \frac{M}{3} \left(a^2 + \frac{2}{3} h^2 \right),$$

其中 $M = 2\pi\rho_0 a^2 h$ 为圆柱的质量.

4154. 求密度为 ρ_0 , 由曲面

$$(x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2)$$

所界的均匀物体对于坐标原点的转动惯量.

解 若令 $x = r\cos\varphi\cos\psi, y = r\sin\varphi\cos\psi, z = r\sin\psi$, 则对坐标原点的转动惯量为

$$\begin{aligned} I_0 &= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{a\cos\psi} \rho_0 \cdot r^2 \cdot r^2 \cos\psi dr \\ &= \frac{4\pi\rho_0 a^5}{5} \int_0^{\frac{\pi}{2}} \cos^5\psi d\psi \\ &= \frac{4\pi\rho_0 a^5}{5} \cdot \frac{5\pi^*}{32} = \frac{\pi^2 a^5 \rho_0}{8}. \end{aligned}$$

*) 利用 2282 题的结果.

4155. 求密度为 ρ_0 的均匀球体 $\xi^2 + \eta^2 + \zeta^2 \leq R^2$ 在点 (x, y, z) 的牛顿位.

解 由对称性显然可知, 所求的牛顿位与 ξ, η, ζ 轴取的方向无关. 今取 $O\zeta$ 轴通过点 $P(x, y, z)$, 即得牛顿位

$$\begin{aligned} u(x, y, z) &= \iiint_{\xi^2 + \eta^2 + \zeta^2 \leq R^2} \rho_0 \frac{d\xi d\eta d\zeta}{\sqrt{\xi^2 + \eta^2 + (\zeta - r)^2}} \\ &= \rho_0 \int_{-R}^R d\zeta \iint_{\xi^2 + \eta^2 \leq R^2 - \zeta^2} \frac{d\xi d\eta}{\sqrt{\xi^2 + \eta^2 + (\zeta - r)^2}}, \end{aligned}$$

其中 $r = \sqrt{x^2 + y^2 + z^2}$.

积分之, 得

$$\begin{aligned} u(x, y, z) &= 2\pi\rho_0 \int_{-R}^R (\sqrt{R^2 - 2r\zeta + r^2} - |\zeta - r|) d\zeta. \end{aligned}$$

由于

$$\begin{aligned} & \int_{-R}^R \sqrt{R^2 - 2r\xi + r^2} d\xi \\ &= \frac{1}{3}r[(R+r)^3 - |R-r|^3] \\ &= \begin{cases} \frac{2}{3}R^3 \frac{1}{r} + 2rR, & (r > R); \\ \frac{2}{3}r^3 - R^2, & (r \leq R), \end{cases} \end{aligned}$$

及

$$\int_{-R}^R |\xi - r| d\xi = \begin{cases} 2Rr & (r > R), \\ r^2 + R^2 & (r \leq R). \end{cases}$$

因而,最后得

$$u(x, y, z) = \begin{cases} \frac{4}{3}\pi R^3 \rho_0 & (r > R), \\ 2\pi\rho_0 \cdot \left(R^2 - \frac{1}{3}r^2 \right) & (r \leq R). \end{cases}$$

由以上结果可以得到下面两个推论:

1. 在球外一点上的牛顿位,与将球的全部质量集中在它的中心处时一样;

2. 如考察一个内半径为 R_1 而外半径为 R_2 的空心球,则它在位于其空隙处的一点 ($r < R$) 上的牛顿位可表示成差

$$\begin{aligned} u(x, y, z) &= u_2(x, y, z) - u_1(x, y, z) \\ &= \left(R_2^2 - \frac{1}{3}r^2 \right) 2\pi\rho_0 \\ &\quad - \left(R_1^2 - \frac{1}{3}r^2 \right) 2\pi\rho_0 \\ &= 2\pi(R_2^2 - R_1^2)\rho_0. \end{aligned}$$

它与 r 无关,故空心球体在其空隙范围内的位势保持一个常数值.

4156. 设密度 $\rho = f(R)$, 其中 f 为已知函数, 且 $R = \sqrt{\xi^2 + \eta^2 + \zeta^2}$, 求球壳层 $R_1^2 \leq \xi^2 + \eta^2 + \zeta^2 \leq R_2^2$ 在点 $P(x, y, z)$ 的牛顿位.

解 取 $O\xi$ 轴通过点 $P(x, y, z)$, 即得牛顿位

$$u(x, y, z) = \iiint_{R_1^2 \leq \xi^2 + \eta^2 + \zeta^2 \leq R_2^2} f(\sqrt{\xi^2 + \eta^2 + \zeta^2}) \cdot \frac{d\xi d\eta d\zeta}{\sqrt{\xi^2 + \eta^2 + (\zeta - r)^2}},$$

其中 $r = x^2 + y^2 + z^2$.

若引入球坐标, 即得

$$\begin{aligned} u(x, y, z) &= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_{R_1}^{R_2} \rho^2 f(\rho) \cos\psi \\ &\quad \cdot \frac{d\rho}{\sqrt{\rho^2 + r^2 - 2\rho r \sin\psi}} \\ &= 2\pi \int_{R_1}^{R_2} d\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho^2 f(\rho) \cdot \frac{\cos\psi d\psi}{\sqrt{\rho^2 + r^2 - 2\rho r \sin\psi}} \\ &= 2\pi \int_{R_1}^{R_2} \rho^2 f(\rho) \\ &\quad \cdot \left(-\frac{1}{\rho r} \sqrt{\rho^2 + r^2 - 2\rho r \sin\psi} \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\rho \\ &= 2\pi \int_{R_1}^{R_2} \rho^2 f(\rho) \left\{ -\frac{1}{\rho r} [|\rho - r| - (\rho + r)] \right\} d\rho \end{aligned}$$

$$= \begin{cases} 4\pi \int_{R_1}^{R_2} \rho f(\rho) d\rho, & \text{当 } \rho > r; \\ 4\pi \int_{R_1}^{R_2} \frac{\rho^2}{r} f(\rho) d\rho, & \text{当 } \rho \leq r. \end{cases}$$

合并之,最后得

$$u(x, y, z) = 4\pi \int_{R_1}^{R_2} f(\rho) \min\left\{\frac{\rho^2}{r}, \rho\right\} d\rho.$$

4157. 求有固定密度 ρ_0 的圆柱 $\xi^2 + \eta^2 \leq a^2, 0 \leq \zeta \leq h$ 在点 $P(0, 0, z)$ 的牛顿位.

解 若引用柱坐标,即得

$$\begin{aligned} u(x, y, z) &= \rho_0 \int_0^{2\pi} d\varphi \int_0^h d\zeta \int_0^a \frac{r dr}{\sqrt{r^2 + (\zeta - z)^2}} \\ &= 2\pi\rho_0 \int_0^h \left. \sqrt{r^2 + (\zeta - z)^2} \right|_0^a d\zeta \\ &= 2\pi\rho_0 \int_0^h [\sqrt{a^2 + (\zeta - z)^2} - |\zeta - z|] d\zeta \\ &= 2\pi\rho_0 \left[\frac{(\zeta - z)}{2} \sqrt{a^2 + (\zeta - z)^2} \right. \\ &\quad \left. + \frac{a^2}{2} \ln |(\zeta - z) + \sqrt{a^2 + (\zeta - z)^2}| \right. \\ &\quad \left. - \frac{(\zeta - z)|\zeta - z|}{2} \right] \Big|_0^h \\ &= \pi\rho_0 \left\{ (h - z) \sqrt{a^2 + (h - z)^2} + z \sqrt{a^2 + z^2} \right. \\ &\quad \left. - [(h - z)|h - z| + z|z|] \right. \\ &\quad \left. + a^2 \ln \left| \frac{h - z + \sqrt{a^2 + (h - z)^2}}{-z + \sqrt{a^2 + z^2}} \right| \right\}. \end{aligned}$$

4158. 半径为 R 和质量为 M 的均匀球体 $\xi^2 + \eta^2 + \zeta^2 \leq R^2$ 以怎样的力吸引质量为 m 的质点 $P(0, 0, a)$?

解 引力在 Ox 轴和 Oy 轴上的射影为零, 即 $X = Y = 0$, 而在 Oz 轴上的射影为

$$\begin{aligned}
 Z &= k\rho_0 m \iiint_{\xi^2 + \eta^2 + \zeta^2 \leq R^2} \frac{(\zeta - a)d\xi d\eta d\zeta}{[\xi^2 + \eta^2 + (\zeta - a)^2]^{\frac{3}{2}}} \\
 &= km\rho_0 \int_{-R}^R (\zeta - a) d\zeta \iint_{\xi^2 + \eta^2 \leq R^2 - \zeta^2} \frac{d\xi d\eta}{[\xi^2 + \eta^2 + (\zeta - a)^2]^{\frac{3}{2}}} \\
 &= km\rho_0 \int_{-R}^R (\zeta - a) d\zeta \int_0^{2\pi} d\varphi \int_0^{\sqrt{R^2 - \zeta^2}} \frac{r dr}{[r^2 + (\zeta - a)^2]^{\frac{3}{2}}} \\
 &= 2\pi km\rho_0 \int_{-R}^R (\zeta - a) \left(\frac{1}{|\zeta - a|} - \frac{1}{\sqrt{R^2 - 2a\zeta + a^2}} \right) d\zeta \\
 &= 2\pi km\rho_0 \int_{-R}^R \operatorname{sgn}(\zeta - a) d\zeta - 2\pi km\rho_0 \int_{-R}^R \frac{(\zeta - a) d\zeta}{\sqrt{R^2 - 2a\zeta + a^2}}
 \end{aligned}$$

其中 $\rho_0 = \frac{3M}{4\pi R^3}$,

分别求上述两个积分:

当 $a \geq R$ 时,

$$\int_{-R}^R \operatorname{sgn}(\zeta - a) d\zeta = - \int_{-R}^R d\zeta = -2R,$$

当 $a < R$ 时,

$$\int_{-R}^R \operatorname{sgn}(\zeta - a) d\zeta = - \int_{-R}^a d\zeta + \int_a^R d\zeta = -2a;$$

而

$$\begin{aligned}
 & \int_{-R}^R \frac{(\zeta - a)d\zeta}{\sqrt{R^2 - 2a\zeta + a^2}} \\
 &= -\frac{1}{2a} \int_{-R}^R \frac{R^2 + a^2 - 2a\zeta - (R^2 + a^2)}{\sqrt{R^2 - 2a\zeta + a^2}} d\zeta \\
 & \quad - a \int_{-R}^R \frac{d\zeta}{\sqrt{R^2 - 2a\zeta + a^2}} \\
 &= -\frac{1}{2a} \int_{-R}^R \sqrt{R^2 + a^2 - 2a\zeta} d\zeta \\
 & \quad + \left(\frac{R^2 + a^2}{2a} - a \right) \int_{-R}^R \frac{d\zeta}{\sqrt{R^2 + a^2 - 2a\zeta}} \\
 &= -\frac{1}{2a} \int_{-R}^R \sqrt{R^2 + a^2 - 2a\zeta} d\zeta \\
 & \quad + \frac{R^2 - a^2}{2a} \int_{-R}^R \frac{d\zeta}{\sqrt{R^2 + a^2 - 2a\zeta}},
 \end{aligned}$$

当 $a \geq R$ 时, 将上式右端分别积分, 得结果:

$$\begin{aligned}
 & \left[\frac{1}{4a^2} (R^2 + a^2 - 2a\zeta)^{\frac{3}{2}} \cdot \frac{2}{3} + \frac{R^2 - a^2}{2a} \left(-\frac{1}{2a} \right) \right. \\
 & \quad \left. \cdot 2 \sqrt{R^2 + a^2 - 2a\zeta} \right] \Big|_{-R}^R \\
 &= \frac{1}{6a^2} [(a - R)^3 - (a + R)^3] \\
 & \quad - \frac{R^2 - a^2}{2a^2} [(a - R) - (a + R)] \\
 &= \frac{2R^3}{3a^2} - 2R;
 \end{aligned}$$

当 $a < R$ 时, 积分得结果:

$$\begin{aligned}
 & \frac{1}{6a^2} [(R - a)^3 - (a + R)^3] \\
 & \quad - \frac{R^2 - a^2}{2a^2} [(R - a) - (R + a)]
 \end{aligned}$$

$$= -\frac{4a}{3}.$$

于是,当 $a \geq R$ 时,则

$$\begin{aligned} Z &= 2\pi km\rho_0 \left(-2R - \frac{2R^3}{3a^2} + 2R \right) \\ &= -\frac{4}{3a^2}\pi km\rho_0 R^3 = -\frac{kMm}{a^2}; \end{aligned}$$

当 $a < R$ 时,则

$$\begin{aligned} Z &= 2\pi km\rho_0 \left(-2a + \frac{4a^3}{3} \right) \\ &= -\frac{4}{3}\pi akm\rho_0 = -\frac{kMm}{R^3}a. \end{aligned}$$

从以上结果可以得到两个推论:

1. 位于球外的一点($a \geq R$)因球体而受到的吸引力相当于将球体的全部质量 $M = \frac{4}{3}\pi R^3\rho_0$ 集中在它的中心处时受到的引力,引力的方向朝向球心;

2. 对于在球里面的一点($a < R$)来说,吸引力与 R 无关,其大小与 $R = a$ 时的情况一样,即在点 P 外面的球壳部分对 P 点的引力为零.

4159. 求密度为 ρ_0 的均匀圆柱 $\xi^2 + \eta^2 \leq a^2, 0 \leq \zeta \leq h$ 对具有单位质量的质点 $P(0, 0, z)$ 的吸引力.

解 由对称性知,引力在 Ox 轴和 Oy 轴上的射影为零,即 $X = Y = 0$. 若引用柱坐标,即得引力在 Oz 轴上的射影为

$$\begin{aligned} Z &= k\rho_0 \iint_{\xi^2 + \eta^2 \leq a^2} d\xi d\eta \int_0^h \frac{(\zeta - z)d\zeta}{[\xi^2 + \eta^2 + (\zeta - z)^2]^{\frac{3}{2}}} \\ &= k\rho_0 \int_0^{2\pi} d\varphi \int_0^a r dr \int_0^h \frac{(\zeta - z)d\zeta}{[r^2 + (\zeta - z)^2]^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned}
&= 2\pi k\rho_0 \int_0^a r \left(-\frac{1}{\sqrt{r^2+z^2}} - \frac{1}{\sqrt{r^2+(h-z)^2}} \right) dr \\
&= 2\pi k\rho_0 [\sqrt{a^2+z^2} - \sqrt{a^2+(h-z)^2} \\
&\quad - |z| + |h-z|].
\end{aligned}$$

易知,

当 $0 \leq z < \frac{h}{2}$ 时, $z > 0$, 此时吸引力朝着向上的铅垂线;

当 $\frac{h}{2} < z \leq h$ 时, $z < 0$, 此时吸引力朝着向下的铅垂线;

当 $Z = \frac{h}{2}$ 时, $Z = 0$, 引力为零.

4160. 求密度为 ρ_0 的均匀球锥体对于在其顶点为一单位质量的质点的吸引力, 设球的半径为 R , 而轴截面的扇形的角等于 $2a$.

解 由对称性知, 引力在 Ox 轴和 Oy 轴上的射影为零, 即 $X = Y = 0$. 若引用球坐标, 即得引力在 Oz 轴上的射影为

$$\begin{aligned}
Z &= \iiint_V \frac{k\rho_0 z}{\sqrt{(x^2+y^2+z^2)^3}} dx dy dz \\
&= k\rho_0 \int_0^{2\pi} d\varphi \int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \cos\psi \sin\psi d\psi \int_0^R dr \\
&= k\pi R\rho_0 \sin^2 a.
\end{aligned}$$

§ 9. 二重和三重广义积分

1° 无界限域的情形 若二维的域 Ω 是无界的及函数 $f(x, y)$ 在域 Ω 上连续, 则定义:

$$\iint_{\Omega} f(x, y) dx dy = \lim_{n \rightarrow \infty} \iint_{\Omega_n} f(x, y) dx dy, \quad (1)$$

其中 Ω_n 为域 Ω 中可求积的有界封闭子域的任意序列, 这个序列可以盖满域 Ω . 若在右端的极限存在且与序列 Ω_n 的选择无关, 则对应的积分称为收敛的; 在相反的情形称为发散的.

同样定义出连续函数展布在无界的三维域上的三重广义积分.

2° 不连续函数的情形 若函数 $f(x, y)$ 在有界封闭域 Ω 内除了点 $P(a, b)$ 而外处处是连续的, 则定义:

$$\iint_{\Omega} f(x, y) dx dy = \lim_{\epsilon \rightarrow 0} \iint_{U_{\epsilon}} f(x, y) dx dy, \quad (2)$$

其中 U_{ϵ} 是点 P 的 ϵ 邻域, 当极限存在的情形, 所研究的积分称为收敛的; 在相反的情形称为发散的.

假定在点 $P(a, b)$ 的邻近有等式

$$f(x, y) = \frac{\varphi(x, y)}{r^{\alpha}},$$

其中函数 $\varphi(x, y)$ 的绝对值是介于二正数 m 和 M 之间, 且

$$r = \sqrt{(x-a)^2 + (y-b)^2},$$

则 1) 当 $\alpha < 2$ 时, 积分(2)收敛; 2) 当 $\alpha \geq 2$ 时, 积分(2)发散.

若函数 $f(x, y)$ 有不连续的线, 同样可定义出广义积分(2).

不连续函数的广义积分定义易于引伸到三重积分的情形.

研究下列具有无界积分域的广义积分的收敛性

$$(0 < m \leq |\varphi(x, y)| \leq M):$$

$$4161. \iint_{x^2+y^2>1} \frac{\varphi(x,y)}{(x^2+y^2)^p} dx dy.$$

解 由于

$$\frac{m}{(x^2+y^2)^p} \leq \frac{|\varphi(x,y)|}{(x^2+y^2)^p} \leq \frac{M}{(x^2+y^2)^p},$$

再注意到广义重积分收敛必绝对收敛, 即知积分

$$\iint_{x^2+y^2>1} \frac{\varphi(x,y)}{(x^2+y^2)^p} dx dy$$

与积分 $\iint_{x^2+y^2>1} \frac{1}{(x^2+y^2)^p} dx dy$ 同时收敛同时发散. 由

于 $\frac{1}{(x^2+y^2)^p}$ 是正的, 故引用极坐标, 得

$$\begin{aligned} & \iint_{x^2+y^2>1} \frac{1}{(x^2+y^2)^p} dx dy \\ &= \int_0^{2\pi} d\varphi \int_1^{+\infty} \frac{r}{r^{2p}} dr = \begin{cases} \frac{\pi}{p-1}, & \text{若 } p > 1; \\ +\infty, & \text{若 } p \leq 1. \end{cases} \end{aligned}$$

由此可知, 原积分 $\iint_{x^2+y^2>1} \frac{\varphi(x,y)}{(x^2+y^2)^p} dx dy$ 当 $p > 1$ 时

收敛, 当 $p \leq 1$ 时发散.

$$4162. \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dx dy}{(1+|x|^p)(1+|y|^q)}.$$

解 由于被积函数是正的, 并且关于 Ox 轴和 Oy 轴都对称, 故

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dx dy}{(1+|x|^p)(1+|y|^q)} \\ &= 4 \int_0^{+\infty} \int_0^{+\infty} \frac{dx dy}{(1+x^p)(1+y^q)} \end{aligned}$$

$$= 4 \left(\int_0^{+\infty} \frac{dx}{1+x^p} \right) \left(\int_0^{+\infty} \frac{dy}{1+y^q} \right).$$

由于 $\lim_{x \rightarrow +\infty} x^p \cdot \frac{1}{1+x^p} = 1$, 故积分 $\int_0^{+\infty} \frac{dx}{1+x^p}$

当 $p > 1$ 时收敛, $p < 1$ 时发散, $p = 1$ 时显然也发散

$\left(\int_0^{+\infty} \frac{dx}{1+x} = +\infty \right)$. 因此,

$$\int_0^{+\infty} \frac{dx}{1+x^p} = \begin{cases} \text{有限数, 当 } p > 1 \text{ 时;} \\ +\infty, & \text{当 } p \leq 1 \text{ 时.} \end{cases}$$

同理有

$$\int_0^{+\infty} \frac{dy}{1+y^q} = \begin{cases} \text{有限数, 当 } q > 1 \text{ 时;} \\ +\infty, & \text{当 } q \leq 1 \text{ 时.} \end{cases}$$

由此可知, $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dxdy}{(1+|x|^p)(1+|y|^q)}$ 仅当 $p > 1$ 且 $q > 1$ 时收敛, 其它情形均发散.

$$4163. \iint_{0 \leq x \leq 1} \frac{\varphi(x, y)}{(1-x^2+y^2)^p} dxdy$$

解 仿 4161 题, 可知积分 $\iint_{0 \leq x \leq 1} \frac{\varphi(x, y)}{(1+x^2+y^2)^p} dxdy$

与积分 $\iint_{0 \leq x \leq 1} \frac{dxdy}{(1+x^2+y^2)^p}$ 同时收敛同时发散. 由于

被积函数是正的, 故

$$\begin{aligned} & \iint_{0 \leq x \leq 1} \frac{dxdy}{(1+x^2+y^2)^p} \\ &= \int_0^1 dy \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2+y^2)^p} \\ &= 2 \int_0^1 dy \int_0^{+\infty} \frac{dx}{(1+x^2+y^2)^p}; \end{aligned}$$

由于, 当 $0 \leq y \leq 1$ 时, 有

$$\begin{aligned}
\int_0^{+\infty} \frac{dx}{(2+x^2)^p} &\leq \int_0^{+\infty} \frac{dx}{(1+x^2+y^2)^p} \\
&\leq \int_0^{+\infty} \frac{dx}{(1+x^2)^p} \text{ (若 } p \geq 0), \\
\int_0^{+\infty} \frac{dx}{(2+x^2)^p} &\geq \int_0^{+\infty} \frac{dx}{(1+x^2+y^2)^p} \\
&\geq \int_0^{+\infty} \frac{dx}{(1+x^2)^p} \text{ (若 } p < 0),
\end{aligned}$$

故

$$\begin{aligned}
2 \int_0^{+\infty} \frac{dx}{(2+x^2)^p} &\leq \iint_{0 \leq y \leq 1} \frac{dx dy}{(1+x^2+y^2)^p} \\
&\leq 2 \int_0^{+\infty} \frac{dx}{(1+x^2)^p} \text{ (} p \geq 0),
\end{aligned}$$

若 $p < 0$, 则有相反的不等式.

对于 $a > 0$, 由于

$$\lim_{x \rightarrow +\infty} x^{2p} \frac{1}{(a^2 + x^2)^p} = 1,$$

故积分 $\int_0^{+\infty} \frac{dx}{(a^2 + x^2)^p}$ 当 $p > \frac{1}{2}$ 时收敛, $p < \frac{1}{2}$ 时发散. 实际上, 此积分当 $p = \frac{1}{2}$ 时也发散, 因为

$$\int_0^{+\infty} \frac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{a^2 + x^2}) \Big|_0^{+\infty} = +\infty.$$

由此可知: 积分 $\iint_{0 \leq y \leq 1} \frac{dx dy}{(1+x^2+y^2)^p}$, 从而积分

$\iint_{0 \leq y \leq 1} \frac{\varphi(x, y)}{(1+x^2+y^2)^p} dx dy$ 当 $p > \frac{1}{2}$ 时收敛, 当 $p \leq \frac{1}{2}$ 时发散.

4164. $\iint_{|x|+|y| \geq 1} \frac{dx dy}{|x|^p + |y|^q} \text{ (} p > 0, q > 0 \text{)}.$

解 由对称性及被积函数的非负性,有

$$\begin{aligned} \iint_{|x|+|y|\geq 1} \frac{dx dy}{|x|^p + |y|^q} &= 4 \iint_{\substack{x\geq 0, y\geq 0 \\ x+y\geq 1}} \frac{dx dy}{x^p + y^q} \\ &= 4 \iint_{\Omega_1} \frac{dx dy}{x^p + y^q} + 4 \iint_{\Omega_2} \frac{dx dy}{x^p + y^q}, \end{aligned}$$

其中 $\Omega_1 = \{(x, y) | x \geq 0, y \geq 0, x + y \geq 1, x^p + y^q \leq 2\}$, $\Omega_2 = \{(x, y) | x \geq 0, y \geq 0, x + y \geq 1, x^p + y^q \geq 2\}$, 令 $\Omega_3 = \{(x, y) | x \geq 0, y \geq 0, x^p + y^q \geq 2\}$, 易知, 当 $x \geq 0, y \geq 0, x^p + y^q \geq 2$ 时必有 $x + y \geq 1$ (因若 $x + y < 1$, 则必有 $0 \leq x < 1, 0 \leq y < 1$, 从而 $0 \leq x^p < 1, 0 \leq y^q < 1$, 这就会得出 $x^p + y^q < 2$), 故 $\Omega_2 = \Omega_3$. 由于 Ω_1 是有界闭区域, 故(1)式右端第一个积分为常义积分, 因此广义积分

$$\iint_{|x|+|y|\geq 1} \frac{dx dy}{|x|^p + |y|^q}$$

的敛散性取决于广义积分 $\iint_{\Omega_3} \frac{dx dy}{x^p + y^q}$ 的敛散性, 在此

积分中作变量代换

$$x = r^{\frac{2}{p}} \cos^{\frac{2}{p}} \theta, y = r^{\frac{2}{q}} \sin^{\frac{2}{q}} \theta,$$

则易知

$$\frac{D(x, y)}{D(r, \theta)} = \frac{4}{pq} r^{\frac{2}{p} - \frac{2}{q} - 1} \sin^{\frac{2}{q} - 1} \theta \cos^{\frac{2}{p} - 1} \theta.$$

于是, 注意到被积函数是非负的, 得

$$\begin{aligned} \iint_{\Omega_3} \frac{dx dy}{x^p + y^q} &= \frac{4}{pq} \int_0^{\frac{\pi}{2}} \sin^{\frac{2}{q} - 1} \theta \cos^{\frac{2}{p} - 1} \theta d\theta \\ &\quad \cdot \int_{\sqrt{2}}^{+\infty} r^{\frac{2}{p} + \frac{2}{q} - 3} dr. \end{aligned}$$

由 3856 题的结果知,右端第一个积分

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{2}{q}-1} \theta \cos^{\frac{2}{p}-1} \theta d\theta \quad (p > 0, q > 0)$$

恒收敛,且其值为 $\frac{1}{2} B(\frac{1}{q}, \frac{1}{p})$; 而第二个积分

$$\int_{\sqrt{2}}^{+\infty} r^{\frac{2}{p} + \frac{2}{q} - 3} dr$$

当 $\frac{2}{p} + \frac{2}{q} - 3 < -1$ (即 $\frac{1}{p} + \frac{1}{q} < 1$) 时收敛, 当 $\frac{2}{p} + \frac{2}{q} - 3 \geq -1$ (即 $\frac{1}{p} + \frac{1}{q} \geq 1$) 时发散.

综上所述,可知广义积分

$$\iint_{|x|+|y| \geq 1} \frac{dxdy}{|x|^p + |y|^q}$$

仅当 $\frac{1}{p} + \frac{1}{q} < 1$ 时收敛.

4165. $\iint_{x+y \geq 1} \frac{\sin x \sin y}{(x+y)^p} dxdy.$

解 设此积分收敛. 以 I 表其值. 先设 $p < 1$.

令 $\Omega_n = \{(x, y) | 1 \leq x - y \leq 2n\pi,$

$$-2n\pi \leq x - y \leq 2n\pi\},$$

$$\Omega'_n = \{(x, y) | 1 \leq x + y \leq 2n\pi - \frac{\pi}{4},$$

$$-2n\pi \leq x - y \leq 2n\pi\},$$

$$\omega_n = \{(x, y) | 2n\pi - \frac{\pi}{4} \leq x + y \leq 2n\pi,$$

$$-2n\pi \leq x - y \leq 2n\pi\},$$

其中 $n = 1, 2, 3, \dots$, 则显然有

$$\lim_{n \rightarrow \infty} \iint_{\Omega_n} \frac{\sin x \sin y}{(x+y)^p} dxdy = I,$$

$$\lim_{n \rightarrow \infty} \iint_{\tilde{D}_n} \frac{\sin x \sin y}{(x+y)^p} dx dy = I.$$

从而

$$\begin{aligned} & \lim_{n \rightarrow \infty} \iint_{\omega_n} \frac{\sin x \sin y}{(x+y)^p} dx dy \\ &= \lim_{n \rightarrow \infty} \left(\iint_{\tilde{D}_n} \frac{\sin x \sin y}{(x+y)^p} dx dy \right. \\ & \quad \left. - \iint_{\tilde{D}_n} \frac{\sin x \sin y}{(x+y)^p} dx dy \right) \\ &= I - I = 0. \end{aligned} \quad (1)$$

由于 $\sin x \sin y = \frac{1}{2} [\cos(x-y) - \cos(x+y)]$, 今在 (1) 式左端的积分中作变量代换 $x+y=u, x-y=v$ (即 $x = \frac{u+v}{2}, y = \frac{u-v}{2}$), 并注意到 $\frac{D(x,y)}{D(u,v)} = -\frac{1}{2}$, 得

$$\begin{aligned} & \iint_{\omega_n} \frac{\sin x \sin y}{(x+y)^p} dx dy \\ &= \frac{1}{4} \int_{2n\pi - \frac{\pi}{4}}^{2n\pi} du \int_{-2n\pi}^{2n\pi} \frac{\cos v - \cos u}{u^p} dv \\ &= -n\pi \int_{2n\pi - \frac{\pi}{4}}^{2n\pi} \frac{\cos u}{u^p} du; \end{aligned}$$

而

$$\int_{2n\pi - \frac{\pi}{4}}^{2n\pi} \frac{\cos u}{u^p} du \geq \frac{1}{\sqrt{2}} \int_{2n\pi - \frac{\pi}{4}}^{2n\pi} \frac{du}{u^p}$$

$$\geq \begin{cases} \frac{\pi}{4\sqrt{2}} \cdot \frac{1}{(2n\pi)^p}, & \text{当 } p > 0 \text{ 时;} \\ \frac{\pi}{4\sqrt{2}}, & \text{当 } p \leq 0 \text{ 时.} \end{cases}$$

由此可知(注意前面假定 $p < 1$)

$$\lim_{n \rightarrow \infty} \iint_{\omega_n} \frac{\sin x \sin y}{(x+y)^p} dx dy = -\infty,$$

此显然与(1)式矛盾.

现设 $p \geq 1$. 令

$$\begin{aligned} \omega'_n = \{ (x, y) \mid 2n\pi - \frac{\pi}{4} \leq x+y \leq 2n\pi, \\ -2\pi n^{[p]+2} \leq x-y \leq 2\pi n^{[p]+2} \}, \end{aligned}$$

仿上, 应有

$$\lim_{n \rightarrow \infty} \iint_{\omega'_n} \frac{\sin x \sin y}{(x+y)^p} dx dy = 0. \quad (2)$$

但另一方面, 和上面一样, 作代换 $x+y=u, x-y=v$ 后, 有

$$\begin{aligned} & \iint_{\omega'_n} \frac{\sin x \sin y}{(x+y)^p} dx dy \\ &= -\pi n^{[p]-2} \int_{2n\pi - \frac{\pi}{4}}^{2n\pi} \frac{\cos u}{u^p} du. \end{aligned}$$

同样, 由

$$\int_{2n\pi - \frac{\pi}{4}}^{2n\pi} \frac{\cos u}{u^p} du \geq \frac{\pi}{4\sqrt{2}} \cdot \frac{1}{(2n\pi)^p},$$

即得

$$\lim_{n \rightarrow \infty} \iint_{\omega'_n} \frac{\sin x \sin y}{(x+y)^p} dx dy = -\infty,$$

此显然与(2)式矛盾.

综上所述,可知:不论 p 为何值,积分

$$\iint_{x+y \geq 1} \frac{\sin x \sin y}{(x+y)^p} dx dy$$

都发散.

4166. 证明:若连续函数 $f(x, y)$ 不为负及 $S_n (n = 1, 2, \dots)$ 为有界闭域的任一叙列,这个叙列可以盖满域 S , 则

$$\iint_S f(x, y) dx dy = \lim_{n \rightarrow \infty} \iint_{S_n} f(x, y) dx dy,$$

这里左端与右端同时有意义或同时无意义.

证 取定一有界闭域的叙列 S'_n , 它盖满 S 并且 $S'_1 \subset S'_2 \subset \dots \subset S'_n \subset \dots \subset S$. 由于 $f(x, y)$ 在 S 上非负, 故积分叙列 $\iint_{S'_n} f(x, y) dx dy$ 是递增的, 从而极限

$$I = \lim_{n \rightarrow \infty} \iint_{S'_n} f(x, y) dx dy \quad (1)$$

存在(是有限数或是 $+\infty$). 我们要证

$$\lim_{n \rightarrow \infty} \iint_{S_n} f(x, y) dx dy = I. \quad (2)$$

先设 I 为有限数. 任给 $\varepsilon > 0$, 由(1)式知, 存在 N , 使当 $n \geq N$ 时, 恒有

$$I - \varepsilon < \iint_{S'_n} f(x, y) dx dy < I + \varepsilon. \quad (3)$$

又存在 n_0 , 使当 $n \geq n_0$ 时, $S_n \supset S'_N$. 从而, 根据 $f(x, y)$ 的非负性以及(3)式, 得

$$\iint_{S_n} f(x, y) dx dy \geq \iint_{S'_N} f(x, y) dx dy > I - \varepsilon.$$

另一方面, 对每个固定的 $n \geq n_0$ 又必存在某个充分大的

$k_n (\geq N)$ 使 $S'_{k_n} \supset S_n$. 于是, 再由 (3) 式得

$$\iint_{\tilde{S}_n} f(x, y) dx dy \leq \iint_{S'_{k_n}} f(x, y) dx dy < I + \varepsilon.$$

由此可知, 当 $n \geq n_0$ 时, 恒有

$$I - \varepsilon < \iint_{\tilde{S}_n} f(x, y) dx dy < I + \varepsilon,$$

故 (2) 式成立.

次设 $I = +\infty$. 任给 $M > 0$, 由 (1) 式知, 存在 N_1 , 使

$$\iint_{S'_{N_1}} f(x, y) dx dy > M.$$

又存在 n_1 , 使当 $n \geq n_1$ 时, 恒有 $S_n \supset S'_{N_1}$, 从而此时

$$\iint_{S_n} f(x, y) dx dy \geq \iint_{S'_{N_1}} f(x, y) dx dy > M,$$

故 (2) 式成立. 证毕.

4167. 证明:

$$\lim_{n \rightarrow \infty} \iint_{\substack{|x| \leq n \\ |y| \leq n}} \sin(x^2 + y^2) dx dy = \pi,$$

但

$$\lim_{n \rightarrow \infty} \iint_{x^2 + y^2 \leq 2n\pi} \sin(x^2 + y^2) dx dy = 0$$

(n 为自然数).

证 利用极坐标, 我们有

$$\iint_{x^2 + y^2 \leq 2n\pi} \sin(x^2 + y^2) dx dy$$

$$\begin{aligned}
&= \int_0^{2\pi} d\theta \int_0^{\sqrt{2\pi n}} r \sin r^2 dr \\
&= \pi(1 - \cos 2n\pi) = 0 \quad (n = 1, 2, \dots),
\end{aligned}$$

故

$$\lim_{n \rightarrow \infty} \iint_{x^2 + y^2 \leq 2\pi n} \sin(x^2 + y^2) dx dy = 0.$$

但由对称性,有

$$\begin{aligned}
&\iint_{\substack{|x| \leq n \\ |y| \leq n}} \sin(x^2 + y^2) dx dy \\
&= 4 \iint_{\substack{0 \leq x \leq n \\ 0 \leq y \leq n}} \sin(x^2 + y^2) dx dy \\
&= 4 \int_0^n dy \int_0^n (\sin x^2 \cos y^2 + \cos x^2 \sin y^2) dx \\
&= 4 \left(\int_0^n \cos y^2 dy \right) \left(\int_0^n \sin x^2 dx \right) \\
&\quad + 4 \left(\int_0^n \cos x^2 dx \right) \left(\int_0^n \sin y^2 dy \right) \\
&= 8 \left(\int_0^n \cos x^2 dx \right) \left(\int_0^n \sin x^2 dx \right).
\end{aligned}$$

根据 3830 题的结果,可知

$$\int_0^{+\infty} \sin x^2 dx = \int_0^{+\infty} \cos x^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}},$$

故

$$\lim_{n \rightarrow \infty} \int_0^n \sin x^2 dx = \lim_{n \rightarrow \infty} \int_0^n \cos x^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}.$$

从而,得

$$\lim_{n \rightarrow \infty} \iint_{\substack{|x| \leq n \\ |y| \leq n}} \sin(x^2 + y^2) dx dy$$

$$= 8 \cdot \frac{1}{2} \sqrt{\frac{\pi}{2}} \cdot \frac{1}{2} \sqrt{\frac{\pi}{2}} = \pi,$$

4168. 证明纵使累次积分

$$\int_1^{+\infty} dx \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dy$$

$$\text{及} \quad \int_1^{+\infty} dy \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx$$

收敛, 但积分

$$\iint_{x \geq 1, y \geq 1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy$$

发散.

证 先证两个累次积分收敛. 我们有

$$\begin{aligned} & \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \\ &= \int_1^{+\infty} \frac{x^2}{2y} \cdot \frac{2y dy}{(x^2 + y^2)^2} - \int_1^{+\infty} \frac{y}{2} \cdot \frac{2y dy}{(x^2 + y^2)^2} \\ &= - \frac{x^2}{2y(x^2 + y^2)} \Big|_{y=1}^{y=+\infty} - \int_1^{+\infty} \frac{x^2 dy}{2y^2(x^2 + y^2)} \\ & \quad + \frac{y}{2(x^2 + y^2)} \Big|_{y=1}^{y=+\infty} - \int_1^{+\infty} \frac{dy}{2(x^2 + y^2)} \\ &= \frac{x^2}{2(x^2 + 1)} - \frac{1}{2} \int_1^{+\infty} \left(\frac{1}{y^2} - \frac{1}{x^2 + y^2} \right) dy \\ & \quad - \frac{1}{2(x^2 + 1)} - \frac{1}{2} \int_1^{+\infty} \frac{dy}{x^2 + y^2} \\ &= \frac{x^2 - 1}{2(x^2 + 1)} - \frac{1}{2} = - \frac{1}{x^2 + 1}, \end{aligned}$$

$$\begin{aligned} \text{故} \quad & \int_1^{+\infty} dx \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \\ &= - \int_1^{+\infty} \frac{dx}{x^2 + 1} = - \frac{\pi}{4}; \end{aligned}$$

同理(利用已算得的结果)

$$\begin{aligned} & \int_1^{+\infty} dy \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \\ &= - \int_1^{+\infty} dy \int_1^{+\infty} \frac{y^2 - x^2}{(x^2 + y^2)^2} dx \\ &= - \int_1^{+\infty} \left(-\frac{1}{y^2 + 1} \right) dy = \frac{\pi}{4}, \end{aligned}$$

故两个累次积分都收敛.

次证积分

$$\iint_{x \geq 1, y \geq 1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy \quad (1)$$

发散. 为此只要证积分

$$\iint_{x \geq 1, 1 \leq y \leq x} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy \quad (2)$$

发散即可(因为如果积分(1)收敛, 则绝对值积分

$$\iint_{x \geq 1, y \geq 1} \left| \frac{x^2 - y^2}{(x^2 + y^2)^2} \right| dx dy \quad (3)$$

必收敛. 从而在小一点的区域上的积分

$$\iint_{x \geq 1, 1 \leq y \leq x} \left| \frac{x^2 - y^2}{(x^2 + y^2)^2} \right| dx dy$$

更收敛. 由此可知, 积分(2)收敛). 由于,

$$\begin{aligned} I_n &= \iint_{\substack{1 \leq x \leq n \\ 1 \leq y \leq x}} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy \\ &= \int_1^n dx \int_1^x \frac{x^2 - y^2}{(x^2 + y^2)^2} dy, \end{aligned}$$

仿上, 利用部分积分法, 容易算得

$$\int_1^x \frac{x^2 - y^2}{(x^2 + y^2)^2} dy$$

$$\begin{aligned}
&= -\frac{x^2}{2y(x^2+y^2)}\Big|_{y=1}^{y=x} - \int_1^x \frac{x^2 dy}{2y^2(x^2+y^2)} \\
&\quad + \frac{y}{2(x^2+y^2)}\Big|_{y=1}^{y=x} - \int_1^x \frac{dy}{2(x^2+y^2)} \\
&= -\frac{1}{x^2+1} + \frac{1}{2x},
\end{aligned}$$

故

$$\begin{aligned}
I_n &= \int_1^n \left(-\frac{1}{x^2+1} + \frac{1}{2x} \right) dx \\
&= \frac{\pi}{4} - \operatorname{arctg} n + \frac{1}{2} \ln n \rightarrow +\infty \text{ (当 } n \rightarrow \infty \text{ 时)},
\end{aligned}$$

由此可知积分(2)发散.

注意,也可用反证法证明积分(1)发散. 假定积分(1)收敛. 于是积分(3)收敛. 但恒有

$$\begin{aligned}
&\iint_{x \geq 1, y \geq 1} \left| \frac{x^2 - y^2}{(x^2 + y^2)^2} \right| dx dy \\
&= \int_1^{+\infty} dx \int_1^{+\infty} \left| \frac{x^2 - y^2}{(x^2 + y^2)^2} \right| dy \\
&= \int_1^{+\infty} dy \int_1^{+\infty} \left| \frac{x^2 - y^2}{(x^2 + y^2)^2} \right| dx, \tag{4}
\end{aligned}$$

故(4)式中两个累次积分都收敛. 又由前面已证不取绝对值的两个累次积分

$$\int_1^{+\infty} dx \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dy$$

与

$$\int_1^{+\infty} dy \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx$$

都收敛, 故知

$$\iint_{x \geq 1, y \geq 1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy$$

$$= \int_1^{+\infty} dx \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dy = -\frac{\pi}{4},$$

$$\iint_{\substack{x \geq 1, y \geq 1}} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy \\ = \int_1^{+\infty} dy \int_1^{+\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2} dx = \frac{\pi}{4},$$

这是不可能的. 证毕.

计算下列积分:

4169. $\iint_{\substack{xy \geq 1 \\ x \geq 1}} \frac{dx dy}{x^p y^q}.$

解 由于被积函数非负, 故

$$I = \iint_{\substack{xy \geq 1 \\ x \geq 1}} \frac{dx dy}{x^p y^q} = \int_1^{+\infty} \frac{dx}{x^p} \int_{\frac{1}{x}}^{+\infty} \frac{dy}{y^q}.$$

而当 $q > 1$ 时,

$$\int_{\frac{1}{x}}^{+\infty} \frac{dy}{y^q} = \frac{x^{q-1}}{q-1}.$$

(注意, 当 $q \leq 1$ 时, 此积分发散, 从而 $I = +\infty$); 又当 $p > q$ 时,

$$I = \frac{1}{q-1} \int_1^{+\infty} x^{q-p-1} dx = \frac{1}{(p-q)(q-1)}.$$

(注意, 当 $p \leq q$ 时, 此积分发散, $I = +\infty$).

综上所述, 可知: 当 $p > q > 1$ 时,

$$\iint_{\substack{xy \geq 1 \\ x \geq 1}} \frac{dx dy}{x^p y^q} = \frac{1}{(p-q)(q-1)}.$$

4170. $\iint_{\substack{x+y \geq 1 \\ 0 \leq x \leq 1}} \frac{dx dy}{(x+y)^p}.$

解 由于被积函数非负, 故

$$\begin{aligned}
 I &= \iint_{\substack{x+y \geq 1 \\ 0 \leq x \leq 1}} \frac{dx dy}{(x+y)^p} \\
 &= \int_0^1 dx \int_{1-x}^{+\infty} \frac{dy}{(x+y)^p}.
 \end{aligned}$$

当 $p > 1$ 时,

$$\begin{aligned}
 &\int_{1-x}^{+\infty} \frac{dy}{(x+y)^p} \\
 &= -\frac{1}{p-1} \cdot \frac{1}{(x+y)^{p-1}} \Big|_{y=1-x}^{y=+\infty} = \frac{1}{p-1}.
 \end{aligned}$$

(注意, 当 $p \leq 1$ 时, 积分发散, $I = +\infty$), 故

$$I = \int_0^1 \frac{dx}{p-1} = \frac{1}{p-1} \quad (\text{当 } p > 1 \text{ 时}).$$

$$4171. \iint_{x^2+y^2 \leq 1} \frac{dx dy}{\sqrt{1-x^2-y^2}}.$$

解 采用极坐标. 由于被积函数非负, 故有

$$\begin{aligned}
 &\iint_{x^2+y^2 \leq 1} \frac{dx dy}{\sqrt{1-x^2-y^2}} \\
 &= \int_0^{2\pi} d\theta \int_0^1 \frac{r}{\sqrt{1-r^2}} dr \\
 &= 2\pi (-\sqrt{1-r^2}) \Big|_{r=0}^{r=1} = 2\pi.
 \end{aligned}$$

$$4172. \iint_{x^2+y^2 \geq 1} \frac{dx dy}{(x^2+y^2)^p}.$$

解 采用极坐标. 由于被积函数非负, 故有

$$\begin{aligned}
 &\iint_{x^2+y^2 \geq 1} \frac{dx dy}{(x^2+y^2)^p} = \int_0^{2\pi} d\theta \int_1^{+\infty} \frac{dr}{r^{2p-1}} \\
 &= \begin{cases} \frac{\pi}{p-1}, & \text{当 } p > 1 \text{ 时;} \\ -\infty, & \text{当 } p \leq 1 \text{ 时.} \end{cases}
 \end{aligned}$$

$$4173. \iint_{y \geq x^2+1} \frac{dx dy}{x^4 + y^2}.$$

解 由于被积函数非负, 故

$$\begin{aligned} I &= \iint_{y \geq x^2+1} \frac{dy}{x^4 + y^2} \\ &= \int_{-\infty}^{+\infty} dx \int_{x^2+1}^{+\infty} \frac{dy}{x^4 + y^2} \\ &= 2 \int_0^{+\infty} dx \int_{x^2+1}^{+\infty} \frac{dx dy}{x^4 + y^2}. \end{aligned}$$

由于

$$\begin{aligned} \int_{x^2+1}^{+\infty} \frac{dy}{x^4 + y^2} &= \frac{1}{x^2} \operatorname{arctg} \frac{y}{x^2} \Big|_{y=x^2+1}^{y=+\infty} \\ &= \frac{1}{x^2} \left(\frac{\pi}{2} - \operatorname{arctg} \left(1 + \frac{1}{x^2} \right) \right), \end{aligned}$$

故

$$\begin{aligned} I &= 2 \int_0^{+\infty} \frac{1}{x^2} \left(\frac{\pi}{2} - \operatorname{arctg} \left(1 + \frac{1}{x^2} \right) \right) dx \\ &= - \frac{2}{x} \left(\frac{\pi}{2} - \operatorname{arctg} \left(1 + \frac{1}{x^2} \right) \right) \Big|_{x=0}^{x=+\infty} \\ &\quad + 2 \int_0^{+\infty} \frac{\frac{1}{x} \cdot \frac{2}{x^3}}{1 + \left(1 + \frac{1}{x^2} \right)^2} dx \\ &= 2 \int_0^{+\infty} \frac{dx}{x^4 + x^2 + \frac{1}{2}}, \end{aligned}$$

$$\text{其中 } \lim_{x \rightarrow +0} \frac{\frac{\pi}{2} - \operatorname{arctg} \left(1 + \frac{1}{x^2} \right)}{x}$$

$$\begin{aligned}
&= \lim_{x \rightarrow +0} \frac{-\frac{2}{x^3}}{1 + \left(1 + \frac{1}{x^2}\right)^2} \\
&= \lim_{x \rightarrow +0} \left(-\frac{x}{x^4 + x^2 + \frac{1}{2}} \right) = 0.
\end{aligned}$$

下面计算积分 $\int_0^{+\infty} \frac{dx}{x^4 + x^2 + \frac{1}{2}}$. 为简单计, 记

$$a = \sqrt{\sqrt{2} - 1}, b = \frac{1}{\sqrt{2}}, \text{ 则}$$

$$\begin{aligned}
&\frac{1}{x^4 + x^2 + \frac{1}{2}} \\
&= \frac{1}{\left(x^2 + \frac{1}{\sqrt{2}}\right)^2 - (\sqrt{2} - 1)x^2} \\
&= \frac{1}{(x^2 + b)^2 - (ax)^2} \\
&= \frac{1}{(x^2 + ax + b)(x^2 - ax + b)} \\
&= \frac{1}{2ab} \left(\frac{x + a}{x^2 + ax + b} - \frac{x - a}{x^2 - ax + b} \right) \\
&= \frac{1}{4ab} \left(\frac{2x + a}{x^2 + ax + b} + \frac{a}{x^2 + ax + b} \right. \\
&\quad \left. - \frac{2x - a}{x^2 - ax + b} + \frac{a}{x^2 - ax + b} \right).
\end{aligned}$$

于是,

$$\begin{aligned}
& \int_0^{+\infty} \frac{dx}{x^4 + x^2 + \frac{1}{2}} \\
&= \frac{1}{4ab} \int_0^{+\infty} \left(\frac{2x+a}{x^2+ax+b} - \frac{2x-a}{x^2-ax+b} \right) dx \\
&\quad + \frac{1}{4b} \int_0^{+\infty} \left(\frac{1}{x^2+ax+b} + \frac{1}{x^2-ax+b} \right) dx \\
&= \frac{1}{4ab} \left(\ln \frac{x^2+ax+b}{x^2-ax+b} \right) \Big|_{x=0}^{x=+\infty} \\
&\quad + \frac{1}{4b} \left(\frac{2}{\sqrt{4b-a^2}} \operatorname{arctg} \frac{2x+a}{\sqrt{4b-a^2}} \right. \\
&\quad \left. + \frac{2}{\sqrt{4b-a^2}} \operatorname{arctg} \frac{2x-a}{\sqrt{4b-a^2}} \right) \Big|_{x=0}^{x=+\infty} \\
&= 0 + \frac{1}{4b} \frac{2\pi}{\sqrt{4b-a^2}} = \frac{\pi}{2b \sqrt{4b-a^2}} \\
&= \frac{\pi}{2 \cdot \frac{1}{\sqrt{2}} \sqrt{\frac{4}{\sqrt{2}} - (\sqrt{2}-1)}} \\
&= \frac{\pi}{\sqrt{2} \cdot \sqrt{\sqrt{2}+1}} \\
&= \frac{\pi \sqrt{\sqrt{2}-1}}{\sqrt{2} \sqrt{\sqrt{2}+1} \sqrt{\sqrt{2}-1}} \\
&= \frac{\pi \sqrt{\sqrt{2}-1} \sqrt{2}}{2} = \frac{\pi \sqrt{2}(\sqrt{2}-1)}{2},
\end{aligned}$$

故

$$I = 2 \int_0^{+\infty} \frac{dx}{x^4 + x^2 + \frac{1}{2}} = \pi \sqrt{2}(\sqrt{2}-1).$$

$$4174. \iint_{0 \leq x \leq y} e^{-(x+y)} dx dy.$$

解 由于被积函数非负, 故

$$\begin{aligned} \iint_{0 \leq x \leq y} e^{-(x+y)} dx dy &= \int_0^{+\infty} dx \int_x^{+\infty} e^{-(x+y)} dy \\ &= \int_0^{+\infty} e^{-x} dx \int_x^{+\infty} e^{-y} dy = \int_0^{+\infty} e^{-2x} dx = \frac{1}{2}. \end{aligned}$$

变换为极坐标而计算积分:

$$4175. \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy.$$

解 由于被积函数非负, 故采用极坐标就有

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy &= \int_0^{2\pi} d\theta \int_0^{+\infty} e^{-r^2} dr \\ &= 2\pi \left(-\frac{1}{2} e^{-r^2} \right) \Big|_{r=0}^{r=+\infty} = \pi. \end{aligned}$$

$$4176. \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \cos(x^2+y^2) dx dy.$$

解 由于

$$|e^{-(x^2+y^2)} \cos(x^2+y^2)| \leq e^{-(x^2+y^2)},$$

而 $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy$ 收敛 (参看 4175 题),

故 $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \cos(x^2+y^2) dx dy$ 收敛. 从而, 采用极坐标就有

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \cos(x^2+y^2) dx dy \\ &= \int_0^{2\pi} d\theta \int_0^{+\infty} r e^{-r^2} \cos r^2 dr = \pi \int_0^{+\infty} e^{-t} \cos t dt \\ &= \pi \left(\frac{\sin t - \cos t}{(-1)^2 + 1} e^{-t} \right) \Big|_{t=0}^{t=+\infty} = \frac{\pi}{2}. \end{aligned}$$

$$4177. \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \sin(x^2+y^2) dx dy.$$

解 由于

$$|e^{-(x^2+y^2)} \sin(x^2+y^2)| \leq e^{-(x^2+y^2)},$$

而积分 $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy$ 收敛(参看 4175 题),

故积分 $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \sin(x^2+y^2) dx dy$ 收敛.

从而,采用极坐标就有

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \sin(x^2+y^2) dx dy \\ &= \int_0^{2\pi} d\theta \int_0^{+\infty} r e^{-r^2} \sin r^2 dr = \pi \int_0^{+\infty} e^{-t} \sin t dt \\ &= \pi \left(\frac{-\sin t - \cos t}{(-1)^2 + 1^2} e^{-t} \right) \Big|_{t=0}^{t=+\infty} = \frac{\pi}{2}. \end{aligned}$$

计算积分:

$$4178. \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ax^2+2bxy+cy^2+2dx+2ey+f} dx dy,$$

其中 $a < 0, ac - b^2 > 0$.

解 我们有(令 $\delta = ac - b^2 > 0, t = x + \frac{b}{a}y$)

$$\begin{aligned} \varphi(x, y) &= ax^2 + 2bxy + cy^2 + 2dx + 2ey + f \\ &= a \left(x^2 + \frac{2b}{a}xy + \frac{b^2}{a^2}y^2 \right) \\ &\quad + \frac{ac - b^2}{a}y^2 + 2dx + 2ey + f \\ &= a \left(x + \frac{b}{a}y \right)^2 + \frac{\delta}{a}y^2 + 2dx + 2ey + f \\ &= at^2 + \frac{\delta}{a}y^2 + 2d \left(t - \frac{b}{a}y \right) + 2ey + f \\ &= a \left(t^2 + \frac{2d}{a}t + \frac{d^2}{a^2} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{d^2}{a} + \frac{\delta}{a} \left[y^2 + \frac{2}{\delta} (ae - bd)y + \frac{(ae - bd)^2}{\delta^2} \right] - \frac{(ae - bd)^2}{a\delta} + f \\
& = a \left(t + \frac{d}{a} \right)^2 + \frac{\delta}{a} \left(y + \frac{ae - bd}{\delta} \right)^2 + \beta,
\end{aligned}$$

其中

$$\begin{aligned}
\beta &= f - \frac{d^2}{a} - \frac{(ae - bd)^2}{a\delta} \\
&= \frac{1}{a\delta} [af(ac - b^2) - d^2(ac - b^2) \\
&\quad - (ae - bd)^2] \\
&= \frac{1}{\delta} (acf - b^2f - cd^2 - ae^2 + 2bde) = \frac{\Delta}{\delta},
\end{aligned}$$

这里

$$\Delta = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}.$$

今作变量代换

$$\begin{cases} u = \sqrt{-a}x + \frac{b\sqrt{-a}}{a}y + \frac{d\sqrt{-a}}{a} \\ v = \sqrt{-\frac{\delta}{a}}y + \sqrt{-\frac{\delta}{a}} \cdot \frac{ae - bd}{\delta}, \end{cases} \quad (1)$$

则 $\varphi(x, y) = -u^2 - v^2 + \beta$. 又

$$\frac{D(x, y)}{D(u, v)} = \frac{1}{\frac{D(u, v)}{D(x, y)}}$$

$$= \frac{1}{\begin{vmatrix} \sqrt{-a} & \frac{b}{a} \sqrt{-a} \\ 0 & \sqrt{-\frac{\delta}{a}} \end{vmatrix}} = \frac{1}{\sqrt{\delta}} > 0.$$

故线性变换(1)是非退化的,它将\$(x,y)\$平面的点与\$(u,v)\$平面的点一一对应.于是,利用4175题的结果,得

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{\phi(x,y)} dx dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-u^2 - v^2 + \beta} \frac{1}{\sqrt{\delta}} du dv \\ &= \frac{1}{\sqrt{\delta}} e^{\frac{\Delta}{2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(u^2+v^2)} du dv = \frac{\pi}{\sqrt{\delta}} e^{\frac{\Delta}{2}}. \end{aligned}$$

4179. $\iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1} e^{-\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)} dx dy.$

解 作广义极坐标变换

$$x = a \cos \theta, y = b \sin \theta,$$

由于被积函数非负,故

$$\begin{aligned} & \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1} e^{-\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)} dx dy \\ &= \int_0^{2\pi} d\theta \int_1^{+\infty} ab r e^{-r^2} dr \\ &= 2\pi ab \left(-\frac{1}{2} e^{-r^2} \right) \Big|_{r=1}^{r=+\infty} = \frac{\pi}{e} ab. \end{aligned}$$

4180. $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x y e^{-\left(\frac{x^2}{a^2} + 2\epsilon \frac{x}{a} \cdot \frac{y}{b} + \frac{y^2}{b^2}\right)} dx dy (0 < |\epsilon| < 1).$

解 作广义极坐标变换

$$x = a \cos \theta, y = b \sin \theta,$$

则有

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy e^{-\left(\frac{x^2}{a^2} + 2\epsilon \frac{x}{a} \cdot \frac{y}{b} + \frac{y^2}{b^2}\right)} dx dy \\ &= \int_0^{2\pi} \int_0^{+\infty} \frac{1}{2} a^2 b^2 r^3 \sin 2\theta e^{-r^2(1+\epsilon \sin 2\theta)} dr d\theta. \end{aligned} \quad (1)$$

由于 $|r^3 \sin 2\theta e^{-r^2(1+\epsilon \sin 2\theta)}| \leq r^3 e^{-r^2(1-|\epsilon|)}$,
而积分

$$\begin{aligned} &\int_0^{2\pi} \int_0^{+\infty} r^3 e^{-r^2(1-|\epsilon|)} dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{+\infty} r^3 e^{-r^2(1-|\epsilon|)} dr \\ &= 2\pi \int_0^{+\infty} r^3 e^{-r^2(1-|\epsilon|)} dr < +\infty, \end{aligned}$$

故(1)式中的二重广义积分收敛. 于是,

$$I = \frac{1}{2} a^2 b^2 \int_0^{2\pi} \sin 2\theta d\theta \int_0^{+\infty} r^3 e^{-r^2(1+\epsilon \sin 2\theta)} dr. \quad (2)$$

但是

$$\begin{aligned} &\int_0^{+\infty} r^3 e^{-r^2(1+\epsilon \sin 2\theta)} dr \\ &= \frac{1}{2} \int_0^{+\infty} t e^{-t(1+\epsilon \sin 2\theta)} dt \\ &= -\frac{1}{2(1+\epsilon \sin 2\theta)} \left[t e^{-t(1+\epsilon \sin 2\theta)} \right]_{t=0}^{t=+\infty} \\ &\quad - \int_0^{+\infty} e^{-t(1+\epsilon \sin 2\theta)} dt \Bigg] \\ &= \frac{1}{2(1+\epsilon \sin 2\theta)} \int_0^{+\infty} e^{-t(1+\epsilon \sin 2\theta)} dt \\ &= \frac{1}{2(1+\epsilon \sin 2\theta)^2}, \end{aligned}$$

故

$$I = \frac{1}{4} a^2 b^2 \int_0^{2\pi} \frac{\sin 2\theta}{(1+\epsilon \sin 2\theta)^2} d\theta$$

$$\begin{aligned}
&= \frac{1}{2}a^2b^2 \int_0^\pi \frac{\sin 2\theta}{(1 + \epsilon \sin 2\theta)^2} d\theta \\
&= \frac{1}{4}a^2b^2 \int_0^{2\pi} \frac{\sin u}{(1 + \epsilon \sin u)^2} du \\
&= \frac{1}{4}a^2b^2 \left[\int_0^{\frac{\pi}{2}} \frac{\sin u}{(1 + \epsilon \sin u)^2} du \right. \\
&\quad + \int_{\frac{\pi}{2}}^\pi \frac{\sin u}{(1 + \epsilon \sin u)^2} du \\
&\quad + \int_\pi^{\frac{3\pi}{2}} \frac{\sin u}{(1 + \epsilon \sin u)^2} du \\
&\quad \left. + \int_{\frac{3\pi}{2}}^{2\pi} \frac{\sin u}{(1 + \epsilon \sin u)^2} du \right] \\
&= \frac{1}{2}a^2b^2 \left[\int_0^{\frac{\pi}{2}} \frac{\sin u du}{(1 + \epsilon \sin u)^2} \right. \\
&\quad \left. - \int_0^{\frac{\pi}{2}} \frac{\sin u du}{(1 - \epsilon \sin u)^2} \right]. \tag{3}
\end{aligned}$$

但是(作代换 $u = \frac{\pi}{2} - v$)

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\sin u du}{(1 + \epsilon \sin u)^2} \\
&= \frac{1}{\epsilon} \int_0^{\frac{\pi}{2}} \left[\frac{1}{1 + \epsilon \sin u} - \frac{1}{(1 + \epsilon \sin u)^2} \right] du \\
&= \frac{1}{\epsilon} \int_0^{\frac{\pi}{2}} \left[\frac{1}{1 + \epsilon \cos v} - \frac{1}{(1 + \epsilon \cos v)^2} \right] dv,
\end{aligned}$$

同理,有

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\sin u du}{(1 - \epsilon \sin u)^2} \\
&= -\frac{1}{\epsilon} \int_0^{\frac{\pi}{2}} \left[\frac{1}{1 - \epsilon \cos v} - \frac{1}{(1 - \epsilon \cos v)^2} \right] dv.
\end{aligned}$$

根据 2028 题(a) 和 2063 题的结果,可知(当 $0 < |\epsilon| < 1$

时)

$$\begin{aligned} & \int \frac{dx}{1 + \epsilon \cos x} \\ &= \frac{2}{\sqrt{1 - \epsilon^2}} \operatorname{arctg} \left[\sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \operatorname{tg} \frac{x}{2} \right] + C, \end{aligned} \quad (4)$$

$$\begin{aligned} & \int \frac{dx}{(1 - \epsilon \cos x)^2} \\ &= -\frac{\epsilon \sin x}{(1 - \epsilon^2)(1 + \epsilon \cos x)} \\ & \quad + \frac{2}{(1 - \epsilon^2)^{\frac{3}{2}}} \operatorname{arctg} \left[\sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \operatorname{tg} \frac{x}{2} \right] + C. \end{aligned} \quad (5)$$

(注意, 2028 题(a) 和 2063 题中假定 $0 < \epsilon < 1$, 但从其推导过程可以看出公式(4)、(5) 当 $-1 < \epsilon < 0$ 时也成立).

于是,

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \frac{\sin u du}{(1 + \epsilon \sin u)^2} \\ &= \frac{1}{\epsilon} \left[\frac{2}{\sqrt{1 - \epsilon^2}} \operatorname{arctg} \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} + \frac{\epsilon}{1 - \epsilon^2} \right. \\ & \quad \left. - \frac{2}{(1 - \epsilon^2)^{\frac{3}{2}}} \operatorname{arctg} \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \right], \\ & \int_0^{\frac{\pi}{2}} \frac{\sin u du}{(1 - \epsilon \sin u)^2} \\ &= \frac{1}{\epsilon} \left[\frac{2}{\sqrt{1 - \epsilon^2}} \operatorname{arctg} \sqrt{\frac{1 + \epsilon}{1 - \epsilon}} - \frac{\epsilon}{1 - \epsilon^2} \right. \\ & \quad \left. - \frac{2}{(1 - \epsilon^2)^{\frac{3}{2}}} \operatorname{arctg} \sqrt{\frac{1 + \epsilon}{1 - \epsilon}} \right]. \end{aligned}$$

从而,由(3)式得

$$I = \frac{1}{\varepsilon} a^2 b^2 \left[\frac{1}{\sqrt{1-\varepsilon^2}} - \frac{1}{(1-\varepsilon^2)^{\frac{3}{2}}} \right] \cdot \left[\operatorname{arctg} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} + \operatorname{arctg} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \right].$$

但对任何的 $x > 0$, 有

$$\operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} = \frac{\pi}{2},$$

故最后得

$$\begin{aligned} I &= \frac{1}{\varepsilon} a^2 b^2 \left[\frac{1}{\sqrt{1-\varepsilon^2}} - \frac{1}{(1-\varepsilon^2)^{\frac{3}{2}}} \right] \cdot \frac{\pi}{2} \\ &= -\frac{\pi \varepsilon a^2 b^2}{2(1-\varepsilon^2)^{\frac{3}{2}}}. \end{aligned}$$

研究不连续函数的二重广义积分的收敛性 ($0 < m \leq |\varphi(x, y)| \leq M$):

4181. $\iint_{\Omega} \frac{dx dy}{x^2 + y^2}$, 式中域 Ω 是由条件 $|y| \leq x^2; x^2 + y^2 \leq 1$ 所确定.

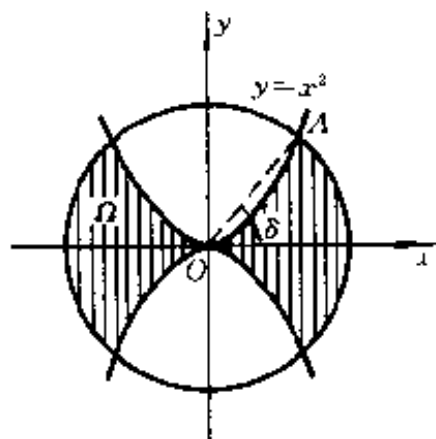


图 8.61

解 显然, Ω 为图 8.61 中的阴影部分. 由于对称性以及

被积函数的非负性,采用极坐标就有

$$\begin{aligned} & \iint_{\Omega} \frac{dx dy}{x^2 + y^2} \\ &= 4 \int_0^{\delta} d\theta \int_{\frac{\sin\theta}{\cos^2\theta}}^1 \frac{dr}{r} = 4 \int_0^{\delta} \ln \frac{\cos^2\theta}{\sin\theta} d\theta, \end{aligned}$$

其中 δ 表图 8.61 中射线 OA 与 Ox 轴之间的夹角, 抛物线 $y = x^2$ 的极坐标方程为 $r = \frac{\sin\theta}{\cos^2\theta}$. 由于

$$\begin{aligned} & \lim_{\theta \rightarrow +0} \theta^{\frac{1}{2}} \ln \frac{\cos^2\theta}{\sin\theta} \\ &= \lim_{\theta \rightarrow +0} \left(\frac{\theta}{\sin\theta} \right)^{\frac{1}{2}} \cdot \cos\theta \cdot \frac{\ln \frac{\cos^2\theta}{\sin\theta}}{\left(\frac{\cos^2\theta}{\sin\theta} \right)^{\frac{1}{2}}} = 0, \end{aligned}$$

故积分 $\int_0^{\delta} \ln \frac{\cos^2\theta}{\sin\theta} d\theta$ 收敛, 从而原积分 $\iint_{\Omega} \frac{dx dy}{x^2 + y^2}$ 收敛.

4182. $\iint_{x^2+y^2 \leq 1} \frac{\varphi(x,y)}{(x^2+xy+y^2)^p} dx dy.$

解 由于

$$\begin{aligned} x^2 + xy + y^2 &= \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(x+y)^2 > 0 \\ & \quad (\text{当 } (x,y) \neq (0,0) \text{ 时}), \end{aligned}$$

故

$$\begin{aligned} & \frac{m}{(x^2 + xy + y^2)^p} \leq \frac{|\varphi(x,y)|}{(x^2 + xy + y^2)^p} \\ & \leq \frac{M}{(x^2 + xy + y^2)^p} \quad (\text{当 } (x,y) \neq (0,0) \text{ 时}), \end{aligned}$$

再注意到广义重积分收敛必绝对收敛, 即知积分

$$\iint_{x^2+y^2 \leq 1} \frac{\varphi(x,y)}{x^2+xy+y^2} dx dy \text{ 与积分}$$

$$\iint_{x^2+y^2 \leq 1} \frac{dx dy}{(x^2+xy+y^2)^p} \text{ 同时收敛或同时发散.}$$

由于 $\frac{1}{(x^2+xy+y^2)^p} > 0$ (当 $(x,y) \neq (0,0)$ 时), 采用极坐标即得

$$\begin{aligned} & \iint_{x^2+y^2 \leq 1} \frac{dx dy}{(x^2+xy+y^2)^p} \\ &= \int_0^{2\pi} \frac{d\theta}{\left(1 + \frac{1}{2}\sin 2\theta\right)^p} \int_0^1 \frac{dr}{r^{2p-1}}, \end{aligned}$$

$$\int_0^{2\pi} \frac{d\theta}{\left(1 + \frac{1}{2}\sin 2\theta\right)^p} \text{ 为常义积分, 其值为有限数,}$$

而

$$\int_0^1 \frac{dr}{r^{2p-1}} = \begin{cases} \frac{1}{2(1-p)}, & \text{当 } p < 1 \text{ 时;} \\ +\infty, & \text{当 } p \geq 1 \text{ 时.} \end{cases}$$

$$\text{由此可知: 原积分 } \iint_{x^2+y^2 \leq 1} \frac{\varphi(x,y)}{(x^2+xy+y^2)^p} dx dy$$

当 $p < 1$ 时收敛, 当 $p \geq 1$ 时发散.

$$4183. \iint_{|x|+|y| \leq 1} \frac{dx dy}{|x|^p + |y|^q} \quad (p > 0, q > 0).$$

解 由对称性及被积函数的非负性, 有

$$\iint_{|x|+|y| \leq 1} \frac{dx dy}{|x|^p + |y|^q}$$

$$\begin{aligned}
&= 4 \iint_{\substack{x \geq 0, y \geq 0 \\ x+y \leq 1}} \frac{dx dy}{x^p + y^q} \\
&= 4 \iint_{\Omega_1} \frac{dx dy}{x^p + y^q} + 4 \iint_{\Omega_2} \frac{dx dy}{x^p + y^q}, \quad (1)
\end{aligned}$$

其中 $\Omega_1 = \{(x, y) | x \geq 0, y \geq 0, x + y \leq 1, x^p + y^q \geq 2^{-p-q}\}$, $\Omega_2 = \{(x, y) | x \geq 0, y \geq 0, x + y \leq 1, x^p + y^q \leq 2^{-p-q}\}$. 令 $\Omega_3 = \{(x, y) | x \geq 0, y \geq 0, x^p + y^q \leq 2^{-p-q}\}$. 易知, 当 $x \geq 0, y \geq 0, x^p + y^q \leq 2^{-p-q}$ 时, 必有 $x + y \leq 1$ (因为 $x \geq 0, y \geq 0, x^p + y^q \leq \frac{1}{2^{p+q}}$, 故 $x^p \leq \frac{1}{2^{p+q}} \leq \frac{1}{2^p}, y^q = \frac{1}{2^{p+q}} \leq \frac{1}{2^q}$, 从而 $x \leq \frac{1}{2}, y \leq \frac{1}{2}$, 由此知 $x + y \leq 1$), 故 $\Omega_3 = \Omega_2$. 由于函数

$\frac{1}{x^p + y^q}$ 在有界闭区域 Ω_1 上连续, 故 (1) 式右端第一个积分为常义积分. 因此, 广义积分 $\iint_{|x|+|y| \leq 1}$

$\frac{dx dy}{|x|^p + |y|^q}$ 的敛散性取决于广义积分 $\iint_{\Omega_3} \frac{dx dy}{x^p + y^q}$ 的

敛散性. 在此积分中作变量代换

$$x = r^{\frac{2}{p}} \cos^{\frac{2}{p}} \theta, y = r^{\frac{2}{q}} \sin^{\frac{2}{q}} \theta,$$

则易知

$$\frac{D(x, y)}{D(r, \theta)} = \frac{4}{pq} r^{\frac{2}{p} + \frac{2}{q} - 1} \sin^{\frac{2}{q} - 1} \theta \cos^{\frac{2}{p} - 1} \theta.$$

于是, 注意到被积函数是非负的, 得

$$\iint_{\Omega_3} \frac{dx dy}{x^p + y^q} = \frac{4}{pq} \int_0^{\frac{\pi}{2}} \sin^{\frac{2}{q} - 1} \theta \cos^{\frac{2}{p} - 1} \theta d\theta$$

$$\cdot \int_0^{(\sqrt{2})^{-p-q}} r^{\frac{2}{p} + \frac{2}{q} - 3} dr.$$

由 3856 题的结果知右端第一个积分

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{2}{q}-1} \theta \cos^{\frac{2}{p}-1} \theta d\theta \quad (p > 0, q > 0)$$

恒收敛, 且其值为 $\frac{1}{2} B(\frac{1}{q}, \frac{1}{p})$; 而第二个积分

$$\int_0^{(\sqrt{2})^{-p-q}} r^{\frac{2}{p} + \frac{2}{q} - 3} dr$$

当 $\frac{2}{p} + \frac{2}{q} - 3 > -1$ (即 $\frac{1}{p} + \frac{1}{q} > 1$) 时收敛, 当 $\frac{2}{p} + \frac{2}{q} - 3 \leq -1$ (即 $\frac{2}{p} + \frac{2}{q} \leq 1$) 时发散.

综上所述, 可知原积分 $\iint_{|x|+|y| \leq 1} \frac{dxdy}{|x|^p + |y|^q}$ 当 $\frac{1}{p}$

$+\frac{1}{q} > 1$ 时收敛, 当

$\frac{1}{p} + \frac{1}{q} \leq 1$ 时发散.

$$4184. \int_0^a \int_0^a \frac{\varphi(x, y)}{|x-y|^p} dxdy.$$

解 由于

$$\frac{m}{|x-y|^p} \leq \frac{|\varphi(x, y)|}{|x-y|^p} \leq \frac{M}{|x-y|^p},$$

并注意到广义重积分收敛必绝对收敛, 可知积分

$$\int_0^a \int_0^a \frac{\varphi(x, y)}{|x-y|^p} dxdy \text{ 与积分 } \int_0^a \int_0^a \frac{dxdy}{|x-y|^p} \text{ 同时收敛或同}$$

时发散. 由对称性及被积函数的非负性, 可知

$$\int_0^a \int_0^a \frac{dxdy}{|x-y|^p} = 2 \iint_{\substack{0 \leq x \leq a \\ 0 \leq y \leq x}} \frac{dxdy}{(x-y)^p}, \quad (1)$$

当 $p < 1$ 时,

$$\begin{aligned} \iint_{\substack{0 \leq x \leq a \\ 0 \leq y \leq x}} \frac{dxdy}{(x-y)^p} &= \int_0^a dx \int_0^x \frac{dy}{(x-y)^p} \\ &= \int_0^a \frac{x^{1-p}}{1-p} dx = \frac{a^{2-p}}{(1-p)(2-p)}. \end{aligned}$$

从而,由(1)式知

$$\int_0^a \int_0^a \frac{dxdy}{|x-y|^p} = \frac{2a^{2-p}}{(1-p)(2-p)}.$$

因此,当 $p < 1$ 时积分 $\int_0^a \int_0^a \frac{dxdy}{|x-y|^p}$ 收敛.

现设 $p \geq 1$. 首先,我们有

$$\begin{aligned} &\iint_{\substack{0 \leq x \leq a \\ 0 \leq y \leq x}} \frac{dxdy}{(x-y)^p} \\ &= \lim_{\epsilon \rightarrow +0} \iint_{\substack{\epsilon \leq x \leq a \\ 0 \leq y \leq x-\epsilon}} \frac{dxdy}{(x-y)^p}. \end{aligned} \quad (2)$$

若 $p = 1$, 则

$$\begin{aligned} \iint_{\substack{\epsilon \leq x \leq a \\ 0 \leq y \leq x-\epsilon}} \frac{dxdy}{(x-y)^p} &= \int_{\epsilon}^a dx \int_0^{x-\epsilon} \frac{dy}{x-y} \\ &= \int_{\epsilon}^a (\ln x - \ln \epsilon) dx = a \ln a - a + \epsilon - a \ln \epsilon, \end{aligned}$$

故

$$\begin{aligned} &\lim_{\epsilon \rightarrow +0} \iint_{\substack{\epsilon \leq x \leq a \\ 0 \leq y \leq x-\epsilon}} \frac{dxdy}{(x-y)^p} \\ &= \lim_{\epsilon \rightarrow +0} (a \ln a - a + \epsilon - a \ln \epsilon) = +\infty. \end{aligned}$$

由此可知,此时 $\int_0^a \int_0^a \frac{dxdy}{|x-y|^p}$ 发散;若 $p = 2$, 则

$$\begin{aligned} \iint_{\substack{\varepsilon \leq x \leq a \\ 0 \leq y \leq x-\varepsilon}} \frac{dxdy}{(x-y)^p} &= \int_{\varepsilon}^a dx \int_0^{x-\varepsilon} \frac{dy}{(x-y)^p} \\ &= \int_{\varepsilon}^a \left(\frac{1}{\varepsilon} - \frac{1}{x} \right) dx = \frac{a}{\varepsilon} - 1 - \ln a + \ln \varepsilon, \end{aligned}$$

故

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \iint_{\substack{\varepsilon \leq x \leq a \\ 0 \leq y \leq x-\varepsilon}} \frac{dxdy}{(x-y)^p} \\ &= \lim_{\varepsilon \rightarrow +0} \left(\frac{a + \varepsilon \ln \varepsilon}{\varepsilon} - 1 - \ln a \right) = +\infty. \end{aligned}$$

由此可知,此时积分 $\int_0^a \int_0^a \frac{dxdy}{|x-y|^p}$ 发散;最后,若 $P > 1, p \neq 2$, 则

$$\begin{aligned} \iint_{\substack{\varepsilon \leq x \leq a \\ 0 \leq y \leq x-\varepsilon}} \frac{dxdy}{(x-y)^p} &= \int_{\varepsilon}^a dx \int_0^{x-\varepsilon} \frac{dy}{(x-y)^p} \\ &= \frac{1}{p-1} \int_{\varepsilon}^a (\varepsilon^{1-p} - x^{1-p}) dx \\ &= \frac{1}{(p-1)\varepsilon^{p-1}} \left(a - \frac{p-1}{p-2} \varepsilon \right) \\ &\quad + \frac{1}{(p-1)(p-2)a^{p-2}}. \end{aligned}$$

从而,

$$\lim_{\varepsilon \rightarrow +0} \iint_{\substack{\varepsilon \leq x \leq a \\ 0 \leq y \leq x-\varepsilon}} \frac{dxdy}{(x-y)^p} = +\infty.$$

由此可知,此时积分 $\int_0^a \int_0^a \frac{dxdy}{|x-y|^p}$ 发散.

综上所述,可知积分 $\int_0^a \int_0^a \frac{dxdy}{|x-y|^p}$ 当 $P < 1$ 时收敛, $p \geq 1$ 时发散.

$$4185. \iint_{x^2+y^2 \leq 1} \frac{\varphi(x, y)}{(1-x^2-y^2)^p} dx dy.$$

解 由于

$$\begin{aligned} \frac{m}{(1-x^2-y^2)^p} &\leq \frac{|\varphi(x, y)|}{(1-x^2-y^2)^p} \\ &\leq \frac{M}{(1-x^2-y^2)^p}, \end{aligned}$$

再注意到广义重积分收敛必绝对收敛, 即知积分

$$\iint_{x^2+y^2 \leq 1} \frac{\varphi(x, y)}{(1-x^2-y^2)^p} dx dy \text{ 与积分}$$

$$\iint_{x^2+y^2 \leq 1} \frac{dx dy}{(1-x^2-y^2)^p} \text{ 同时收敛同时发散. 采用极坐}$$

标, 由于被积函数 $\frac{1}{(1-x^2-y^2)^p}$ 是正的, 故

$$\begin{aligned} &\iint_{x^2+y^2 \leq 1} \frac{dx dy}{(1-x^2-y^2)^p} \\ &= \int_0^{2\pi} d\theta \int_0^1 \frac{r}{(1-r^2)^p} dr \\ &= 2\pi \int_0^1 \frac{r dr}{(1-r)^p (1+r)^p}. \end{aligned}$$

由于

$$\lim_{r \rightarrow 1-0} (1-r)^p \cdot \frac{r}{(1-r)^p (1+r)^p} = 2^{-p},$$

故积分 $\int_0^1 \frac{r dr}{(1-r)^p (1+r)^p}$ 当 $p < 1$ 时收敛, $p > 1$ 时发散; 当 $p = 1$ 时, 有

$$\int_0^1 \frac{r dr}{1-r^2} = -\frac{1}{2} \ln(1-r^2) \Big|_0^1 = +\infty,$$

故积分也发散. 由此可知, 积分

$\iint_{x^2+y^2 \leq 1} \frac{\varphi(x,y)}{(1-x^2-y^2)^p} dx dy$ 当 $p < 1$ 时收敛; 当

$p \geq 1$ 时发散.

4186. 证明, 如果: 1) 函数 $\varphi(x, y)$ 在有界域 $a \leq x \leq A, b \leq y \leq B$ 内是连续的; 2) 函数 $f(x)$ 在闭区间 $a \leq x \leq A$ 上连续; 3) $p < 1$, 则积分

$$\int_a^A dx \int_b^B \frac{\varphi(x, y)}{|f(x) - y|^p} dy$$

收敛.

证 首先注意, 由于 $p < 1$, 故积分 $\int_b^B \frac{dy}{|f(x) - y|^p}$ 对每个固定的 $x \in [a, A]$ 恒收敛 (若 $f(x) \in [b, B]$, 此为瑕积分, 点 $f(x)$ 是瑕点, 由于 $p < 1$, 它收敛; 若 $f(x) \notin [b, B]$, 则为常义积分, 当然收敛). 再根据 $\varphi(x, y)$ 的有界性, 即知: 对每个固定的 $x \in [a, A]$, 积分 $\int_b^B$

$\frac{\varphi(x, y)}{|f(x) - y|^p} dy$ 都收敛. 令

$$F(x) = \int_b^B \frac{\varphi(x, y)}{|f(x) - y|^p} dy \quad (a \leq x \leq A).$$

下面我们证明 $F(x)$ 是 $a \leq x \leq A$ 上的连续函数. 若已获证, 则积分

$$\int_a^A dx \int_b^B \frac{\varphi(x, y)}{|f(x) - y|^p} dy = \int_a^A F(x) dx$$

显然是收敛的 (右端为常义积分), 于是本题即获证. 令 $c = \max_{a \leq x \leq A} |f(x)|$. 今将函数 $\varphi(x, y)$ 连续地延拓到有界闭矩形 $R(a \leq x \leq A, b - 2c \leq y \leq B + 2c)$ 上 (只要规定

$$\varphi(x, y) = \begin{cases} \varphi(x, B), & \text{当 } a \leq x \leq A, \\ & B < y \leq B + 2c \text{ 时,} \\ \varphi(x, b), & \text{当 } a \leq x \leq A, \\ & b - 2c \leq y < b \text{ 时} \end{cases}$$

即可). 延拓后的函数仍记为 $\varphi(x, y)$. 由于 $\varphi(x, y)$ 及 $|f(x) - y|^{1-p}$ 都在 R 上连续, 故有界且一致连续:

存在常数 M , 使对一切 $(x, y) \in R$, 有

$$|\varphi(x, y)| \leq M, |f(x) - y|^{1-p} \leq M. \quad (1)$$

任给 $\varepsilon > 0$, 存在 $\delta_1 > 0$ (取 $\delta_1 < (\frac{\varepsilon}{2})^{\frac{1}{1-p}}$), 使当 $|x_1 - x_2| < \delta_1, |y_1 - y_2| < \delta_1 ((x_1, y_1) \in R, (x_2, y_2) \in R)$ 时, 恒有

$$|\varphi(x_1, y_1) - \varphi(x_2, y_2)| < \varepsilon, \quad (2)$$

$$||f(x_1) - y_1|^{1-p} - |f(x_2) - y_2|^{1-p}| < \varepsilon. \quad (3)$$

又由 $f(x)$ 在 $[a, A]$ 上的一致连续性可知, 存在 $\delta_2 > 0$, 使当 $|x_1 - x_2| < \delta_2 (x_1, x_2 \in [a, A])$ 时, 恒有

$$|f(x_1) - f(x_2)| < \delta_1. \quad (4)$$

令 $\delta = \min\{\delta_1, \delta_2\}$. 于是, 由 (2) 式可知: 当 $|x_1 - x_2| < \delta (x_1, x_2 \in [a, A])$ 时, 对一切 $b - c \leq y \leq B + c$, 恒有 $|\varphi(x_1, y + f(x_1)) - \varphi(x_2, y + f(x_2))| < \varepsilon. \quad (5)$

现设 $|x_1 - x_2| < \delta, (x_1, x_2 \in [a, A])$. 不失一般性, 设 $f(x_1) \geq f(x_2)$, 我们有

$$\begin{aligned} & F(x_1) - F(x_2) \\ &= \int_a^B \frac{\varphi(x_1, y)}{|f(x_1) - y|^p} dy - \int_a^B \frac{\varphi(x_2, y)}{|f(x_2) - y|^p} dy \\ &= \int_{b-f(x_1)}^{B-f(x_1)} \frac{\varphi(x_1, u + f(x_1))}{|u|^p} du \end{aligned}$$

$$\begin{aligned}
& - \int_{b-f(x_2)}^{B-f(x_2)} \frac{\varphi(x_2, u + f(x_2))}{|u|^p} du \\
& = \int_{b-f(x_1)}^{B-f(x_2)} \frac{\varphi(x_1, u + f(x_1))}{|u|^p} - \frac{\varphi(x_2, u + f(x_2))}{|u|^p} du \\
& \quad - \int_{B-f(x_1)}^{B-f(x_2)} \frac{\varphi(x_1, u + f(x_1))}{|u|^p} du \\
& \quad + \int_{b-f(x_1)}^{b-f(x_2)} \frac{\varphi(x_2, u + f(x_2))}{|u|^p} du \\
& = I_1 - I_2 + I_3, \tag{6}
\end{aligned}$$

其中 I_1, I_2, I_3 分别表上式中的三个积分。易知 ($p < 1$)

$$\begin{aligned}
& \int_a^\beta \frac{du}{|u|^p} \\
& = \begin{cases} \frac{1}{1-p} [\beta^{1-p} - \alpha^{1-p}], & \text{当 } 0 \leq \alpha \leq \beta \text{ 时;} \\ \frac{1}{1-p} [(-\alpha)^{1-p} - (-\beta)^{1-p}], & \text{当 } \alpha \leq \beta \leq 0 \text{ 时;} \\ \frac{1}{1-p} [\beta^{1-p} + (-\alpha)^{1-p}], & \text{当 } \alpha < 0 < \beta \text{ 时.} \end{cases}
\end{aligned}$$

从而,在任何情形下均有

$$\int_a^\beta \frac{du}{|u|^p} \leq \frac{1}{1-p} (|\beta|^{1-p} + |\alpha|^{1-p}); \tag{7}$$

而当 α, β 同号时,有

$$\int_a^\beta \frac{du}{|u|^p} = \frac{1}{1-p} |\beta|^{1-p} - |\alpha|^{1-p}. \tag{8}$$

于是,由(5)式、(1)式及(7)式,得

$$\begin{aligned}
|I_1| & < \epsilon \int_{b-f(x_1)}^{B-f(x_2)} \frac{du}{|u|^p} \\
& \leq \frac{\epsilon}{1-p} (|B-f(x_2)|^{1-p} + |b-f(x_1)|^{1-p})
\end{aligned}$$

$$\leq \frac{2M\varepsilon}{1-p}. \quad (9)$$

下面估计 I_2 : 若 $B - f(x_2)$ 与 $B - f(x_1)$ 同号, 则由(1)式、(8)式及(3)式, 有

$$\begin{aligned} |I_2| &\leq M \int_{B-f(x_1)}^{B-f(x_2)} \frac{du}{|u|^p} \\ &= \frac{M}{1-p} (|B-f(x_2)|^{1-p} - |B-f(x_1)|^{1-p}) \\ &< \frac{M\varepsilon}{1-p}; \end{aligned}$$

若 $B - f(x_2)$ 与 $B - f(x_1)$ 异号, 即 $B - f(x_1) < 0 < B - f(x_2)$, 由于

$[B - f(x_2)] - [B - f(x_1)] = f(x_1) - f(x_2) < \delta_1$, 故有 $|B - f(x_1)| < \delta_1$, $|B - f(x_2)| < \delta_1$.

于是, 由(7)式并注意到 $\delta_1 < \left(\frac{\varepsilon}{2}\right)^{\frac{1}{1-p}}$, 即得

$$\begin{aligned} |I_2| &\leq M \int_{B-f(x_1)}^{B-f(x_2)} \frac{du}{|u|^p} \\ &\leq \frac{M}{1-p} (|B-f(x_2)|^{1-p} + |B-f(x_1)|^{1-p}) \\ &< \frac{M}{1-p} (\delta_1^{1-p} + \delta_1^{1-p}) < \frac{M\varepsilon}{1-p}. \end{aligned}$$

所以, 在任何情形下均有

$$|I_2| < \frac{M\varepsilon}{1-p}. \quad (10)$$

同理, 可得(在任何情形下)

$$|I_3| < \frac{M\varepsilon}{1-p}. \quad (11)$$

于是, 由(6)式、(9)式、(10)式及(11)式, 即得

$$|F(x_1) - F(x_2)| < |I_1| + |I_2| + |I_3|$$

$$< \frac{4M\epsilon}{1-p}.$$

由此可知, $F(x)$ 在 $a \leq x \leq A$ 上(一致)连续, 证毕.

计算下列积分:

$$4187. \iint_{x^2+y^2 \leq 1} \ln \frac{1}{\sqrt{x^2+y^2}} dx dy.$$

解 采用极坐标, 由于被积函数非负, 故有

$$\begin{aligned} & \iint_{x^2+y^2 \leq 1} \ln \frac{1}{\sqrt{x^2+y^2}} dx dy \\ &= \int_0^{2\pi} d\theta \int_0^1 r \ln \frac{1}{r} dr = -2\pi \int_0^1 r \ln r dr \\ &= -2\pi \left(\frac{r^2}{2} \ln r \Big|_0^1 - \int_0^1 \frac{r}{2} dr \right) = \frac{\pi}{2}. \end{aligned}$$

$$4188. \int_0^a dx \int_0^x \frac{dy}{\sqrt{(a-x)(x-y)}} \quad (a > 0).$$

$$\text{解} \quad \int_0^a dx \int_0^x \frac{dy}{\sqrt{(a-x)(x-y)}}$$

$$= \int_0^a \frac{2\sqrt{x}}{\sqrt{a-x}} dx.$$

作变量代换 $x = au$, 则

$$\int_0^a \frac{2\sqrt{x}}{\sqrt{a-x}} dx = 2a \int_0^1 u^{\frac{1}{2}} (1-u)^{-\frac{1}{2}} du$$

$$= 2a B\left(\frac{3}{2}, \frac{1}{2}\right) = 2a \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}$$

$$= 2a \cdot \frac{1}{2} \left[\Gamma\left(\frac{1}{2}\right) \right]^2 = \pi a.$$

$$4189. \iint_{\Omega} \ln \sin(x-y) dx dy, \text{ 这里域 } \Omega \text{ 是由直线 } y=0, y=x,$$

$x = \pi$ 所界.

解 作变量代换 $x = u + v, y = u - v$, 则 Oxy 平面上的域 Ω 变为 uv 平面上的域 Ω' . 显然 Ω' 由直线 $u = v, v = 0, u + v = \pi$ 所界. 又有 $\frac{D(x, y)}{D(u, v)} = -2$.

于是, 再注意到被积函数非正, 即有

$$\begin{aligned} & \iint_{\Omega} \ln \sin(x - y) dx dy \\ &= 2 \iint_{\Omega'} \ln \sin 2v du dv = 2 \int_0^{\frac{\pi}{2}} dv \int_v^{\pi-v} \ln \sin 2v du \\ &= 2 \int_0^{\frac{\pi}{2}} (\pi - 2v) \ln \sin 2v dv = 2 \ln 2 \int_0^{\frac{\pi}{2}} (\pi - 2v) dv \\ &\quad + 2 \int_0^{\frac{\pi}{2}} (\pi - 2v) \ln \sin v dv + 2 \int_0^{\frac{\pi}{2}} (\pi - 2v) \ln \cos v dv \\ &= \pi^2 \ln 2 - \frac{\pi^2}{2} \ln 2 + 2 \int_0^{\frac{\pi}{2}} (\pi - 2v) \ln \sin v dv \\ &\quad + 2 \int_0^{\frac{\pi}{2}} 2t \ln \sin t dt \\ &= \frac{\pi^2}{2} \ln 2 + 2\pi \int_0^{\frac{\pi}{2}} \ln \sin v dv \\ &= \frac{\pi^2}{2} \ln 2 + 2\pi \left(-\frac{\pi}{2} \ln 2 \right)^{*}) = -\frac{\pi^2}{2} \ln 2. \end{aligned}$$

*) 利用 2353 题(a) 的结果.

4190. $\iint_{x^2+y^2 \leq x} \frac{dx dy}{\sqrt{x^2+y^2}}.$

解 由关于 Ox 轴的对称性与被积函数的非负性, 采用极坐标, 有

$$\begin{aligned}
& \iint_{x^2+y^2 \leq r} \frac{dx dy}{\sqrt{x^2+y^2}} \\
&= 2 \iint_{\substack{x^2+y^2 \leq r \\ y > 0}} \frac{dx dy}{\sqrt{x^2+y^2}} \\
&= 2 \int_0^{\frac{\pi}{2}} d\theta \int_0^{\cos \theta} dr = 2 \int_0^{\frac{\pi}{2}} \cos \theta d\theta = 2.
\end{aligned}$$

研究下列三重积分的收敛性:

4191. $\iiint_{x^2+y^2+z^2>1} \frac{\varphi(x,y,z)}{(x^2+y^2+z^2)^p} dx dy dz$, 这里 $0 < m \leq |\varphi(x,y,z)| \leq M$.

解 由于

$$\begin{aligned}
\frac{m}{(x^2+y^2+z^2)^p} &\leq \frac{|\varphi(x,y,z)|}{(x^2+y^2+z^2)^p} \\
&\leq \frac{M}{(x^2+y^2+z^2)^p},
\end{aligned}$$

再注意到广义重积分收敛必绝对收敛, 可知积分

$$\begin{aligned}
& \iiint_{x^2+y^2+z^2>1} \frac{\varphi(x,y,z)}{(x^2+y^2+z^2)^p} dx dy dz \text{ 与积分} \\
& \iiint_{x^2+y^2+z^2>1} \frac{dx dy dz}{(x^2+y^2+z^2)^p} \text{ 同时收敛或同时发散.}
\end{aligned}$$

由于被积函数 $\frac{1}{(x^2+y^2+z^2)^p}$ 是正的, 采用球坐标

$x = r \cos \varphi \cos \psi, y = r \sin \varphi \cos \psi, z = r \sin \psi$, 得

$$\begin{aligned}
& \iiint_{x^2+y^2+z^2>1} \frac{dx dy dz}{(x^2+y^2+z^2)^p} \\
&= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \psi d\psi \int_1^{\infty} \frac{dr}{r^{2p-2}}
\end{aligned}$$

$$= 4\pi \int_1^{+\infty} \frac{dr}{r^{2p-2}}.$$

显然, $\int_1^{+\infty} \frac{dr}{r^{2p-2}}$ 当 $p > \frac{3}{2}$ 时收敛, $p \leq \frac{3}{2}$ 时发散;

由此可知, $\iiint_{x^2+y^2+z^2>1} \frac{\varphi(x,y,z)}{(x^2+y^2+z^2)^p} dx dy dz$

当 $p > \frac{3}{2}$ 时收敛, 当 $p \leq \frac{3}{2}$ 时发散.

4192. $\iiint_{x^2+y^2+z^2 \leq 1} \frac{\varphi(x,y,z)}{(x^2+y^2+z^2)^p} dx dy dz$, 这里 $0 < m \leq |\varphi(x, y, z)| \leq M$.

解 和 4191 题完全类似(请参看 4191 题的解题过程), 易得

$$\begin{aligned} & \iiint_{x^2+y^2+z^2 \leq 1} \frac{dx dy dz}{(x^2+y^2+z^2)^p} \\ &= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi d\phi \int_0^1 \frac{dr}{r^{2p-2}} \\ &= 4\pi \int_0^1 \frac{dr}{r^{2p-2}}. \end{aligned}$$

显然, $\int_0^1 \frac{dr}{r^{2p-2}}$ 当 $p < \frac{3}{2}$ 时收敛, 当 $p \geq \frac{3}{2}$ 时发散;

故 $\iiint_{x^2+y^2+z^2 \leq 1} \frac{\varphi(x,y,z)}{(x^2+y^2+z^2)^p} dx dy dz$ 当 $p < \frac{3}{2}$ 时收

敛, 当 $p \geq \frac{3}{2}$ 时发散.

4193. $\iiint_{|x|+|y|+|z|>1} \frac{dx dy dz}{|x|^p + |y|^q + |z|^r} (p > 0, q > 0, r > 0).$

解 由对称性及被积函数的非负性, 有

$$\begin{aligned}
& \iint_{|x|+|y|+|z|>1} \frac{dx dy dz}{|x|^p + |y|^q + |z|^r} \\
= & 8 \iint_{\substack{x \geq 0, y \geq 0, z \geq 0 \\ x+y+z > 1}} \frac{dx dy dz}{x^p + y^q + z^r} \\
= & 8 \iint_{\Omega_1} \frac{dx dy dz}{x^p + y^q + z^r} + 8 \iint_{\Omega_2} \frac{dx dy dz}{x^p + y^q + z^r}.
\end{aligned}$$

其中 $\Omega_1 = \{(x, y, z) | x \geq 0, y \geq 0, z \geq 0, x + y + z > 1, x^p + y^q + z^r \leq 3\}$, $\Omega_2 = \{(x, y, z) | x \geq 0, y \geq 0, z \geq 0, x + y + z > 1, x^p + y^q + z^r > 3\}$. 令 $\Omega_3 = \{(x, y, z) | x \geq 0, y \geq 0, z \geq 0, x^p + y^q + z^r > 3\}$. 由于当 $x \geq 0, y \geq 0, z \geq 0, x^p + y^q + z^r > 3$ 时必有 $x + y + z > 1$ (否则, $x + y + z \leq 1$, 就有 $x \leq 1, y \leq 1, z \leq 1$, 从而 $x^p \leq 1, y^q \leq 1, z^r \leq 1$, 于是 $x^p + y^q + z^r \leq 3$), 故 $\Omega_2 = \Omega_3$. 显然,

$\iint_{\Omega_1} \frac{dx dy dz}{x^p + y^q + z^r}$ 为常义积分, 故积分

$\iint_{|x|+|y|+|z|>1} \frac{dx dy dz}{|x|^p + |y|^q + |z|^r}$ 的敛散性取决于

$\iint_{\Omega_1} \frac{dx dy dz}{x^p + y^q + z^r}$ 的敛散性. 对此积分, 作变量代换

$$x = R^{\frac{2}{p}} \cos^{\frac{2}{p}} \varphi \cos^{\frac{2}{r}} \psi,$$

$$y = R^{\frac{2}{q}} \sin^{\frac{2}{q}} \varphi \cos^{\frac{2}{r}} \psi,$$

$$z = R^{\frac{2}{r}} \sin^{\frac{2}{r}} \psi,$$

则易知

$$\frac{D(x, y, z)}{D(r, \varphi, \psi)}$$

$$= \frac{8}{pqr} R^{\frac{2}{p} + \frac{2}{q} + \frac{2}{r} - 1} \cos^{\frac{2}{p}-1} \varphi \sin^{\frac{2}{q}-1} \varphi \\ \cdot \sin^{\frac{2}{r}-1} \psi \cos^{\frac{2}{p} + \frac{2}{q} - 1} \psi.$$

于是, 由被积函数的非负性, 并利用 3856 题的结果, 得

$$\begin{aligned} & \iiint_{\Omega_1} \frac{dx dy dz}{x^p + y^q + z^r} \\ &= \frac{8}{pqr} \int_0^{\frac{\pi}{2}} \sin^{\frac{2}{r}-1} \psi \cos^{\frac{2}{p} + \frac{2}{q} - 1} \psi d\psi \\ & \quad \cdot \int_0^{\frac{\pi}{2}} \sin^{\frac{2}{q}-1} \varphi \cos^{\frac{2}{p}-1} \varphi d\varphi \\ & \quad \cdot \int_{\sqrt{3}}^{+\infty} R^{\frac{2}{p} + \frac{2}{q} + \frac{2}{r} - 3} dR \\ &= \frac{8}{pqr} \cdot \frac{1}{2} B\left(\frac{1}{r}, \frac{1}{p} + \frac{1}{q}\right) \\ & \quad \cdot \frac{1}{2} B\left(\frac{1}{q}, \frac{1}{p}\right) \int_{\sqrt{3}}^{+\infty} R^{\frac{2}{p} + \frac{2}{q} + \frac{2}{r} - 3} dR \\ &= \frac{2}{pqr} B\left(\frac{1}{r}, \frac{1}{p} + \frac{1}{q}\right) B\left(\frac{1}{q}, \frac{1}{p}\right) \\ & \quad \cdot \int_{\sqrt{3}}^{+\infty} R^{\frac{2}{p} + \frac{2}{q} + \frac{2}{r} - 3} dR. \end{aligned}$$

由于积分 $\int_{\sqrt{3}}^{+\infty} R^{\frac{2}{p} + \frac{2}{q} + \frac{2}{r} - 3} dR$ 当 $\frac{2}{p} + \frac{2}{q} + \frac{2}{r} - 3 < -1$ (即 $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$) 时收敛, 当 $\frac{2}{p} + \frac{2}{q} + \frac{2}{r} - 3 \geq -1$ 时发散, 故积分 $\iiint_{\Omega_1} \frac{dx dy dz}{x^p + y^q + z^r}$ (从

而积分 $\iiint_{|x|^p + |y|^q + |z|^r > 1} \frac{dx dy dz}{|x|^p + |y|^q + |z|^r}$) 当 $\frac{1}{p} + \frac{1}{q}$

+ $\frac{1}{r} < 1$ 时收敛, 当 $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$ 时发散.

4194. $\int_0^a \int_0^a \int_0^a \frac{f(x, y, z) dx dy dz}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}}$, 其中 $0 < m \leq$
 $|f(x, y, z)| \leq M$, 而 $\varphi(x)$ 和 $\psi(x)$ 是在闭区间 $[0, a]$ 上的连续函数.

解 由于

$$\begin{aligned} & \frac{1}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}} \\ & \leq \frac{|f(x, y, z)|}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}} \\ & \leq \frac{M}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}} \end{aligned}$$

并注意到广义重积分收敛必绝对收敛, 即知积分

$$\int_0^a \int_0^a \int_0^a \frac{f(x, y, z) dx dy dz}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}} \text{ 与积分}$$

$$\int_0^a \int_0^a \int_0^a \frac{dx dy dz}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}} \text{ 同时收敛或同}$$

时发散. 由被积函数 $\frac{1}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}}$ 的非负性, 我们有

$$\begin{aligned} & \int_0^a \int_0^a \int_0^a \frac{dx dy dz}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}} \\ & = \int_0^a F(x) dx, \end{aligned}$$

其中

$$F(x) = \int_0^a \int_0^a \frac{dy dz}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^{\frac{m}{2}}} \quad (0 \leq x \leq a).$$

作变量代换

$$u = y - \varphi(x), v = z - \psi(x) \quad (x \text{ 固定}),$$

则

$$\frac{D(y, z)}{D(u, v)} = \frac{1}{\frac{D(u, v)}{D(y, z)}} = 1.$$

从而,有

$$F(x) = \iint_{\substack{\varphi(x) \leq v \leq a - \varphi(x) \\ \psi(x) \leq u \leq a - \psi(x)}} \frac{dudv}{(u^2 + v^2)^p}. \quad (1)$$

先设 $p < 1$. 令 $c = \max_{0 \leq x \leq a} (|\varphi(x)| + |\psi(x)|)$, 则由

(1) 式知

$$\begin{aligned} 0 < F(x) &\leq \int_{\substack{-c \leq u \leq a+c \\ -c \leq v \leq a+c}} \frac{dudv}{(u^2 + v^2)^p} \\ &< \iint_{u^2 + v^2 \leq 2(a+c)^2} \frac{dudv}{(u^2 + v^2)^p} \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}(a+c)} \frac{dr}{r^{2p-1}} \\ &= \frac{\pi}{1-p} [\sqrt{2}(a+c)]^{2-2p}, \end{aligned}$$

即 $F(x)$ 有界 (实际上, 仿 4186 题的证明过程还可证明

$F(x)$ 在 $0 \leq x \leq a$ 上连续), 从而 $\int_0^a F(x) dx$ 是常义积分, 显然收敛. 由此可知, 此时积分

$$\int_0^a \int_0^a \int_0^a \frac{dx dy dz}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^p} \quad (2)$$

收敛.

次设 $p \geq 1$, 这时积分 (2) 可能收敛也可能发散, 分两种情况讨论:

i) 若不存在这样的 $x \in [0, a]$ 使 $0 \leq \varphi(x) \leq a, 0 \leq$

$\varphi(x) \leq a$ 同时成立(例如, $\varphi(x)$ 或 $\psi(x)$ 的值完全位于 $[0, a]$ 之外;这时,对一切 $0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a$, 均有:连续函数 $\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^p > 0$. 从而,积分(2)收敛(这时是常义积分).

ii) 若存在这样的点 $x \in [0, a]$ 使 $0 < \varphi(x) < a, 0 < \psi(x) < a$ 同时成立;由 $\varphi(x)$ 与 $\psi(x)$ 的连续性,必存在正数 ε 及闭区间 $I_0 \subset [0, a]$, 使当 $x \in I_0$ 时,恒有 $\varepsilon \leq \varphi(x) \leq a - \varepsilon, \varepsilon \leq \psi(x) \leq a - \varepsilon$. 从而由(1)式知:当 $x \in I_0$ 时,有

$$\begin{aligned} F(x) &\geq \iint_{\substack{\varepsilon \leq u \leq a-\varepsilon \\ \varepsilon \leq v \leq a-\varepsilon}} \frac{dudv}{(u^2 + v^2)^p} \\ &\geq \iint_{u^2 + v^2 \leq \varepsilon^2} \frac{dudv}{(u^2 + v^2)^p} \\ &= \int_0^{2\pi} d\theta \int_0^\varepsilon \frac{dr}{r^{2p-1}} \\ &= 2\pi \int_0^\varepsilon \frac{dr}{r^{2p-1}} = +\infty \quad (\text{注意 } p \geq 1), \end{aligned}$$

即当 $x \in I_0$ 时恒有 $F(x) = +\infty$, 由此可知,积分 $\int_0^a F(x)dx$ 发散. 于是,积分(2)发散.

综上所述,可知:积分

$$\int_0^a \int_0^a \int_0^a \frac{f(x, y, z) dx dy dz}{\{[y - \varphi(x)]^2 + [z - \psi(x)]^2\}^p}$$

当 $p < 1$ 时收敛;当 $p \geq 1$ 时,若不存在 $x \in [0, a]$ 使 $0 \leq \varphi(x) \leq a, 0 \leq \psi(x) \leq a$, 则收敛;若存在 $x \in [0, a]$, 使 $0 < \varphi(x) < a, 0 < \psi(x) < a$, 则发散.

$$4195. \iiint_{\substack{|x| \leq 1 \\ |y| \leq 1 \\ |z| \leq 1}} \frac{dx dy dz}{|x + y - z|^p}.$$

解 我们有(注意被积函数的非负性)

$$\begin{aligned} & \iiint_{\substack{|x| \leq 1 \\ |y| \leq 1 \\ |z| \leq 1}} \frac{dx dy dz}{|x + y - z|^p} \\ &= 2 \iiint_{\substack{|x| \leq 1, |y| \leq 1, |z| \leq 1 \\ x+y-z \geq 0}} \frac{dx dy dz}{(x + y - z)^p} \\ &= 2 \iint_{\substack{|x| \leq 1, |y| \leq 1 \\ -1 \leq x+y \leq 1}} dx dy \int_{-1}^{x+y} \frac{dz}{(x + y - z)^p} \\ &\quad + 2 \iint_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1 \\ x+y \geq 1}} dx dy \int_{-1}^1 \frac{dz}{(x + y - z)^p} \\ &= 2I_1 + 2I_2, \end{aligned}$$

其中 I_1 表第一个积分, I_2 表第二个积分.

若 $p < 1$, 则

$$\begin{aligned} & \int_{-1}^{x+y} \frac{dz}{(x + y - z)^p} = \frac{(x + y + 1)^{1-p}}{1-p}, \\ & \int_{-1}^1 \frac{dz}{(x + y - z)^p} \\ &= \frac{(x + y + 1)^{1-p} - (x + y - 1)^{1-p}}{p-1} (x + y \geq 1), \end{aligned}$$

故

$$I_1 = \frac{1}{1-p} \iint_{\substack{|x| \leq 1, |y| \leq 1 \\ -1 \leq x+y \leq 1}} (x + y + 1)^{1-p} dx dy,$$

$$I_2 = \frac{1}{1-p} \iint_{\substack{0 \leq x \leq 1, 0 \leq y \leq 1 \\ x+y \leq 1}} [(x+y+1)^{1-p} - (x+y-1)^{1-p}] dx dy.$$

显然, I_1 与 I_2 均为常义(二重)积分, 当然收敛. 因此,

$$\text{积分 } \iiint_{\substack{|x| \leq 1 \\ |y| \leq 1 \\ |z| \leq 1}} \frac{dx dy dz}{|x+y-z|^p} \text{ 收敛.}$$

若 $p \geq 1$, 则当 $x+y > -1$ 时,

$$\int_{-1}^{x+y} \frac{dz}{|x+y-z|^p} = +\infty,$$

故 $I_1 = +\infty$, 又显然有 $I_2 > 0$, 故此时积分

$$\iiint_{\substack{|x| \leq 1 \\ |y| \leq 1 \\ |z| \leq 1}} \frac{dx dy dz}{|x+y-z|^p}$$

发散.

计算积分:

$$4196. \int_0^1 \int_0^1 \int_0^1 \frac{dx dy dz}{x^p y^q z^r}.$$

解 由于被积函数非负, 故

$$\begin{aligned} & \int_0^1 \int_0^1 \int_0^1 \frac{dx dy dz}{x^p y^q z^r} \\ &= \int_0^1 \frac{dx}{x^p} \int_0^1 \frac{dy}{y^q} \int_0^1 \frac{dz}{z^r} \\ &= \frac{1}{(1-p)(1-q)(1-r)} \quad (\text{若 } p < 1, q < 1, r < 1). \end{aligned}$$

注意, 若 $p \geq 1$ 或 $q \geq 1$ 或 $r \geq 1$, 则

$$\int_0^1 \int_0^1 \int_0^1 \frac{dx dy dz}{x^p y^q z^r} = +\infty.$$

$$4197. \iiint_{x^2+y^2+z^2 \geq 1} \frac{dx dy dz}{(x^2+y^2+z^2)^3}.$$

解 采用球坐标. 由于被积函数的非负性, 有

$$\begin{aligned} & \iiint_{x^2+y^2+z^2 \geq 1} \frac{dx dy dz}{(x^2+y^2+z^2)^3} \\ &= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi d\phi \int_1^{+\infty} \frac{dr}{r^4} \\ &= 2\pi \cdot 2 \cdot \frac{1}{3} = \frac{4\pi}{3}. \end{aligned}$$

$$4198. \iiint_{x^2+y^2+z^2 \leq 1} \frac{dx dy dz}{(1-x^2-y^2-z^2)^p}.$$

解 采用球坐标. 由于被积函数的非负性, 有

$$\begin{aligned} & \iiint_{x^2+y^2+z^2 \leq 1} \frac{dx dy dz}{(1-x^2-y^2-z^2)^p} \\ &= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi d\phi \int_0^1 \frac{r^2}{(1-r^2)^p} dr \\ &= 4\pi \int_0^1 \frac{r^2}{(1-r^2)^p} dr. \end{aligned}$$

作代换 $t = r^2$, 则当 $p < 1$ 时有

$$\begin{aligned} & \int_0^1 \frac{r^2}{(1-r^2)^p} dr = \frac{1}{2} \int_0^1 t^{\frac{1}{2}} (1-t)^{-p} dt \\ &= \frac{1}{2} B\left(\frac{3}{2}, 1-p\right). \end{aligned}$$

从而, 当 $p < 1$ 时有

$$\begin{aligned} & \iiint_{x^2+y^2+z^2 \leq 1} \frac{dx dy dz}{(1-x^2-y^2-z^2)^p} \\ &= 2\pi B\left(\frac{3}{2}, 1-p\right). \end{aligned}$$

注意, 若 $p \geq 1$, 则 $\int_0^1 t^{\frac{1}{2}}(1-t)^{-p} dt = +\infty$, 故

$$\iiint_{x^2+y^2+z^2 \leq 1} \frac{dx dy dz}{(1-x^2-y^2-z^2)^p} = +\infty.$$

4199. $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2+z^2)} dx dy dz.$

解 采用球坐标. 由被积函数的非负性, 有

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2+z^2)} dx dy dz \\ &= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi d\phi \int_0^{+\infty} r^2 e^{-r^2} dr \\ &= 4\pi \int_0^{+\infty} r^2 e^{-r^2} dr. \end{aligned}$$

作代换 $r^2 = t$, 则

$$\begin{aligned} \int_0^{+\infty} r^2 e^{-r^2} dr &= \frac{1}{2} \int_0^{+\infty} t^{\frac{1}{2}} e^{-t} dt \\ &= \frac{1}{2} \Gamma\left(\frac{3}{2}\right) = \frac{1}{4} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}. \end{aligned}$$

于是,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2+z^2)} dx dy dz = \pi^{\frac{3}{2}}.$$

4200. 计算积分

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-P(x_1, x_2, x_3)} dx_1 dx_2 dx_3,$$

其中 $P(x_1, x_2, x_3) = \sum_{i=1}^3 \sum_{j=1}^3 a_{ij} x_i x_j$ ($a_{ij} = a_{ji}$) 为正定形.

解 用 A 表矩阵

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

由于二次型 $\sum_{i=1}^3 \sum_{j=1}^3 a_{ij}x_i x_j$ 是正定的, 故由高等代数中关于二次型的理论知: 存在正交矩阵

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad (1)$$

$$\text{使 } B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad (2)$$

其中 $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$; 也即在线性(正交)变换

$$\begin{cases} x_1 = b_{11}x'_1 + b_{12}x'_2 + b_{13}x'_3 \\ x_2 = b_{21}x'_1 + b_{22}x'_2 + b_{23}x'_3 \\ x_3 = b_{31}x'_1 + b_{32}x'_2 + b_{33}x'_3 \end{cases} \quad (3)$$

之下, 二次型 $P(x_1, x_2, x_3)$ 化为平方和:

$$\begin{aligned} P(x_1, x_2, x_3) &= \sum_{i=1}^3 \sum_{j=1}^3 a_{ij}x_i x_j \\ &= \lambda_1 x_1'^2 + \lambda_2 x_2'^2 + \lambda_3 x_3'^2. \end{aligned} \quad (4)$$

注意, 由于 B 是正交矩阵, 故 $B^{-1} = B'$ (B' 表 B 的转置矩阵), 从而 $|B| = |b_{ij}| = \pm 1$. 显然,

$$\frac{D(x_1, x_2, x_3)}{D(x'_1, x'_2, x'_3)} = |b_{ij}| = \pm 1.$$

由(4)式, 有

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-P(x_1, x_2, x_3)} dx_1 dx_2 dx_3 \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\lambda_1 x_1'^2 - \lambda_2 x_2'^2 - \lambda_3 x_3'^2} dx'_1 dx'_2 dx'_3. \end{aligned} \quad (5)$$

再作变量代换 $x'_1 = \frac{1}{\sqrt{\lambda_1}}u_1, x'_2 = \frac{1}{\sqrt{\lambda_2}}u_2, x'_3 =$

$$\frac{1}{\sqrt{\lambda_1}} u_3, \text{ 则 } \frac{D(x'_1, x'_2, x'_3)}{D(u_1, u_2, u_3)} = \frac{1}{\sqrt{\lambda_1 \lambda_2 \lambda_3}}.$$

于是(注意 4199 题的结果)

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\lambda_1 x'^2_1 - \lambda_2 x'^2_2 - \lambda_3 x'^2_3} dx'_1 dx'_2 dx'_3 \\ &= \frac{1}{\sqrt{\lambda_1 \lambda_2 \lambda_3}} \\ & \quad \cdot \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(u_1^2 + u_2^2 + u_3^2)} du_1 du_2 du_3 \\ &= \frac{\pi^{\frac{3}{2}}}{\sqrt{\lambda_1 \lambda_2 \lambda_3}}. \end{aligned} \quad (6)$$

但由(2)式知(记 $\Delta = |a_{ij}| = |A|$, 注意, 由于

$$\sum_{i=1}^3 \sum_{j=1}^3 a_{ij} x_i x_j \text{ 是正定的, 故 } \Delta > 0)$$

$$\Delta = |A| = |B^{-1}| \cdot |A| \cdot |B| = \lambda_1 \lambda_2 \lambda_3. \quad (7)$$

于是, 根据(5), (6), (7) 诸式, 最后得

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-P(x_1, x_2, x_3)} dx_1 dx_2 dx_3 \\ &= \sqrt{\frac{\pi^3}{\Delta}}. \end{aligned}$$

§ 10. 多重积分

1° 多重积分的直接算法 若函数 $f(x_1, x_2, \dots, x_n)$ 在由下列不等式所确定的有界域 Ω 内是连续的:

$$\begin{cases} x_1 \leq x_1 \leq x_1'', \\ x_2'(x_1) \leq x_2 \leq x_2''(x_1) \\ \dots\dots\dots \\ x_n'(x_1, x_2, \dots, x_{n-1}) \leq x_n \leq x_n''(x_1, x_2, \dots, x_{n-1}), \end{cases}$$

其中 x_1' 和 x_1'' 为常数及 $x_2'(x_1), x_2''(x_1), \dots, x_n'(x_1, x_2, \dots, x_{n-1}), x_n''(x_1, x_2, \dots, x_{n-1})$ 为连续函数, 则对应的多重积分可按下列公式来计算:

$$\begin{aligned} & \iint_{\Omega} \dots \int f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ &= \int_{x_1'}^{x_1''} dx_1 \int_{x_2'(x_1)}^{x_2''(x_1)} dx_2 \dots \\ & \int_{x_n'(x_1, \dots, x_{n-1})}^{x_n''(x_1, \dots, x_{n-1})} f(x_1, x_2, \dots, x_n) dx_n. \end{aligned}$$

2° 重积分中的变量代换 若 1) 函数 $f(x_1, x_2, \dots, x_n)$ 在有界可测的域 Ω 内是均匀连续的; 2) 连续可微分的函数

$$x_i = \varphi_i(\xi_1, \xi_2, \dots, \xi_n) (i = 1, 2, \dots, n),$$

把 $Ox_1, x_2, \dots, x_n$ 空间的域 Ω 双方单值地映射成 $O\xi_1, \xi_2, \dots, \xi_n$ 空间内的有界域 Ω' ; 3) 在域 Ω' 内雅可比式

$$J = \frac{D(x_1, x_2, \dots, x_n)}{D(\xi_1, \xi_2, \dots, \xi_n)} \neq 0,$$

则下面的公式正确

$$\begin{aligned} & \int_{\Omega} \dots \int f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ &= \int_{\Omega'} \dots \int f(\xi_1, \xi_2, \dots, \xi_n) |J| d\xi_1 d\xi_2 \dots d\xi_n. \end{aligned}$$

特别是, 根据公式

$$\begin{aligned} x_1 &= r \cos \varphi_1, \\ x_2 &= r \sin \varphi_1 \cos \varphi_2, \\ &\dots\dots\dots \\ x_{n-1} &= r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \cos \varphi_{n-1}, \\ x_n &= r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \sin \varphi_{n-1} \end{aligned}$$

变换成极坐标时 $(r, \varphi_1, \varphi_2, \dots, \varphi_{n-1})$, 有

$$I = \frac{D(x_1, x_2, \dots, x_n)}{D(r, \varphi_1, \dots, \varphi_{n-1})} \\ = r^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2}.$$

4201. 设 $K(x, y)$ 为域 $R[a \leq x \leq b, a \leq y \leq b]$ 内的连续函数及

$$K_n(x, y) = \int_a^b \int_a^b \cdots \int_a^b K(x, t_1) K(t_1, t_2) \cdots \\ \cdot K(t_n, y) dt_1 dt_2 \cdots dt_n,$$

证明:

$$K_{n+m+1}(x, y) = \int_a^b K_n(x, t) K_m(t, y) dt.$$

证

$$K_{n+m+1}(x, y) = \int_a^b \int_a^b \cdots \int_a^b K(x, t_1) K(t_1, t_2) \\ \cdots K(t_n, t) K(t, z_1) K(z_1, z_2) \cdots K(z_m, y) dt_1 dt_2 \\ \cdots dt_n dt dz_1 dz_2 \cdots dz_m \\ = \int_x^b \left\{ \left[\int_a^b \int_a^b \cdots \int_a^b K(x, t_1) K(t_1, t_2) \right. \right. \\ \left. \cdots K(t_n, t) dt_1 dt_2 \cdots dt_n \right] \\ \cdot \left[\int_a^b \int_a^b \cdots \int_a^b K(t, z_1) K(z_1, z_2) \right. \\ \left. \cdots K(z_m, y) dz_1 dz_2 \cdots dz_m \right] \right\} dt \\ = \int_x^b K_n(x, t) K_m(t, y) dt.$$

4202. 设 $f = f(x_1, x_2, \dots, x_n)$ 为域 $0 \leq x_i \leq x (i = 1, 2, \dots, n)$ 内的连续函数, 证明等式

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f dx_n \\ = \int_0^x dx_n \int_{x_n}^x dx_{n-1} \cdots \int_{x_2}^x f dx_1 (n \geq 2).$$

证 考虑下面三个有界闭域:

$$\Omega = \{(x_1, x_2, \dots, x_n) | 0 \leq x_i \leq x, \\ i = 1, 2, \dots, n\},$$

$$\Omega_1 = \{(x_1, x_2, \dots, x_n) | 0 \leq x_1 \leq x, \\ 0 \leq x_2 \leq x_1, \dots, 0 \leq x_n \leq x_{n-1}\},$$

$$\Omega_2 = \{(x_1, x_2, \dots, x_n) | 0 \leq x_n \leq x, \\ x_n \leq x_{n-1} \leq x, \dots, x_2 \leq x_1 \leq x\}.$$

由假定 $f(x_1, \dots, x_n)$ 在域 Ω 上连续, 显然, $\Omega_1 \subset \Omega, \Omega_2 \subset \Omega$, 故 $f(x_1, \dots, x_n)$ 在 Ω_1 与 Ω_2 上连续. 根据化 n 重积分为累次积分的公式, 我们有

$$\begin{aligned} & \iint_{\Omega_1} \dots \int f dx_1 \dots dx_n \\ &= \int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} f dx_n, \end{aligned} \quad (1)$$

$$\begin{aligned} & \iint_{\Omega_2} \dots \int f dx_1 \dots dx_n \\ &= \int_0^x dx_n \int_{x_n}^x dx_{n-1} \dots \int_{x_2}^x f dx_1. \end{aligned} \quad (2)$$

下证 $\Omega_1 = \Omega_2$, 事实上, 若 $(x_1, \dots, x_n) \in \Omega_1$, 则

$$0 \leq x_1 \leq x, 0 \leq x_2 \leq x_1, \dots, 0 \leq x_n \leq x_{n-1}, \quad (3)$$

从而

$$0 \leq x_n \leq x_{n-1} \leq x_{n-2} \leq \dots \leq x_2 \leq x_1 \leq x. \quad (4)$$

于是,

$$0 \leq x_n \leq x, x_n \leq x_{n-1} \leq x, \dots, x_2 \leq x_1 \leq x. \quad (5)$$

由此可知 $(x_1, \dots, x_n) \in \Omega_2$. 反之, 若 $(x_1, \dots, x_n) \in \Omega_2$, 则 (5) 式成立, 从而 (4) 式显然成立, 由此又知 (3) 式成立, 故 $(x_1, \dots, x_n) \in \Omega_1$, 于是 $\Omega_1 = \Omega_2$ 获证. 由此, 再根

据(1)式与(2)式,即得

$$\begin{aligned} & \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f dx_n \\ &= \int_0^x dx_n \int_{x_n}^x dx_{n-1} \cdots \int_{x_n}^x f dx_1. \end{aligned}$$

证毕.

4203. 证明

$$\begin{aligned} & \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} f(t_1) f(t_2) \cdots f(t_n) dt_n \\ &= \frac{1}{n!} \left\{ \int_0^1 f(\tau) d\tau \right\}^n, \end{aligned}$$

其中 f 为连续函数.

证 证法一:

我们有

$$\begin{aligned} & \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} f(t_1) f(t_2) \cdots f(t_n) dt_n \\ &= \int_0^1 f(t_1) dt_1 \int_0^{t_1} f(t_2) dt_2 \\ & \quad \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \int_0^{t_{n-1}} f(t_n) dt_n. \end{aligned}$$

令 $F(s) = \int_0^s f(\tau) d\tau$. 由于 f 是连续函数, 故 $F'(s) = f(s)$. 我们有 (注意到 $F(0) = 0$)

$$\begin{aligned} & \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \int_0^{t_{n-1}} f(t_n) dt_n \\ &= \int_0^{t_{n-2}} F(t_{n-1}) f(t_{n-1}) dt_{n-1} \\ &= \int_0^{t_{n-2}} F(t_{n-1}) F'(t_{n-1}) dt_{n-1} \\ &= \frac{1}{2} [F(t_{n-1})]^2 \Big|_{t_{n-1}=0}^{t_{n-1}=t_{n-2}} \end{aligned}$$

$$= \frac{1}{2} [F(t_{n-2})]^2,$$

由此

$$\int_0^{t_{n-3}} f(t_{n-2}) dt_{n-2} \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1}$$

$$\int_0^{t_{n-1}} f(t_n) dt_n$$

$$= \int_0^{t_{n-3}} \frac{1}{2} [F(t_{n-2})]^2 F(t_{n-2}) dt_{n-2}$$

$$= \frac{1}{3!} [F(t_{n-3})]^3,$$

.....,

这样继续下去,显然有

$$\int_0^{t_1} f(t_2) dt_2 \cdots \int_0^{t_{n-2}} f(t_{n-1}) dt_{n-1} \int_0^{t_{n-1}} f(t_n) dt_n$$

$$= \frac{1}{(n-1)!} [F(t_1)]^{n-1}.$$

于是,

$$\int_0^t f(t_1) dt_1 \int_0^{t_1} f(t_2) dt_2 \cdots \int_0^{t_{n-1}} f(t_n) dt_n$$

$$= \int_0^t \frac{1}{(n-1)!} [F(t_1)]^{n-1} f(t_1) dt_1$$

$$= \frac{1}{(n-1)!} \int_0^t [F(t_1)]^{n-1} F(t_1) dt_1$$

$$= \frac{1}{n!} [F(t)]^n = \frac{1}{n!} \left\{ \int_0^t f(\tau) d\tau \right\}^n.$$

从而

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} f(t_1) f(t_2) \cdots f(t_n) dt_n$$

$$= \frac{1}{n!} \left\{ \int_0^t f(\tau) d\tau \right\}^n. \text{证毕.}$$

证法二:

用归纳法证明所述公式. 当 $n = 1$ 时此公式显然成立, 今设 $n = k$ 时成立, 要证 $n = k + 1$ 时也成立, 我们有

$$\begin{aligned} & \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_k} f(t_1) f(t_2) \cdots f(t_{k+1}) dt_{k+1} \\ &= \int_0^t f(t_1) \left[\int_0^{t_1} dt_2 \cdots \int_0^{t_k} f(t_2) \cdots f(t_{k+1}) dt_{k+1} \right] dt_1. \end{aligned}$$

由于假定公式当 $n = k$ 时成立, 故

$$\begin{aligned} & \int_0^{t_1} dt_2 \cdots \int_0^{t_k} f(t_2) \cdots f(t_{k+1}) dt_{k+1} \\ &= \frac{1}{k!} \left\{ \int_0^{t_1} f(\tau) d\tau \right\}^k. \end{aligned}$$

从而 (令 $F(s) = \int_0^s f(\tau) d\tau$, 则 $F'(s) = f(s)$)

$$\begin{aligned} & \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_k} f(t_1) f(t_2) \cdots f(t_{k+1}) dt_{k+1} \\ &= \int_0^t f(t_1) \cdot \frac{1}{k!} \left\{ \int_0^{t_1} f(\tau) d\tau \right\}^k dt_1 \\ &= \frac{1}{k!} \int_0^t [F(t_1)]^k F'(t_1) dt_1 \\ &= \frac{1}{(k+1)!} [F(t)]^{k+1} \\ &= \frac{1}{(k+1)!} \left\{ \int_0^t f(\tau) d\tau \right\}^{k+1}, \end{aligned}$$

因此, 所述公式当 $n = k + 1$ 时成立. 于是, 由归纳法知所述公式对一切自然数 n 均成立. 证毕.

计算下列多重积分:

$$\begin{aligned} 4204. (a) & \int_0^1 \int_0^1 \cdots \int_0^1 (x_1^2 + x_2^2 + \cdots + x_n^2) dx_1 dx_2 \cdots dx_n; \\ (b) & \int_0^1 \int_0^1 \cdots \int_0^1 (x_1 + x_2 + \cdots + x_n)^2 dx_1 dx_2 \cdots dx_n. \end{aligned}$$

$$\begin{aligned}
\text{解} \quad (a) & \int_0^1 \int_0^1 \cdots \int_0^1 (x_1^2 + x_2^2 + \cdots x_n^2) dx_1 dx_2 \cdots dx_n \\
&= \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 (x_1^2 + x_2^2 + \cdots + x_n^2) dx_n \\
&= \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 (x_1^2 + x_2^2 + \cdots \\
&\quad + x_{n-1}^2 + \frac{1}{3}) dx_{n-1} \\
&= \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 \left(x_1^2 + x_2^2 + \cdots \right. \\
&\quad \left. + x_{n-2}^2 + \frac{1}{3} + \frac{1}{3} \right) dx_{n-2} \\
&= \cdots \cdots \\
&= \frac{n}{3}.
\end{aligned}$$

$$\begin{aligned}
(b) & \int_0^1 \int_0^1 \cdots \int_0^1 (x_1 + x_2 + \cdots + x_n)^2 dx_1 dx_2 \cdots dx_n \\
&= \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 [(x_1^2 + x_2^2 + \cdots + x_n^2) \\
&\quad + 2(x_1 x_2 + x_1 x_3 + \cdots + x_1 x_n + x_2 x_3 \\
&\quad + \cdots + x_2 x_n + x_3 x_4 + \cdots + x_3 x_n + \cdots \\
&\quad + x_{n-1} x_n)] dx_n \\
&= \frac{n}{3} + 2 \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 [(x_1 x_2 + \cdots \\
&\quad + x_1 x_n) + (x_2 x_3 + \cdots + x_2 x_n) \\
&\quad + \cdots + x_{n-1} x_n] dx_n \\
&= \frac{n}{3} + 2 \left(\frac{n-1}{4} + \frac{n-2}{4} + \cdots + \frac{1}{4} \right) \\
&= \frac{n(3n+1)}{12}.
\end{aligned}$$

*) 利用本题(a) 的结果.

$$4205. I_n = \int_0^a \int_0^{a-x_1} \cdots \int_0^{a-x_1-x_2-\cdots-x_{n-1}} dx_1 dx_2 \cdots dx_n,$$

解 解法一:

化为累次积分,有

$$I_n = \int_0^a dx_1 \int_0^{a-x_1} dx_2 \cdots \int_0^{a-x_1-\cdots-x_{n-2}} dx_{n-1} \int_0^{a-x_1-\cdots-x_{n-1}} dx_n,$$

我们又知

$$\begin{aligned} & \int_0^{a-x_1-\cdots-x_{n-2}} dx_{n-1} \int_0^{a-x_1-\cdots-x_{n-1}} dx_n \\ &= \int_0^{a-x_1-\cdots-x_{n-2}} (a-x_1-\cdots-x_{n-2}-x_{n-1}) dx_{n-1} \\ &= -\frac{1}{2} (a-x_1-\cdots-x_{n-2}-x_{n-1})^2 \Big|_{x_{n-1}=0}^{x_{n-1}=a-x_1-\cdots-x_{n-2}} \\ &= \frac{1}{2} (a-x_1-\cdots-x_{n-2})^2, \\ & \int_0^{a-x_1-\cdots-x_{n-3}} dx_{n-2} \int_0^{a-x_1-\cdots-x_{n-2}} dx_{n-1} \\ & \quad \int_0^{a-x_1-\cdots-x_{n-1}} dx_n \\ &= \int_0^{a-x_1-\cdots-x_{n-3}} \frac{1}{2} (a-x_1-\cdots-x_{n-2})^2 dx_{n-2} \\ &= \frac{1}{3!} (a-x_1-\cdots-x_{n-3})^3, \\ & \cdots \cdots \cdots \end{aligned}$$

这样继续下去,显然有

$$\int_0^{a-x_1} dx_2 \int_0^{a-x_1-x_2} dx_3$$

$$\cdots \int_0^{a-x_1-\cdots-x_{n-2}} dx_{n-1} \int_0^{a-x_1-\cdots-x_{n-1}} dx_n \\ = \frac{1}{(n-1)!} (a-x_1)^{n-1},$$

于是,

$$I_n = \int_0^a \frac{1}{(n-1)!} (a-x_1)^{n-1} dx_1 = \frac{a^n}{n!}.$$

解法二:

我们有

$$I_n = \int_0^a dx_1 \int_0^{a-x_1} dx_2 \cdots \int_0^{a-x_1-\cdots-x_{n-1}} dx_n.$$

在右端的逐次积分中作代换:

$$x_1 = a\xi_1, x_2 = a\xi_2, \cdots, x_n = a\xi_n,$$

即得

$$\begin{aligned} I_n &= a^n \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2 \cdots \int_0^{1-\xi_1-\cdots-\xi_{n-1}} d\xi_n \\ &= a^n \int\limits_{\substack{\xi_1 \geq 0, \xi_2 \geq 0, \cdots, \xi_n \geq 0 \\ \xi_1 + \xi_2 + \cdots + \xi_n \leq 1}} \cdots \int d\xi_1 d\xi_2 \cdots d\xi_n \\ &= a^n \cdot I_n(1), \end{aligned}$$

其中 $I_n(1)$ 表示当 $a=1$ 时积分 I_n 的值.

另一方面, 我们有

$$\begin{aligned} I_n(1) &= \int_0^1 d\xi_n \int\limits_{\substack{\xi_1 \geq 0, \xi_2 \geq 0, \cdots, \xi_n \geq 0 \\ \xi_1 + \xi_2 + \cdots + \xi_{n-1} \leq 1-\xi_n}} \cdots \int d\xi_1 d\xi_2 \cdots d\xi_{n-1} \\ &= I_{n-1}(1) \int_0^1 (1-\xi_n)^{n-1} d\xi_n \\ &= \frac{I_{n-1}(1)}{n}. \end{aligned}$$

反复运用上述循环公式, 可得

则由 $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0, x_1 + x_2 + \dots + x_n \leq 1$ 知

$$0 \leq u_i \leq 1 \quad (i = 1, 2, \dots, n),$$

且有

$$I = \begin{vmatrix} 1-u_2 & u_2(1-u_3) & \cdots & u_2 u_3 \cdots u_{n-1}(1-u_n) & u_2 u_3 \cdots u_n \\ -u_1 & u_1(1-u_3) & \cdots & u_1 u_3 \cdots u_{n-1}(1-u_n) & u_1 u_3 \cdots u_n \\ 0 & -u_1 u_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_1 u_2 \cdots u_{n-2}(1-u_n) & u_1 \cdots u_{n-2} u_n \\ 0 & 0 & \cdots & -u_1 \cdots u_{n-1} & u_1 \cdots u_{n-1} \end{vmatrix}.$$

如在每一列的元素上加上所有以后各列相应的元素, 则在对角线下面的全部元素都等于零, 而在对角线上的元素就等于 $1, u_1, u_1 u_2, \dots, u_1 \cdots u_{n-1}$. 因此, 得

$$I = u_1^{n-1} u_2^{n-2} \cdots u_{n-1}.$$

于是, 最后得

$$\begin{aligned} & \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} \cdots \int \sqrt{x_1 + x_2 + \dots + x_n} dx_1 dx_2 \cdots dx_n \\ &= \int_0^1 \int_0^1 \cdots \int_0^1 u_1^{n-1} u_2^{n-2} \cdots u_{n-1} du_1 du_2 \cdots du_{n-1} du_n \\ &= \frac{2}{(n-1)!(2n+1)}. \end{aligned}$$

4208. 求由平面

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \pm h_i \quad (i = 1, 2, \dots, n)$$

所界的 n 维平行 $2n$ 面体的体积, 这里设 $\Delta = |a_{ij}| \neq 0$.

解 令 $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \xi_i \quad (i = 1, 2, \dots, n)$,

即得 $2n$ 面体的体积

$$\begin{aligned} V &= \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} \cdots \int_{-h_n}^{h_n} \frac{1}{|\Delta|} d\xi_1 d\xi_2 \cdots d\xi_n \\ &= \frac{2^n h_1 \cdots h_n}{|\Delta|}. \end{aligned}$$

4209. 求 n 维角锥

$$\begin{aligned} \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} &\leq 1, x_i \geq 0 (i = 1, 2, \cdots, n) \\ (a_i > 0, i = 1, 2, \cdots, n) \end{aligned}$$

的体积.

解 令 $x_i = a_i \xi_i, (i = 1, 2, \cdots, n)$ 即得体积

$$\begin{aligned} V &= a_1 a_2 \cdots a_n \int_{\substack{\xi_1 \geq 0, \xi_2 \geq 0, \cdots, \xi_n \geq 0 \\ \xi_1 + \xi_2 + \cdots + \xi_n \leq 1}} d\xi_1 d\xi_2 \cdots d\xi_n \\ &= \frac{a_1 a_2 \cdots a_n}{n!} * \end{aligned}$$

*) 利用 4205 题的结果.

4210. 求由曲面

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_{n-1}^2}{a_{n-1}^2} = \frac{x_n^2}{a_n^2}, x_n = a_n$$

所界的 n 维锥的体积.

解 作代换:

$$x_1 = a_1 r \cos \varphi_1,$$

$$x_2 = a_2 r \sin \varphi_1 \cos \varphi_2,$$

$$\cdots \cdots \cdots$$

$$x_{n-2} = a_{n-2} r \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-3} \cos \varphi_{n-2},$$

$$x_{n-1} = a_{n-1} r \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-3} \sin \varphi_{n-2},$$

$$x_n = a_n x'_n,$$

则域 V 为

$$0 \leq r \leq 1, 0 \leq \varphi_1 \leq \pi, 0 \leq \varphi_2 \leq \pi, \dots,$$

$$0 \leq \varphi_{n-3} \leq \pi, 0 \leq \varphi_{n-2} \leq 2\pi, r \leq x'_n \leq 1,$$

并且 $|I| = a_1 a_2 \cdots a_n r^{n-2} \sin^{n-3} \varphi_1 \sin^{n-4} \varphi_2 \sin \varphi_{n-3}$.

于是, 体积为

$$\begin{aligned} V &= a_1 a_2 \cdots a_n \int_0^1 r^{n-2} dr \int_0^\pi \sin^{n-3} \varphi_1 d\varphi_1 \\ &\quad \cdots \int_0^\pi \sin \varphi_{n-3} d\varphi_{n-3} \int_0^{2\pi} d\varphi_{n-2} \int_r^1 dx'_n \\ &= \frac{2\pi a_1 a_2 \cdots a_n}{n(n-1)} \int_0^\pi \sin^{n-3} \varphi_1 d\varphi_1 \\ &\quad \cdots \int_0^\pi \sin \varphi_{n-3} d\varphi_{n-3} \\ &= \frac{2\pi a_1 a_2 \cdots a_n}{n(n-1)} \cdot 2 \int_0^{\frac{\pi}{2}} \sin^{n-3} \varphi_1 d\varphi_1 \\ &\quad \cdots 2 \int_0^{\frac{\pi}{2}} \sin \varphi_{n-3} d\varphi_{n-3} \cdots \\ &= \frac{2\pi a_1 a_2 \cdots a_n}{n(n-1)} \\ &\quad \cdot B\left(\frac{n-2}{2}, \frac{1}{2}\right) B\left(\frac{n-3}{2}, \frac{1}{2}\right) \\ &\quad \cdots B\left(\frac{2}{2}, \frac{1}{2}\right) \cdots \\ &= \frac{2\pi a_1 a_2 \cdots a_n}{n(n-1)} \cdot \frac{\Gamma\left(\frac{n-2}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \\ &\quad \cdot \frac{\Gamma\left(\frac{n-3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \cdots \frac{\Gamma\left(\frac{2}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \end{aligned}$$

$$\begin{aligned}
&= \frac{2\pi a_1 a_2 \cdots a_n}{n(n-1)} \cdot \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \\
&= \frac{2\pi a_1 a_2 \cdots a_n}{n(n-1)} \cdot \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \\
&= \frac{a_1 a_2 \cdots a_n \pi^{\frac{n-1}{2}}}{n} \cdot \frac{1}{\frac{n-1}{2} \Gamma\left(\frac{n-1}{2}\right)} \\
&= \frac{a_1 a_2 \cdots a_n \pi^{\frac{n-1}{2}}}{n} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2} + 1\right)} \\
&= \frac{\pi^{\frac{n-1}{2}}}{n \Gamma\left(\frac{n+1}{2}\right)} a_1 a_2 \cdots a_n.
\end{aligned}$$

*) 利用等式 $\int_0^{\frac{\pi}{2}} \sin^{a-1} \varphi d\varphi = \int_0^{\frac{\pi}{2}} \cos^{a-1} \varphi d\varphi (a > 0)$,

即得

$$\begin{aligned}
&\int_0^{\pi} \sin^{a-1} \varphi d\varphi \\
&= \int_0^{\frac{\pi}{2}} \sin^{a-1} \varphi d\varphi + \int_{\frac{\pi}{2}}^{\pi} \sin^{a-1} \varphi d\varphi \\
&= \int_0^{\frac{\pi}{2}} \sin^{a-1} \varphi d\varphi + \int_0^{\frac{\pi}{2}} \cos^{a-1} \varphi d\varphi \\
&= 2 \int_0^{\frac{\pi}{2}} \sin^{a-1} \varphi d\varphi.
\end{aligned}$$

* *) 利用 3856 题的结果.

4211. 求 n 维球体

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq a^2$$

的体积.

解 令 $x_i = a\xi_i (i = 1, 2, \dots, n)$, 即得体积

$$V_n = \iiint_{x_1^2 + x_2^2 + \dots + x_n^2 \leq a^2} \dots \int dx_1 dx_2 \dots dx_n = a^n V_n(1),$$

其中 $V_n(1)$ 表示 $a = 1$ 时的 n 维球体的体积. 但是,

$$\begin{aligned} V_n(1) &= \iiint_{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2 \leq 1} \dots \int d\xi_1 d\xi_2 \dots d\xi_n \\ &= \int_{-1}^1 d\xi_n \iiint_{\xi_1^2 + \dots + \xi_{n-1}^2 \leq 1 - \xi_n^2} \dots \int d\xi_1 d\xi_2 \dots d\xi_{n-1} \\ &= V_{n-1}(1) \int_{-1}^1 (1 - \xi_n^2)^{\frac{n-1}{2}} d\xi_n \\ &= 2V_{n-1}(1) \int_0^{\frac{\pi}{2}} \sin^n \varphi d\varphi \\ &= 2V_{n-1}(1) \cdot \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \\ &= V_{n-1}(1) \sqrt{\pi} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}, \end{aligned}$$

因为 $V_1(1) = 2$, 故由上述循环公式可得

$$V_n(1) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

因而, 所求的体积为

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} a^n.$$

对于 n 为偶数及奇数, 分别可得公式

$$V_{2m} = \frac{\pi^m}{m!} a^{2m},$$

$$V_{2m+1} = \frac{2 \cdot (2\pi)^m}{(2m+1)!} a^{2m+1}.$$

特别是, 对于 V_1, V_2, V_3 可求得熟知的值: $2a, \pi a^2,$

$$\frac{4}{3} \pi a^3.$$

4212. 求 $\iint_{\Omega} \cdots \int x_n^2 dx_1 dx_2 \cdots dx_n$, 其中域 Ω 是由下列不等式所确定:

$$x_1^2 + x_2^2 + \cdots x_{n-1}^2 \leq a^2, \quad -\frac{h}{2} \leq x_n \leq \frac{h}{2}.$$

$$\begin{aligned} \text{解} \quad & \iint_{\Omega} \cdots \int x_n^2 dx_1 dx_2 \cdots dx_n \\ &= \int_{-\frac{h}{2}}^{\frac{h}{2}} x_n^2 dx_n \iint_{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq a^2} \cdots \int dx_1 dx_2 \cdots dx_{n-1} \\ &= \frac{h^3}{12} \cdot \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} a^{n-1} *). \end{aligned}$$

*) 利用 4211 题的结果.

4213. 计算

$$\begin{aligned} & \iint_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} \cdots \int \frac{dx_1 dx_2 \cdots dx_n}{\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2}}. \\ \text{解} \quad & \iint_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} \cdots \int \frac{dx_1 dx_2 \cdots dx_n}{\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2}} \\ &= \iint_{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 1} \cdots \int dx_1 dx_2 \cdots dx_{n-1} \end{aligned}$$

$$\begin{aligned}
& \int_0^1 \frac{\sqrt{1-x_1^2-\cdots-x_{n-1}^2}}{\sqrt{1-x_1^2-\cdots-x_{n-1}^2}} \frac{dx_n}{\sqrt{1-x_1^2-x_2^2-\cdots-x_n^2}} \\
&= \pi \int_0^1 \cdots \int_0^1 dx_1 dx_2 \cdots dx_{n-1} \\
& \quad x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 1 \\
&= \pi \cdot \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2} + 1\right)} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.
\end{aligned}$$

*) 利用 4211 题的结果.

4214. 证明等式

$$\begin{aligned}
& \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_n) dx_n \\
&= \int_0^x f(u) \frac{(x-u)^{n-1}}{(n-1)!} du.
\end{aligned}$$

$$\begin{aligned}
\text{证} \quad & \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_n) dx_n \\
&= \int_0^x f(x_n) dx_n \int_{x_n}^x dx_{n-1} \int_{x_{n-1}}^x dx_{n-2} \cdots \int_{x_2}^x dx_1^{*}) \\
&= \int_0^x f(x_n) dx_n \int_{x_n}^x dx_{n-1} \int_{x_{n-1}}^x dx_{n-2} \\
& \quad \cdots \int_{x_2}^x (x-x_2) dx_2 \\
&= \int_0^x f(x_n) dx_n \int_{x_n}^x dx_{n-1} \int_{x_{n-1}}^x dx_{n-2} \\
& \quad \cdots \int_{x_3}^x \frac{1}{2} (x-x_3)^2 dx_3 \\
&= \int_0^x f(x_n) dx_n \int_{x_n}^x dx_{n-1} \int_{x_{n-1}}^x dx_{n-2} \\
& \quad \cdots \int_{x_4}^x \frac{1}{2 \cdot 3} (x-x_4)^3 dx_4
\end{aligned}$$

.....

$$\begin{aligned}
 &= \int_0^x f(x_n) dx_n \int_{x_n}^x \frac{1}{(n-2)!} (x - x_{n-1})^{n-2} dx_{n-1} \\
 &= \int_0^x \frac{(x - x_n)^{n-1}}{(n-1)!} f(x_n) dx_n.
 \end{aligned}$$

在上述积分中,将 x_n 代之以 u , 不影响积分的值, 故得

$$\begin{aligned}
 &\int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_n) dx_n \\
 &= \int_0^x f(u) \frac{(x-u)^{n-1}}{(n-1)!} du.
 \end{aligned}$$

*) 利用 4202 题的结果.

4215. 证明等式

$$\begin{aligned}
 &\int_0^x x_1 dx_1 \int_0^{x_1} x_2 dx_2 \cdots \int_0^{x_n} f(x_{n+1}) dx_{n+1} \\
 &= \frac{1}{2^n n!} \int_0^x (x^2 - u^2)^n f(u) du.
 \end{aligned}$$

证 利用 4202 题的结果, 即得

$$\begin{aligned}
 &\int_0^x x_1 dx_1 \int_0^{x_1} x_2 dx_2 \cdots \int_0^{x_n} f(x_{n+1}) dx_{n+1} \\
 &= \int_0^x f(x_{n+1}) dx_{n+1} \int_{x_{n+1}}^x x_n dx_n \int_{x_n}^x x_{n-1} dx_{n-1} \\
 &\quad \cdots \int_{x_2}^x x_1 dx_1 \\
 &= \int_0^x f(x_{n+1}) dx_{n+1} \int_{x_{n+1}}^x x_n dx_n \int_{x_n}^x x_{n-1} dx_{n-1} \\
 &\quad \cdots \int_{x_3}^x \frac{1}{2} (x^2 - x_2^2) x_2 dx_2 \\
 &= \int_0^x f(x_{n+1}) dx_{n+1} \int_{x_{n+1}}^x x_n dx_n \int_{x_n}^x x_{n-1} dx_{n-1}
 \end{aligned}$$

$$\begin{aligned}
& \cdots \int_{x_4}^x \frac{1}{2^2 \cdot 2} (x^2 - x_3^2)^2 x_3 dx_3 \\
& = \cdots \cdots \cdots \\
& = \int_0^x f(x_{n+1}) dx_{n+1} \int_{x_{n+1}}^x \frac{1}{2^{n-1} (n-1)!} \\
& \quad \cdot (x^2 - x_n^2)^{n-1} x_n dx_n \\
& = \int_0^x \frac{1}{2^n n!} f(x_{n+1}) (x^2 - x_{n+1})^n dx_{n+1}.
\end{aligned}$$

于是, 将 x_{n+1} 代之以 u , 不影响积分的值, 故得

$$\begin{aligned}
& \int_0^x x_1 dx_1 \int_0^{x_1} x_2 dx_2 \cdots \int_0^{x_n} f(x_{n+1}) dx_{n+1} \\
& = \frac{1}{2^n n!} \int_0^x (x^2 - u^2)^n f(u) du.
\end{aligned}$$

4216. 证明迪里黑里公式

$$\begin{aligned}
& \iint \cdots \int_{\substack{x_1, x_2, \dots, x_n \geq 0, \\ x_1 + x_2 + \cdots + x_n \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \\
& \quad \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n \\
& = \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n + 1)} \\
& \quad (p_1, p_2, \dots, p_n > 0).
\end{aligned}$$

证 我们应用数学归纳法证明之.

当 $n = 1$ 时, 公式显然成立, 即

$$\int_{0 \leq x_1 \leq 1} x_1^{p_1-1} dx_1 = \frac{1}{p_1} = \frac{\Gamma(p_1)}{\Gamma(p_1 + 1)}.$$

其次, 设公式对 $n - 1$ 成立, 今证公式对 n 也成立.

为此, 将公式左端写为

$$\int_0^1 x_n^{p_n-1} dx_n \iint \cdots \int_{\substack{x_1, x_2, \dots, x_{n-1} \geq 0, \\ x_1 + x_2 + \cdots + x_{n-1} \leq 1 - x_n}} x_1^{p_1-1} x_2^{p_2-1}$$

$$\cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_{n-1}.$$

在里面的 $n-1$ 重积分中进行代换:

$$x_1 = (1-x_n)\xi_1, x_2 = (1-x_n)\xi_2, \cdots,$$

$$x_{n-1} = (1-x_n)\xi_{n-1},$$

$$\begin{aligned} \text{即得} & \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_{n-1})}{\Gamma(p_1+p_2+\cdots+p_{n-1}+1)} \\ & \cdot \int_0^1 x_n^{p_n-1} (1-x_n)^{p_1+p_2+\cdots+p_{n-1}} dx_n \\ & = \frac{\Gamma(p_1)\cdots\Gamma(p_{n-1})}{\Gamma(p_1+\cdots+p_{n-1}+1)} \\ & \quad \cdot B(p_n, p_1+\cdots+p_{n-1}+1) \\ & = \frac{\Gamma(p_1)\cdots\Gamma(p_{n-1})}{\Gamma(p_1+\cdots+p_{n-1}+1)} \\ & \quad \cdot \frac{\Gamma(p_n) \cdot \Gamma(p_1+\cdots+p_{n-1}+1)}{\Gamma(p_1+\cdots+p_n+1)} \\ & = \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n+1)}. \end{aligned}$$

这样一来,我们得知公式对 n 重积分也正确. 从而对 n 为任意的自然数时,迪里黑里公式均成立.

4217. 证明柳维耳公式

$$\begin{aligned} & \iint_{\substack{x_1, x_2, \dots, x_n \geq 0 \\ x_1+x_2+\cdots+x_n \leq 1}} \cdots \int f(x_1+x_2+\cdots+x_n) \\ & \quad \cdot x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n \\ & = \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)} \\ & \quad \cdot \int_0^1 f(u) u^{p_1+p_2+\cdots+p_n-1} du \end{aligned}$$

$$(p_1, p_2, \dots, p_n > 0),$$

式中 $f(u)$ 为连续函数.

证 我们应用数学归纳法证明之.

当 $n = 1$ 时, 公式显然成立, 当 $n = 2$ 时, 公式也成立, 即

$$\begin{aligned} & \int_{\substack{x_1 \geq 0, x_2 \geq 0, \\ x_1 + x_2 \leq 1}} f(x_1 + x_2) x_1^{p_1-1} x_2^{p_2-1} dx_1 dx_2 \\ &= \frac{\Gamma(p_1)\Gamma(p_2)}{\Gamma(p_1 + p_2)} \int_0^1 f(u) u^{p_1+p_2-1} du. \end{aligned}$$

事实上, 令 Ω 表域: $x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1$.

作代换:

$$x_1 = \xi_1, x_1 + x_2 = \xi_2, \text{ 及 } t = \frac{\xi_1}{\xi_2},$$

则有

$$\begin{aligned} & \iint_{\Omega} f(x_1 + x_2) x_1^{p_1-1} x_2^{p_2-1} dx_1 dx_2 \\ &= \int_0^1 f(\xi_2) d\xi_2 \int_0^{\xi_2} \xi_1^{p_1-1} (\xi_2 - \xi_1)^{p_2-1} d\xi_1 \\ &= \int_0^1 f(\xi_2) d\xi_2 \int_0^1 t^{p_1-1} (1-t)^{p_2-1} \xi_2^{p_1+p_2-1} dt \\ &= \frac{\Gamma(p_1)\Gamma(p_2)}{\Gamma(p_1 + p_2)} \int_0^1 f(\xi_2) \xi_2^{p_1+p_2-1} d\xi_2 \\ &= \frac{\Gamma(p_1)\Gamma(p_2)}{\Gamma(p_1 + p_2)} \int_0^1 f(u) u^{p_1+p_2-1} du. \end{aligned}$$

其次, 设公式对于 $n-1$ 成立, 今证对于 n 公式也成立, 为此, 将公式左端写为

$$\iint \dots \int_{\substack{x_1, x_2, \dots, x_{n-1} \geq 0 \\ x_1 + x_2 + \dots + x_{n-1} \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \dots x_{n-1}^{p_{n-1}-1} dx_1 dx_2$$

$$\cdots dx_n \int_0^{1-(x_1+x_2+\cdots+x_{n-1})} f(x_1+x_2+\cdots+x_n) \\ \cdot x_n^{p_n-1} dx_n.$$

如令

$$\psi(t) = \int_0^{1-t} f(t+x_n) x_n^{p_n-1} dx_n$$

代入上式, 并利用公式对 $n-1$ 成立的假定, 得知上式为

$$\frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_{n-1})}{\Gamma(p_1+p_2+\cdots+p_{n-1})} \\ \cdot \int_0^1 \psi(t) t^{p_1+p_2+\cdots+p_{n-1}-1} dt.$$

利用上面已证的 $n=2$ 时的公式, 于是即得

$$\begin{aligned} & \iint_{\substack{x_1, x_2, \dots, x_n \geq 0 \\ x_1+x_2+\cdots+x_n \leq 1}} \cdots \int f(x_1+x_2+\cdots+x_n) \\ & \cdot x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n \\ &= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_{n-1})}{\Gamma(p_1+p_2+\cdots+p_{n-1})} \int_0^1 dt \int_0^{1-t} f(t+x_n) \\ & \cdot t^{p_1+p_2+\cdots+p_{n-1}-1} x_n^{p_n-1} dx_n \\ &= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_{n-1})}{\Gamma(p_1+p_2+\cdots+p_{n-1})} \iint_{\substack{t, x_n \geq 0 \\ t+x_n \leq 1}} f(t+x_n) \\ & \cdot t^{p_1+p_2+\cdots+p_{n-1}-1} x_n^{p_n-1} dt dx_n \\ &= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_{n-1})}{\Gamma(p_1+p_2+\cdots+p_{n-1})} \\ & \cdot \frac{\Gamma(p_1+p_2+\cdots+p_{n-1})\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)} \\ & \cdot \int_0^1 f(u) u^{p_1+p_2+\cdots+p_n-1} du \end{aligned}$$

$$= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)} \\ \cdot \int_0^1 f(u)u^{p_1+p_2+\cdots+p_n-1}du,$$

即公式对于 n 成立. 从而, 公式对于任意自然数均成立.

4218.† 将展布于域 $x_1^2+x_2^2+\cdots+x_n^2\leq R^2$ 上的 n 重积分 ($n\geq 2$)

$$\int_0^R\cdots\int_0^R f(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})dx_1dx_2\cdots dx_n$$

化为单积分, 其中 $f(u)$ 为连续函数

解 作代换:

$$x_1 = Rr\cos\varphi,$$

$$x_2 = Rr\sin\varphi_1\cos\varphi_2,$$

$$\dots\dots\dots$$

$$x_{n-1} = Rr\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\cos\varphi_{n-1},$$

$$x_n = Rr\sin\varphi\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}.$$

则有

$$I = R^n r^{n-1} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin\varphi_{n-2}.$$

于是,

$$\begin{aligned} & \int_0^R \cdots \int_0^R f(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})dx_1dx_2\cdots dx_n \\ &= R^n \int_0^1 r^{n-1} f(Rr)dr \int_0^\pi \sin^{n-2}\varphi_1 d\varphi_1 \\ & \quad \cdot \int_0^\pi \sin^{n-3}\varphi_2 d\varphi_2 \cdots \int_0^\pi \sin\varphi_{n-2} d\varphi_{n-2} \int_0^{2\pi} d\varphi_{n-1} \\ &= 2\pi R^n \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{\Gamma\left(\frac{n-2}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdots \frac{\Gamma\left(\frac{2}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \\
& \cdot \int_0^1 r^{n-1} f(Rr) dr \cdot \\
& = R^n \frac{2\pi \cdot \pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 r^{n-1} f(Rr) dr \\
& = R^n \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 (r^2)^{\frac{n}{2}-1} f(Rr) d(r^2) \\
& = R^n \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 u^{\frac{n}{2}-1} f(R\sqrt{u}) du.
\end{aligned}$$

*) 参看 4210 题的计算过程.

4219. 计算半径为 R , 密度为 ρ_0 的均匀球对自己的位, 即求积分

$$u = \frac{\rho_0^2}{2} \iiint_{x_1^2+y_1^2+z_1^2 \leq R^2} \iiint_{x_2^2+y_2^2+z_2^2 \leq R^2} \frac{dx_1 dy_1 dz_1 dx_2 dy_2 dz_2}{r_{1,2}},$$

式中 $r_{1,2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

解 我们有

$$\begin{aligned}
u &= \frac{\rho_0^2}{2} \iiint_{x_1^2+y_1^2+z_1^2 \leq R^2} dx_1 dy_1 dz_1 \\
& \quad \iiint_{x_2^2+y_2^2+z_2^2 \leq R^2} \frac{dx_2 dy_2 dz_2}{r_{1,2}}.
\end{aligned}$$

由 4155 题的结果可知

$$\iiint_{x_1^2 + y_1^2 + z_1^2 \leq R^2} \frac{dx_1 dy_1 dz_1}{r_{1,2}} = 2\pi R^2 - \frac{2}{3}\pi r_1^2,$$

其中 $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$. 于是(利用球坐标)

$$\begin{aligned} u &= \frac{\rho_0^2}{2} \iiint_{x_1^2 + y_1^2 + z_1^2 \leq R^2} (2\pi R^2 - \frac{2}{3}\pi r_1^2) dx_1 dy_1 dz_1 \\ &= \frac{\rho_0^2}{2} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\psi d\psi \\ &\quad \cdot \int_0^R \left(2\pi R^2 - \frac{2}{3}\pi r^2 \right) r^2 dr \\ &= \frac{16}{15}\pi^2 \rho_0^2 R^5. \end{aligned}$$

4220. 设 $\sum_{i,j=1}^n a_{ij}x_i x_j$ ($a_{ij} = a_{ji}$) 为正定形, 计算 n 重积分

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\left(\sum_{i,j=1}^n a_{ij}x_i x_j + 2\sum_{i=1}^n b_i x_i + c\right)} dx_1 dx_2 \cdots dx_n.$$

解 作变量代换

$$x_i = y_i + a_i \quad (i = 1, 2, \dots, n), \quad (1)$$

其中诸常数 a_i 以下再确定. 于是易得(注意到 $a_{ij} = a_{ji}$)

$$\begin{aligned} &\sum_{i,j=1}^n a_{ij}x_i x_j + 2\sum_{i=1}^n b_i x_i + c \\ &= \sum_{i,j=1}^n a_{ij}y_i y_j + 2\sum_{i=1}^n \left[\left(\sum_{j=1}^n a_{ij}a_j \right) + b_i \right] y_i \\ &\quad + \sum_{i,j=1}^n a_{ij}a_i a_j + 2\sum_{i=1}^n b_i a_i + c. \end{aligned}$$

由于 $\sum_{i,j=1}^n a_{ij}x_i x_j$ 是正定形, 故必有 $\delta = |a_{ij}| > 0$, 从而线性方程组

$$\sum_{j=1}^n a_{ij}x_j + b_i = 0 \quad (i = 1, 2, \dots, n) \quad (2)$$

有唯一的一组解 $\alpha_1, \dots, \alpha_n$, 今取变换(1)式中的诸 α_i 即为方程组(2)的解. 于是,

$$\begin{aligned} & \sum_{i,j=1}^n a_{ij}x_ix_j + 2 \sum_{i=1}^n b_ix_i + c \\ &= \sum_{i,j=1}^n a_{ij}y_iy_j + c', \end{aligned} \quad (3)$$

$$\begin{aligned} \text{其中 } c' &= \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}\alpha_j \right) \alpha_i + 2 \sum_{i=1}^n b_i\alpha_i + c \\ &= - \sum_{i=1}^n b_i\alpha_i + 2 \sum_{i=1}^n b_i\alpha_i + c \\ &= \sum_{i=1}^n b_i\alpha_i + c. \end{aligned} \quad (4)$$

下面我们用诸 a_{ij} 和 b_i 及 c 来表出 c' 令

$$\Delta = \begin{vmatrix} a_{11} & \vdots & b_1 \\ \cdots & \cdots & \cdots \\ a_{n1} & \vdots & b_n \\ b_1 & \vdots & c \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} & b_n \\ b_1 & \cdots & b_n & c \end{vmatrix}$$

($n+1$ 阶行列式, 即 $|a_{ij}|$ 的加边行列式). 将此行列式的行 i 列乘上 α_i , 第二列乘上 $\alpha_2, \dots$, 第 n 列乘上 α_n 都加到第 $n+1$ 列上去, 并注意到(2)式与(4)式, 得

$$\Delta = \begin{vmatrix} a_{11} & \vdots & b_1 \\ \cdots & \cdots & \cdots \\ a_{n1} & \vdots & b_n \\ b_1 & \vdots & c \end{vmatrix} = \begin{vmatrix} a_{11} & \vdots & \sum_{j=1}^n a_{1j}\alpha_j + b_1 \\ \cdots & \cdots & \cdots \\ a_{n1} & \vdots & \sum_{j=1}^n a_{nj}\alpha_j + b_n \\ b_1 & \vdots & \sum_{j=1}^n b_j\alpha_j + c \end{vmatrix}$$

$$= \begin{vmatrix} a_{rr} & \vdots & 0 \\ & \cdots & \\ b_r & \vdots & c \end{vmatrix} = c' |a_{rr}| = c' \delta,$$

故

$$c' = \frac{\Delta}{\delta}. \quad (5)$$

由于 $\sum_{i,j=1}^n a_{ij}y_iy_j$ 是正定二次型,故由高等代数中二次型的理论知,存在正交矩阵

$$P = (p_{ij}) = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix},$$

使在线性变换

$$y_i = \sum_{j=1}^n p_{ij}z_j, (i = 1, 2, \cdots, n) \quad (6)$$

下,二次型变为平方和:

$$\sum_{i,j=1}^n a_{ij}y_iy_j = \sum_{i=1}^n \lambda_i z_i^2, \quad (7)$$

其中 $\lambda_i > 0 (i = 1, 2, \cdots, n)$; 也即

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad (8)$$

其中 $A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$. 由于 P 为正交矩阵,

故 $P^{-1} = P'$ (P' 表 P 的转置矩阵), 且 $|P| = |p_{ij}| = \pm$

1. 由(8)式又知

$$\delta = |a_{ij}| = |P^{-1}| = |A| = |P| = \lambda_1 \lambda_2 \cdots \lambda_n. \quad (9)$$

根据(1)式与(6)式,可知

$$\frac{D(x_1, \cdots, x_n)}{D(y_1, \cdots, y_n)} = 1,$$

$$\frac{D(y_1, \cdots, y_n)}{D(z_1, \cdots, z_n)} = |p_{ij}| = |P| = \pm 1.$$

于是,利用广义 n 重积分的变量代换公式,并注意到被积函数的非负性,得(注意(3)式、(5)式与(7)式)

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\left\{ \sum_{i,j=1}^n a_{ij} x_i x_j + 2 \sum_{i=1}^n b_i x_i \right\}} \\ &\quad \cdot dx_1 dx_2 \cdots dx_n \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\left\{ \sum_{i,j=1}^n a_{ij} y_i y_j + c \right\}} \\ &\quad \left| \frac{D(x_1, \cdots, x_n)}{D(y_1, \cdots, y_n)} \right| dy_1 dy_2 \cdots dy_n \\ &= e^{-\frac{4}{\delta}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\left\{ \sum_{i,j=1}^n a_{ij} y_i y_j \right\}} dy_1 dy_2 \cdots dy_n \\ &= e^{-\frac{4}{\delta}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\sum_{i=1}^n \lambda_i z_i^2} \\ &\quad \left| \frac{D(y_1, \cdots, y_n)}{D(z_1, \cdots, z_n)} \right| dz_1 dz_2 \cdots dz_n \\ &= e^{-\frac{4}{\delta}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\sum_{i=1}^n \lambda_i z_i^2} dz_1 dz_2 \cdots dz_n \\ &= e^{-\frac{4}{\delta}} \left(\int_{-\infty}^{+\infty} e^{-\lambda_1 z_1^2} dz_1 \right) \left(\int_{-\infty}^{+\infty} e^{-\lambda_2 z_2^2} dz_2 \right) \\ &\quad \cdots \left(\int_{-\infty}^{+\infty} e^{-\lambda_n z_n^2} dz_n \right). \end{aligned}$$

作代换 $z_i = -\frac{u}{\sqrt{\lambda_i}}$ (i 固定), 得

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-u^2} dz_i &= \frac{1}{\sqrt{\lambda_i}} \int_{-\infty}^{+\infty} e^{-u^2} du \\ &= \frac{2}{\sqrt{\lambda_i}} \int_0^{+\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{\sqrt{\lambda_i}} \quad (i = 1, 2, \dots, n).\end{aligned}$$

以此代入上式,并注意到(9)式,最后得

$$\begin{aligned}I &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\left(\sum_{i,j=1}^n a_{ij} x_i x_j + 2 \sum_{i=1}^n b_i x_i + c\right)} \\ &\quad \cdot dx_1 dx_2 \cdots dx_n \\ &= e^{-\frac{A}{\delta}} \cdot \frac{\pi^{\frac{n}{2}}}{\sqrt{\lambda_1 \lambda_2 \cdots \lambda_n}} = \sqrt{\frac{\pi^n}{\delta}} e^{-\frac{A}{\delta}}.\end{aligned}$$

§ 11. 曲线积分

1° 第一型的曲线积分 若 $f(x, y, z)$ 在平滑曲线 C

$$x = x(t), y = y(t), z = z(t) \quad (t_0 \leq t \leq T) \quad (1)$$

的各点上有定义并且是连续的函数, ds 为弧的微分, 则

$$\begin{aligned}&\int_C f(x, y, z) ds \\ &= \int_{t_0}^T f[x(t), y(t), z(t)] \cdot \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} dt.\end{aligned}$$

这个积分的特性在于它与曲线 C 的方向无关.

2° 第一型曲线积分在力学方面的应用 若 $\rho = \rho(x, y, z)$ 为曲线 C 在流动点 (x, y, z) 的线密度, 则 曲线 C 的质量 等于

$$M = \int_C \rho(x, y, z) ds.$$

此曲线的 重心坐标 (x_0, y_0, z_0) 由下面的公式来表示

$$x_0 = \frac{1}{M} \int_C x \rho(x, y, z) ds,$$

$$y_0 = \frac{1}{M} \int_C y \rho(x, y, z) ds,$$

$$z_0 = \frac{1}{M} \int_C z \rho(x, y, z) ds.$$

3° 第二型的曲线积分 若函数 $P = P(x, y, z), Q = Q(x, y, z), R = R(x, y, z)$ 在曲线(1)上的各点是连续的, 这曲线的方向是使参数 t 增加的方向, 则

$$\begin{aligned} & \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz \\ &= \int_{t_0}^T \{P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) \\ &+ R(x(t), y(t), z(t))z'(t)\} dt. \end{aligned} \quad (2)$$

当曲线 C 环行的方向变更时此积分的符号也变更. 在力学上积分(2)是当其作用点描绘出曲线 C 时 变力 (P, Q, R) 所作的功.

4° 全微分的情形 若

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = du,$$

式中 $u = u(x, y, z)$ 为域 V 内的单值函数, 则与完全位于域 V 内的曲线 C 的形状无关, 而有:

$$\int_C Pdx + Qdy + Rdz = u(x_2, y_2, z_2) - u(x_1, y_1, z_1),$$

式中 (x_1, y_1, z_1) 为路径的始点, (x_2, y_2, z_2) 为路径的终点. 最简单的情況是域 V 是单联通的而函数 P, Q, R 有连续的一级偏导函数, 对于此事的充分而且必要的条件为: 在域 V 内, 下列条件恒满足:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

这时, 函数 u 可按下面的公式来求得

$$\begin{aligned} u(x, y, z) &= \int_{x_0}^x P(x, y, z) dx + \int_{y_0}^y Q(x_0, y, z) dy \\ &+ \int_{z_0}^z R(x_0, y_0, z) dz, \end{aligned}$$

其中 (x_0, y_0, z_0) 为域 V 内某一固定的点.

在力学上这个情况对应于位力所作的功.

计算下列第一型的曲线积分:

4221. $\int_C (x+y)ds$, 其中 C 为以 $O(0,0)$, $A(1,0)$ 和 $B(0,1)$ 为顶点的三角形围线.

$$\begin{aligned} \text{解} \quad & \int_C (x+y)ds \\ &= \int_{OA} (x+y)ds + \int_{AB} (x+y)ds + \int_{BO} (x+y)ds \\ &= \int_0^1 xdx + \int_0^1 \sqrt{2} dx + \int_0^1 ydy = 1 + \sqrt{2}. \end{aligned}$$

4222. $\int_C y^2 ds$, 其中 C 为摆线 $x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq 2\pi$) 的一拱.

解 弧长的微分为

$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} dt \\ &= 2a \sin \frac{t}{2} dt. \end{aligned}$$

于是,

$$\begin{aligned} \int_C y^2 ds &= 2a^3 \int_0^{2\pi} \sin \frac{t}{2} (1 - \cos t)^2 dt \\ &= 8a^3 \int_0^{2\pi} \sin^5 \frac{t}{2} dt = 32a^3 \int_0^{\pi} \sin^5 u du \\ &= \frac{256}{15} a^3. \end{aligned}$$

4223. $\int_C (x^2 + y^2)ds$, 其中 C 为曲线 $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ ($0 \leq t \leq 2\pi$).

解 弧长的微分为

$$ds = \sqrt{a^2 t^2 \cos^2 t + a^2 t^2 \sin^2 t} dt = at dt.$$

于是,

$$\begin{aligned}& \int_c (x^2 + y^2) ds \\&= \int_0^{2\pi} [a^2(\cos t + t \sin t)^2 + a^2(\sin t - t \cos t)^2] a dt \\&= \int_0^{2\pi} a^3 t(1 + t^2) dt = 2\pi^2 a^3(1 + 2\pi^2).\end{aligned}$$

4224. $\int_c xy ds$, 其中 C 为双曲线 $x = a \cosh t, y = a \sinh t (0 \leq t \leq t_0)$ 的弧.

解 弧长的微分为

$$ds = \sqrt{a^2 \sinh^2 t + a^2 \cosh^2 t} dt = a \sqrt{\cosh 2t} dt.$$

于是,

$$\begin{aligned}\int_c xy ds &= a^3 \int_0^{t_0} \cosh t \sinh t \sqrt{\cosh 2t} dt \\&= \frac{a^3}{2} \int_0^{t_0} \sinh 2t \sqrt{\cosh 2t} dt \\&= \frac{a^3}{4} \int_0^{t_0} \sqrt{\cosh 2t} d(\cosh 2t) \\&= \frac{a^3}{6} (\sqrt{\cosh^3 2t_0} - 1).\end{aligned}$$

4225. $\int_c (x^{\frac{4}{3}} + y^{\frac{4}{3}}) ds$, 其中 C 为内摆线 $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ 的弧.

解 方法一:

按直角坐标方程计算, 弧长的微分为

$$ds = \sqrt{1 + y'^2} dx = \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}} dx.$$

于是,

$$\int_c (x^{\frac{4}{3}} + y^{\frac{4}{3}}) ds$$

$$\begin{aligned}
&= 4 \int_0^a \left[x^{\frac{4}{3}} + (a^{\frac{2}{3}} - x^{\frac{2}{3}})^2 \right] \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}} dx \\
&= 4a^{\frac{1}{3}} \int_0^a (2x + a^{\frac{4}{3}}x^{-\frac{1}{3}} - 2a^{\frac{2}{3}}x^{\frac{1}{3}}) dx = 4a^{\frac{7}{3}}.
\end{aligned}$$

方法二:

按参数方程计算. 若令 $x = a\cos^3 t, y = a\sin^3 t$,

则

$$\begin{aligned}
ds &= \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2\sin^4 t \cos^2 t} dt \\
&= 3a\cos t \sin t dt \left(0 \leq t \leq \frac{\pi}{2} \right).
\end{aligned}$$

于是,

$$\begin{aligned}
&\int_C (x^{\frac{4}{3}} + y^{\frac{4}{3}}) ds \\
&= 4a^{\frac{4}{3}} \int_0^{\frac{\pi}{2}} (\cos^4 t + \sin^4 t) \cdot 3a\cos t \sin t dt \\
&= 24a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} \sin^5 t d(\sin t) = 4a^{\frac{7}{3}}.
\end{aligned}$$

4226. $\int_C e^{\sqrt{x^2+y^2}} ds$, 其中 C 为由曲线 $r = a, \varphi = 0, \varphi = \frac{\pi}{4}$ (r 和 φ 为极坐标) 所界的凸围线.

解 凸围线由三段组成, 分别是: 直线段 $\varphi = 0 (0 \leq r \leq a)$; 圆弧段 $r = a (0 \leq \varphi \leq \frac{\pi}{4})$; 直线段 $\varphi = \frac{\pi}{4} (0 \leq r \leq a)$, 弧长的微分相应地是: $ds = dr; ds = \sqrt{r^2 + r'^2} d\varphi = ad\varphi; ds = dr$. 于是,

$$\begin{aligned}
\int_C e^{\sqrt{x^2+y^2}} ds &= \int_0^a e^r dr + \int_0^{\frac{\pi}{4}} e^a a d\varphi + \int_0^a e^r dr \\
&= 2(e^a - 1) + \frac{\pi a e^a}{4}.
\end{aligned}$$

4227. $\int_C |y| ds$, 其中 C 为双纽线 $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ 的弧.

解 双纽线的极坐标方程为 $r^2 = a^2 \cos 2\varphi$. 弧长的微分为

$$ds = \sqrt{r^2 + r'^2} d\varphi = \frac{a}{\sqrt{\cos 2\varphi}} d\varphi.$$

于是,

$$\begin{aligned} \int_C |y| ds &= 4 \int_0^{\frac{\pi}{4}} a \sqrt{\cos 2\varphi} \sin \varphi \cdot \frac{a}{\sqrt{\cos 2\varphi}} d\varphi \\ &= 4a^2 (-\cos \varphi) \Big|_0^{\frac{\pi}{4}} = 2a^2 (2 - \sqrt{2}). \end{aligned}$$

4228. $\int_C x ds$, 其中 C 为对数螺线 $r = ae^{k\varphi} (k > 0)$ 在圆 $r = a$ 内的部分.

解 弧长的微分为

$$ds = ae^{k\varphi} \sqrt{1 + k^2} d\varphi \quad (-\infty < \varphi < 0).$$

于是,

$$\begin{aligned} \int_C x ds &= \int_{-\infty}^0 ae^{k\varphi} \cos \varphi \cdot ae^{k\varphi} \sqrt{1 + k^2} d\varphi \\ &= a^2 \sqrt{1 + k^2} \frac{2k \cos \varphi + \sin \varphi}{1 + 4k^2} e^{2k\varphi} \Big|_{-\infty}^0 \\ &= \frac{2ka^2 \sqrt{1 + k^2}}{1 + 4k^2}. \end{aligned}$$

4229. $\int_C \sqrt{x^2 + y^2} ds$, 其中 C 为圆周 $x^2 + y^2 = ax$.

解 对于上半圆周, 弧长的微分为

$$ds = \sqrt{1 + \left(\frac{a - 2x}{2y} \right)^2} dx = \frac{a}{2y} dx$$

$$= \frac{a}{2\sqrt{ax-x^2}} dx (0 \leq x \leq a).$$

于是,

$$\begin{aligned} \int_C \sqrt{x^2+y^2} ds &= 2 \int_0^a \sqrt{ax} \cdot \frac{a}{2\sqrt{ax-x^2}} dx \\ &= a \sqrt{a} \int_0^a \frac{dx}{\sqrt{a-x}} = 2a^2. \end{aligned}$$

4230. $\int_C \frac{ds}{y^2}$, 其中 C 为悬链线 $y = a \operatorname{ch} \frac{x}{a}$.

解 弧长的微分为

$$ds = \sqrt{1 + \operatorname{sh}^2 \frac{x}{a}} dx = \operatorname{ch} \frac{x}{a} dx.$$

于是,

$$\begin{aligned} \int_C \frac{ds}{y^2} &= \int_{-\infty}^{+\infty} \frac{\operatorname{ch} \frac{x}{a}}{a^2 \operatorname{ch}^2 \frac{x}{a}} dx \\ &= \frac{1}{a} \int_{-\infty}^{+\infty} \frac{d\left(\operatorname{sh} \frac{x}{a}\right)}{1 + \operatorname{sh}^2 \frac{x}{a}} \\ &= \frac{1}{a} \operatorname{arctg} \left(\operatorname{sh} \frac{x}{a} \right) \Big|_{-\infty}^{+\infty} = \frac{\pi}{a}. \end{aligned}$$

求下列空间曲线的弧长 (参数是正的):

4231. $x = 3t, y = 3t^2, z = 2t^3$ 从 $O(0,0,0)$ 到 $A(3,3,2)$.

解 弧长的微分为

$$ds = \sqrt{x'^2 + y'^2 + z'^2} dt = 3(2t^2 + 1) dt.$$

于是, 弧长为

$$s = \int_0^1 3(2t^2 + 1) dt = 5.$$

4232. $x = e^{-t} \cos t, y = e^{-t} \sin t, z = e^{-t}$, 当 $0 < t < +\infty$.

解 弧长的微分为

$$\begin{aligned} ds &= \sqrt{e^{-2t}(\cos t - \sin t)^2 + e^{-2t}(\cos t + \sin t)^2 + e^{-2t}} dt \\ &= \sqrt{3} e^{-t} dt. \end{aligned}$$

于是, 弧长为

$$s = \sqrt{3} \int_0^{+\infty} e^{-t} dt = \sqrt{3}.$$

4233. $y = a \arcsin \frac{x}{a}, z = \frac{a}{4} \ln \frac{a-x}{a+x}$ 从 $O(0, 0, 0)$ 到 $A(x_0, y_0, z_0)$.

解 弧长的微分为

$$\begin{aligned} ds &= \sqrt{1 + \frac{a^2}{a^2 - x^2} + \frac{a^4}{4(a^2 - x^2)^2}} dx \\ &= \frac{3a^2 - 2x^2}{2(a^2 - x^2)} dx (|x_0| < a). \end{aligned}$$

于是, 当 $x_0 \geq 0$ 时, 有

$$\begin{aligned} s &= \int_0^{x_0} \frac{3a^2 - 2x^2}{2(a^2 - x^2)} dx \\ &= \frac{a}{4} \ln \frac{a+x_0}{a-x_0} + x_0 = |z_0| + |x_0|; \end{aligned}$$

当 $x_0 < 0$ 时, 有

$$\begin{aligned} s &= \int_{x_0}^0 \frac{3a^2 - 2x^2}{2(a^2 - x^2)} dx \\ &= -\frac{a}{4} \ln \frac{a+x_0}{a-x_0} - x_0 = |z_0| + |x_0|. \end{aligned}$$

总之, 当 $|x_0| < a$, 有 $s = |z_0| + |x_0|$.

4234. $(x-y)^2 = a(x+y), x^2 - y^2 = \frac{9}{8}z^2$ 从 $O(0, 0, 0)$ 到 $A(x_0, y_0, z_0)$.

解 由 $(x-y)^2 = a(x+y)$, $x^2 - y^2 = \frac{9}{8}x^2$ 可解得

$$x = \frac{1}{2} \left[\frac{1}{a} \sqrt[3]{\left(\frac{9a}{8}\right)^2} \sqrt[3]{x^4} + \sqrt[3]{\frac{9a}{8}} \sqrt[3]{x^2} \right],$$

$$y = \frac{1}{2} \left[\frac{1}{a} \sqrt[3]{\left(\frac{9a}{8}\right)^2} \sqrt[3]{x^4} - \sqrt[3]{\frac{9a}{8}} \sqrt[3]{x^2} \right].$$

由于

$$\begin{aligned} & \left(\frac{dx}{dz} \right)^2 + \left(\frac{dy}{dz} \right)^2 \\ &= \frac{8}{9a^2} \sqrt[3]{\left(\frac{9a}{8}\right)^4} \sqrt[3]{x^2} + \frac{2}{9} \sqrt[3]{\left(\frac{9a}{8}\right)^2} \sqrt[3]{x^{-2}} \\ &= \frac{\sqrt[3]{9a}}{2a} \sqrt[3]{x^2} + \frac{\sqrt[3]{3a^2}}{6} \sqrt[3]{x^{-2}}, \end{aligned}$$

故弧长为

$$\begin{aligned} s &= \int_0^{z_0} \sqrt{\frac{\sqrt[3]{9a}}{2a} \sqrt[3]{x^2} + \frac{\sqrt[3]{3a^2}}{6} \sqrt[3]{x^{-2}} + 1} dz \\ &= \int_0^{\sqrt[3]{z_0^2}} \sqrt{\frac{\sqrt[3]{9a}}{2a} t + \frac{\sqrt[3]{3a^2}}{6} \frac{1}{t} + 1} \\ &\quad \cdot \frac{3\sqrt{t}}{2} dt \\ &= \frac{3}{2} \int_0^{\sqrt[3]{z_0^2}} \sqrt{\frac{\sqrt[3]{9a}}{2a} t^2 + t + \frac{\sqrt[3]{3a^2}}{6}} dt \\ &= \frac{3}{2} \int_0^{\sqrt[3]{z_0^2}} \left[\frac{1}{\sqrt{2}} \sqrt{\frac{3}{a}} t + \frac{1}{\sqrt{2}} \sqrt{\frac{a}{3}} \right] dt \\ &= \frac{3}{4\sqrt{2}} \left[\sqrt{\frac{3z_0^4}{a}} + 2\sqrt{\frac{az_0^2}{3}} \right]. \end{aligned}$$

4235. $x^2 + y^2 = cz$, $\frac{y}{x} = \operatorname{tg} \frac{z}{c}$ 从 $O(0,0,0)$ 到 $A(x_0, y_0, z_0)$.

解 取曲线的参数方程为

$$x = \sqrt{cz} \cos \frac{z}{c}, y = \sqrt{cz} \sin \frac{z}{c}, z = z,$$

则弧长的微分为

$$\begin{aligned} ds &= \left[\left(\frac{\sqrt{c}}{2\sqrt{z}} \cos \frac{z}{c} - \sqrt{\frac{z}{c}} \sin \frac{z}{c} \right)^2 \right. \\ &\quad \left. + \left(\frac{\sqrt{c}}{2\sqrt{z}} \sin \frac{z}{c} + \sqrt{\frac{z}{c}} \cos \frac{z}{c} \right)^2 + 1 \right]^{\frac{1}{2}} dz \\ &= \sqrt{\frac{c}{4z} + \frac{z}{c} + 1} dz = \frac{2z+c}{\sqrt{4cz}} dz. \end{aligned}$$

于是,弧长为

$$\begin{aligned} s &= \int_0^{z_0} \frac{2z+c}{\sqrt{4cz}} dz = \int_0^{z_0} \sqrt{\frac{z}{c}} dz + \int_0^{z_0} \frac{\sqrt{c}}{2\sqrt{z}} dz \\ &= \sqrt{cz_0} \left(1 + \frac{2z_0}{3c} \right). \end{aligned}$$

4236. $x^2 + y^2 + z^2 = a^2$, $\sqrt{x^2 + y^2} \operatorname{ch} \left(\operatorname{arctg} \frac{y}{x} \right) = a$ 从点 $A(a, 0, 0)$ 到点 $B(x, y, z)$.

解 令 $x = \sqrt{a^2 - z^2} \cos \varphi$, $y = \sqrt{a^2 - z^2} \sin \varphi$, 不妨设 $z > 0$, 则有

$$\begin{aligned} z &= \sqrt{a^2 - (x^2 + y^2)} \\ &= \sqrt{a^2 \left(1 - \frac{1}{\operatorname{ch}^2 \varphi} \right)} = a \operatorname{th} \varphi. \end{aligned}$$

而 $\sqrt{a^2 - z^2} = \sqrt{a^2 (1 - \operatorname{th}^2 \varphi)} = \frac{a}{\operatorname{ch} \varphi}$, 故

$x = \frac{a \cos \varphi}{\operatorname{ch} \varphi}$, $y = \frac{a \sin \varphi}{\operatorname{ch} \varphi}$, $z = a \operatorname{th} \varphi$ 为曲线的参数方程, 弧长的微分为

$$\begin{aligned}
 ds &= \sqrt{\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 + \left(\frac{dz}{d\varphi}\right)^2} d\varphi \\
 &= a \sqrt{\frac{\operatorname{ch}^2 \varphi + \operatorname{sh}^2 \varphi + 1}{\operatorname{ch}^4 \varphi}} d\varphi \\
 &= \sqrt{2} a \frac{d\varphi}{\operatorname{ch} \varphi}.
 \end{aligned}$$

于是,弧长为

$$\begin{aligned}
 s &= \int_0^{\varphi} \sqrt{2} a \frac{d\varphi}{\operatorname{ch} \varphi} = \sqrt{2} a \int_0^{\varphi} \frac{2}{e^{\varphi} + e^{-\varphi}} d\varphi \\
 &= 2 \sqrt{2} a \int_0^{\varphi} \frac{1}{1 + (e^{\varphi})^2} d(e^{\varphi}) \\
 &= 2 \sqrt{2} a \operatorname{arctg} e^{\varphi} \Big|_0^{\varphi} \\
 &= 2 \sqrt{2} a \left(\operatorname{arctg} \frac{a+z}{\sqrt{a^2-z^2}} - \frac{\pi}{4} \right)^{*)} \\
 &= \sqrt{2} a \operatorname{arctg} \frac{z}{\sqrt{a^2-z^2}}.^{**})
 \end{aligned}$$

容易推证,当 $z < 0$ 时,弧长为

$$s = \sqrt{2} a \operatorname{arctg} \frac{-z}{\sqrt{a^2-z^2}}.$$

总之,最后得

$$s = \sqrt{2} a \operatorname{arctg} \frac{|z|}{\sqrt{a^2-z^2}}.$$

*) 由 $z = a \operatorname{th} \varphi$ 知:

$$z(e^{\varphi} + e^{-\varphi}) = a(e^{\varphi} - e^{-\varphi}),$$

$$z(e^{2\varphi} - 1) = a(e^{2\varphi} - 1),$$

从而

$$e^{2\varphi} = \frac{a+z}{a-z} \text{ 或 } e^{\varphi} = \frac{a+z}{\sqrt{a^2-z^2}}.$$

**) 由于

$$\operatorname{tg}\left(\operatorname{arctg} \frac{a+z}{\sqrt{a^2-z^2}} - \frac{\pi}{4}\right)$$

$$= \frac{a - \sqrt{a^2 - z^2}}{z},$$

$$\operatorname{tg} \frac{1}{2} \left(\operatorname{arctg} \frac{z}{\sqrt{a^2 - z^2}} \right)$$

$$= \frac{a - \sqrt{a^2 - z^2}}{z},$$

故在主值范围内有

$$\operatorname{arctg} \frac{a+z}{\sqrt{a^2-z^2}} - \frac{\pi}{4}$$

$$= \frac{1}{2} \operatorname{arctg} \frac{z}{\sqrt{a^2 - z^2}}.$$

计算沿空间曲线所取的第一型曲线积分:

4237. $\int_C (x^2 + y^2 + z^2) ds$, 其中 C 为螺线 $x = a \cos t, y = a \sin t, z = bt$ ($0 \leq t \leq 2\pi$) 的一段.

解 弧长的微分为

$$ds = \sqrt{a^2 + b^2} dt.$$

于是,

$$\begin{aligned} & \int_C (x^2 + y^2 + z^2) ds \\ &= \sqrt{a^2 + b^2} \int_0^{2\pi} (a^2 + b^2 t^2) dt \\ &= \frac{2\pi}{3} (3a^2 + 4\pi^2 b^2) \sqrt{a^2 + b^2}. \end{aligned}$$

4238. $\int_C x^2 ds$, 其中 C 为圆周 $x^2 + y^2 + z^2 = a^2, x + y + z = 0$.

解 方法一

作代换:

$$u = \frac{x+y}{\sqrt{2}}, v = \frac{x+y-2z}{\sqrt{6}},$$

$$w = \frac{x+y+z}{\sqrt{3}},$$

则圆周 C 化为

$$u^2 + v^2 + w^2 = a^2, w = 0.$$

于是,

$$\begin{aligned} \int_C x^2 ds &= \int_C \left(\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{6}} + \frac{w}{\sqrt{3}} \right)^2 ds \\ &= \int_C \left(\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{6}} \right)^2 ds \\ &= \frac{1}{6} \int_C (3u^2 + v^2) ds + \frac{1}{\sqrt{3}} \int_C uv ds \\ &= \frac{1}{6} \int_C a^2 ds + \frac{1}{3} \int_C u^2 ds + \frac{1}{\sqrt{3}} \int_C uv ds \\ &= \frac{1}{3} \pi a^3 + \frac{1}{3} \int_0^{2\pi} a^3 \cos^2 \varphi d\varphi \\ &\quad + \frac{1}{\sqrt{3}} \int_0^{2\pi} a^3 \cos \varphi \sin \varphi d\varphi \\ &= \frac{1}{3} \pi a^3 + \frac{1}{3} \pi a^3 = \frac{2}{3} \pi a^3. \end{aligned}$$

方法二

由对称性知:

$$\int_C x^2 ds = \int_C y^2 ds = \int_C z^2 ds.$$

于是,

$$\int_C x^2 ds = \frac{1}{3} \int_C (x^2 + y^2 + z^2) ds$$

$$= \frac{a^2}{3} \int_c ds = \frac{2\pi a^3}{3}.$$

4239. $\int_c z ds$, 其中 C 为圆锥螺线 $x = t \cos t, y = t \sin t, z = t (0 \leq t \leq t_0)$.

解 弧长的微分为

$$\begin{aligned} ds &= \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 1} dt \\ &= \sqrt{2 + t^2} dt. \end{aligned}$$

于是,

$$\int_c z ds = \int_0^{t_0} t \sqrt{2 + t^2} dt = \frac{1}{3} [(2 + t_0^2)^{\frac{3}{2}} - 2^{\frac{3}{2}}].$$

4240. $\int_c z ds$, 其中 C 为曲线 $x^2 + y^2 = z^2, y^2 = ax$ 上从点 $O(0, 0, 0)$ 到点 $A(a, a, a\sqrt{2})$ 的弧.

解 由曲线方程得

$$z = \sqrt{x^2 + y^2} = \sqrt{\frac{y^4}{a^2} + y^2} = \frac{y}{a} \sqrt{y^2 + a^2}.$$

从而, 曲线的参数方程可取为

$$x = \frac{y^2}{a}, y = y, z = \frac{y}{a} \sqrt{y^2 + a^2}.$$

弧长的微分为

$$\begin{aligned} ds &= \sqrt{\left(\frac{2y}{a}\right)^2 + 1 + \left(\frac{2y^2 + a^2}{a\sqrt{y^2 + a^2}}\right)^2} dy \\ &= \sqrt{\frac{8y^4 + 9a^2y^2 + 2a^4}{a^2(y^2 + a^2)}} dy. \end{aligned}$$

于是,

$$\int_c z ds$$

$$\begin{aligned}
&= \int_0^a \frac{y}{a} \sqrt{y^2 + a^2} \sqrt{\frac{8y^4 + 9a^2y^2 + 2a^4}{a^2(y^2 + a^2)}} dy \\
&= \frac{\sqrt{8}}{a^2} \int_0^a y \sqrt{y^4 + \frac{9}{8}a^2y^2 + \frac{1}{4}a^4} dy \\
&= \frac{\sqrt{2}}{a^2} \int_0^a \sqrt{\left(y^2 + \frac{9a^2}{16}\right)^2 - \frac{17a^4}{16^2}} \\
&\quad \cdot d\left(y^2 + \frac{9a^2}{16}\right) \\
&= \frac{\sqrt{2}}{a^2} \left[\frac{y^2 + \frac{9a^2}{16}}{2} \sqrt{y^4 + \frac{9}{8}a^2y^2 + \frac{1}{4}a^4} \right. \\
&\quad \left. - \frac{17a^4}{2 \cdot 16^2} \ln\left(y^2 + \frac{9a^2}{16} + \sqrt{y^4 + \frac{9}{8}a^2y^2 + \frac{1}{4}a^4}\right) \right] \Big|_0^a \\
&= \frac{\sqrt{2}}{a^2} \left[\left(\frac{25a^4}{64} \sqrt{\frac{19}{2}} \right. \right. \\
&\quad \left. \left. - \frac{17a^4}{2 \cdot 16^2} \ln \frac{25a^2 + 8\sqrt{\frac{19}{2}}a^2}{16} \right) \right. \\
&\quad \left. - \left(\frac{9a^4}{64} - \frac{17a^4}{2 \cdot 16^2} \ln \frac{17a^2}{16} \right) \right] \\
&= \frac{\sqrt{2}}{a^2} \frac{25a^4 \sqrt{38} - 18a^4}{128} \\
&\quad + \frac{\sqrt{2}}{a^2} \frac{17a^4}{2 \cdot 16^2} \ln \frac{\frac{17a^2}{16}}{\frac{25a^2 + 8\sqrt{\frac{19}{2}}a^2}{16}}
\end{aligned}$$

$$= \frac{a^2}{256 \sqrt{2}} \left[100 \sqrt{38} - 72 \right. \\ \left. - 17 \ln \frac{25 + 4 \sqrt{38}}{17} \right].$$

4241⁺. 设曲线 $x = a \cos t, y = b \sin t (0 \leq t \leq 2\pi)$ 在点 (x, y) 的线密度等于 $\rho = |y|$, 求其质量.

解 质量 $m = \int_C |y| ds$, 其中 C 为椭圆 $x = a \cos t, y = b \sin t (0 \leq t \leq 2\pi)$.

先设 $a > b$. 这时

$$ds = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt \\ = a \sqrt{1 - \epsilon^2 \cos^2 t} dt,$$

其中 $\epsilon = \frac{\sqrt{a^2 - b^2}}{a}$. 于是,

$$\begin{aligned} m &= \int_0^\pi a b \sin t \sqrt{1 - \epsilon^2 \cos^2 t} dt \\ &\quad + \int_\pi^{2\pi} a (-b \sin t) \sqrt{1 - \epsilon^2 \cos^2 t} dt \\ &= -ab \int_0^\pi \sqrt{1 - \epsilon^2 \cos^2 t} d(\cos t) \\ &\quad + ab \int_\pi^{2\pi} \sqrt{1 - \epsilon^2 \cos^2 t} d(\cos t) \\ &= ab \int_{-1}^1 \sqrt{1 - \epsilon^2 u^2} du + ab \int_{-1}^1 \sqrt{1 - \epsilon^2 u^2} du \\ &= 4ab \int_0^1 \sqrt{1 - \epsilon^2 u^2} du \\ &= \frac{4ab}{\epsilon} \left[\frac{1}{2} \epsilon u \sqrt{1 - \epsilon^2 u^2} + \frac{1}{2} \arcsin(\epsilon u) \right] \Big|_{u=0}^{u=1} \\ &= 2b^2 + 2ab \frac{\arcsin \epsilon}{\epsilon}. \end{aligned}$$

次设 $a < b$. 这时

$$\begin{aligned} ds &= \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt \\ &= a \sqrt{1 + \epsilon_1^2 \cos^2 t} dt, \end{aligned}$$

其中 $\epsilon_1 = \frac{\sqrt{b^2 - a^2}}{a}$. 仿前, 有

$$\begin{aligned} m &= \int_0^\pi a b \sin t \sqrt{1 + \epsilon_1^2 \cos^2 t} dt \\ &\quad + \int_\pi^{2\pi} a (-b \sin t) \sqrt{1 + \epsilon_1^2 \cos^2 t} dt \\ &= 4ab \int_0^1 \sqrt{1 + \epsilon_1^2 u^2} du \\ &= \frac{4ab}{\epsilon_1} \left[\frac{1}{2} \epsilon_1 u \sqrt{1 + \epsilon_1^2 u^2} \right. \\ &\quad \left. + \frac{1}{2} \ln(\epsilon_1 u + \sqrt{1 + \epsilon_1^2 u^2}) \right] \Big|_{u=0}^{u=1} \\ &= 2b^2 + 2ab \frac{\ln(\epsilon_1 + \sqrt{1 + \epsilon_1^2})}{\epsilon_1}. \end{aligned}$$

最后, 若 $a = b$, 则椭圆退化成圆, 这时 $ds = a dt$, 故

$$m = \int_0^\pi a^2 \sin t dt + \int_\pi^{2\pi} (-a \sin t) a dt = 4a^2$$

综上所述, 可知

$$m = \begin{cases} 2b^2 + 2ab \frac{\operatorname{arcsin} \epsilon}{\epsilon}, & \text{若 } a > b; \\ 2b^2 + 2ab \frac{\ln(\epsilon_1 + \sqrt{1 + \epsilon_1^2})}{\epsilon_1}, & \text{若 } a < b; \\ 4a^2, & \text{若 } a = b, \end{cases}$$

其中 $\epsilon = \frac{\sqrt{a^2 - b^2}}{a} (a > b)$,

$$\epsilon_1 = \frac{\sqrt{b^2 - a^2}}{a} (a < b).$$

4242. 求曲线 $x = at, y = \frac{a}{2}t^2, z = \frac{a}{3}t^3 (0 \leq t \leq 1)$ 的弧之质

量, 其密度依规律 $\rho = \sqrt{\frac{2y}{a}}$ 而变化.

解 弧长的微分为

$$\begin{aligned} ds &= \sqrt{a^2 + a^2 t^2 + a^2 t^4} dt \\ &= a \sqrt{1 + t^2 + t^4} dt, \end{aligned}$$

而密度 $\rho = \sqrt{\frac{2y}{a}} = t$. 于是, 质量为 (作代换 $u = t^2$)

$$\begin{aligned} m &= \int_0^1 \sqrt{\frac{2y}{a}} ds = a \int_0^1 t \sqrt{1 + t^2 + t^4} dt \\ &= \frac{a}{2} \int_0^1 \sqrt{1 + u + u^2} du \\ &= \frac{a}{2} \left[\frac{u + \frac{1}{2}}{2} \sqrt{1 + u + u^2} \right. \\ &\quad \left. + \frac{3}{8} \ln \left(u + \frac{1}{2} + \sqrt{1 + u + u^2} \right) \right] \Big|_0^1 \\ &= \frac{a}{8} \left[(3\sqrt{3} - 1) + \frac{3}{2} \ln \frac{3 + 2\sqrt{3}}{3} \right]. \end{aligned}$$

4243. 计算均匀的曲线 $y = a \operatorname{ch} \frac{x}{a}$ 从点 $A(0, a)$ 到点 $B(b, h)$ 的弧的重心的坐标.

解 弧长的微分为

$$ds = \sqrt{1 + \operatorname{sh}^2 \frac{x}{a}} dx = \operatorname{ch} \frac{x}{a} dx.$$

质量为

$$m = \rho_0 \int_0^b \operatorname{ch} \frac{x}{a} dx = a \rho_0 \operatorname{sh} \frac{b}{a} = \rho_0 \sqrt{h^2 - a^2}. *$$

于是,重心的坐标为

$$\begin{aligned} x_0 &= \frac{\rho_0}{m} \int_0^b x \operatorname{ch} \frac{x}{a} dx \\ &= \frac{\rho_0}{m} \left[ab \operatorname{sh} \frac{b}{a} - a^2 \left(\operatorname{ch} \frac{b}{a} - 1 \right) \right] \\ &= \frac{1}{\sqrt{h^2 - a^2}} \left[b \sqrt{h^2 - a^2} - a^2 \left(\frac{h}{a} - 1 \right) \right] \\ &= b - a \sqrt{\frac{h-a}{h+a}}; \end{aligned}$$

$$\begin{aligned} y_0 &= \frac{\rho_0}{m} \int_0^b y \operatorname{ch} \frac{x}{a} dx = \frac{a \rho_0}{m} \int_0^b \operatorname{ch}^2 \frac{x}{a} dx \\ &= \frac{a \rho_0}{m} \int_0^b \frac{1 + \operatorname{ch} 2x}{2} dx \\ &= \frac{a \rho_0}{m} \left[\frac{x}{2} + \frac{a}{4} \operatorname{sh} \frac{2x}{a} \right] \Big|_0^b \\ &= \frac{a \rho_0}{m} \left(\frac{b}{2} + \frac{a}{4} \operatorname{sh} \frac{2b}{a} \right) \\ &= \frac{a}{\sqrt{h^2 - a^2}} \left(\frac{b}{2} + \frac{h}{2} \frac{\sqrt{h^2 - a^2}}{a} \right) \\ &= \frac{h}{2} + \frac{ab}{2 \sqrt{h^2 - a^2}}. \end{aligned}$$

*) 由 $h = a \operatorname{ch} \frac{b}{a}$ 知: $\operatorname{ch} \frac{b}{a} = \frac{h}{a}$. 从而

$$\operatorname{sh} \frac{b}{a} = \sqrt{\operatorname{ch}^2 \frac{b}{a} - 1} = \frac{\sqrt{h^2 - a^2}}{a}.$$

4244. 求摆线

$x = a(t - \sin t)$, $y = a(1 - \cos t)$ ($0 \leq t \leq \pi$) 的弧的重心.

解 弧长的微分为

$$\begin{aligned} ds &= \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} dt \\ &= 2a \sin \frac{t}{2} dt. \end{aligned}$$

质量为

$$m = 2a\rho_0 \int_0^\pi \sin \frac{t}{2} dt = 4a\rho_0.$$

于是,重心的坐标为

$$\begin{aligned} x_0 &= \frac{1}{m} \int_0^\pi \rho_0 a (t - \sin t) \cdot 2a \sin \frac{t}{2} dt \\ &= \frac{a}{2} \int_0^\pi t \sin \frac{t}{2} dt - \frac{a}{2} \int_0^\pi \sin t \sin \frac{t}{2} dt \\ &= -at \cos \frac{t}{2} \Big|_0^\pi + a \int_0^\pi \cos \frac{t}{2} dt \\ &\quad + \frac{a}{4} \int_0^\pi \left(\cos \frac{3t}{2} - \cos \frac{t}{2} \right) dt \\ &= \frac{4a}{3}. \end{aligned}$$

$$\begin{aligned} y_0 &= \frac{1}{m} \int_0^\pi \rho_0 a (1 - \cos t) \cdot 2a \sin \frac{t}{2} dt \\ &= \frac{a}{2} \int_0^\pi \sin \frac{t}{2} dt - \frac{a}{4} \int_0^\pi \left(\sin \frac{3t}{2} - \sin \frac{t}{2} \right) dt \\ &= \frac{4a}{3}. \end{aligned}$$

4245. 计算球面上的三角形 $x^2 + y^2 + z^2 = a^2; x > 0, y > 0, z > 0$ 的围线的重心的坐标.

解 作球坐标变换:

$$\begin{aligned} x &= r \cos \varphi \cos \psi, y = r \sin \varphi \cos \psi, \\ z &= r \sin \psi, \end{aligned}$$

则球面上的三角形三条曲边的方程分别是:

$$x = a \cos \varphi, y = a \sin \varphi, z = 0; 0 \leq \varphi \leq \frac{\pi}{2};$$

$$x = a \cos \psi, y = 0, z = a \sin \psi; 0 \leq \psi \leq \frac{\pi}{2};$$

$$x = 0, y = a \cos \phi, z = a \sin \phi, 0 \leq \phi \leq \frac{\pi}{2}.$$

又因围线的周长为

$$s = 3 \cdot \frac{\pi a}{2} = \frac{3\pi a}{2}.$$

于是,重心的坐标为

$$\begin{aligned} x_0 &= \frac{\int_0^{\frac{\pi}{2}} a \cos \varphi \cdot a d\varphi + \int_0^{\frac{\pi}{2}} a \cos \psi \cdot a d\psi}{\frac{3\pi a}{2}} \\ &= \frac{2a^2}{\frac{3\pi a}{2}} = \frac{4a}{3\pi}. \end{aligned}$$

利用对称性知: $x_0 = y_0 = z_0 = \frac{4a}{3\pi}$.

4246. 求均匀的弧 $x = e^t \cos t, y = e^t \sin t, z = e^t (-\infty < t \leq 0)$ 的重心的坐标.

解 弧长的微分为

$$\begin{aligned} ds &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + e^{2t}} dt \\ &= \sqrt{3} e^t dt. \end{aligned}$$

质量为

$$m = \int_{-\infty}^0 \sqrt{3} e^t dt = \sqrt{3}.$$

于是,重心的坐标为

$$\begin{aligned} x_0 &= \frac{1}{m} \int_{-\infty}^0 e^t \cos t \cdot \sqrt{3} e^t dt = \int_{-\infty}^0 e^{2t} \cos t dt \\ &= \left. \frac{2 \cos t + \sin t}{5} e^{2t} \right|_{-\infty}^0 = \frac{2}{5}. \end{aligned}$$

$$\begin{aligned}
 y_3 &= \frac{1}{m} \int_{-\infty}^0 e^t \sin t \cdot \sqrt{3} e^t dt = \int_{-\infty}^0 e^{2t} \sin t dt \\
 &= \left. \frac{2 \sin t - \cos t}{5} e^{2t} \right|_{-\infty}^0 = -\frac{1}{5}. \\
 z_0 &= \frac{1}{m} \int_{-\infty}^0 e^t \cdot \sqrt{3} e^t dt = \int_{-\infty}^0 e^{2t} dt = \frac{1}{2}.
 \end{aligned}$$

4247. 求螺线 $x = a \cos t, y = a \sin t, z = \frac{h}{2\pi} t$ ($0 \leq t \leq 2\pi$) 的一枝对于坐标轴的转动惯量.

解 弧长的微分为

$$\begin{aligned}
 ds &= \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + \frac{h^2}{4\pi^2} dt} \\
 &= \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} dt.
 \end{aligned}$$

于是, 转动惯量为

$$\begin{aligned}
 I_x &= \int_0^{2\pi} (y^2 + z^2) ds \\
 &= \int_0^{2\pi} \left(a^2 \sin^2 t + \frac{h^2}{4\pi^2} t^2 \right) \frac{\sqrt{4\pi^2 a^2 + h^2}}{2\pi} dt \\
 &= \frac{a^2}{2\pi} \sqrt{4\pi^2 a^2 + h^2} \cdot \pi \\
 &\quad + \frac{h^2}{4\pi^2} \cdot \frac{1}{2\pi} \sqrt{4\pi^2 a^2 + h^2} \cdot \frac{1}{3} (2\pi)^3 \\
 &= \left(\frac{a^2}{2} + \frac{h^2}{3} \right) \sqrt{4\pi^2 a^2 + h^2}.
 \end{aligned}$$

$$\begin{aligned}
 I_y &= \int_0^{2\pi} (x^2 + z^2) ds \\
 &= \int_0^{2\pi} \left(a^2 \cos^2 t + \frac{h^2}{4\pi^2} t^2 \right) \\
 &\quad \cdot \frac{1}{2\pi} \sqrt{4\pi^2 a^2 + h^2} dt
 \end{aligned}$$

$$\begin{aligned}
&= \frac{a^2}{2\pi} \sqrt{4\pi^2 a^2 + h^2} \cdot \pi \\
&\quad + \frac{h^2}{4\pi^2} \cdot \frac{1}{2\pi} \sqrt{4\pi^2 a^2 + h^2} \cdot \frac{1}{3} (2\pi)^3 \\
&= \left(\frac{a^2}{2} + \frac{h^2}{3} \right) \sqrt{4\pi^2 a^2 + h^2}. \\
I_2 &= \int_C (x^2 + y^2) ds \\
&= \int_0^{2\pi} a^2 \cdot \frac{1}{2\pi} \sqrt{4\pi^2 a^2 + h^2} dt \\
&= a^2 \sqrt{4\pi^2 a^2 + h^2}.
\end{aligned}$$

4248. 计算第二型的曲线积分

$$\int_{OA} xdy - ydx,$$

式中 O 为坐标原点, A 点的坐标为 $(1, 2)$ 并设: (a) OA 为直线段; (b) OA 为抛物线, 其轴为 Oy ; (c) OA 为由 Ox 轴上的线段 OB 和平行于 Oy 轴的线段 BA 所组成的折线.

解 (a) 直线段的方程为 $y = 2x$. 于是,

$$\int_{OA} xdy - ydx = \int_0^1 (2x - 2x)dx = 0.$$

(b) 抛物线的方程为 $y = 2x^2$. 于是,

$$\int_{OA} xdy - ydx = \int_0^1 (4x^2 - 2x^2)dx = \frac{2}{3}.$$

(c) 线段 OB 的方程为 $y = 0$, BA 的方程为 $x = 1$. 于是,

$$\int_{OA} xdy - ydx = \int_0^1 0 \cdot dx + \int_0^2 dy = 2.$$

4249. 对于上题中所指示的路径(a), (b), (c), 计算

$$\int_{OA} xdy + ydx.$$

$$\text{解} \quad (\text{a}) \int_{OA} xdy + ydx = \int_0^1 (2x + 2x)dx = 2.$$

$$(\text{b}) \int_{OA} xdy + ydx = \int_0^1 (4x^2 + 2x^2)dx = 2.$$

$$(\text{B}) \int_{OA} xdy + ydx = \int_0^2 dy = 2.$$

在参数增加的方向,沿所指示的曲线来计算下列第二型曲线积分:

$$4250. \int_C (x^2 - 2xy)dx + (y^2 - 2xy)dy, \text{其中 } C \text{ 为抛物线 } y = x^2 (-1 \leq x \leq 1).$$

解 由题设 $y = x^2$, 从而 $dy = 2xdx$. 于是,

$$\begin{aligned} & \int_C (x^2 - 2xy)dx + (y^2 - 2xy)dy \\ &= \int_{-1}^1 [(x^2 - 2x^3) + 2x(x^4 - 2x^3)]dx \\ &= -\frac{14}{15}. \end{aligned}$$

$$4251. \int_C (x^2 + y^2)dx + (x^2 - y^2)dy, \text{其中 } C \text{ 为曲线} \\ y = 1 - |1 - x| (0 \leq x \leq 2).$$

解 当 $0 \leq x \leq 1$ 时, $y = 1 - (1 - x) = x$, 从而 $dy = dx$; 当 $1 \leq x \leq 2$ 时, $y = 1 - (x - 1) = 2 - x$, 从而 $dy = -dx$. 于是,

$$\begin{aligned} & \int_C (x^2 + y^2)dx + (x^2 - y^2)dy \\ &= \int_0^1 2x^2 dx + \int_1^2 [x^2 + (2 - x)^2 - x^2 + (2 - x)^2]dx \\ &= \frac{4}{3}. \end{aligned}$$

4252. $\oint_C (x+y)dx + (x-y)dy$, 其中 C 为依反时针方向通过的椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

解 利用椭圆的参数方程

$$x = acost, y = asint (0 \leq t \leq 2\pi),$$

则有

$$\begin{aligned} & \oint_C (x+y)dx + (x-y)dy \\ &= \int_0^{2\pi} [(acost + bsint)(-asint) \\ & \quad + (acost - bsint)bcost]dt \\ &= \int_0^{2\pi} \left(ab\cos 2t - \frac{a^2 + b^2}{2} \sin 2t \right) dt = 0. \end{aligned}$$

4253. $\int_C (2a-y)dx + xdy$, 其中 C 为摆线 $x = a(t - \sin t), y = a(1 - \cos t) (0 \leq t \leq 2\pi)$ 的一拱.

解 由题设知: $dx = a(1 - \cos t)dt, dy = asintdt$.

于是,

$$\begin{aligned} & \int_C (2a-y)dx + xdy \\ &= \int_0^{2\pi} \{ [2a - a(1 - \cos t)]a(1 - \cos t) \\ & \quad + a(t - \sin t)asint \} dt \\ &= \int_0^{2\pi} a^2 t \sin t dt \\ &= -a^2 (t \cos t - \sin t) \Big|_0^{2\pi} = -2\pi a^2. \end{aligned}$$

4254. $\oint_C \frac{(x+y)dx - (x-y)dy}{x^2 + y^2}$, 其中 C 为依反时针方向通过的圆周 $x^2 + y^2 = a^2$.

解 利用圆的参数方程

$$x = a \cos t, y = a \sin t (0 \leq t \leq 2\pi).$$

则有

$$\begin{aligned} & \oint \frac{(x+y)dx - (x-y)dy}{x^2 + y^2} \\ &= \int_0^{2\pi} \frac{(a \cos t + a \sin t)a \sin t - (a \cos t - a \sin t)a \cos t}{a^2} dt \\ &= - \int_0^{2\pi} dt = -2\pi. \end{aligned}$$

4255. $\oint_{ABCD} \frac{dx + dy}{|x| + |y|}$, 其中 $ABCD$ 为以 $A(1,0), B(0,1), C(-1,0), D(0,-1)$ 为顶点的正方形的围线.

解 正方形各边的方程分别为

$$\begin{aligned} AB: y &= 1 - x; & BC: y &= 1 + x; \\ CD: y &= -1 - x; & DA: y &= -1 + x. \end{aligned}$$

于是,

$$\begin{aligned} & \oint \frac{dx + dy}{|x| + |y|} \\ &= \int_{AB} \frac{dx + dy}{x + y} + \int_{BC} \frac{dx + dy}{-x + y} \\ & \quad + \int_{CD} \frac{dx + dy}{-x - y} + \int_{DA} \frac{dx + dy}{x - y} \\ &= \int_1^0 (1 - 1)dx + \int_0^{-1} 2dx \\ & \quad + \int_{-1}^0 (1 - 1)dx + \int_0^1 2dx \\ &= 0. \end{aligned}$$

4256. $\int_{AB} \sin y dx + \sin x dy$, 其中 AB 为界于点 $A(0, \pi)$ 和点 $B(\pi, 0)$ 之间的直线段.

解 AB 的方程为 $y = \pi - x$. 于是,

$$\begin{aligned}
& \int_{AB} \sin y dx + \sin x dy \\
&= \int_0^\pi \sin(\pi - x) dx + \sin x dx \\
&= \int_0^\pi (\sin x + \sin x) dx = 0.
\end{aligned}$$

注：原题为 $\int_{AB} dx \sin y - dy \sin x$ ，若把它理解为

$\int_{AB} d(x \sin y) + d(y \sin x)$ ，其值仍为零，与原答案也符合。

4257. $\oint_{OmAnO} \operatorname{arctg} \frac{y}{x} dy - dx,$

其中 OmA 为抛物线段 $y = x^2$, OnA 为直线段 $y = x$.

解 如图 8.62 所示，我们有

$$\begin{aligned}
& \oint_{OmAnO} \operatorname{arctg} \frac{y}{x} dy - dx \\
&= \int_{OmA} \operatorname{arctg} \frac{y}{x} dy - dx \\
&+ \int_{AnO} \operatorname{arctg} \frac{y}{x} dy - dx \\
&= \int_0^1 2x \operatorname{arctg} x dx - \int_0^1 dx \\
&\quad + \int_1^0 (\operatorname{arctg} 1 - 1) dx \\
&= x^2 \operatorname{arctg} x \Big|_0^1 - \int_0^1 \frac{x^2}{1+x^2} dx
\end{aligned}$$

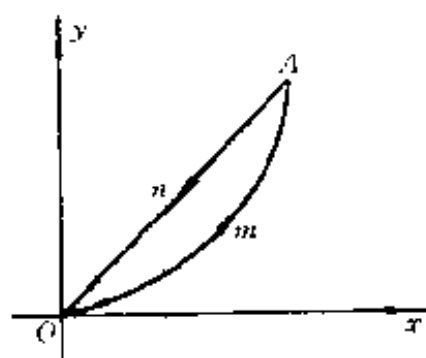


图 8.62

$$\begin{aligned}
&= 1 + \left(\frac{\pi}{4} - 1 \right) x \Big|_1^0 \\
&= \frac{\pi}{4} - (x - \arctg x) \Big|_0^1 = 1 - \left(\frac{\pi}{4} - 1 \right) \\
&= \frac{\pi}{4} - 1.
\end{aligned}$$

注：原题为 $\oint_{OmAnO} d\text{yarctg } \frac{y}{x} - dx$ ，若把它理解为

$\oint_{OmAnO} d\left(\text{yarctg } \frac{y}{x}\right) - dx$ ，则其值为零，与原答案不符。

验证被积函数为全微分，并计算下列曲线积分：

$$4258. \int_{(-1,2)}^{(2,3)} xdy + ydx.$$

解 显然， $xdy + ydx = d(xy)$ 是全微分。于是，

$$\begin{aligned}
&\int_{(-1,2)}^{(2,3)} xdy + ydx \\
&= \int_{(-1,2)}^{(2,3)} d(xy) = xy \Big|_{(-1,2)}^{(2,3)} = 8.
\end{aligned}$$

$$4259. \int_{(0,1)}^{(3,-4)} xdx + ydy.$$

解 显然， $xdx + ydy = d\left(\frac{x^2 + y^2}{2}\right)$ 是全微分。

于是，

$$\begin{aligned}
&\int_{(0,1)}^{(3,-4)} xdx + ydy \\
&= \int_{(0,1)}^{(3,-4)} d\left(\frac{x^2 + y^2}{2}\right) \\
&= \frac{x^2 + y^2}{2} \Big|_{(0,1)}^{(3,-4)} = 12.
\end{aligned}$$

$$4260. \int_{(0,1)}^{(2,3)} (x+y)dx + (x-y)dy.$$

解 显然，我们有

$$\begin{aligned}
& (x+y)dx + (x-y)dy \\
&= (ydx + xdy) + (xdx - ydy) \\
&= d(xy) + d\left(\frac{x^2 - y^2}{2}\right) \\
&= d\left(xy + \frac{x^2 - y^2}{2}\right),
\end{aligned}$$

即是全微分, 于是,

$$\begin{aligned}
& \int_{(0,1)}^{(2,3)} (x+y)dx + (x-y)dy \\
&= \int_{(0,1)}^{(2,3)} d\left(xy + \frac{x^2 - y^2}{2}\right) \\
&= \left(xy + \frac{x^2 - y^2}{2}\right) \Big|_{(0,1)}^{(2,3)} = 4.
\end{aligned}$$

4261. $\int_{(1,-1)}^{(1,1)} (x-y)(dx-dy).$

解 显然, $(x-y)(dx-dy) = d\frac{(x-y)^2}{2}$ 是全微分, 于是,

$$\begin{aligned}
& \int_{(1,-1)}^{(1,1)} (x-y)(dx-dy) \\
&= \int_{(1,-1)}^{(1,1)} d\frac{(x-y)^2}{2} \\
&= \frac{(x-y)^2}{2} \Big|_{(1,-1)}^{(1,1)} = -2.
\end{aligned}$$

4262. $\int_{(0,0)}^{(a,b)} f(x+y)(dx+dy)$, 其中 $f(u)$ 为连续函数.

解 令 $F(x,y) = \int_0^{x+y} f(u)du$. 由于 $f(u)$ 连续, 故

$$F'_x(x,y) = f(x+y), F'_y(x,y) = f(x+y),$$

并且它们都是 x, y 的连续函数. 因此, $F(x,y)$ 可微, 且

$$dF(x,y) = F'_x(x,y)dx + F'_y(x,y)dy$$

$$= f(x+y)(dx+dy),$$

故 $f(x+y)(dx+dy)$ 是全微分, 并且

$$\begin{aligned} & \int_{(0,0)}^{(a,b)} f(x,y)(dx+dy) \\ &= F(a,b) - F(0,0) = \int_c^{a+c} f(u)du. \end{aligned}$$

4263. $\int_{(2,1)}^{(1,2)} \frac{ydx - xdy}{x^2}$ 沿着不与 Oy 轴相交的路径.

解 显然, 当 $x \neq 0$ 时,

$$\frac{ydx - xdy}{x^2} = d\left(-\frac{y}{x}\right)$$

是全微分. 于是,

$$\begin{aligned} & \int_{(2,1)}^{(1,2)} \frac{ydx - xdy}{x^2} \\ &= \int_{(2,1)}^{(1,2)} d\left(-\frac{y}{x}\right) = -\frac{y}{x} \Big|_{(2,1)}^{(1,2)} = -\frac{3}{2}. \end{aligned}$$

4264. $\int_{(1,0)}^{(6,8)} \frac{xdx + ydy}{\sqrt{x^2 + y^2}}$ 沿着不通过坐标原点的路径.

解 显然, 当 $(x,y) \neq (0,0)$ 时,

$$\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = d(\sqrt{x^2 + y^2})$$

是全微分. 于是,

$$\begin{aligned} & \int_{(1,0)}^{(6,8)} \frac{xdx + ydy}{\sqrt{x^2 + y^2}} \\ &= \int_{(1,0)}^{(6,8)} d(\sqrt{x^2 + y^2}) \\ &= \sqrt{x^2 + y^2} \Big|_{(1,0)}^{(6,8)} = 9. \end{aligned}$$

4265. $\int_{(x_1, y_1)}^{(x_2, y_2)} \varphi(x)dx + \psi(y)dy$, 其中 φ 和 ψ 为连续函数.

解 由于 φ, ψ 是连续函数, 故显然有

$$\begin{aligned} & \varphi(x)dx + \psi(y)dy \\ &= dF(x) + dG(y) = d[F(x) + G(y)], \end{aligned}$$

其中 $F(x) = \int_{x_1}^x \varphi(u)du, G(y) = \int_{y_1}^y \psi(v)dv$. 于是, $\varphi(x)dx + \psi(y)dy$ 是函数 $F(x) + G(y)$ 的全微分, 从而有

$$\begin{aligned} & \int_{(x_1, y_1)}^{(x_2, y_2)} \varphi(x)dx + \psi(y)dy \\ &= [F(x) + G(y)] \Big|_{(x_1, y_1)}^{(x_2, y_2)} \\ &= [F(x_2) + G(y_2)] - [F(x_1) + G(y_1)] \\ &= \int_{x_1}^{x_2} \varphi(u)du + \int_{y_1}^{y_2} \psi(v)dv. \end{aligned}$$

$$4266. \int_{(-2, -1)}^{(3, 0)} (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy.$$

解 $P = x^4 + 4xy^3, Q = 6x^2y^2 - 5y^4$.

显然, P, Q 在全平面上具有连续偏导数, 并且

$$\frac{\partial Q}{\partial x} = 12xy^2, \frac{\partial P}{\partial y} = 12xy^2.$$

故 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. 由于全平面是单连通区域, 故在整个平面上表达式 $Pdx + Qdy$ 是某函数 $u(x, y)$ 的全微分, 并且线积分 $\int Pdx + Qdy$ 与路径无关, 因而可按平行于坐标轴的直线段来计算所给积分, 得

$$\begin{aligned} & \int_{(-2, -1)}^{(3, 0)} (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy \\ &= \int_{-2}^3 (x^4 + 4x \cdot 0^3)dx + \int_{-1}^0 [6(-2)^2y^2 - 5y^4]dy \\ &= 55 + 7 = 62. \end{aligned}$$

注:也可利用简单的技巧求出函数 $u(x, y)$ 来, 我们有

$$\begin{aligned} & (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy \\ &= d\left(\frac{x^5}{5}\right) + 2y^3d(x^2) + 2x^2d(y^3) - d(y^5) \\ &= d\left(\frac{x^5}{5} + 2x^2y^3 - y^5\right), \end{aligned}$$

从而

$$\begin{aligned} & \int_{(-2, -1)}^{(3, 0)} (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy \\ &= \left(\frac{x^5}{5} + 2x^2y^3 - y^5\right) \Big|_{(-2, -1)}^{(3, 0)} = 62. \end{aligned}$$

4267. $\int_{(0, -1)}^{(1, 0)} \frac{xdy - ydx}{(x - y)^2}$ 沿着不与直线 $y = x$ 相交的路径.

解 $P = -\frac{y}{(x - y)^2}, Q = \frac{x}{(x - y)^2} (x \neq y).$

容易验证

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = -\frac{x + y}{(x - y)^3} \quad (x \neq y).$$

考虑平面上的区域 $\Omega = \{(x, y) | x > y\}$. 由于 Ω 是单连通区域且在其上 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, 故在 Ω 上, $Pdx + Qdy$ 是某

函数 $u = u(x, y)$ 的全微分, 从而在 Ω 上线积分 $\int_c Pdx + Qdy$ 与路径无关. 因此, 可按平行于坐标轴的直线段来计算所给积分, 得

$$\begin{aligned} & \int_{(0, -1)}^{(1, 0)} \frac{xdy - ydx}{(x - y)^2} \\ &= \int_0^1 \frac{-(-1)dx}{(x + 1)^2} + \int_{-1}^0 \frac{dy}{(1 - y)^2} \\ &= \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

注:也可利用简单的技巧求出函数 $u(x, y)$ 来. 我们有

$$\begin{aligned}\frac{xdy - ydx}{(x - y)^2} &= \frac{(x - y)dy - yd(x - y)}{(x - y)^2} \\ &= d\left(\frac{y}{x - y}\right),\end{aligned}$$

从而

$$\int_{(0, -1)}^{(1, 0)} \frac{xdy - ydx}{(x - y)^2} = \frac{y}{x - y} \Big|_{(0, -1)}^{(1, 0)} = 1.$$

4268. $\int_{(1, \pi)}^{(2, \pi)} \left(1 - \frac{y^2}{x^2} \cos \frac{y}{x}\right) dx + \left(\sin \frac{y}{x} + \frac{y}{x} \cos \frac{y}{x}\right) dy$ 沿着不与 Oy 轴相交的路径.

解 当 $x \neq 0$ 时, 有

$$P = 1 - \frac{y^2}{x^2} \cos \frac{y}{x}, Q = \sin \frac{y}{x} + \frac{y}{x} \cos \frac{y}{x},$$

$$\frac{\partial P}{\partial y} = -\frac{2y}{x^2} \cos \frac{y}{x} + \frac{y^2}{x^3} \sin \frac{y}{x},$$

$$\frac{\partial Q}{\partial x} = -\frac{y}{x^2} \cos \frac{y}{x} - \frac{y}{x^2} \cos \frac{y}{x} + \frac{y^2}{x^3} \sin \frac{y}{x}$$

$$= -\frac{2y}{x^2} \cos \frac{y}{x} + \frac{y^2}{x^3} \sin \frac{y}{x}.$$

考虑右半平面 $\Omega = \{(x, y) | x > 0\}$. 由于 Ω 是单连通区域, 且在其上 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, 故在 Ω 上必是某函数 $u(x, y)$ 的全微分, 且可取

$$\begin{aligned}u(x, y) &= \int_1^x \left(1 - \frac{y^2}{x^2} \cos \frac{y}{x}\right) dx \\ &\quad + \int_\pi^y (\sin y + y \cos y) dy \\ &= \left(x + y \sin \frac{y}{x}\right) \Big|_1^x + y \sin y \Big|_\pi^y\end{aligned}$$

$$= x - 1 + y \sin \frac{y}{x}.$$

于是,

$$\begin{aligned} & \int_{(1,0)}^{(2,\pi)} \left(1 - \frac{y^2}{x^3} \cos \frac{y}{x} \right) dx \\ & + \left[\sin \frac{y}{x} + \frac{y}{x} \cos \frac{y}{x} \right] dy \\ & = \left[x - 1 + y \sin \frac{y}{x} \right] \Big|_{(1,0)}^{(2,\pi)} = \pi + 1. \end{aligned}$$

4269. $\int_{(0,0)}^{(a,b)} e^x (\cos y dx - \sin y dy).$

解 显然,有

$$e^x (\cos y dx - \sin y dy) = d(e^x \cos y),$$

于是,

$$\begin{aligned} & \int_{(0,0)}^{(a,b)} e^x (\cos y dx - \sin y dy) \\ & = \int_{(0,0)}^{(a,b)} d(e^x \cos y) = (e^x \cos y) \Big|_{(0,0)}^{(a,b)} \\ & = e^a \cos b - 1. \end{aligned}$$

4270. 证明:若 $f(u)$ 为连续函数且 C 为逐段光滑的封闭围线, 则

$$\oint_C f(x^2 + y^2)(x dx + y dy) = 0.$$

证 令 $F(x, y) = \frac{1}{2} \int_0^{x^2+y^2} f(u) du$. 由于 $f(u)$ 是连续函数, 故

$$F'_x(x, y) = x f(x^2 + y^2),$$

$$F'_y(x, y) = y f(x^2 + y^2),$$

并且显然 $F'_x(x, y), F'_y(x, y)$ 都是 x, y 的连续函数. 因此, $F(x, y)$ 可微, 且

$$\begin{aligned} dF(x, y) &= F'_x(x, y)dx + F'_y(x, y)dy \\ &= f(x^2 + y^2)(xdx - ydy). \end{aligned}$$

于是,任取 C 上一点 (x_0, y_0) , 有

$$\begin{aligned} &\oint_C f(x^2 + y^2)(xdx + ydy) \\ &= F(x, y) \Big|_{(x_0, y_0)}^{(x_0, y_0)} \\ &= F(x_0, y_0) - F(x_0, y_0) = 0. \end{aligned}$$

证毕.

求原函数 z , 设

$$4271. dz = (x^2 + 2xy - y^2)dx + (x^2 - 2xy - y^2)dy.$$

$$\begin{aligned} \text{解} \quad z &= \int_0^x (x^2 + 2xy - y^2)dx \\ &\quad + \int_0^y (0 - 0 - y^2)dy + C \\ &= \frac{x^3}{3} + x^2y - xy^2 - \frac{1}{3}y^3 + C. \end{aligned}$$

$$4272. dz = \frac{ydx - xdy}{3x^2 - 2xy + 3y^2}.$$

$$\begin{aligned} \text{解} \quad z &= \int_0^x \frac{ydx}{3x^2 - 2xy + 3y^2} + \int_1^y 0dy + C \\ &= \frac{y}{3} \int_0^x \frac{dx}{\left(x - \frac{1}{3}y\right)^2 + \frac{8y^2}{9}} + C \\ &= \frac{y}{3} \cdot \frac{3}{2\sqrt{2}y} \operatorname{arctg} \frac{3\left(x - \frac{y}{3}\right)}{2\sqrt{2}y} \Big|_0^x + C \\ &= \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{3x - y}{2\sqrt{2}y} + C_1. \end{aligned}$$

$$4273. dz = \frac{(x^2 + 2xy + 5y^2)dx + (x^2 - 2xy + y^2)dy}{(x + y)^3}$$

$$\begin{aligned}
\text{解} \quad z &= \int_0^x \frac{x^2 + 2xy + 5y^2}{(x+y)^3} dx \\
&\quad + \int_1^y \frac{0 - 0 + y^2}{(0+y)^3} dy + C \\
&= \int_0^x \frac{(x+y)^2 + 4y^2}{(x+y)^3} dx + \int_1^y \frac{dy}{y} + C \\
&= [\ln|x+y|] \Big|_0^x - \frac{2y^2}{(x+y)^2} \Big|_0^x \\
&\quad + [\ln|y|] \Big|_1^y + C \\
&= \ln|x+y| - \frac{2y^2}{(x+y)^2} + C_1.
\end{aligned}$$

$$4274. \quad dz = e^x[e^y(x-y+2) + y]dx + e^x(e^y(x-y) + 1)dy.$$

$$\begin{aligned}
\text{解} \quad z &= \int_0^x [(x-y+2)e^{x+y} + ye^x] dx \\
&\quad + \int_0^y (1 - ye^y) dy + C \\
&= [(x-y+1)e^{x+y} + ye^x] \Big|_0^x \\
&\quad + [y - ye^y + e^y] \Big|_0^y + C \\
&= (x-y+1)e^{x+y} + ye^x + C_1.
\end{aligned}$$

$$4275. \quad dz = \frac{\partial^{m+1}u}{\partial x^{n+1}\partial y^m} dx + \frac{\partial^{m+1}u}{\partial x^n\partial y^{m+1}} dy.$$

解 因为

$$\begin{aligned}
dz &= \frac{\partial^{m+1}u}{\partial x^{n+1}\partial y^m} dx + \frac{\partial^{m+1}u}{\partial x^n\partial y^{m+1}} dy \\
&= d\left(\frac{\partial^{m+1}u}{\partial x^n\partial y^m}\right)
\end{aligned}$$

故有

$$z = \frac{\partial^{m+1}u}{\partial x^n\partial y^m} + C.$$

$$4276. dz = \frac{\partial^{n+m+1}}{\partial x^{n+2} \partial y^{m-1}} \left(\ln \frac{1}{r} \right) dx \\ - \frac{\partial^{n+m+1}}{\partial x^{n+1} \partial y^{m+2}} \left(\ln \frac{1}{r} \right) dy, \text{ 其中 } r = \sqrt{x^2 + y^2}.$$

解 易知(当 $(x, y) \neq (0, 0)$ 时)

$$\frac{\partial}{\partial x} \left(\ln \frac{1}{r} \right) = -\frac{x}{r^2}, \frac{\partial}{\partial y} \left(\ln \frac{1}{r} \right) = -\frac{y}{r^2},$$

$$\frac{\partial^2}{\partial x^2} \left(\ln \frac{1}{r} \right) = -\frac{r^2 - 2x^2}{r^4},$$

$$\frac{\partial^2}{\partial y^2} \left(\ln \frac{1}{r} \right) = -\frac{r^2 - 2y^2}{r^4},$$

故(当 $(x, y) \neq (0, 0)$ 时)

$$\frac{\partial^2}{\partial x^2} \left(\ln \frac{1}{r} \right) + \frac{\partial^2}{\partial y^2} \left(\ln \frac{1}{r} \right) = 0. \quad (1)$$

令

$$P = \frac{\partial^{n+m+1}}{\partial x^{n+2} \partial y^{m-1}} \left(\ln \frac{1}{r} \right),$$

$$Q = -\frac{\partial^{n+m+1}}{\partial x^{n+1} \partial y^{m+2}} \left(\ln \frac{1}{r} \right),$$

则当 $(x, y) \neq (0, 0)$ 时, 由(1)式知

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \\ = \frac{\partial^{n+m}}{\partial x^n \partial y^m} \left[\frac{\partial^2}{\partial x^2} \left(\ln \frac{1}{r} \right) + \frac{\partial^2}{\partial y^2} \left(\ln \frac{1}{r} \right) \right] = 0.$$

因此, 在任何不含原点 $(0, 0)$ 的单连通区域中, $Pdx + Qdy$ 都是某函数 z 的全微分, 并且对上半平面的点 (x, y) (即 $y > 0$), 可取

$$z(x, y) = \int_0^x P(x, y) dx + \int_1^y Q(0, y) dy + C$$

$$\begin{aligned}
&= \int_0^x \frac{\partial^{n+m-1}}{\partial x^{n+1/2} \partial y^{m-1}} \left(\ln \frac{1}{r} \right) dx \\
&\quad - \int_{-1}^y \left[\frac{\partial^{n+m-1}}{\partial x^{n+1} \partial y^{m+1/2}} \left(\ln \frac{1}{r} \right) \right] \Big|_{x=0} dy + C \\
&= \frac{\partial^{n+m}}{\partial x^{n+1} \partial y^m} \left(\ln \frac{1}{r} \right) \\
&\quad - \left[\frac{\partial^{n+m}}{\partial x^{n+1} \partial y^{m+1}} \left(\ln \frac{1}{r} \right) \right] \Big|_{x=0} \\
&\quad - \left[\frac{\partial^{n+m}}{\partial x^n \partial y^{m+1}} \left(\ln \frac{1}{r} \right) \right] \Big|_{x=0} \\
&\quad + \left[\frac{\partial^{n+m}}{\partial x^n \partial y^{m+1}} \left(\ln \frac{1}{r} \right) \right] \Big|_{\substack{x=0 \\ y=1}} + C \\
&= \frac{\partial^{n+m-1}}{\partial x^n \partial y^{m-1}} \left(\frac{\partial}{\partial x} \ln \frac{1}{r} \right) \\
&\quad - \frac{\partial^{n+m-2}}{\partial x^{n-1} \partial y^{m-1}} \left[\frac{\partial^2}{\partial x^2} \left(\ln \frac{1}{r} \right) \right. \\
&\quad \left. + \frac{\partial^2}{\partial y^2} \left(\ln \frac{1}{r} \right) \right] \Big|_{x=0} + C_1 \\
&= \frac{\partial^{n+m-1}}{\partial x^n \partial y^{m-1}} \left(-\frac{x}{r^2} \right) + C_1 \\
&= \frac{\partial^{n+m-1}}{\partial x^n \partial y^{m-1}} \left(\frac{\partial}{\partial y} \operatorname{arctg} \frac{x}{y} \right) + C_1 \\
&= \frac{\partial^{n+m}}{\partial x^n \partial y^m} \left(\operatorname{arctg} \frac{x}{y} \right) + C_1,
\end{aligned}$$

其中 $C_1 = \left[\frac{\partial^{n+m}}{\partial x^n \partial y^{m+1}} \left(\ln \frac{1}{r} \right) \right] \Big|_{\substack{x=0 \\ y=1}} + C$ 是任意常数.

同理, 对下半平面的点 (x, y) (即 $y < 0$), 可取

$$z(x, y) = \int_0^x P(x, y) dx + \int_{-1}^y Q(0, y) dy + C.$$

经过和前面完全类似的计算, 可得

$$z(x, y) = \frac{\mathcal{F}^{+m}}{\partial x^n \partial y^m} \left(\operatorname{arctg} \frac{x}{y} \right) + C_2,$$

其中

$$C_2 = \left[\frac{\mathcal{F}^{+m}}{\partial x^{n-1} \partial y^{m+1}} \left(\ln \frac{1}{r} \right) \right] \Big|_{\substack{x=0 \\ y=-1}} + C$$

也是任意常数.

4277. 证明下面的估计对于曲线积分是正确的:

$$\left| \int_C P dx + Q dy \right| \leq LM,$$

式中 L 为积分路径的长及 $M = \max \sqrt{P^2 + Q^2}$ (在弧 C 上).

证 由于

$$\begin{aligned} & \left| \int_C P dx + Q dy \right| \\ &= \left| \int_C (P \cos \alpha + Q \sin \alpha) ds \right| \\ &\leq \int_C |P \cos \alpha + Q \sin \alpha| ds, \end{aligned}$$

又因

$$\begin{aligned} & (P \cos \alpha + Q \sin \alpha)^2 \\ &= P^2 \cos^2 \alpha + Q^2 \sin^2 \alpha + 2PQ \sin \alpha \cos \alpha, \\ &0 \leq (P \sin \alpha - Q \cos \alpha)^2 \\ &= P^2 \sin^2 \alpha + Q^2 \cos^2 \alpha - 2PQ \sin \alpha \cos \alpha, \end{aligned}$$

故有 $(P \cos \alpha + Q \sin \alpha)^2 \leq P^2 + Q^2$. 从而

$$|P \cos \alpha + Q \sin \alpha| \leq \sqrt{P^2 + Q^2} \leq M.$$

于是,

$$\left| \int_C P dx + Q dy \right| \leq M \int_C ds = LM.$$

4278. 估计积分

$$I_R = \oint_{x^2+y^2=R^2} \frac{ydx - xdy}{(x^2 + xy + y^2)^2}.$$

证明 $\lim_{R \rightarrow \infty} I_R = 0$.

解 在圆 $x^2 + y^2 = R^2$ 上, 有

$$\begin{aligned} P^2 + Q^2 &= \frac{y^2}{(x^2 + xy + y^2)^4} \\ &\quad + \frac{x^2}{(x^2 + xy + y^2)^4} \\ &= \frac{x^2 + y^2}{(x^2 + xy + y^2)^4} \\ &= \frac{R^2}{(R^2 + xy)^4} \leq \frac{R^2}{(R^2 - |xy|)^4} \\ &\leq \frac{R^2}{\left(R^2 - \frac{x^2 + y^2}{2}\right)^4} \\ &= \frac{16}{R^6}. \end{aligned}$$

于是, $M \leq \frac{4}{R^3}$. 利用 4277 题的结果, 即得 I_R 的估计式:

$$|I_R| \leq \frac{4}{R^3} \cdot 2\pi R = \frac{8\pi}{R^2}.$$

由此可知: $\lim_{R \rightarrow \infty} I_R = 0$.

计算沿空间曲线所取的线积分(假定坐标系是右手的):

4279. $\int_C (y^2 - z^2)dx + 2yzdy - x^2dz$, 式中 C 为依参数增加的方向进行的曲线 $x = t, y = t^2, z = t^3 (0 \leq t \leq 1)$.

$$\begin{aligned} \text{解} \quad &\int_C (y^2 - z^2)dx + 2yzdy - x^2dz \\ &= \int_0^1 [t^4 - t^6] + 2t^5 \cdot 2t - t^2 \cdot 3t^2]dt \\ &= \int_0^1 (3t^6 - 2t^4)dt = \frac{1}{35}. \end{aligned}$$

4280. $\int_C ydx + zdy - xdz$, 式中 C 为依参数增加方向进行的
 纽形螺线 $x = acost, y = asint, z = bt (0 \leq t \leq 2\pi)$.

$$\begin{aligned} \text{解} \quad & \int_C ydx + zdy + xdz \\ &= \int_0^{2\pi} (-a^2 \sin t \cos t + abt \cos t + ab \cos t) dt \\ &= \left(-\frac{at^2}{2} + \frac{a^2 \sin 2t}{4} + abt \sin t \right. \\ &\quad \left. + ab \cos t + ab \sin t \right) \Big|_0^{2\pi} \\ &= \pi a^2. \end{aligned}$$

4281. $\int_C (y-z)dx - (z-x)dy + (x-y)dz$, 式中 C 为圆
 周 $x^2 + y^2 + z^2 = a^2, y = x \operatorname{tg} \alpha (0 < \alpha < \pi)$, 若从 Ox 轴
 的正向看去, 这圆周是沿逆时针方向进行的.

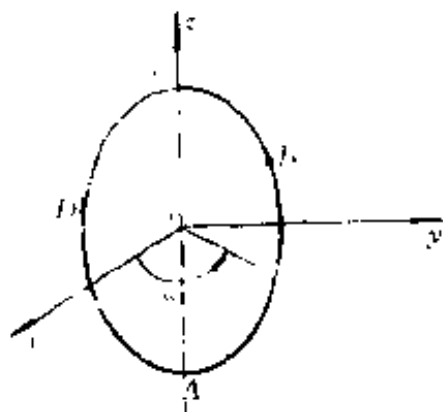


图 8.63

解 如图 8.63 所示. 利于球面的参数方程 $x = a \cos \varphi \cos \psi, y = a \sin \varphi \cos \psi, z = a \sin \psi$. 在 $\widehat{ABC}$ 上, $\varphi = \alpha$, 因而有

$$\begin{aligned} x &= a \cos \alpha \cos \psi, dx = -a \cos \alpha \sin \psi d\psi, \\ y &= a \sin \alpha \cos \psi, dy = -a \sin \alpha \sin \psi d\psi, \\ z &= a \sin \psi, dz = a \cos \psi d\psi, \end{aligned}$$

且

$$\begin{aligned}
 & \int_{\widehat{ABC}} (y-z)dx + (z-x)dy + (x-y)dz \\
 &= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [-(\sin\alpha\cos\psi - \sin\psi)\cos\alpha\sin\psi \\
 &\quad - (\sin\psi - \cos\alpha\cos\psi)\sin\alpha\sin\psi \\
 &\quad + (\cos\alpha\cos\psi - \sin\alpha\cos\psi)\cos\psi] d\psi \\
 &= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\alpha - \sin\alpha) d\psi = \pi a^2 (\cos\alpha - \sin\alpha) \\
 &= \sqrt{2} a^2 \pi \sin\left(\frac{\pi}{4} - \alpha\right).
 \end{aligned}$$

在 $\widehat{CDA}$ 上, $\varphi = \alpha + \pi$, 同样可得

$$\begin{aligned}
 & \int_{\widehat{CDA}} (y-z)dx + (z-x)dy + (x-y)dz \\
 &= -a^2 \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\alpha - \cos\alpha) d\psi \\
 &= \sqrt{2} \pi a^2 \sin\left(\frac{\pi}{4} - \alpha\right).
 \end{aligned}$$

于是, 最后得

$$\begin{aligned}
 & \int_C (y-z)dx + (z-x)dy + (x-y)dz \\
 &= 2\sqrt{2} \pi a^2 \sin\left(\frac{\pi}{4} - \alpha\right).
 \end{aligned}$$

4282. $\int_C y^2 dx + z^2 dy + x^2 dz$, 式中 C 为维维安尼曲线 $x^2 + y^2 + z^2 = a^2, x^2 + y^2 = ax (z \geq 0, a > 0)$, 若从 Ox 轴的正的部分 ($x > a$) 看去, 此曲线是沿逆时针方向进行的.

解 柱面 $x^2 + y^2 = ax$ 可变为

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2,$$

故若令 $x = \frac{a}{2} = \frac{a}{2}\cos t, y = \frac{a}{2}\sin t (0 \leq t \leq 2\pi)$, 则

$$\begin{aligned} z &= \sqrt{a^2 - (x^2 + y^2)} \\ &= \sqrt{a^2 - \left[\frac{a^2(1 + \cos t)^2}{4} + \frac{a^2 \sin^2 t}{4} \right]} \\ &= a \sin \frac{t}{2}. \end{aligned}$$

从而, 曲线的参线方程为

$$x = \frac{a(1 + \cos t)}{2}, y = \frac{a \sin t}{2},$$

$$z = a \sin \frac{t}{2} \quad (0 \leq t \leq 2\pi),$$

于是,

$$\begin{aligned} & \int y^2 dx + z^2 dy + x^2 dz \\ &= \int_0^{2\pi} \left[\frac{a^3 \sin^3 t}{8} + \frac{a^3 \sin^2 \frac{t}{2} \cos t}{2} - \frac{a^3 \cos^3 \frac{t}{2}}{2} \right] dt \\ &= \int_0^{2\pi} \frac{a^3}{8} (1 - \cos^3 t) d(\cos t) \\ &\quad + \frac{a^3}{2} \int_0^{2\pi} \frac{1 - \cos t}{2} \cos t dt \\ &\quad + a^3 \int_0^{2\pi} \left(1 - \sin^2 \frac{t}{2} \right) d\left(\sin \frac{t}{2} \right) \\ &= \frac{a^3}{8} \left[\cos t - \frac{1}{3} \cos^3 t \right] \Big|_0^{2\pi} \\ &\quad + \frac{a^3}{4} \left[\sin t - \left(\frac{t}{2} + \frac{1}{4} \sin 2t \right) \right] \Big|_0^{2\pi} \end{aligned}$$

$$+ a^3 \left(\sin \frac{t}{2} - \frac{1}{3} \sin^3 \frac{t}{2} \right) \Big|_0^{2\pi}$$

$$= -\frac{\pi a^3}{4}.$$

4283. $\int_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz$, 式中 C 为球面的一部分 $x^2 + y^2 + z^2 = 1, x \geq 0, y \geq 0, z \geq 0$ 的围线, 当沿着它的正向进行时该曲面的外面保持在左方.

解 围线在 Oxy 平面部分的方程为

$$x = \cos \varphi, y = \sin \varphi, z = 0 \quad \left(0 \leq \varphi \leq \frac{\pi}{2} \right).$$

根据轮换对称性知, 只要沿这部分计算线积分, 再三倍之, 便得要求的结果, 即

$$\begin{aligned} & \int_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz \\ &= 3 \int_0^{\frac{\pi}{2}} [\sin^2 \varphi \cdot (-\sin \varphi) - \cos^2 \varphi \cdot \cos \varphi] d\varphi \\ &= 3 \int_0^{\frac{\pi}{2}} (1 - \cos^2 \varphi) d(\cos \varphi) \\ &\quad - \int_0^{\frac{\pi}{2}} (1 - \sin^2 \varphi) d(\sin \varphi) \\ &= 3 \left(\cos \varphi - \frac{1}{3} \cos^3 \varphi - \sin \varphi + \frac{1}{3} \sin^3 \varphi \right) \Big|_0^{\frac{\pi}{2}} \\ &= -4. \end{aligned}$$

利用全微分计算下列曲线积分:

$$4284. \int_{(1,1,1)}^{(2,3,-4)} xdx + y^2dy - z^3dz.$$

$$\text{解} \quad \int_{(1,1,1)}^{(2,3,-4)} xdx + y^2dy - z^3dz$$

$$= \left(\frac{1}{2}x^2 + \frac{1}{3}y^3 - \frac{1}{4}z^4 \right) \Big|_{(1,1,1)}^{(2,3,-4)}$$

$$= -53\frac{7}{12}.$$

4285. $\int_{(1,2,3)}^{(6,1,1)} yzdx + xzdy + xydz.$

解 $\int_{(1,2,3)}^{(6,1,1)} yzdx + xzdy + xydz$
 $= xyz \Big|_{(1,2,3)}^{(6,1,1)} = 0.$

4286. $\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}},$ 其中点 (x_1, y_1, z_1) 位于球 $x^2 + y^2 + z^2 = a^2$ 之上, 而点 (x_2, y_2, z_2) 位于球 $x^2 + y^2 + z^2 = b^2$ 之上 ($a > 0, b > 0$).

解 由题设知:

$$x_1^2 + y_1^2 + z_1^2 = a^2, x_2^2 + y_2^2 + z_2^2 = b^2.$$

于是,

$$\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \frac{xdx + ydy + zdz}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \sqrt{x^2 + y^2 + z^2} \Big|_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)}$$

$$= \sqrt{x_2^2 + y_2^2 + z_2^2} - \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$= b - a.$$

4287. $\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \varphi(x)dx + \psi(y)dy + \chi(z)dz,$ 式中 φ, ψ, χ 为连续函数.

解 因为

$$\varphi(x)dx + \psi(y)dy + \chi(z)dz$$

$$= d \left(\int_{x_1}^x \varphi(u)du + \int_{y_1}^y \psi(v)dv + \int_{z_1}^z \chi(\omega)d\omega \right),$$

故有

$$\begin{aligned} & \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \varphi(x) dx + \psi(y) dy + \chi(z) dz \\ &= \left[\int_{x_1}^x \varphi(u) du + \int_{y_1}^y \psi(v) dv \right. \\ & \quad \left. + \int_{z_1}^z \chi(\omega) d\omega \right] \Big|_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \\ &= \int_{x_1}^{x_2} \varphi(u) du + \int_{y_1}^{y_2} \psi(v) dv + \int_{z_1}^{z_2} \chi(\omega) d\omega. \end{aligned}$$

4288. $\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} f(x+y+z)(dx+dy+dz)$, 其中 f 为连续函数.

解 令 $F(x, y, z) = \int_0^{x+y+z} f(u) du$. 由于 $f(u)$ 是连续函数, 故

$$F'_x(x, y, z) = f(x+y+z),$$

$$F'_y(x, y, z) = f(x+y+z),$$

$$F'_z(x, y, z) = f(x+y+z),$$

并且这些偏导数都是连续的. 因此, $F(x, y, z)$ 可微, 且

$$\begin{aligned} dF(x, y, z) &= F'_x(x, y, z)dx + F'_y(x, y, z)dy \\ & \quad + F'_z(x, y, z)dz \\ &= f(x+y+z)(dx+dy+dz). \end{aligned}$$

于是,

$$\begin{aligned} & \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} f(x+y+z)(dx+dy+dz) \\ &= F(x, y, z) \Big|_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \\ &= F(x_2, y_2, z_2) - F(x_1, y_1, z_1) \end{aligned}$$

$$\begin{aligned}
&= \int_0^{x_2+y_2+z_2} f(u)du - \int_0^{x_1+y_1+z_1} f(u)du \\
&= \int_{x_1+y_1+z_1}^{x_2+y_2+z_2} f(u)du.
\end{aligned}$$

4289. $\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} f(\sqrt{x^2+y^2+z^2})xdx + ydy + zdz$, 式中 f 为连续函数.

解 令 $F(x, y, z) = \frac{1}{2} \int_0^{x^2+y^2+z^2} f(\sqrt{v})dv$. 由于 f 是连续函数, 故

$$F'_x(x, y, z) = xf(\sqrt{x^2+y^2+z^2}),$$

$$F'_y(x, y, z) = yf(\sqrt{x^2+y^2+z^2}),$$

$$F'_z(x, y, z) = zf(\sqrt{x^2+y^2+z^2}),$$

并且这些偏导数都是连续的. 因此, $F(x, y, z)$ 可微, 且

$$\begin{aligned}
&dF(x, y, z) \\
&= F'_x(x, y, z)dx + F'_y(x, y, z)dy \\
&\quad + F'_z(x, y, z)dz \\
&= f(\sqrt{x^2+y^2+z^2})(xdx + ydy + zdz).
\end{aligned}$$

于是,

$$\begin{aligned}
&\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} f(\sqrt{x^2+y^2+z^2}) \\
&\quad \cdot (xdx + ydy + zdz) \\
&= F(x_2, y_2, z_2) - F(x_1, y_1, z_1) \\
&= \frac{1}{2} \int_{x_1^2+y_1^2+z_1^2}^{x_2^2+y_2^2+z_2^2} f(\sqrt{v})dv^*) \\
&= \int_{\sqrt{x_1^2+y_1^2+z_1^2}}^{\sqrt{x_2^2+y_2^2+z_2^2}} uf(u)du,
\end{aligned}$$

*) 这里已作代换 $\sqrt{v} = u (v = u^2, dv = 2udu)$. 求原

函数 u , 若:

$$4290. du = (x^2 - 2yz)dx + (y^2 - 2xz)dy + (z^2 - 2xy)dz.$$

$$\begin{aligned}\text{解 } du &= (x^2dx + y^2dy + z^2dz) \\ &\quad - 2(yzdx + xzdy + xydz) \\ &= d\left(\frac{x^3 + y^3 + z^3}{3} - 2xyz\right).\end{aligned}$$

于是,

$$u = \frac{1}{3}(x^3 + y^3 + z^3) - 2xyz + C.$$

$$4291. du = \left(1 - \frac{1}{y} + \frac{y}{z}\right)dx + \left(\frac{x}{z} + \frac{x}{y^2}\right)dy - \frac{xy}{z^2}dz.$$

$$\begin{aligned}\text{解 } du &= dx + \left(-\frac{1}{y}dx + \frac{x}{y^2}dy\right) \\ &\quad + \frac{1}{z}(ydx + xdy) - \frac{xy}{z^2}dz \\ &= dx + \left[-\frac{1}{y}dx + xd\left(-\frac{1}{y}\right)\right] \\ &\quad + \frac{1}{z}d(xy) + xyd\left(\frac{1}{z}\right) \\ &= dx + d\left(-\frac{x}{y}\right) + d\left(\frac{xy}{z}\right) \\ &= d\left(x - \frac{x}{y} + \frac{xy}{z}\right).\end{aligned}$$

$$\text{于是, } u = x - \frac{x}{y} + \frac{xy}{z} + C.$$

$$4292. du = \frac{(x+y-z)dx + (x+y-z)dy + (x+y+z)dz}{x^2 + y^2 + z^2 + 2xy}.$$

解 由于

$$\begin{aligned}&(x+y-z)dx + (x+y-z)dy + (x+y+z)dz \\ &= (xdx + ydy) + (ydx + xdy) + (x+y)dz \\ &\quad - z(dx + dy) + zdz\end{aligned}$$

$$= \frac{1}{2} d[(x^2 + y^2 + 2xy) + z^2] \\ + (x + y)dz - zd(x + y),$$

故

$$du = \frac{1}{2} \frac{d[(x + y)^2 + z^2]}{(x + y)^2 + z^2} \\ + \frac{(x + y)dz - zd(x + y)}{(x + y)^2 + z^2} \\ = \frac{1}{2} d \ln [(x + y)^2 + z^2] \\ + d \left(\operatorname{arctg} \frac{z}{x + y} \right) \\ = d \left[\ln \sqrt{(x + y)^2 + z^2} \right. \\ \left. + \operatorname{arctg} \frac{z}{x + y} \right].$$

于是,

$$u = \ln \sqrt{(x + y)^2 + z^2} \\ + \operatorname{arctg} \frac{z}{x + y} + C.$$

4293. 求当质量为 m 的点从位置 (x_1, y_1, z_1) 移动到位置 (x_2, y_2, z_2) 时, 重力所产生的功 (Oz 轴的方向垂直向上).

解 设 $\vec{i}, \vec{j}, \vec{k}$ 为各坐标轴上的单位矢量, 则重力

$\vec{F} = -mg\vec{k}$, 而

$$d\vec{s} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$

从而功的微分为

$$dA = \vec{F} \cdot d\vec{s} = -mgdz = d(-mgz).$$

于是, 重力的功为

$$A = \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} -mgdz$$

$$\begin{aligned}
 &= (-mgz) \Big|_{\langle x_1, y_1, z_1 \rangle}^{\langle x_2, y_2, z_2 \rangle} \\
 &= -mg(z_2 - z_1).
 \end{aligned}$$

4294⁺. 弹性力的方向向着坐标原点, 力的大小与质点距坐标原点的距离成比例. 设此点依反时针方向描绘出椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 的正四分之一, 求弹性力所作的功.

解 弹性力

$$\vec{F} = -k(x\vec{i} + y\vec{j}),$$

功的微分为

$$\begin{aligned}
 dA &= \vec{F} \cdot d\vec{s} \\
 &= -k(x\vec{i} + y\vec{j}) \cdot (dx\vec{i} + dy\vec{j}) \\
 &= -k(xdx + ydy) \\
 &= d\left[-\frac{k}{2}(x^2 + y^2)\right].
 \end{aligned}$$

于是, 功为

$$\begin{aligned}
 A &= -k \int_{(a,0)}^{(0,b)} xdx + ydy \\
 &= -\frac{k}{2}(x^2 + y^2) \Big|_{(a,0)}^{(0,b)} \\
 &= \frac{k}{2}(a^2 - b^2).
 \end{aligned}$$

4295⁺. 当单位质量从点 $M_1(x_1, y_1, z_1)$ 移动到点 $M_2(x_2, y_2, z_2)$ 时, 求作用于单位质量的引力 $F = \frac{k}{r^2}$ (其中 $r = \sqrt{x^2 + y^2 + z^2}$) 所做的功.

解 引力指向坐标原点, 故它的方向余弦为

$$\cos\alpha = -\frac{x}{r} \quad \cos\beta = -\frac{y}{r},$$

$$\cos \gamma = -\frac{z}{r},$$

而引力的射影为

$$X = -\frac{kx}{r^3}, Y = -\frac{ky}{r^3},$$

$$Z = -\frac{kz}{r^3}.$$

于是, 功为

$$\begin{aligned} A &= -k \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \frac{xdx + ydy + zdz}{r^3} \\ &= -\frac{k}{2} \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \frac{d(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \\ &= \frac{k}{\sqrt{x^2 + y^2 + z^2}} \Big|_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} \\ &= k \left[\frac{1}{\sqrt{x_2^2 + y_2^2 + z_2^2}} - \frac{1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} \right]. \end{aligned}$$

当然, 这里假设从 M_1 点到 M_2 点的路径是不经过原点的, 上式表明功与路径无关, 仅决定于起始点的坐标.

§ 12. 格林公式

1° 曲线积分与二重积分的关系 设 C 为逐段光滑的简单封闭围线, 它围成单联通的有界域 S , 这围线的方向是这样的: 域 S 保持在左边, 函数 $P(x, y), Q(x, y)$ 与它们自己的一阶偏导函数在域 S 内及其边缘上皆是连续的, 则有格林公式

$$\begin{aligned} &\oint_C P(x, y)dx + Q(x, y)dy \\ &= \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy. \end{aligned} \quad (1)$$

若把域 S 的边界 C 了解为一切边界围线的和, 而围线绕转的方向是选择来使得域 S 保持在左边, 则公式(1) 对于由几个简单围线所界的有界域 S 也真确.

2° 平面域的面积 由逐段光滑的简单围线 C 所界的面积 S 等于:

$$S = \frac{1}{2} \oint_C xdy - ydx.$$

在这一节中, 若没有相反的约定, 则假定积分的封闭围线是简单的 (无自交点), 并选择它们的正方向使所界不含无穷远点的域是保持在曲线的左边.

4296. 利用格林公式变换曲线积分

$$I = \oint_C \sqrt{x^2 + y^2} dx + y[xy + \ln(x + \sqrt{x^2 + y^2})] dy,$$

式中围线 C 包含有界的域 S .

解 此处 $P = \sqrt{x^2 + y^2}$, $Q = xy^2 + y \ln(x + \sqrt{x^2 + y^2})$. 从而

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + \frac{y}{\sqrt{x^2 + y^2}} - \frac{y}{\sqrt{x^2 + y^2}} = y^2.$$

于是,

$$I = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_S y^2 dx dy.$$

注: 这里应假定 C 不与 Ox 轴的左半部分 (即 $x \leq 0, y = 0$) 相交, 从而这时在 S 中 $x + \sqrt{x^2 + y^2} > 0$.

4297. 应用格林公式, 计算曲线积分

$$I = \oint_L (x + y)^2 dx - (x^2 + y^2) dy.$$

其中 k 依正方向经过以 $A(1,1), B(3,2), C(2,5)$ 为顶点的三角形 ABC 的围线. 直接计算积分, 以验证所求得的结果.

解 如图 8.64 所示. AB, BC 及 CA 的方程分别为 $y = \frac{1}{2}(x+1), y = -3x+11, y = 4x-3$.

由于 $P = (x+y)^2, Q = -(x^2+y^2)$, 故

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2x - 2 \cdot (x+y) = -4x - 2y.$$

通过顶点 C 引直线垂直于 Ox 轴, 它把三角形域 S 分成 S_1 和 S_2 两部分. 于是,

$$\begin{aligned} I &= \iint_S (-4x - 2y) dx dy \\ &= \iint_{S_1} (-4x - 2y) dx dy + \iint_{S_2} (-4x - 2y) dx dy \\ &= \int_1^2 dx \int_{\frac{1}{2}(x+1)}^{4x-3} (-4x - 2y) dy \\ &\quad + \int_2^3 dx \int_{\frac{1}{2}(x+1)}^{3x+11} (-4x - 2y) dy \\ &= \int_1^2 \left(-\frac{119}{4}x^2 + \frac{77}{2}x - \frac{35}{4} \right) dx \end{aligned}$$

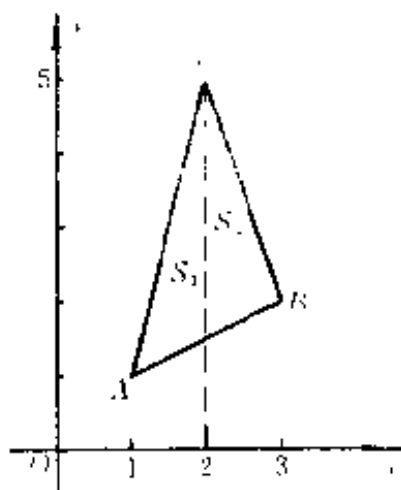


图 8.64

$$\begin{aligned}
& + \int_2^3 \left(\frac{21}{4}x^2 + \frac{49}{2}x - \frac{483}{4} \right) dx \\
& = -\frac{245}{12} - \frac{105}{4} = -46\frac{2}{3}.
\end{aligned}$$

如果直接计算,则

$$\begin{aligned}
I &= \int_{AB} + \int_{BC} + \int_{CA} \\
&= \int_1^3 \left[\left(x + \frac{x}{2} + \frac{1}{2} \right)^2 - \frac{1}{2} \left(x^2 + \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4} \right) \right] dx \\
&\quad + \int_3^2 [(x - 3x + 11)^2 - (-3)(x^2 + 9x^2 \\
&\quad - 66x + 121)] dx \\
&\quad + \int_2^1 [(x + 4x - 3)^2 - 4(x^2 + 16x^2 - 24x + 9)] dx \\
&= \int_1^3 \left(\frac{13}{8}x^2 + \frac{5}{4}x - \frac{1}{8} \right) dx \\
&\quad + \int_3^2 (34x^2 - 242x + 484) dx \\
&\quad + \int_2^1 (-43x^2 + 66x - 27) dx \\
&= \frac{58}{3} - \frac{283}{3} + \frac{5}{3} = -46\frac{2}{3}.
\end{aligned}$$

应用格林公式计算下列曲线积分:

4298. $\oint_C xy^2 dy - x^2 y dx$, 式中 C 为圆周 $x^2 + y^2 = a^2$.

解 由于 $P = -x^2 y, Q = xy^2$, 故有

$$\begin{aligned}
\oint_C xy^2 dy - x^2 y dx &= \iint_{x^2 + y^2 \leq a^2} (x^2 + y^2) dx dy \\
&= \int_0^{2\pi} d\varphi \int_0^a r^3 dr = \frac{\pi a^4}{2}.
\end{aligned}$$

如果直接计算,可令 $x = a \cos t, y = a \sin t$, 则

$$\begin{aligned}\oint_C xy^2 dy - x^2 y dx &= a^4 \int_0^{2\pi} (\cos^2 t \sin^2 t \\ &\quad + \cos^2 t \sin^2 t) dt \\ &= \frac{a^4}{2} \int_0^{2\pi} \sin^2 2t dt = \frac{\pi a^4}{2}.\end{aligned}$$

4299. $\oint_C (x+y)dy - (x-y)dx$, 式中 C 为椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

解 由于 $P = x+y, Q = -(x-y)$, 故有

$$\begin{aligned}\oint_C (x+y)dx - (x-y)dy \\ = \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} (-1-1)dxdy = -2\pi ab.\end{aligned}$$

如果直接计算, 则

$$\begin{aligned}\oint_C (x+y)dy - (x-y)dx \\ = \int_0^{2\pi} [(a\cos t + b\sin t)(-a\sin t) - (a\cos t - b\sin t) \\ \quad \cdot (b\cos t)]dt \\ = \int_0^{2\pi} [(b^2 - a^2)\cos t \sin t - ab]dt = -2\pi ab.\end{aligned}$$

4300. $\oint_C e^x [(1-\cos y)dx - (y-\sin y)dy]$, 其中 C 为域 $0 < x < \pi, 0 < y < \sin x$ 的正方向的围线.

解 由于

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^x (\sin y - y) - e^x \sin y = -ye^x,$$

故有

$$\oint_C e^x [(1-\cos y)dx - (y-\sin y)dy]$$

$$\begin{aligned}
&= - \iint_{\substack{0 \leq x \leq \pi \\ 0 \leq y \leq \sin x}} ye^x dx dy = - \int_0^{\pi} e^x dx \int_0^{\sin x} y dy \\
&= - \frac{1}{2} \int_0^{\pi} e^x \sin^2 x dx \\
&= - \frac{1}{4} \left(\int_0^{\pi} e^x dx - \int_0^{\pi} e^x \cos 2x dx \right) \\
&= - \frac{1}{4} \left[(e^x - 1) - \frac{\cos 2x}{5} + \frac{2 \sin 2x}{5} e^x \right]_0^{\pi} \\
&= - \frac{1}{5} (e^{\pi} - 1).
\end{aligned}$$

4301. $\oint_{x^2+y^2=R^2} e^{-(x^2+y^2)} (\cos 2xy dx + \sin 2xy dy).$

解 由于

$$\begin{aligned}
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= e^{-(x^2+y^2)} [(-2x \sin 2xy + 2y \cos 2xy) \\
&\quad - (2y \cos 2xy - 2x \sin 2xy)] = 0,
\end{aligned}$$

故有

$$\begin{aligned}
&\oint_{x^2+y^2=R^2} e^{-(x^2+y^2)} (\cos 2xy dx + \sin 2xy dy) \\
&= \iint_{x^2+y^2 \leq R^2} 0 dx dy = 0.
\end{aligned}$$

4302. 积分

$$I_1 = \int_{AmB} (x+y)^2 dx - (x-y)^2 dy$$

和

$$I_2 = \int_{AnB} (x+y)^2 dx - (x-y)^2 dy$$

(其中 AmB 为连接点 $A(1,1)$ 和点 $B(2,6)$ 的直线, AnB 是其轴为垂直的抛物线, 并通过 A, B 及坐标原点) 相差多少?

解 由于

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2(x-y) - 2(x+y) = -4x,$$

故 I_1 与 I_2 之差为(利用格林公式)

$$\begin{aligned} I_2 - I_1 &= \oint_{AmBmA} (x+y)^2 dx - (x-y)^2 dy \\ &= \iint_S (-4x) dx dy = \int_1^2 dx \int_{2x^2}^{3x^2-4} (-4x) dy \\ &= - \int_1^2 4x(-2x^2 + 6x - 4) dx \\ &= (2x^4 - 8x^3 + 8x^2) \Big|_1^2 = -2, \end{aligned}$$

或 $I_1 - I_2 = 2$.

4303. 计算曲线积分

$$\int_{AmO} (e^x \sin y - my) dx + (e^x \cos y - m) dy,$$

其中 AmO 为由点 $A(a, 0)$ 至点 $O(0, 0)$ 的上半圆周 $x^2 + y^2 = ax$.

解 在 Ox 轴上连接点 $O(0, 0)$ 与点 $A(a, 0)$, 这样, 便构成封闭的半圆形 $AmOA$, 且在线段 OA 上,

$$\int_{OA} (e^x \sin y - my) dx + (e^x \cos y - m) dy = 0.$$

从而

$$\oint_{AmOA} = \int_{AmO} + \int_{OA} = \int_{AmO}.$$

另一方面, 利用格林公式可得

$$\begin{aligned} \oint_{AmOA} (e^x \sin y - my) dx + (e^x \cos y - m) dy \\ = \iint_{x^2 + y^2 \leq ax} m dx dy = \frac{\pi m a^2}{8}. \end{aligned}$$

于是,

$$\begin{aligned} & \int_{AmB} (e' \sin y - my) dx + (e' \cos y - m) dy \\ &= \frac{\pi m a^2}{8}. \end{aligned}$$

4304. 计算曲线积分

$$\int_{AmB} [\varphi(y)e' - my] dx + [\varphi'(y)e' - m] dy,$$

式中 $\varphi(y)$ 和 $\varphi'(y)$ 为连续函数, AmB 为连接点 $A(x_1, y_1)$ 和点 $B(x_2, y_2)$ 的任何路径, 但与线段 AB 围成已知大小为 S 的面积 $AmBA$.

解 首先, 我们有

$$\oint_{AmBA} = \int_{AmB} + \int_{BA},$$

而

$$\begin{aligned} & \oint_{AmBA} [\varphi(y)e' - my] dx + [\varphi'(y)e' - m] dy \\ &= \iint_S m dx dy = mS. \end{aligned}$$

另一方面,

$$\begin{aligned} & \int_{BA} [\varphi(y)e' - my] dx + [\varphi'(y)e' - m] dy \\ &= \int_{BA} d[e' \varphi(y)] - \int_{BA} m(y dx + dy) \\ &= e' \varphi(y) \Big|_{(x_2, y_2)}^{(x_1, y_1)} - m \int_{x_2}^{x_1} \left[y_1 + \frac{y_2 - y_1}{x_2 - x_1} \right. \\ & \quad \left. \cdot (x - x_1) + \frac{y_2 - y_1}{x_2 - x_1} \right] dx \\ &= e' \varphi(y_1) - e' \varphi(y_2) - m \left(y_1 + \frac{y_2 - y_1}{x_2 - x_1} \right) \end{aligned}$$

$$\begin{aligned}
& \cdot (x_2 - x_1) + \frac{m}{2} \cdot \frac{y_2 - y_1}{x_2 - x_1} (x_2 - x_1)^2 \\
& = e^{i_1} \varphi(y_1) - e^{i_2} \varphi(y_2) + m(y_2 - y_1) \\
& = -\frac{m}{2} (x_2 - x_1)(y_2 + y_1).
\end{aligned}$$

于是,

$$\begin{aligned}
& \int_{AmR} [\varphi(y)e^x - my]dx + [\varphi'(y)e^x - m]dy \\
& = mS + e^{i_2} \varphi(y_2) - e^{i_1} \varphi(y_1) - m(y_2 - y_1) \\
& = \frac{m}{2} (x_2 - x_1)(y_2 + y_1).
\end{aligned}$$

注:利用此题的结果可计算 4303 题.事实上,由于 $\varphi(y) = \sin y, x_1 = a, y_1 = 0, x_2 = y_2 = 0, S = \frac{\pi a^2}{8}$, 代入即得

$$\int_{AmC} (e^x \sin y - my)dx + (e^x \cos y - m)dy = \frac{\pi m a^2}{8}.$$

4305. 求两个二次连续地可微分的函数 $P(x, y)$ 和 $Q(x, y)$, 使得线积分

$$I = \oint_C P(x + \alpha, y + \beta)dx + Q(x + \alpha, y + \beta)dy$$

对于任何封闭的围线 C 与常数 α 和 β 无关.

解 由格林公式,得

$$\begin{aligned}
I = \iint_S \left\{ \frac{\partial Q(x + \alpha, y + \beta)}{\partial x} - \frac{\partial P(x + \alpha, y + \beta)}{\partial y} \right\} \\
\cdot dxdy = \tau.
\end{aligned} \tag{1}$$

由假定 τ 为一常数,它与 α, β 无关(只与围线 C 有关),上式中的 S 表围线 C 所围成的闭区域.由假定 P, Q 具有连续的二阶偏导数,故(1)式中二重积分的被积函数具有关于 α, β 的一阶连续偏导数.因此,可以在积分

号下关于 α, β 求偏导数, 得

$$\iint_S \left\{ \frac{\partial^2 Q(x + \alpha, y + \beta)}{\partial \alpha \partial x} - \frac{\partial^2 P(x + \alpha, y + \beta)}{\partial \alpha \partial y} \right\} \\ \cdot dx dy = \frac{\partial}{\partial \alpha} \tau = 0. \quad (2)$$

$$\iint_S \left\{ \frac{\partial^2 Q(x + \alpha, y + \beta)}{\partial \beta \partial x} - \frac{\partial^2 P(x + \alpha, y + \beta)}{\partial \beta \partial y} \right\} \\ \cdot dx dy = \frac{\partial}{\partial \beta} \tau = 0. \quad (3)$$

于是, (2) 式和 (3) 式对任何 α, β 以及任何 S 都成立. 再注意到 (2) 式和 (3) 式中二重积分的被积函数都是连续的, 故被积函数必恒为零 (参看 4097 题, 此题对二重积分也成立):

$$\frac{\partial^2 Q(x + \alpha, y + \beta)}{\partial \alpha \partial x} - \frac{\partial^2 P(x + \alpha, y + \beta)}{\partial \alpha \partial y} \equiv 0, \quad (4)$$

$$\frac{\partial^2 Q(x + \alpha, y + \beta)}{\partial \beta \partial x} - \frac{\partial^2 P(x + \alpha, y + \beta)}{\partial \beta \partial y} \equiv 0 \quad (5)$$

(对任何 x, y, α, β). 记 $x + \alpha = u, y + \beta = v$, 显然有

$$\frac{\partial^2 Q(x + \alpha, y + \beta)}{\partial \alpha \partial x} = \frac{\partial^2 Q(u, v)}{\partial u^2},$$

$$\frac{\partial^2 P(x + \alpha, y + \beta)}{\partial \alpha \partial y} = \frac{\partial^2 P(u, v)}{\partial u \partial v},$$

$$\frac{\partial^2 Q(x + \alpha, y + \beta)}{\partial \beta \partial x} = \frac{\partial^2 Q(u, v)}{\partial v \partial u},$$

$$\frac{\partial^2 P(x + \alpha, y + \beta)}{\partial \beta \partial y} = \frac{\partial^2 P(u, v)}{\partial v^2}.$$

于是, (4) 式与 (5) 式为

$$\frac{\partial}{\partial u} \left\{ \frac{\partial Q(u, v)}{\partial u} - \frac{\partial P(u, v)}{\partial v} \right\}$$

$$\begin{aligned}
&= \frac{\partial^2 Q(u, v)}{\partial u^2} - \frac{\partial^2 P(u, v)}{\partial u \partial v} = 0, \\
&\frac{\partial}{\partial v} \left\{ \frac{\partial Q(u, v)}{\partial u} - \frac{\partial P(u, v)}{\partial v} \right\} \\
&= \frac{\partial^2 Q(u, v)}{\partial v \partial u} - \frac{\partial^2 P(u, v)}{\partial v^2} = 0
\end{aligned}$$

(对任何 u, v), 由此可知:

$$\frac{\partial Q(u, v)}{\partial u} - \frac{\partial P(u, v)}{\partial v} = k(\text{常数}).$$

将 u, v 改记为 x, y , 则上式为

$$\frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} = k(\text{常数}). \quad (6)$$

令 $u(x, y) = \int_0^x P(t, y) dt$, 则 $u(x, y)$ 具有连续的二阶偏导数, 且

$$\frac{\partial u(x, y)}{\partial x} = P(x, y). \quad (7)$$

由(6)式知:

$$\begin{aligned}
\frac{\partial Q(x, y)}{\partial x} &= k + \frac{\partial P(x, y)}{\partial y} \\
&= k + \frac{\partial}{\partial y} \left\{ \frac{\partial u(x, y)}{\partial x} \right\} \\
&= k + \frac{\partial}{\partial x} \left\{ \frac{\partial u(x, y)}{\partial y} \right\}.
\end{aligned}$$

两端积分, 得

$$Q(x, y) = kx + \frac{\partial u(x, y)}{\partial y} + \varphi(y), \quad (8)$$

其中 $\varphi(y)$ 为具有二阶连续导数的任意函数. 由(7), (8)两式又知 $u(x, y)$ 具有连续的三阶偏导数.

反之, 若 $u(x, y)$ 是任一具有三阶连续偏导数的函数, 而 $\varphi(y)$ 是任一具二阶连续导数的函数, 则由(7)

式和(8)式确定的 $P(x, y)$ 与 $Q(x, y)$ 必具连续二阶偏导数, 且使(6)式成立, 从而使

$$\begin{aligned} I &= \oint_C P(x + \alpha, y + \beta) dx + Q(x + \alpha, y + \beta) dy \\ &= \iint_S \left\{ \frac{\partial Q(x + \alpha, y + \beta)}{\partial x} \right. \\ &\quad \left. - \frac{\partial P(x + \alpha, y + \beta)}{\partial y} \right\} dx dy \\ &= \iint_S k dx dy = kS, \end{aligned}$$

故 I 是与 α, β 无关的常数(对于任意固定的 C).

综上所述, 可知: 使线积分 I 对于任何封闭围线 C 与常数 α, β 无关的二阶连续地可微的函数 $P(x, y)$ 与 $Q(x, y)$ 的全体由公式(7)与(8)给出, 其中 k 为常数, $u(x, y)$ 为三阶连续地可微的任一函数, $\varphi(y)$ 为二阶连续地可微的任意一个一元函数.

4306. 为了使线积分

$$\int_{A \text{ 至 } B} F(x, y)(y dx + x dy)$$

与积分路径的形状无关, 则可微分函数 $F(x, y)$ 应满足怎样的条件?

解 由于 $P = yF(x, y), Q = xF(x, y)$, 故由格林公式知所求的条件为

$$\frac{\partial}{\partial x} [xF(x, y)] = \frac{\partial}{\partial y} [yF(x, y)],$$

即

$$xF'_x(x, y) = yF'_y(x, y).$$

4307. 计算

$$I = \oint_C \frac{xdy - ydx}{x^2 + y^2},$$

其中 C 为依正方向进行而不经坐标原点的简单封闭围线.

解 令 $P = -\frac{y}{x^2 + y^2}$, $Q = \frac{x}{x^2 + y^2}$. 易知, 当 $(x, y) \neq (0, 0)$ 时, 恒有

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

今分两种情况讨论:

(1) 坐标原点在围线 C 之外, 这时, 在由 C 围成的有界闭区域 S 上, P 与 Q 以及它们的偏导数都连续, 故可应用格林公式, 得

$$I = \oint_C Pdx + Qdy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0.$$

(2) 围线 C 包围坐标原点. 这时, 由于 P, Q 在原点无定义, 故不能直接对由 C 围成的区域应用格林公式. 今取 $a > 0$ 充分小, 使中心在原点半径为 a 的圆周 $L_a (L_a: x^2 + y^2 = a^2)$ 完全位于围线 C 之内. 用 S_a 表界于 C 和 L_a 之间的环形闭区域. 显然, 在 S_a 上, P, Q 及其偏导数均连续, 故可应用格林公式, 得

$$\begin{aligned} & \left(\oint_C + \oint_{-L_a} \right) Pdx + Qdy \\ &= \iint_{S_a} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0, \end{aligned}$$

其中 $-L_a$ 表沿 L_a 的负方向 (即顺时针方向).

于是,

$$I = \oint_C Pdx + Qdy = \oint_{L_a} Pdx + Qdy,$$

其中 L_+ 沿正方向(即逆时针方向). 利用 L_+ 的参数方程 $x = a \cos t, y = a \sin t (0 \leq t \leq 2\pi)$, 即得

$$\begin{aligned} I &= \oint_{L_+} P dx + Q dy = \oint_{L_+} \frac{x dy - y dx}{x^2 + y^2} \\ &= \frac{1}{a^2} \int_0^{2\pi} [(a \cos t)(a \cos t) - a \sin t(-a \sin t)] dt \\ &= \int_0^{2\pi} dt = 2\pi. \end{aligned}$$

利用曲线积分计算由下列曲线所界的面积:

4308. 椭圆 $x = a \cos t, y = b \sin t (0 \leq t \leq 2\pi)$.

解 面积为

$$\begin{aligned} S &= \frac{1}{2} \oint_C x dy - y dx \\ &= \frac{1}{2} \int_0^{2\pi} ab(\cos^2 t + \sin^2 t) dt = \pi ab. \end{aligned}$$

4309. 星形线 $x = a \cos^3 t, y = b \sin^3 t (0 \leq t \leq 2\pi)$.

解 面积为

$$\begin{aligned} S &= \frac{1}{2} \oint_C x dy - y dx \\ &= \frac{3ab}{2} \int_0^{2\pi} (\cos^4 t \sin^2 t + \cos^2 t \sin^4 t) dt \\ &= \frac{3}{8} ab \int_0^{2\pi} \sin^2 2t dt = \frac{3}{8} \pi ab. \end{aligned}$$

4310. 抛物线 $(x+y)^2 = ax (a > 0)$ 和轴 Ox .

解 作代换 $y = tx$, 则原方程化为 $x^2(1+t)^2 = ax$.

从而得曲线的参数方程为

$$x = \frac{a}{(1+t)^2}, y = \frac{at}{(1+t)^2} (0 \leq t < +\infty).$$

它与 Ox 轴的交点为 $(a, 0)$ 与 $(0, 0)$. 在 Ox 轴上从点 $(0, 0)$ 到点 $(a, 0)$ 的一段上, 有

$$xdy - ydx = 0.$$

在抛物线上,有

$$xdy - ydx = \frac{a^2}{(1+t)^3} dt.$$

于是,面积为

$$\begin{aligned} S &= \frac{1}{2} \oint_C xdy - ydx = \frac{a^2}{2} \int_0^{+\infty} \frac{dt}{(1+t)^3} \\ &= -\frac{a^2}{6} \cdot \frac{1}{(1+t)^2} \Big|_0^{+\infty} = \frac{a^2}{6}. \end{aligned}$$

4311. 笛卡儿叶形线 $x^3 + y^3 = 3axy (a > 0)$.

解 作代换 $y = tx$, 则得曲线的参数方程为

$$x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}.$$

由于

$$dx = \frac{3a(1-2t^3)}{(1+t^3)^2} dt, dy = \frac{3at(2-t^3)}{(1+t^3)^2} dt,$$

从而

$$xdy - ydx = \frac{9a^2 t^2}{(1+t^3)^2} dt.$$

于是,面积为

$$\begin{aligned} S &= \frac{1}{2} \oint_C xdy - ydx = \frac{9a^2}{2} \int_0^{+\infty} \frac{t^2}{(1+t^3)^2} dt \\ &= \frac{3a^2}{2} \left[-\frac{1}{1+t^3} \right]_0^{+\infty} = \frac{3a^2}{2}. \end{aligned}$$

4312. 双纽线 $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

解 利用极坐标 $x = r\cos\varphi, y = r\sin\varphi$, 得双纽线的方程为 $r^2 = a^2\cos 2\varphi$, 故

$$x = a\cos\varphi \sqrt{\cos 2\varphi}, y = a\sin\varphi \sqrt{\cos 2\varphi}.$$

从而 $xdy - ydx = a^2\cos 2\varphi d\varphi$. 于是,面积为

$$\begin{aligned}
 S &= 4 \cdot \frac{1}{2} \oint_C xdy - ydx \\
 &= 2 \int_0^{\frac{\pi}{4}} a^2 \cos 2\varphi d\varphi = a^2.
 \end{aligned}$$

4313. 曲线 $x^3 + y^3 = x^2 + y^2$ 及坐标轴.

解 作代换 $y = tx$, 即得曲线的参数方程为

$$x = \frac{1+t^2}{1+t^3}, y = \frac{t(1+t^2)}{1+t^3} \quad (0 \leq t < +\infty).$$

曲线的起点为 $(1, 0)$, 终点为 $(0, 1)$. 在曲线上,

$$xdy - ydx = \frac{(1+t^2)^2}{(1+t^3)^2} dt \quad (0 \leq t < +\infty).$$

在 Ox 轴上从点 $(0, 1)$ 到点 $(0, 0)$ 一段, 以及在 Ox 轴上从点 $(0, 0)$ 到点 $(1, 0)$ 一段上, 均有

$$xdy - ydx = 0.$$

于是, 面积为

$$\begin{aligned}
 S &= \frac{1}{2} \oint_C xdy - ydx = \frac{1}{2} \int_0^{+\infty} \frac{(1+t^2)^2}{(1+t^3)^2} dt \\
 &= \frac{1}{2} \left[\int_0^{+\infty} \frac{t^4}{(1+t^3)^2} dt + 2 \int_0^{+\infty} \frac{t^2}{(1+t^3)^2} dt \right. \\
 &\quad \left. + \int_0^{+\infty} \frac{1}{(1+t^3)^2} dt \right] \\
 &= \frac{1}{2} \left[\frac{1}{3} B\left(2 - \frac{1}{3}, \frac{1}{3}\right) + \frac{2}{3} B(1, 1) \right. \\
 &\quad \left. + \frac{1}{3} B\left(\frac{1}{3}, 2 - \frac{1}{3}\right) \right] \\
 &= \frac{1}{3} + \frac{1}{3} \frac{\Gamma\left(2 - \frac{1}{3}\right) \Gamma\left(\frac{1}{3}\right)}{\Gamma(2)} \\
 &= \frac{1}{3} + \frac{1}{3} \left(1 - \frac{1}{3}\right) \Gamma\left(1 - \frac{1}{3}\right) \Gamma\left(\frac{1}{3}\right)
 \end{aligned}$$

$$= \frac{1}{3} + \frac{2}{9} \frac{\pi}{\sin \frac{\pi}{3}} = \frac{1}{3} + \frac{4\pi}{9\sqrt{3}}.$$

*) 利用 3853 题的结果.

4314. 计算由曲线

$$(x+y)^{n+m+1} = ax^n y^m (a > 0, n > 0, m > 0)$$

所界的面积.

解 作代换 $y = tx$, 即得曲线的参数方程为

$$x = \frac{at^m}{(1+t)^{n+m+1}}, y = \frac{at^{m+1}}{(1+t)^{n+m+1}} (0 \leq t < +\infty).$$

从而

$$xdy - ydx = \frac{a^2 t^{2m}}{(1+t)^{2n+2m+2}} dt (0 \leq t < +\infty)$$

于是, 面积为

$$\begin{aligned} S &= \frac{1}{2} \oint_C xdy - ydx = \frac{a^2}{2} \int_0^{+\infty} \frac{t^{2m}}{(1+t)^{2n+2m+2}} dt \\ &= \frac{a^2}{2} B(2m+1, 2n+1). \end{aligned}$$

*) 利用 3852 题的结果.

4315. 计算由曲线

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1 \quad (a > 0, b > 0, n > 0)$$

和坐标轴所界的面积.

解 作代换 $x = a \cos^{\frac{2}{n}} \varphi, y = b \sin^{\frac{2}{n}} \varphi (0 \leq \varphi \leq \frac{\pi}{2})$,

即得

$$xdy - ydx = \frac{2ab}{n} \cos^{\frac{2}{n}-1} \varphi \sin^{\frac{2}{n}-1} \varphi d\varphi.$$

曲线与坐标轴交于点 $(a, 0)$ 和点 $(0, b)$. 在 Oy 轴上, 从点 $(0, b)$ 到点 $(0, 0)$ 一段, 以及在 Ox 轴上从点 $(0, 0)$ 到

点 $(a, 0)$ 一段上, 显然有

$$xdy - ydx = 0.$$

于是, 面积为

$$\begin{aligned} S &= \frac{1}{2} \oint_C xdy - ydx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{2ab}{n} \cos^{\frac{2}{n}-1} \varphi \sin^{\frac{2}{n}-1} \varphi d\varphi \\ &= \frac{ab}{n} \cdot \frac{1}{2} B\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{ab}{2n} \frac{\Gamma^2\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}. \end{aligned}$$

*) 利用 3856 题的结果.

4316. 计算由曲线

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = \left(\frac{x}{a}\right)^{n-1} + \left(\frac{y}{b}\right)^{n-1}$$

$(a > 0, b > 0, n > 0)$ 和坐标轴所界的面积.

解 作代换 $y = \frac{b}{a}t$, 即得曲线的参数方程为

$$x = \frac{a(1+t^{n-1})}{1+t^n}, y = \frac{bt(1+t^{n-1})}{1+t^n} \quad (0 < t < +\infty).$$

易知

$$xdy - ydx = ab \frac{(1+t^{n-1})^2}{(1+t^n)^2} dt.$$

又在两坐标轴上, 显然有 $xdy - ydx = 0$. 于是, 面积为

$$\begin{aligned} S &= \frac{1}{2} \oint_C xdy - ydx = \frac{ab}{2} \int_0^{+\infty} \frac{(1+t^{n-1})^2}{(1+t^n)^2} dt \\ &= \frac{ab}{2} \left(\int_0^{+\infty} \frac{t^{2n-2}}{(1+t^n)^2} dt + 2 \int_0^{+\infty} \frac{t^{n-1}}{(1+t^n)^2} dt \right. \\ &\quad \left. + \int_0^{+\infty} \frac{1}{(1+t^n)^2} dt \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{ab}{2} \left[\frac{1}{n} B\left(2 - \frac{1}{n}, \frac{1}{n}\right) + \frac{2}{n} B(1, 1) \right. \\
&\quad \left. + \frac{1}{n} B\left(\frac{1}{n}, 2 - \frac{1}{n}\right) \right] \\
&= \frac{ab}{n} \left[1 + B\left(2 - \frac{1}{n}, \frac{1}{n}\right) \right] \\
&= \frac{ab}{n} \left[1 + \left(1 - \frac{1}{n}\right) \frac{\Gamma\left(1 - \frac{1}{n}\right) \Gamma\left(\frac{1}{n}\right)}{\Gamma(1)} \right] \\
&= \frac{ab}{n} \left[1 + \frac{\left(1 - \frac{1}{n}\right) \pi}{\sin \frac{\pi}{n}} \right].
\end{aligned}$$

*) 利用 3853 题的结果.

4317. 计算由纽形曲线

$$\left(\frac{x}{a}\right)^{2n+1} + \left(\frac{y}{b}\right)^{2n+1} = c \left(\frac{x}{a}\right)^n \left(\frac{y}{b}\right)^n$$

($a > 0, b > 0, c > 0, n > 0$) 所界的面积.

解 作代换 $y = \frac{b}{a}xt$, 即得曲线的参数方程为

$$x = \frac{act^n}{1+t^{2n+1}}, y = \frac{bct^{n+1}}{1+t^{2n+1}} \quad (0 \leq t < +\infty).$$

易知

$$xdy - ydx = \frac{abc^2 t^{2n}}{(1+t^{2n+1})^2} dt.$$

于是, 面积为

$$\begin{aligned}
S &= \frac{1}{2} \oint_C xdy - ydx = \frac{abc^2}{2} \int_0^{+\infty} \frac{t^{2n}}{(1+t^{2n+1})^2} dt \\
&= \frac{abc^2}{2(2n+1)} \cdot \frac{1}{1+t^{2n+1}} \Big|_0^{+\infty} = \frac{abc^2}{2(2n+1)}.
\end{aligned}$$

4318. 一个半径为 r 的圆沿着半径为 R 的定圆外面圆周滚动 (而不滑动) 时, 由动圆上的一点所描绘出来的曲线称

为外摆线. 假定比值 $\frac{R}{r} = n$ 是整数 ($n \geq 1$), 求外摆线所界的面积. 研究特殊情况 $r = R$ (心脏形线).

解 取定圆的中心 O 作坐标原点, 取 Ox 轴通过点 A , 点 A 是动点的始点, 即为两圆的公切点时的位置 (图 8.65). 当动圆滚到如图的新位置时, 点 A 移到点 M . 动点 M 的轨迹便是外摆线, 其方程推导如下: 设动圆的圆心为 C , 两圆的切点为 B , 记 $\angle MCB = t$ (运动开始时, 设 t 等于零). 切点在定圆上所移过的弧 $\widehat{AB}$ 应等于它在动圆上所移过的弧 $\widehat{MB}$, 即

$$R \cdot \angle AOB = \frac{R}{n} \cdot \angle MCB = \frac{R}{n} t.$$

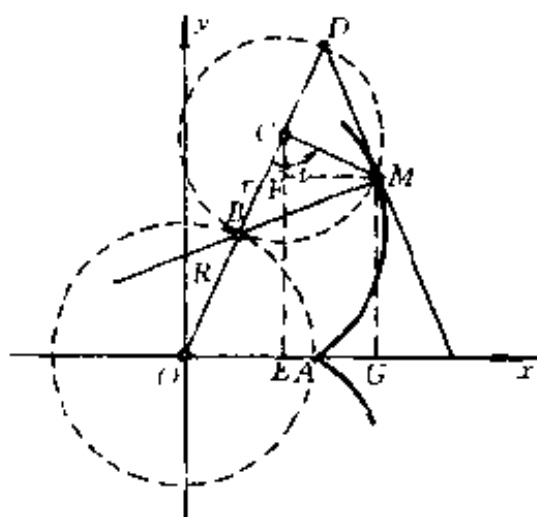


图 8.65

从而 $\angle AOB = \frac{t}{n}$, 设动点 M 的坐标为 (x, y) , 则

$$\begin{aligned}x &= OG = OE + FM \\&= \left(R + \frac{R}{n}\right) \cos \frac{t}{n} + \frac{R}{n} \sin \angle FCM,\end{aligned}$$

但 $\angle FCM = \angle BCM - \angle OCE$, 且 $\angle OCE = \frac{\pi}{2} - \frac{t}{n}$,
从而

$$\begin{aligned}\angle FCM &= \left(1 + \frac{1}{n}\right)t - \frac{\pi}{2}, \\ \sin \angle FCM &= -\cos \left(1 + \frac{1}{n}\right)t.\end{aligned}$$

于是,最后得

$$x = R \left(1 + \frac{1}{n}\right) \cos \frac{t}{n} - \frac{R}{n} \cos \left(1 + \frac{1}{n}\right)t.$$

类似地,可求得

$$y = R \left(1 + \frac{1}{n}\right) \sin \frac{t}{n} - \frac{R}{n} \sin \left(1 + \frac{1}{n}\right)t.$$

若记 $\varphi = \frac{t}{n}$, 并注意到 $R = nr$, 则外摆线可用如下的参数方程表示:

$$\begin{aligned}x &= (n+1)r \cos \varphi - r \cos(n+1)\varphi, \\ y &= (n+1)r \sin \varphi - r \sin(n+1)\varphi.\end{aligned}$$

由 $R = nr$ 知, 当动圆滚 n 圈后, 起点与终点重合, 即 φ 的变化范围为 $0 \leq \varphi \leq 2\pi$. 由于

$$x dy - y dx = r^2(n+1)(n+2)(1 - \cos n\varphi) d\varphi,$$

故所求的面积为

$$\begin{aligned}S &= \frac{1}{2} \oint_C x dy - y dx \\&= \frac{r^2(n+1)(n+2)}{2} \int_0^{2\pi} (1 - \cos n\varphi) d\varphi \\&= \pi r^2(n+1)(n+2).\end{aligned}$$

特别是,当 $r = R$ 时,即 $n = 1$,则得心脏形线的面积为 $S = 6\pi r^2$.

4319. 一个半径为 r 的圆沿着半径为 R 的定圆内面圆周滚动(而不滑动)时,由动圆上的一点所描绘出来的曲线称为内摆线. 假定比值 $\frac{R}{r} = n$ 是整数($n \geqslant 2$),求内摆线所界的面积. 研究特殊情况 $r = \frac{R}{4}$ (星形线).

解 仿上题,容易求得内摆线的参数方程为

$$x = R \left(1 - \frac{1}{n} \right) \cos \frac{t}{n} + \frac{R}{n} \cos \left(1 - \frac{1}{n} \right) t,$$

$$y = R \left(1 - \frac{1}{n} \right) \sin \frac{t}{n} - \frac{R}{n} \sin \left(1 - \frac{1}{n} \right) t$$

若以 $\varphi = \frac{t}{n}$ 为参数,并注意到 $R = nr$,则得

$$x = r(n-1)\cos\varphi + r\cos(n-1)\varphi,$$

$$y = r(n-1)\sin\varphi - r\sin(n-1)\varphi.$$

由于

$$xdy - ydx = r^2(n-1)(n-2)(1 - \cos n\varphi)d\varphi,$$

故面积为

$$\begin{aligned} S &= \frac{1}{2} \oint_C xdy - ydx \\ &= \frac{r^2(n-1)(n-2)}{2} \int_0^{2\pi} (1 - \cos n\varphi)d\varphi \\ &= \pi r^2(n-1)(n-2). \end{aligned}$$

特别是,当 $\frac{R}{r} = 4$ 时,即 $n = 4$,则得星形线所界的面积为 $S = 6\pi r^2$.

4320. 计算圆柱面 $x^2 + y^2 = ax$ 被曲面 $x^2 + y^2 + z^2 = a^2$ 所截那部分的面积.

解 两曲面的交线为

$$x^2 + y^2 = ax, z^2 = a^2 - ax.$$

若将 Oxy 平面上的圆周 $x^2 + y^2 = ax$ 记以 C , 其弧长记以 s , 则所求的面积显然可表为

$$S = 2 \oint_C \sqrt{a^2 - ax} ds.$$

由于 $x^2 + y^2 = ax$ 即为 $\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$, 故令

$$x = \frac{a}{2} + \frac{a}{2} \cos \varphi, y = \frac{a}{2} \sin \varphi,$$

从而弧长的微分为 $ds = \frac{a}{2} d\varphi$, 于是, 面积为

$$\begin{aligned} S &= 2 \oint_C \sqrt{a^2 - ax} ds = 2 \int_0^{2\pi} \sqrt{\frac{a^2}{2} (1 - \cos \varphi)} \cdot \frac{a}{2} d\varphi \\ &= 2 \int_0^{2\pi} a^2 \sin \frac{\varphi}{2} d\left(\frac{\varphi}{2}\right) = 4a^2. \end{aligned}$$

4321. 计算

$$I = \frac{1}{2\pi} \oint_C \frac{XdY - YdX}{X^2 + Y^2},$$

若 $X = ax + by, Y = cx + dy$, 且 C 为包围坐标原点的简单的封闭围线 ($ad - bc \neq 0$).

解 首先注意, 由于 $ad - bc \neq 0$, 故只有原点 $(0, 0)$ 使 $X^2 + Y^2 = 0$. 易知

$$\begin{aligned} XdY - YdX &= (ax + by)(cdx + ddy) \\ &\quad - (cx + dy)(adx + bdy) \\ &= (ad - bc)(xdy - ydx), \end{aligned}$$

故

$$I = \frac{1}{2\pi} \oint_C \frac{XdY - YdX}{X^2 + Y^2}.$$

$$= \frac{1}{2\pi} \oint_{C'} P(x, y) dx + Q(x, y) dy,$$

其中

$$P = - \frac{(ad - bc)y}{(ax + by)^2 + (cx + dy)^2},$$

$$Q = \frac{(ad - bc)x}{(ax + by)^2 + (cx + dy)^2}.$$

容易算得

$$\begin{aligned} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= - \frac{(ad - bc)[(a^2 + c^2)x^2 - (b^2 + d^2)y^2]}{[(ax + by)^2 + (cx + dy)^2]^2} \\ & \quad ((x, y) \neq (0, 0) \text{ 时}), \end{aligned}$$

故由格林公式知

$$\begin{aligned} & \oint_C P(x, y) dx + Q(x, y) dy \\ &= \oint_{C'} P(x, y) dx + Q(x, y) dy, \end{aligned}$$

其中 C' 可为包围原点 $(0, 0)$ 的任一位于 C 内的围线. 特别是, 可取 C' 为围线 $(ax + by)^2 + (cx + dy)^2 = r^2$ (即 $X^2 + Y^2 = r^2$), $r > 0$ 充分小. 于是, 得 (利用格林公式)

$$\begin{aligned} I &= \frac{1}{2\pi} \oint_{C'} \frac{XdY - YdX}{X^2 + Y^2} \\ &= \frac{1}{2\pi} \oint_{X^2 + Y^2 = r^2} \frac{XdY - YdX}{X^2 + Y^2} \\ &= \frac{1}{2\pi r^2} \oint_{X^2 + Y^2 = r^2} XdY - YdX \\ &= \frac{ad - bc}{2\pi r^2} \oint_{X^2 + Y^2 = r^2} xdy - ydx \\ &= \frac{ad - bc}{2\pi r^2} \iint_{X^2 + Y^2 \leq r^2} 2dxdy \end{aligned}$$

$$= \frac{ad - bc}{\pi r^2} \iint_{X^2 + Y^2 \leq r^2} \left| \frac{D(x, y)}{D(X, Y)} \right| dXdY.$$

由于 $\frac{D(X, Y)}{D(x, y)} = ad - bc$, 故 $\frac{D(x, y)}{D(X, Y)} = \frac{1}{ad - bc}$. 于是, 代入上式得

$$\begin{aligned} I &= \frac{ad - bc}{\pi r^2} \iint_{X^2 + Y^2 \leq r^2} \frac{1}{|ad - bc|} dXdY \\ &= \frac{ad - bc}{\pi r^2} \cdot \frac{1}{|ad - bc|} \cdot \pi r^2 = \operatorname{sgn}(ad - bc). \end{aligned}$$

4322. 若简单的围线 C 包围坐标原点, $X = \varphi(x, y)$, $Y = \psi(x, y)$, 而曲线 $\varphi(x, y) = 0$ 和 $\psi(x, y) = 0$ 在围线 C 内面有几个单交点, 计算积分 I (参阅前题).

解 设 $\varphi(x, y) = 0, \psi(x, y) = 0$ 在 C 内的交点为 $P_i(x_i, y_i) (i = 1, 2, \dots, m)$. 首先注意, 本题应假定函数 $\varphi(x, y)$ 与 $\psi(x, y)$ 在 C 围成的区域内具有连续的二阶偏导数, 并且在各点 $P_i (i = 1, 2, \dots, m)$ 处有 $\frac{D(X, Y)}{D(x, y)} = \varphi'_x \psi'_y - \varphi'_y \psi'_x \neq 0$. 容易算得

$$XdY - YdX = (\varphi\psi'_x - \varphi'_x\psi)dx + (\varphi\psi'_y - \varphi'_y\psi)dy,$$

从而

$$\begin{aligned} I &= \frac{1}{2\pi} \oint_C \frac{XdY - YdX}{X^2 + Y^2} \\ &= \frac{1}{2\pi} \oint_C P(x, y)dx + Q(x, y)dy, \end{aligned}$$

其中

$$P = \frac{\varphi\psi'_x - \varphi'_x\psi}{\varphi^2 + \psi^2}, Q = \frac{\varphi\psi'_y - \varphi'_y\psi}{\varphi^2 + \psi^2}.$$

又可算得

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{(\varphi^2 + \psi^2)^2} [(\varphi\psi''_{xy} - \varphi''_{xy}\psi)(\varphi^2 + \psi^2)$$

$$- (\varphi_r' \psi_s' + \varphi_s' \psi_r') \varphi^2 + (\varphi_r' \psi_r' + \varphi_s' \psi_s') \psi^2 \\ + 2(\varphi_r' \varphi_s' - \psi_r' \psi_s') \varphi \psi]$$

$$((x, y) \neq (x_i, y_i) (i = 1, 2, \dots, m)).$$

围绕点 $P_i(x_i, y_i)$ 作围线 $C_i: [\varphi(x, y)]^2 + [\psi(x, y)]^2 = r^2$ (即 $X^2 + Y^2 = r^2$), 取 $r > 0$ 充分小, 使诸 C_i 互不相交且都位于 C 内 (这是办得到的, 因为在各点 $P_i, \frac{D(X, Y)}{D(x, y)} \neq 0$. 从而由连续性知在 P_i 的某邻域内 $\frac{D(X, Y)}{D(x, y)} \neq 0$ 且保持定号, 于是根据隐函数存在定理知变换 $X = \varphi(x, y), Y = \psi(x, y)$ 在点 $(x, y) = (x_i, y_i)$ 邻近及点 $(X, Y) = (0, 0)$ 邻近是双方单值双方连续的) 并使 $\frac{D(X, Y)}{D(x, y)}$ 在 P_i 的邻近 $X^2 + Y^2 \leq r^2$ (记为 S_i) 上保持定号, 将格林公式应用于诸围线 $C, C_1, \dots, C_m$ 之间的区域, 可得

$$\oint_C P(x, y) dx + Q(x, y) dy \\ = \sum_{i=1}^m \oint_{C_i} P(x, y) dx + Q(x, y) dy,$$

故

$$I = \frac{1}{2\pi} \sum_{i=1}^n \oint_{C_i} \frac{XdY - YdX}{X^2 + Y^2}. \quad (1)$$

但

$$\oint_{C_i} \frac{XdY - YdX}{X^2 + Y^2} \\ = \frac{1}{r^2} \oint_{C_i} XdY - YdX \\ = \frac{1}{r^2} \oint_{C_i} (\varphi\psi_r' - \varphi_r'\psi)dx + (\varphi\psi_s' - \varphi_s'\psi)dy$$

$$\begin{aligned}
&= \frac{1}{r^2} \iint_{S_r} 2(\phi_x \psi'_y - \phi'_x \psi_y) dx dy \\
&= \frac{2}{r^2} \iint_S \frac{D(X, Y)}{D(x, y)} dx dy \\
&= \frac{2}{r^2} \left(\operatorname{sgn} \frac{D(X, Y)}{D(x, y)} \right)_{P_1} \iint_{S_r} \frac{D(X, Y)}{D(x, y)} dx dy \\
&= \frac{2}{r^2} \left(\operatorname{sgn} \frac{D(X, Y)}{D(x, y)} \right)_{P_1} \iint_{X^2+Y^2 \leq r^2} dX dY \\
&= \frac{2}{r^2} \left(\operatorname{sgn} \frac{D(X, Y)}{D(x, y)} \right)_{P_1} \cdot \pi r^2 \\
&= 2\pi \left(\operatorname{sgn} \frac{D(\varphi, \psi)}{D(x, y)} \right)_{P_1},
\end{aligned}$$

代入(1)式,即得

$$I = \sum_{i=1}^n \left(\operatorname{sgn} \frac{D(\varphi, \psi)}{D(x, y)} \right)_{P_i},$$

或写为

$$I = \sum \operatorname{sgn} \frac{D(\varphi, \psi)}{D(x, y)},$$

其中的 $\sum$ 是对曲线 $\varphi(x, y) = 0$ 与 $\psi(x, y) = 0$ 在 C 内的各交点相加.

注:显然,4321题是4322题的特例.这时,曲线 $ax + by = 0$ 与 $cx + dy = 0$ 在 C 内只有一个交点,即原点 $(0, 0)$, 而 $\frac{D(\varphi, \psi)}{D(x, y)} = ad - bc$.

4323. 证明,若 C 为封闭的围线且 $\vec{l}$ 为任意的方向,有

$$\oint_C \cos(\vec{l}, \vec{n}) ds = 0,$$

式中 n 为围线 C 的外法线.

证 如图 8.66 所示, 不妨规定 C 的方向为逆时针的, 以 $\vec{l}$ 表示. 由于夹角

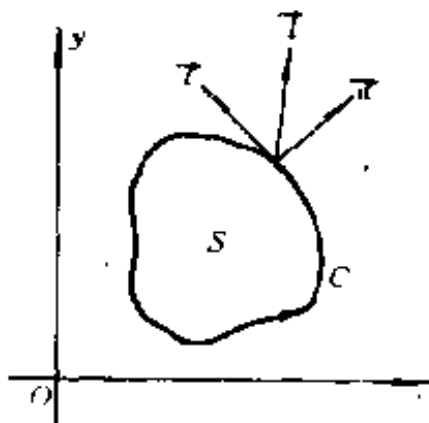


图 8.66

$$(\vec{l}, \vec{n}) = (\vec{l}, x) - (\vec{n}, x),$$

故得

$$\begin{aligned} \cos(\vec{l}, \vec{n}) &= \cos(\vec{l}, x)\cos(\vec{n}, x) \\ &\quad - \sin(\vec{l}, x)\sin(\vec{n}, x), \end{aligned}$$

$$\text{但 } \sin(\vec{n}, x) = \sin\left[(\vec{l}, x) - \frac{\pi}{2}\right] = -\cos(\vec{l}, x),$$

$$\cos(\vec{n}, x) = \cos\left[(\vec{l}, x) - \frac{\pi}{2}\right] = \sin(\vec{l}, x), \text{ 且}$$

$$\cos(\vec{l}, x) = \frac{dx}{ds}, \sin(\vec{l}, x) = \frac{dy}{ds},$$

因此, 有

$$\cos(\vec{l}, \vec{n})ds = \cos(\vec{l}, x)dy - \sin(\vec{l}, x)dx.$$

再利用格林公式, 并注意到 $\sin(\vec{l}, x)$ 和 $\cos(\vec{l}, x)$ 均为常数, 即得

$$\begin{aligned}
& \oint_C \cos(\vec{l}, \vec{n}) ds \\
&= \oint_C [-\sin(\vec{l}, x) dx + \cos(\vec{l}, x)] dy \\
&= \iint_S 0 dx dy = 0.
\end{aligned}$$

4324. 求积分

$$I = \oint_C [x \cos(\vec{n}, x) + y \cos(\vec{n}, y)] ds$$

之值, 式中 C 为包围有界域 S 的简单封闭曲线, $\vec{n}$ 为它的外法线.

解 如上题所述, 已知

$$\begin{aligned}
\cos(\vec{n}, x) &= \cos\left[(\vec{l}, x) - \frac{\pi}{2}\right] \\
&= \sin(\vec{l}, x) = \frac{dy}{ds}, \\
\cos(\vec{n}, y) &= \cos\left[\frac{\pi}{2} - (\vec{n}, x)\right] = \sin(\vec{n}, x) \\
&= \sin\left[(\vec{l}, x) - \frac{\pi}{2}\right] = -\cos(\vec{l}, x) = -\frac{dx}{ds}.
\end{aligned}$$

于是,

$$I = \oint_C x dy - y dx = 2 \cdot \frac{1}{2} \oint_C x dy - y dx = 2S,$$

这里 S 表示有界域 S 面积的数值.

4325. 求

$$\lim_{d(S) \rightarrow 0} \frac{1}{S} \oint_C (\vec{F}, \vec{n}) ds,$$

其中 S 为包含点 (x_0, y_0) 的围线 C 所界的面积, $d(S)$ 为域 S 的直径, $\vec{n}$ 为围线 C 的外法线上的单位向量, $\vec{F}(x, y)$ 为在 $S + C$ 上连续地可微分的向量.

解 由 4321 题的推导过程中知, 矢量 $\vec{n}$ 在坐标轴上的射影为

$$n_x = \cos(\vec{n}, x) = \frac{dy}{ds}, n_y = \cos(\vec{n}, y) = -\frac{dx}{ds}.$$

于是,

$$(\vec{F}, \vec{n})ds = (Xn_x + Yn_y)dx = Xdy - Ydx,$$

因此, 利用格林公式知

$$\begin{aligned} \oint_C (\vec{F}, \vec{n})ds &= \oint_C Xdy - Ydx \\ &= \iint_S \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) dxdy \\ &= \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \Big|_{(\xi, \eta)} \cdot S, \end{aligned}$$

其中点 $(\xi, \eta) \in$ 域 S . 于是,

$$\begin{aligned} \lim_{d(S) \rightarrow 0} \frac{1}{S} \oint_C (\vec{F}, \vec{n})ds &= \lim_{d(S) \rightarrow 0} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \Big|_{(\xi, \eta)} \\ &= X'_{,i}(x_0, y_0) + Y'_{,i}(x_0, y_0). \end{aligned}$$

§ 13. 曲线积分的物理应用

4326. 均匀分布在圆 $x^2 + y^2 = a^2, y \geq 0$ 的上半部的质量 M 以怎样的力吸引质量为 m 位于 $(0, 0)$ 的质点?

解 由对称性知, 引力在 Ox 轴上的射影 $X = 0$, 故只要计算引力在 Oy 轴上的射影.

设圆心角为 θ , 由 $ds = ad\theta$ 知, 对于长为 ds 一段圆弧吸引质量为 m 的质点的力在 Oy 轴上的射影为

$$dY = -\frac{km}{a^2} \frac{M}{\pi a} \sin\theta \cdot a d\theta = -\frac{kmM}{\pi a^2} \sin\theta d\theta,$$

其中 k 为引力常数.

于是, 所求的引力在 Oy 轴上的射影为

$$Y = -\frac{kmM}{\pi a^2} \int_0^\pi \sin\theta d\theta = -\frac{2kmM}{\pi a^2}.$$

4327. 计算单层的对数位

$$u(x, y) = \oint_C k \ln \frac{1}{r} ds,$$

式中 k = 常数 = 密度, $r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$,
设围线 C 是圆周 $\xi^2 + \eta^2 = R^2$.

解 由对称性知, 对数位

$$\begin{aligned} u(x, y) &= 2k \int_0^\pi \ln \frac{1}{r} \cdot R d\theta \\ &= 2Rk \int_0^\pi \ln \frac{1}{\sqrt{R^2 - 2R\rho \cos\theta + \rho^2}} d\theta \\ &= -Rk \int_0^\pi \ln R^2 \left[1 - 2 \frac{\rho}{R} \cos\theta + \left(\frac{\rho}{R} \right)^2 \right] d\theta, \end{aligned}$$

其中 $\rho = \sqrt{x^2 + y^2}$, $\xi x + \eta y = R\rho \cos\theta$, 而 θ 是矢量 $\vec{r} = x\vec{i} + y\vec{j}$ 与 $\vec{r}_0 = \xi\vec{i} + \eta\vec{j}$ 的正向夹角.

利用 3733 题(或 2192 题)的结果, 可得

$$\begin{aligned} &\int_0^\pi \ln \left[1 - 2 \frac{\rho}{R} \cos\theta + \left(\frac{\rho}{R} \right)^2 \right] d\theta \\ &= \begin{cases} 0, & \rho \leq R; \\ 2\pi \ln \frac{\rho}{R}, & \rho > R. \end{cases} \end{aligned}$$

于是, 我们有

$$u(x, y) = -2Rk \int_0^\pi \ln R d\theta$$

$$\begin{aligned}
&= Rk \int_0^\pi \left(1 - 2 \frac{\rho}{R} \cos \theta + \left| \frac{\rho}{R} \right|^2 \right) d\theta \\
&= \begin{cases} 2\pi Rk \ln \frac{1}{R}, & \rho \leq R, \\ 2\pi Rk \ln \frac{1}{\rho}, & \rho > R. \end{cases}
\end{aligned}$$

4328. 采用极坐标 ρ 和 φ , 计算单层的对数位

$$I_1 = \int_0^{2\pi} \cos m\phi \ln \frac{1}{r} d\phi$$

和

$$I_2 = \int_0^{2\pi} \sin m\phi \ln \frac{1}{r} d\phi.$$

式中 r 为点 (ρ, φ) 与动点 $(1, \psi)$ 间的距离, m 为自然数.

解 由于

$$\begin{aligned}
r &= \sqrt{(\rho \cos \varphi - \cos \psi)^2 + (\rho \sin \varphi - \sin \psi)^2} \\
&= \sqrt{1 - 2\rho \cos(\psi - \varphi) + \rho^2},
\end{aligned}$$

于是, 当 $\rho < 1$ 时, 我们有

$$\begin{aligned}
I_1 &= -\frac{1}{2} \int_0^{2\pi} \cos m\phi \ln(1 - 2\rho \cos(\psi - \varphi) + \rho^2) d\phi \\
&= -\frac{1}{2} \int_{\varphi}^{\varphi+2\pi} \cos(mu + m\varphi) \ln(1 - 2\rho \cos u + \rho^2) du \\
&= -\frac{1}{2} \int_{\varphi}^{\varphi+2\pi} \cos m\varphi \cos mu \ln(1 - 2\rho \cos u + \rho^2) du \\
&\quad + \frac{1}{2} \int_{\varphi}^{\varphi+2\pi} \sin m\varphi \sin mu \ln(1 - 2\rho \cos u + \rho^2) du.
\end{aligned}$$

因为上述右端两个积分中被积函数均为以 2π 为周期的函数, 并注意到奇偶函数在对称区间上的积分性质, 则有

$$I_1 = -\cos m\varphi \int_0^\pi \cos mu \ln(1 - 2\rho \cos u + \rho^2) du$$

$$\begin{aligned}
& + \frac{\sin m\varphi}{2} \int_{-\pi}^{\pi} \sin mu \ln(1 - 2\rho \cos u + \rho^2) du \\
& = -\cos m\varphi \int_0^{\pi} \cos mu \ln(1 - 2\rho \cos u + \rho^2) du \\
& = -(\cos m\varphi) \left(-\frac{\pi}{m} \rho^m \right) = \frac{\pi}{m} \rho^m \cos m\varphi.
\end{aligned}$$

同理,我们有

$$\begin{aligned}
I_2 & = -\frac{1}{2} \int_0^{2\pi} \sin m\psi \ln[1 - 2\rho \cos(\psi - \varphi) + \rho^2] d\psi \\
& = -\frac{1}{2} \int_{-\varphi}^{\varphi-2\pi} \sin(mu + m\varphi) \ln(1 - 2\rho \cos u + \rho^2) du \\
& = -\frac{\cos m\varphi}{2} \int_{-\pi}^{\pi} \sin mu \ln(1 - 2\rho \cos u + \rho^2) du \\
& \quad - \sin m\varphi \int_0^{\pi} \cos mu \ln(1 - 2\rho \cos u + \rho^2) du \\
& = -\sin m\varphi \int_0^{\pi} \cos mu \ln(1 - 2\rho \cos u + \rho^2) du \\
& = -(\sin m\varphi) \left(-\frac{\pi}{m} \rho^m \right) = \frac{\pi}{m} \rho^m \sin m\varphi.
\end{aligned}$$

当 $\rho > 1$ 时,则有

$$\begin{aligned}
I_1 & = -\cos m\varphi \int_0^{\pi} \cos mu \ln(1 - 2\rho \cos u + \rho^2) du \\
& = -\cos m\varphi \int_0^{\pi} \cos mu \ln \rho \left(1 - 2 \frac{1}{\rho} \cos u + \frac{1}{\rho^2} \right) du \\
& = -\cos m\varphi \int_0^{\pi} \cos mu \ln \left(1 - 2 \frac{1}{\rho} \cos u + \frac{1}{\rho^2} \right) du \\
& = -(\cos m\varphi) \left(-\frac{\pi}{m\rho^m} \right) = \frac{\pi}{m} \rho^{-m} \cos m\varphi.
\end{aligned}$$

同理,我们有

$$\begin{aligned}
I_2 & = -\sin m\varphi \int_0^{\pi} \cos mu \ln \left(1 - 2 \frac{1}{\rho} \cos u + \frac{1}{\rho^2} \right) du \\
& = -(\sin m\varphi) \left(-\frac{\pi}{m\rho^m} \right) = \frac{\pi}{m} \rho^{-m} \sin m\varphi.
\end{aligned}$$

对于 $\rho = 0$, 显然有

$$I_1 = I_2 = 0.$$

现在来研究当 $\rho = 1$ 的情况. 首先, 积分

$$I_1 = \int_0^\pi \cos mu \ln(1 - 2\rho \cos u + \rho^2) du$$

对于 ρ 在区间 $[1, 1 + \delta]$ 上是一致收敛的, 其中 δ 为很小的正数. 事实上, 对于充分小的 η , 当 u 在 $(0, \eta)$ 内取值时, 有

$$\begin{aligned} 1 &> 1 - 2\rho \cos u + \rho^2 = (1 - \rho)^2 + 2\rho(1 - \cos u) \\ &\geq 2(1 - \cos u) > 0. \end{aligned}$$

于是, 当 $1 \leq \rho \leq 1 + \delta, u \in (0, \eta)$ 时, 有

$$|\cos mu \ln(1 - 2\rho \cos u + \rho^2)| \leq |\ln 2(1 - \cos u)|.$$

而积分

$$\int_0^\eta |\ln 2(1 - \cos u)| du$$

是收敛的. 这是由于当 $0 < 2\beta < 1$, 有

$$\begin{aligned} &\lim_{u \rightarrow +0} u^{2\beta} |\ln 2(1 - \cos u)| \\ &= \lim_{u \rightarrow +0} [2(1 - \cos u)]^\beta \ln[2(1 - \cos u)] \\ &\quad \cdot \frac{u^{2\beta}}{2^\beta (1 - \cos u)^\beta} \\ &= 0 \cdot 1 = 0. \end{aligned}$$

于是, 积分

$$\int_0^\eta \cos mu \ln(1 - 2\rho \cos u + \rho^2) du$$

在 $1 \leq \rho \leq 1 + \delta$ 上一致收敛, 故知积分

$$I_1 = \int_0^\pi \cos mu \ln(1 - 2\rho \cos u + \rho^2) du$$

在 $1 \leq \rho \leq 1 + \delta$ 上一致收敛从而, I_1 作为参数 ρ 的函

数在 $\rho = 1$ 是右连续的. 由此, 根据上面已求出 $\rho > 1$ 时

$$I_1 = \frac{\pi}{m} \rho^{-m} \cos m\varphi, \text{ 得知: 当 } \rho = 1 \text{ 时,}$$

$$I_1 = \lim_{\rho \rightarrow 1+0} \frac{\pi}{m} \rho^{-m} \cos m\varphi = \frac{\pi}{m} \cos m\varphi.$$

同理, 可得

$$I_2 = \lim_{\rho \rightarrow 1+0} \frac{\pi}{m} \rho^{-m} \sin m\varphi = \frac{\pi}{m} \sin m\varphi.$$

综上所述, 得

$$I_1 = \frac{\pi}{m} \rho^m \cos m\varphi, \quad I_2 = \frac{\pi}{m} \rho^m \sin m\varphi, \quad \text{当 } 0 \leq \rho \leq 1;$$

$$I_1 = \frac{\pi}{m} \rho^{-m} \cos m\varphi, \quad I_2 = \frac{\pi}{m} \rho^{-m} \sin m\varphi, \quad \text{当 } \rho > 1.$$

*) 参看 И. В. 雷日克、И. С. 格拉德什坦编著“函数表与积分表”3.765 公式 1.

$$\int_0^\pi \ln(1 - 2p \cos x + p^2) \cos \alpha x dx = -\frac{\pi}{\alpha} p^\alpha (p^2 < 1).$$

* *) 根据上面公式, 当 $p^2 > 1$ 时, 有

$$\begin{aligned} & \int_0^\pi \ln(1 - 2p \cos x + p^2) \cos \alpha x dx \\ &= \int_0^\pi \ln p^2 \left(1 - 2 \frac{1}{p} \cos x + \frac{1}{p^2} \right) \cos \alpha x dx \\ &= \int_0^\pi 2 \ln p \cdot \cos \alpha x dx + \int_0^\pi \ln \left(1 - 2 \frac{1}{p} \cos x \right. \\ & \quad \left. + \frac{1}{p^2} \right) \cos \alpha x dx \\ &= \int_0^\pi \ln \left(1 - 2 \frac{1}{p} \cos x + \frac{1}{p^2} \right) \cos \alpha x dx \\ &= -\frac{\pi}{\alpha} p^{-\alpha}, \end{aligned}$$

其中 α 为自然数.

4329. 计算高斯积分

$$u(x, y) = \oint_C \frac{\cos(\vec{r}, \vec{n})}{r} ds,$$

式中 $r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$ 为向量 $\vec{r}$ 的长度, 此向量是连接点 $A(x, y)$ 和简单封闭光滑围线 C 上的动点 $M(\xi, \eta)$ 而得的, $(\vec{r}, \vec{n})$ 为向量 $\vec{r}$ 与在曲线 C 上 M 点的外法线 $\vec{n}$ 所夹的角.

解 设 $\vec{n}$ 与 Ox 轴的夹角为 α , $\vec{r}$ 与 Ox 轴的夹角为 β , 则 $(\vec{r}, \vec{n}) = \alpha - \beta$. 于是,

$$\begin{aligned} \cos(\vec{r}, \vec{n}) &= \cos\alpha \cos\beta + \sin\alpha \sin\beta \\ &= \frac{\xi - x}{r} \cos\alpha + \frac{\eta - y}{r} \sin\alpha. \end{aligned}$$

代入高斯积分, 得

$$\begin{aligned} u(x, y) &= \oint_C \left(\frac{\eta - y}{r^2} \sin\alpha + \frac{\xi - x}{r^2} \cos\alpha \right) ds \\ &= \oint_C \frac{\xi - x}{r^2} d\eta - \frac{\eta - y}{r^2} d\xi. \end{aligned}$$

令

$$P = -\frac{\eta - y}{r^2}, \quad Q = \frac{\xi - x}{r^2},$$

则有

$$\begin{aligned} \frac{\partial P}{\partial \eta} &= \frac{-(\xi - x)^2 + (\eta - y)^2}{r^4}, \\ \frac{\partial Q}{\partial \xi} &= \frac{-(\xi - x)^2 + (\eta - y)^2}{r^4}, \end{aligned}$$

因而 P, Q 的偏导数除去点 A (此处 $r = 0$) 外, 在全平面上是连续的, 并且 $\frac{\partial Q}{\partial \xi} = \frac{\partial P}{\partial \eta}$. 于是, 利用格林公式知: 当点 A 在曲线 C 之外时, 有

$$u(x, y) = \oint_C \frac{\cos(\vec{r}, \vec{n})}{r} ds = 0.$$

当点 A 在曲线 C 之内时, 则在曲线 C 内以 A 为圆心, R 为半径作一圆 l , 即得

$$u(x, y) = \oint_l \frac{\cos(\vec{r}, \vec{n})}{r} ds = \oint_l \frac{1}{R} ds = 2\pi.$$

当点 A 在曲线 C 上时, 不妨利用关系式

$$\frac{\cos(\vec{r}, \vec{n})}{r} ds = d\varphi^{**},$$

其中 $d\varphi$ 为从点 A 看曲线 C 上弧长的微分 ds 所张的角度. 今以 A 为圆心, r_1 为半径作一小圆, 交 C 于 B_1 及 B_2 两点, 将曲线 C 除去小圆内的部分记以 $\widehat{B_1 B_2}$, 则有

$$\int_{\widehat{B_1 B_2}} \frac{\cos(\vec{r}, \vec{n})}{r} ds = \int_{\widehat{B_1 B_2}} d\varphi = \angle B_1 A B_2.$$

于是, 我们有

$$\begin{aligned} u(x, y) &= \oint_C \frac{\cos(\vec{r}, \vec{n})}{r} ds \\ &= \lim_{r_1 \rightarrow +0} \int_{\widehat{B_1 B_2}} \frac{\cos(\vec{r}, \vec{n})}{r} ds \\ &= \lim_{r_1 \rightarrow +0} \angle B_1 A B_2 = \pi. \end{aligned}$$

综上所述, 得高斯积分

$$u(x, y) = \oint_C \frac{\cos(\vec{r}, \vec{n})}{r} ds = \begin{cases} 0, & \text{点 } A \text{ 在 } C \text{ 外;} \\ \pi, & \text{点 } A \text{ 在 } C \text{ 上;} \\ 2\pi, & \text{点 } A \text{ 在 } C \text{ 内.} \end{cases}$$

*) 参看 Г. М. 菲赫金哥尔茨著《微积分学教程》538 目.

4330. 采用极坐标系 ρ 和 φ , 计算双层的对数位

$$K_1 = \int_0^{2\pi} \cos m\varphi \frac{\cos(\vec{r}, \vec{n})}{r} d\varphi$$

和

$$K_2 = \int_0^{2\pi} \sin m\psi \frac{\cos(\vec{r}, \vec{n})}{r} d\psi.$$

式中 r 为点 $A(\rho, \varphi)$ 和动点 $M(1, \psi)$ 之间的距离, $(\vec{r}, \vec{n})$ 为方向 $\overrightarrow{AM} = \vec{r}$ 与从点 $O(0, 0)$ 所引的半径 $\overrightarrow{OM} = \vec{n}$ 二者之间的夹角, m 为自然数.

解 由题意知:

$$\begin{aligned} & \frac{\cos(\vec{r}, \vec{n})}{r} \\ &= \frac{(\cos\psi - \rho\cos\varphi)\cos\psi + (\sin\psi - \rho\sin\varphi)\sin\psi}{(\cos\psi - \rho\cos\varphi)^2 + (\sin\psi - \rho\sin\varphi)^2} \\ &= \frac{1 - \rho\cos(\psi - \varphi)}{1 + \rho^2 - 2\rho\cos(\psi - \varphi)}. \end{aligned}$$

从而, 当 $\rho = 1$ 时, $\frac{\cos(\vec{r}, \vec{n})}{r} = \frac{1}{2}$. 又因 m 为自然数, 故此时有

$$K_1 = \frac{1}{2} \int_0^{2\pi} \cos m\psi d\psi = 0,$$

$$K_2 = \frac{1}{2} \int_0^{2\pi} \sin m\psi d\psi = 0.$$

当 $\rho < 1$ 时, 因为级数 (利用 2968 题的结果)

$$\frac{1 - \rho\cos(\psi - \varphi)}{1 + \rho^2 - 2\rho\cos(\psi - \varphi)} = 1 + \sum_{n=1}^{\infty} \rho^n \cos n(\psi - \varphi)$$

在 $[0, 2\pi]$ 上一致收敛, 乘 $\cos m(\psi - \varphi)$ 和 $\sin m(\psi - \varphi)$ 以后在 $[0, 2\pi]$ 上也一致收敛, 故可逐项积分. 于是

$$\begin{aligned} K_1 &= \int_0^{2\pi} \cos m\psi \frac{1 - \rho\cos(\psi - \varphi)}{1 + \rho^2 - 2\rho\cos(\psi - \varphi)} d\psi \\ &= \int_0^{2\pi} [\cos m(\psi - \varphi)\cos m\varphi - \sin m(\psi - \varphi)\sin m\varphi] \end{aligned}$$

$$\begin{aligned}
& \cdot \left(1 + \sum_{n=1}^{+\infty} \rho^n \cos n(\psi - \varphi) \right) d\psi \\
&= \cos m\varphi \int_0^{2\pi} \cos m(\psi - \varphi) \rho^m \cos m(\psi - \varphi) d\psi \\
&= \rho^m \cos m\varphi \int_0^{2\pi} \cos^2 m(\psi - \varphi) d\psi \\
&= \pi \rho^m \cos m\varphi.
\end{aligned}$$

同理, 容易求得

$$\begin{aligned}
K_2 &= \int_0^{2\pi} \sin m\psi \frac{1 - \rho \cos(\psi - \varphi)}{1 + \rho^2 - 2\rho \cos(\psi - \varphi)} d\psi \\
&= \pi \rho^m \sin m\varphi.
\end{aligned}$$

当 $\rho > 1$ 时, 我们有

$$\begin{aligned}
K_1 &= \int_0^{2\pi} \cos m\psi \frac{1 - \rho \cos(\psi - \varphi)}{1 + \rho^2 - 2\rho \cos(\psi - \varphi)} d\psi \\
&= \frac{1}{2} \int_0^{2\pi} \cos m\psi \frac{2 - 2\rho \cos(\psi - \varphi)}{1 + \rho^2 - 2\rho \cos(\psi - \varphi)} d\psi \\
&= \frac{1}{2} \int_0^{2\pi} \cos m\psi \\
&\quad \cdot \frac{[1 + \rho^2 - 2\rho \cos(\psi - \varphi)] + (1 - \rho^2)}{1 + \rho^2 - 2\rho \cos(\psi - \varphi)} d\psi \\
&= \frac{1}{2} \int_0^{2\pi} \cos m\psi \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\psi - \varphi)} d\psi \\
&= -\frac{1}{2} \int_0^{2\pi} \cos m\psi \frac{1 - r^2}{1 + r^2 - 2r \cos(\psi - \varphi)} d\psi \\
&= -\frac{1}{2} \int_0^{2\pi} \cos m\psi \\
&\quad \cdot \frac{(1 - r^2) + [1 + r^2 - 2r \cos(\psi - \varphi)]}{1 + r^2 - 2r \cos(\psi - \varphi)} d\psi \\
&= -\int_0^{2\pi} \cos m\psi \frac{1 - r \cos(\psi - \varphi)}{1 + r^2 - 2r \cos(\psi - \varphi)} d\psi \\
&= -\pi r^m \cos m\varphi = -\frac{\pi}{\rho^m} \cos m\varphi,
\end{aligned}$$

其中 $r = \rho^{-1} < 1$.

同理, 可求得

$$\begin{aligned} K_2 &= \int_0^{2\pi} \sin m\phi \frac{1 - \rho \cos(\phi - \varphi)}{1 + \rho^2 - 2\rho \cos(\phi - \varphi)} d\phi \\ &= -\frac{\pi}{\rho^m} \sin m\varphi. \end{aligned}$$

综上所述, 得

$$K_1 = \pi \rho^m \cos m\varphi, \quad K_2 = \pi \rho^m \sin m\varphi, \quad \text{当 } \rho < 1,$$

$$K_1 = K_2 = 0, \quad \text{当 } \rho = 1,$$

$$K_1 = -\frac{\pi}{\rho^m} \cos m\varphi, \quad K_2 = -\frac{\pi}{\rho^m} \sin m\varphi, \quad \text{当 } \rho > 1.$$

4331. 若 $\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, 则称可微分两次的函数 $u = u(x, y)$ 为调和函数. 证明: 当且仅当

$$\oint_C \frac{\partial u}{\partial n} ds = 0$$

(式中 C 为任意封闭围线, $\frac{\partial u}{\partial n}$ 为沿此围线之外法线的导函数) 时, u 是调和函数.

证 由于

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \cos(\vec{n}, x) + \frac{\partial u}{\partial y} \sin(\vec{n}, x),$$

而(参看 4324 题的推导)

$$\cos(\vec{n}, x) = \frac{dy}{ds}, \quad \sin(\vec{n}, x) = -\frac{dx}{ds},$$

故利用格林公式(注意, 题中应假定 $u(x, y)$ 具有连续的二阶偏导数), 得

$$\oint_C \frac{\partial u}{\partial n} ds = \oint_C \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$$

$$\begin{aligned}
&= \iint_S \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy \\
&= \iint_S (\Delta u) dx dy,
\end{aligned}$$

其中 S 表由封闭围线 C 围成的区域. 由此式知:

$\oint_C \frac{\partial u}{\partial n} ds = 0$ (对任何封闭围线 C) 当且仅当 $\iint_S (\Delta u) \cdot dx dy = 0$ (对任何区域 S). 但易知这又相当于 $\Delta u \equiv 0$. 事实上, 若 $\Delta u \equiv 0$, 则对任何 S , 有 $\iint_S (\Delta u) \cdot dx dy = 0$;

反之, 若对任何 S , 有 $\iint_S (\Delta u) dx dy = 0$, 则必 $\Delta u \equiv 0$. 因为, 若不然, 在某点 (x_0, y_0) , $\Delta u \neq 0$. 例如, 设在此点, $\Delta u > 0$, 则由连续性可知, 必存在以 (x_0, y_0) 为中心, 半径为 r_0 (充分小) 的圆域 S_0 , 使在其上每一点, 都有 $\Delta u > 0$. 由此可知, $\iint_{S_0} (\Delta u) dx dy > 0$. 矛盾, 证毕.

4332. 证明:

$$\begin{aligned}
&\iint_S \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy \\
&= - \iint_S u \Delta u dx dy + \oint_C u \frac{\partial u}{\partial n} ds,
\end{aligned}$$

式中光滑围线 C 包围着有界域 S .

证 由于

$$\begin{aligned}
\oint_C u \frac{\partial u}{\partial n} ds &= \oint_C u \left[\frac{\partial u}{\partial x} \cos(\vec{n}, x) \right. \\
&\quad \left. + \frac{\partial u}{\partial y} \sin(\vec{n}, x) \right] ds
\end{aligned}$$

$$\begin{aligned}
&= \oint_C u \frac{\partial u}{\partial x} dy - u \frac{\partial u}{\partial y} dx \\
&= \iint_S \left[\frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(u \frac{\partial u}{\partial y} \right) \right] dxdy \\
&= \iint_S u \Delta u dxdy + \iint_S \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dxdy,
\end{aligned}$$

故得

$$\begin{aligned}
\iint_S \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dxdy &= - \iint_S u \Delta u dxdy \\
&+ \oint_C u \frac{\partial u}{\partial n} ds.
\end{aligned}$$

4333. 证明:在有界域 S 内及其周界 C 为调和的函数,则此函数单值地由它在围线 C 上的数值确定(参照习题 4332).

证 由题意知,我们只要证明:如有界域 S 上的两个调和函数 u_1 和 u_2 ,在其周界 C 上有相同的数值,则它们在整个域上恒等.这也就是要证明:若调和函数 $u = u_1 - u_2$ 在周界 C 上等于零,则它在整个域上恒为零.事实上,利用 4332 题的结果,得

$$\iint_S \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dxdy = 0.$$

于是,在整个域 S 上,有

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

这表明,在 S 上 u 为常数.但在周界 C 上 $u = 0$,故在域 S 上 $u \equiv 0$,即 $u_1 = u_2$.

4334. 证明平面上的格林第二公式

$$\iint_S \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx dy = \oint_C \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial v}{\partial n} \\ u & v \end{vmatrix} ds,$$

式中光滑的围线 C 包围着有界线 S , $\frac{\partial}{\partial n}$ 为沿 C 的外法线方向的导函数.

证 我们有

$$\begin{aligned} \oint_C v \frac{\partial u}{\partial n} ds &= \oint_C v \left(\frac{\partial u}{\partial x} \cos(\vec{n}, x) \right. \\ &\quad \left. + \frac{\partial u}{\partial y} \sin(\vec{n}, x) \right) ds \\ &= \oint_C v \frac{\partial u}{\partial x} dy - v \frac{\partial u}{\partial y} dx \\ &= \iint_S \left[\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(-v \frac{\partial u}{\partial y} \right) \right] dx dy \\ &= \iint_S \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy + \iint_S v \Delta u dx dy. \end{aligned}$$

同理, 有

$$\begin{aligned} \oint_C u \frac{\partial v}{\partial n} ds &= \oint_C u \frac{\partial v}{\partial x} dy - u \frac{\partial v}{\partial y} dx \\ &= \iint_S \left[\frac{\partial}{\partial x} \left(u \frac{\partial v}{\partial x} \right) - \frac{\partial}{\partial y} \left(-u \frac{\partial v}{\partial y} \right) \right] dx dy \\ &= \iint_S \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy + \iint_S u \Delta v dx dy. \end{aligned}$$

于是,

$$\oint_C \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial v}{\partial n} \\ u & v \end{vmatrix} ds = \oint_C \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds$$

$$\begin{aligned}
&= \iint_S v \Delta u dx dy - \iint_S u \Delta v dx dy \\
&= \iint_S \left| \begin{array}{cc} \Delta u & \Delta v \\ u & v \end{array} \right| dx dy.
\end{aligned}$$

4335. 利用格林第二公式证明, 若 $u = u(x, y)$ 是有界闭域 S 内的调和函数, 则

$$u(x, y) = \frac{1}{2\pi} \oint_C \left(u \frac{\partial \ln r}{\partial n} - \ln r \frac{\partial u}{\partial n} \right) ds,$$

式中 C 为域 S 的边界, $\vec{n}$ 为围线 C 的外法线方向, (x, y) 为域 S 内的点, $r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$ 为点 (x, y) 与围线 C 上的动点 (ξ, η) 间的距离.

证 先证 $v = \ln r$ 为 (ξ, η) ($(\xi, \eta) \neq (x, y)$) 的调和函数. 事实上, 我们有

$$\begin{aligned}
\frac{\partial v}{\partial \xi} &= \frac{\xi - x}{(\xi - x)^2 + (\eta - y)^2}, \\
\frac{\partial^2 v}{\partial \xi^2} &= \frac{(\eta - y)^2 - (\xi - x)^2}{[(\xi - x)^2 + (\eta - y)^2]^2}, \\
\frac{\partial v}{\partial \eta} &= \frac{\eta - y}{(\xi - x)^2 + (\eta - y)^2}, \\
\frac{\partial^2 v}{\partial \eta^2} &= \frac{(\xi - x)^2 - (\eta - y)^2}{[(\xi - x)^2 + (\eta - y)^2]^2}.
\end{aligned}$$

因此,

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = 0, \text{ 即 } \Delta v = 0.$$

今以点 (x, y) (当 $(\xi, \eta) \neq (x, y)$ 时) 为中心, ρ 为半径画一圆 C_0 , 使此圆包含在围线 C 内, C_0 及 C 的正向如图 8.67 所示. 曲线 C 的法线向外, C_0 的法线指向点 (x, y) . 因此, 在 C_0 上, 我们有

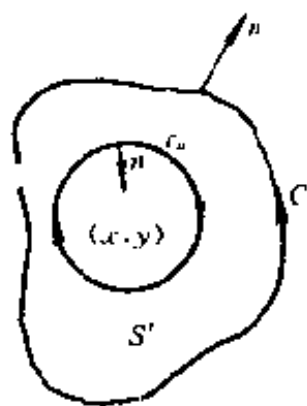


图 8.67

$$\begin{aligned}\frac{\partial \ln r}{\partial n} \Big|_{r=\rho} &= - \frac{\partial \ln r}{\partial r} \Big|_{r=\rho} \\ &= - \frac{1}{r} \Big|_{r=\rho} = - \frac{1}{\rho}.\end{aligned}$$

现将格林第二公式应用到由 C_0 及 C 所界的域 S' 上去, 即得

$$\iint_{S'} \begin{vmatrix} \Delta u & \Delta \ln r \\ u & \ln r \end{vmatrix} d\xi d\eta = \oint_{C_0+C} \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial \ln r}{\partial n} \\ u & \ln r \end{vmatrix} ds.$$

由于 $\Delta \ln r = 0, \Delta u = 0$, 故得

$$\oint_{C_0+C} \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial \ln r}{\partial n} \\ u & \ln r \end{vmatrix} ds = 0.$$

将行列式展开, 并利用线积分性质, 即得

$$\begin{aligned}& \oint_C \left(\ln r \frac{\partial u}{\partial n} - u \frac{\partial \ln r}{\partial n} \right) ds \\ &= - \oint_{C_0} \left(\ln r \frac{\partial u}{\partial n} - u \frac{\partial \ln r}{\partial n} \right) ds.\end{aligned}$$

但由于

$$\begin{aligned}& \oint_{C_0} \left(\ln r \frac{\partial u}{\partial n} - u \frac{\partial \ln r}{\partial n} \right) ds \\ &= \oint_{C_0} \ln \rho \frac{\partial u}{\partial n} ds - \oint_{C_0} u \left(- \frac{1}{\rho} \right) ds \\ &= 0 \cdot \ln \rho^{**} + \frac{1}{\rho} \oint_{C_0} u ds \\ &= \frac{1}{\rho} u(\xi', \eta') \oint_{C_0} ds^{**} = 2\pi u(\xi', \eta'),\end{aligned}$$

其中 $u(\xi', \eta')$ 为 u 在圆 C_0 上某点的值, 故得

$$u(\xi', \eta') = \frac{1}{2\pi} \oint_C \left(u \frac{\partial \ln r}{\partial n} - \ln r \frac{\partial u}{\partial n} \right) ds.$$

两端令 $\rho \rightarrow +0$ 取极限, 并注意到函数 u 在点 (x, y) 的连续性, 即得

$$u(x, y) = \frac{1}{2\pi} \oint_C \left(u \frac{\partial \ln r}{\partial n} - \ln r \frac{\partial u}{\partial n} \right) ds.$$

*) 利用 1331 题的结果.

**) 利用第一型曲线积分的中值定理, 其证明方法与普通定积分的中值定理类似.

4336.*) 证明对于调和函数 $u(M) = u(x, y)$ 的中值定理:

$$u(M) = \frac{1}{2\pi\rho} \oint_{C_\rho} u(\xi, \eta) ds;$$

式中 C_ρ 是以 M 点为中心 ρ 为半径的圆周.

证 利用 4335 题的结果(取 C 为 C_ρ), 得

$$u(M) = \frac{1}{2\pi} \oint_{C_\rho} \left(u \frac{\partial \ln r}{\partial n} - \ln r \frac{\partial u}{\partial n} \right) ds;$$

但在 C_ρ 上, 有

$$r = \rho,$$

$$\left. \frac{\partial \ln r}{\partial n} \right|_{r=\rho} = \left. \frac{\partial \ln r}{\partial r} \right|_{r=\rho} = \frac{1}{r} \Big|_{r=\rho} = \frac{1}{\rho},$$

由此, 再注意到 $\oint_{C_\rho} \frac{\partial u}{\partial n} ds = 0$ (这是利用 4331 题的结果), 得

$$\begin{aligned} u(M) &= \frac{1}{2\pi} \oint_{C_\rho} \left(\frac{u}{\rho} - \ln \rho \frac{\partial u}{\partial n} \right) ds \\ &= \frac{1}{2\pi\rho} \oint_{C_\rho} u ds - \frac{\ln \rho}{2\pi} \oint_{C_\rho} \frac{\partial u}{\partial n} ds \\ &= \frac{1}{2\pi\rho} \oint_{C_\rho} u(\xi, \eta) ds. \end{aligned}$$

证毕.

*) 原题中漏掉了 ρ , 即应将 $\frac{1}{2\pi}$ 改为 $\frac{1}{2\pi\rho}$.

4337. 证明在有界闭域内是调和的且于此域内不为常数的函数 $u(x, y)$ 在此域的内点不能达到其最大或最小值(极大值原则).

证 设有界闭域为 $\bar{\Omega}$, 它是由有界开域 Ω 及其边界 $\partial\Omega$ 构成. 我们要证明: 如果 $u(x, y)$ 在 $\bar{\Omega}$ 的某内点 $P_0(x_0, y_0)$ 达到其最大值或最小值(例如, 设达到最大值), 则 $u(x, y)$ 在 $\bar{\Omega}$ 上必为常数. 下分三步证之.

i) 先证: 若圆域 $S_\rho = \{(x, y) | (x - x_0)^2 + (y - y_0)^2 \leq \rho^2\}$ 完全属于 Ω , 则 $u(x, y)$ 在 S_ρ 上为常数.

对任何 $0 < r \leq \rho$, 用 C_r 表圆周 $\{(x, y) | (x - x_0)^2 + (y - y_0)^2 = r^2\}$. 由 4336 题的结果可知

$$u(x_0, y_0) = \frac{1}{2\pi r} \oint_{C_r} u(\xi, \eta) ds,$$

故

$$\frac{1}{2\pi r} \oint_{C_r} [u(x_0, y_0) - u(\xi, \eta)] ds = 0. \quad (1)$$

但因 $u(x_0, y_0)$ 为最大值, 故在 C_r 上恒有

$$u(x_0, y_0) - u(\xi, \eta) \geq 0.$$

由此, 根据(1), 即易知在 C_r 上 $u(x_0, y_0) - u(\xi, \eta) \equiv 0$. 因为, 若有某点 $(\xi_0, \eta_0) \in C_r$ 使 $u(x_0, y_0) - u(\xi_0, \eta_0) = \tau > 0$, 则由 $u(x, y)$ 的连续性可知, 必有以 (ξ_0, η_0) 为中心的某小圆域 σ 存在, 使当 $(\xi, \eta) \in \sigma$ 时, 恒有 $u(x_0, y_0)$

$u(\xi, \eta) \geq \frac{\tau}{2}$. 用 C' 表 C_r 含于 σ 内的部分, 则

$$\oint_{C_r} [u(x_0, y_0) - u(\xi, \eta)] ds \geq \int_{C'} [u(x_0, y_0)$$

$$-u(\xi, \eta)] ds \geq \int_{C_r'} \frac{\tau}{2} ds = \frac{1}{2} \tau l_r > 0,$$

其中 l_r 表圆弧 C_r' 之长, 此显然与 (1) 式矛盾.

于是, 在 C_r' 上恒有 $u(x_0, y_0) - u(\xi, \eta) \equiv 0$. 再根据 r 的任意性 ($0 < r \leq \rho$), 即知对任何 $(\xi, \eta) \in S_\rho$, 都有 $u(\xi, \eta) = u(x_0, y_0)$. 换句话说, $u(x, y)$ 在 S_ρ 上是常数.

ii) 次证: 设 $P^*(x^*, y^*)$ 为 $\bar{\Omega}$ 的任一内点 (即 $P^* \in \Omega$), 则必 $u(x^*, y^*) = u(x_0, y_0)$.

用完全含于 Ω 内的折线 l 将点 $P_0(x_0, y_0)$ 与点 $P^*(x^*, y^*)$ 连接起来 (图 8.68). 用 δ 表 $\partial\Omega$ 与 l 之间的

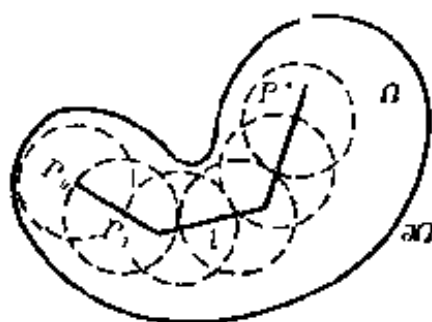


图 8.68

距离, 即 $\delta = \min \sqrt{(x - x')^2 + (y - y')^2}$, 其中的 $\min$ 是对一切 $(x, y) \in \partial\Omega, (x', y') \in l$ 来取的 (由于 $\partial\Omega, l$ 是互不相交的有界闭集, 可证 $\min$ 一定能达到, 从而 $\delta > 0$). 取 $0 < \delta' < \delta$. 以点 P_0 为中心, δ' 为半径作一圆, 得圆域 $S_{\delta'} = \{(x, y) | (x - x_0)^2 + (y - y_0)^2 \leq \delta'^2\}$, 此圆域完全含于 Ω 内, 由 i) 段已证的结论知 $u(x, y)$ 在 $S_{\delta'}$ 中为常数. 特别 $u(x_1, y_1) = u(x_0, y_0)$, 这里点 $P_1(x_1, y_1)$

代表圆周 $C_0 = \{(x, y) | (x - x_0)^2 + (y - y_0)^2 = \delta'^2\}$ 与折线 l 的交点. 又以点 P_1 为中心, δ' 为半径作一圆, 得圆域 $S_1 = \{(x, y) | (x - x_1)^2 + (y - y_1)^2 \leq \delta'^2\}$. 由于 $u(x, y)$ 在点 $P_1(x_1, y_1)$ 也达到最大值, 而 S_1 完全含于 Ω 内, 故将 i) 段结果用于 S_1 可知 $u(x, y)$ 在 S_1 上为常数, 特别 $u(x_2, y_2) = u(x_1, y_1)$, 这里点 $P_2(x_2, y_2)$ 表圆周 $C_1 = \{(x, y) | (x - x_1)^2 + (y - y_1)^2 = \delta'^2\}$ 与 l 的交点 (除 P_0 外的另一交点). 再以点 P_2 为中心, δ' 为半径作一圆域 $S_2, \dots$, 这样继续作下去, 显然, 至多经过 n 次 (n 表大于 $\frac{s}{\delta'}$ 的最小正整数, s 表 l 的长), 点 $P^*(x^*, y^*)$ 必属于 S_{n-1} , 从而

$$\begin{aligned} u(x^*, y^*) &= u(x_{n-1}, y_{n-1}) = \dots = u(x_1, y_1) \\ &= u(x_0, y_0). \end{aligned}$$

iii) 由 ii) 段的结果可知, $u(x, y)$ 在 Ω 上是常数; 根据 $u(x, y)$ 在 $\bar{\Omega}$ 上的连续性, 通过由 Ω 的点趋向 $\partial\Omega$ 的点取极限, 即知 $u(x, y)$ 在 $\bar{\Omega}$ 上是常数. 证毕.

注: 从证明过程中看出, 需假定区域 Ω (从而 $\bar{\Omega}$) 是连通的. 事实上, 若 Ω 不连通, 则结论不一定成立. 例如, 设 $\bar{\Omega} = S_1 + S_2$, 其中 S_1 与 S_2 是两个互无公共点的闭圆域, 而令

$$u(x, y) = \begin{cases} c_1, & (x, y) \in S_1; \\ c_2, & (x, y) \in S_2, \end{cases}$$

其中 $c_1 \neq c_2$ 是两个常数, 则显然 $u(x, y)$ 是 $\bar{\Omega}$ 上的调和函数且在 $\bar{\Omega}$ 上不是常数, 但它却在其内点达到最大值与最小值.

4338. 证明黎曼公式

$$\iint_S \begin{vmatrix} L(u) & M(v) \\ u & v \end{vmatrix} dx dy = \oint_C P dx + Q dy,$$

式中

$$L(u) = \frac{\partial^2 u}{\partial x \partial y} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} - cu,$$

$$M(v) = \frac{\partial^2 v}{\partial x \partial y} - a \frac{\partial v}{\partial x} - b \frac{\partial v}{\partial y} + cv$$

(a, b, c 为常数), P 和 Q 为某些确定的函数, 围线 C 包围着有界域 S .

证 因为

$$\begin{aligned} \begin{vmatrix} L(u) & M(v) \\ u & v \end{vmatrix} &= vL(u) - uM(v) \\ &= v \frac{\partial^2 u}{\partial x \partial y} + av \frac{\partial u}{\partial x} + bv \frac{\partial u}{\partial y} + cuv \\ &\quad - u \frac{\partial^2 v}{\partial x \partial y} + au \frac{\partial v}{\partial x} + bu \frac{\partial v}{\partial y} - cuv \\ &= \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} \right) + a \frac{\partial}{\partial x} (vu) \\ &\quad + b \frac{\partial}{\partial y} (uv) \\ &= \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} + auv \right) - \frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} - buv \right), \end{aligned}$$

故利用格林公式, 即得

$$\iint_S \begin{vmatrix} L(u) & M(v) \\ u & v \end{vmatrix} dx dy = \oint_C P dx + Q dy,$$

其中

$$P = u \frac{\partial v}{\partial x} - buv, \quad Q = v \frac{\partial u}{\partial y} + auv.$$

4339. 设 $u = u(x, y)$ 和 $v = v(x, y)$ 为液体的速度的分量.

求在单位时间内流过以围线 C 为界的域 S 的液体的量 (即液体流出量与流入量的差). 若液体不能压缩且在域 S 内没有源泉和漏孔, 则函数 u 和 v 满足怎样的方程式?

解 设液体的速度为 $\vec{W}$, 则 $\vec{W} = u\vec{i} + v\vec{j}$, 又 $d\vec{s} = dx\vec{i} + dy\vec{j}$. 于是, 所求的液体量

$$\begin{aligned} Q &= \oint_C \vec{w} \cdot \vec{n} ds = \oint_C [u \cos(\vec{n}, x) + v \sin(\vec{n}, x)] ds \\ &= \oint_C u dy - v dx = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx dy, \end{aligned}$$

其中 $\vec{n}$ 表示曲线 C 的外法线上的单位矢量, 并且此处已假定流体的面密度等于 1. 若液体是不可压缩的, 且在域 S 内无源泉和漏孔, 则液体流出量与流入量的差 Q 应等于零, 即

$$\iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx dy = 0.$$

又显然, 对于任意的围线 C , 上述结果均正确. 于是, 连续函数 u, v 应满足方程:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

*) 参看 4323 题的题解.

4340⁺ 根据比奥沙伐耳 (BIO — SAVART) 定律通过线元 ds 的电流 i 在空间的点 $M(x, y, z)$ 处产生一磁场, 其应力为

$$d\vec{H} = ki \frac{(\vec{r} \times d\vec{s})}{r^3},$$

其中 $\vec{r}$ 为连接元素 $d\vec{s}$ 与点 M 的向量, k 为比例系数. 对于封闭导线 C 的情形求磁场 $\vec{H}$ 在点 M 之应力的射影 H_x, H_y, H_z .

解 由题意知:若设导线 C 上的动点为 (ξ, η, ζ) , 则

$$\vec{r} = (\xi - x)\vec{i} + (\eta - y)\vec{j} + (\zeta - z)\vec{k},$$

又 $d\vec{s} = d\xi\vec{i} + d\eta\vec{j} + d\zeta\vec{k}$. 于是, 磁场强度

$$\begin{aligned}\vec{H} &= ki \oint_C \frac{1}{r^3} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \xi - x & \eta - y & \zeta - z \\ d\xi & d\eta & d\zeta \end{vmatrix} \\ &= ki \oint_C \frac{1}{r^3} [(\eta - y)d\zeta - (\zeta - z)d\eta] \vec{i} \\ &\quad + ki \oint_C \frac{1}{r^3} [(\zeta - z)d\xi - (\xi - x)d\zeta] \vec{j} \\ &\quad + ki \oint_C \frac{1}{r^3} [(\xi - x)d\eta - (\eta - y)d\xi] \vec{k},\end{aligned}$$

从而射影

$$H_x = ki \oint_C \frac{1}{r^3} [(\eta - y)d\zeta - (\zeta - z)d\eta],$$

$$H_y = ki \oint_C \frac{1}{r^3} [(\zeta - z)d\xi - (\xi - x)d\zeta],$$

$$H_z = ki \oint_C \frac{1}{r^3} [(\xi - x)d\eta - (\eta - y)d\xi],$$

§ 14. 曲面积分

1° 第一型的曲面积分 若 S 为逐片光滑的双面曲面

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

$$((u, v) \in Q), \quad (1)$$

而 $f(x, y, z)$ 为在曲面 S 上的各点上有定义并且是连续的函数, 则

$$\iint_S f(x, y, z) dS = \iint_Q f(x(u, v), y(u, v), z(u, v))$$

$$\cdot \sqrt{EG} \cdot F^2 du dv, \quad (2)$$

式中

$$E = \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2,$$

$$G = \left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2,$$

$$F = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v}.$$

特别情形,若曲面的方程式的形状为

$$z = z(x, y) \quad ((x, y) \in \sigma),$$

其中 $z(x, y)$ 为单值连续地可微分函数,则

$$\begin{aligned} \iint_{\Sigma} f(x, y, z) ds &= \iint_{\sigma} f(x, y, z(x, y)) \\ &\cdot \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dx dy. \end{aligned}$$

此积分与曲面 S 的方向的选择无关.

若把函数 $f(x, y, z)$ 当作曲面 S 在点 (x, y, z) 的密度,则积分(2)是此曲面的质量.

2° 第二型的曲面积分 若 S 为平滑的双面曲面; S^+ 为它的正面,由法线的方向 $\vec{n} = \{\cos\alpha, \cos\beta, \cos\gamma\}$ 所确定的一面, $P = P(x, y, z)$, $Q = Q(x, y, z)$, $R = R(x, y, z)$ 为在曲面 S 上有定义而且连续的三个函数,则

$$\begin{aligned} &\iint_{S^+} P dy dz + Q dx dz + R dx dy \\ &= \iint_S (P \cos\alpha + Q \cos\beta + R \cos\gamma) dS. \end{aligned} \quad (3)$$

若曲面 S 的方程为参数式(1),则法线 $\vec{n}$ 的方向余弦由下列公式来确定:

$$\cos\alpha = \frac{A}{\pm \sqrt{A^2 + B^2 + C^2}},$$

$$\cos \beta = \frac{B}{\pm \sqrt{A^2 + B^2 + C^2}},$$

$$\cos \gamma = \frac{C}{\pm \sqrt{A^2 + B^2 + C^2}},$$

$$\text{其中 } A = \frac{\partial(y, z)}{\partial(u, v)}, \quad B = \frac{\partial(z, x)}{\partial(u, v)},$$

$$C = \frac{\partial(x, y)}{\partial(u, v)},$$

且方根前的符号用适当的方法来选择.

当变换为曲面 S 的另一面 S^- 时, 积分 (3) 的符号变为相反的符号.

4341. 积分

$$I_1 = \iint_S (x^2 + y^2 + z^2) dS$$

$$\text{和 } I_2 = \iint_P (x^2 + y^2 + z^2) dP,$$

(式中 S 为球 $x^2 + y^2 + z^2 = a^2$ 的表面, P 为内接于此球的八面体 $|x| + |y| + |z| = a$ 的表面) 相差若干?

解 若令

$$x = a \sin \varphi \cos \theta, \quad y = a \sin \varphi \sin \theta, \quad z = a \cos \varphi,$$

则有

$$\begin{aligned} I_1 &= \iint_S (x^2 + y^2 + z^2) dS = \int_0^\pi d\varphi \int_0^{2\pi} a^2 \cdot a^2 \sin \varphi d\theta \\ &= 4\pi a^4. \end{aligned}$$

为求 I_2 , 只要注意到 $|z| = a - (|x| + |y|)$, 并利用对称性, 即得

$$\begin{aligned} I_2 &= \iint_P (x^2 + y^2 + z^2) dP = 8 \int_0^a dx \int_0^{a-x} \sqrt{3} \\ &\quad \cdot (x^2 + y^2 + (a - x - y)^2) dy \end{aligned}$$

$$\begin{aligned}
&= 16 \sqrt{3} \int_0^a dx \int_0^a [x^2 + y^2 + xy + \frac{a^2}{2} \\
&\quad - a(x + y)] dy \\
&= 16 \sqrt{3} \int_0^a [x^2(a - x) - \frac{1}{6}(a - x)^3 \\
&\quad - ax(a - x) + \frac{a^2}{2}(a - x)] dx \\
&= 16 \sqrt{3} \left(\frac{1}{3} - \frac{1}{4} - \frac{1}{24} - \frac{1}{2} + \frac{1}{3} + \frac{1}{2} - \frac{1}{4} \right) a^4 \\
&= 2 \sqrt{3} a^4.
\end{aligned}$$

于是,两积分之差

$$I_1 - I_2 = 2(2\pi - \sqrt{3})a^4.$$

4342. 计算

$$\iint_S z dS,$$

式中 S 为曲面 $x^2 + z^2 = 2ax (a > 0)$ 被曲面 $z = \sqrt{x^2 + y^2}$ 所割下的部分.

解 作变换

$$x = a \sin \theta, \quad y = y, \quad z = a + a \cos \theta,$$

则两曲面分别化为

$$r = 1, \text{ 和 } y^2 = 2a^2 \cos \theta (1 + \cos \theta)$$

两曲面交线的参数方程为

$$\begin{aligned}
x &= a \sin \theta, \quad y = \pm \sqrt{2} a \sqrt{\cos \theta (1 + \cos \theta)}, \\
z &= a + a \cos \theta \quad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right).
\end{aligned}$$

于是,

$$\iint_S z dS = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{-\frac{\sqrt{2}a\sqrt{\cos\theta(1+\cos\theta)}}{\sqrt{2}a\sqrt{\cos\theta(1+\cos\theta)}}}^{\frac{\sqrt{2}a\sqrt{\cos\theta(1+\cos\theta)}}{\sqrt{2}a\sqrt{\cos\theta(1+\cos\theta)}}} (a + a \cos \theta) a dy$$

$$\begin{aligned}
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \sqrt{2} a^3 \sqrt{\cos \theta} \sqrt{(1 + \cos \theta)^3} d\theta \\
&= -4 \sqrt{2} a^3 \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos \theta} \sqrt{(1 + \cos \theta)^3}}{\sin \theta} d(\cos \theta) \\
&= -4 \sqrt{2} a^3 \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos \theta} (1 + \cos \theta)}{\sqrt{1 - \cos \theta}} d(\cos \theta) \\
&= 4 \sqrt{2} a^3 \int_0^1 [t^{\frac{1}{2}} (1 - t)^{-\frac{1}{2}} + t^{\frac{3}{2}} (1 - t)^{-\frac{1}{2}}] dt \\
&= 4 \sqrt{2} a^3 \left[B\left(\frac{3}{2}, \frac{1}{2}\right) + B\left(\frac{5}{2}, \frac{1}{2}\right) \right] \\
&= \frac{7}{2} \sqrt{2} \pi a^3.
\end{aligned}$$

计算下列第一型曲面积分:

4343. $\iint_S (x + y + z) dS$, 式中 S 为曲面 $x^2 + y^2 + z^2 = a^2$,
 $z \geq 0$.

解 由于

$$\begin{aligned}
\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} \\
&= \frac{a}{\sqrt{a^2 - x^2 - y^2}},
\end{aligned}$$

故有

$$\begin{aligned}
\iint_S (x + y + z) dS &= \int_{-a}^a dx \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \\
&\quad \cdot [x + y + \sqrt{a^2 - x^2 - y^2}] dy \\
&= \int_{-a}^a (\pi a x + 2a \sqrt{a^2 - x^2}) dx \\
&= 4a \int_0^a \sqrt{a^2 - x^2} dx
\end{aligned}$$

$$= 4a \cdot \frac{\pi a^2}{4} = \pi a^3.$$

4344. $\iint_S (x^2 + y^2) dS$, 式中 S 为体积 $\sqrt{x^2 + y^2} \leq z \leq 1$ 的边界.

解 面积 S 由两部分组成. 一部分为 $S_1: z = \sqrt{x^2 + y^2}$, 它在 Oxy 平面上的射影为 $x^2 + y^2 = 1$; 另一部分为 $S_2: z = 1$, 它在 Oxy 平面上的射影也是 $x^2 + y^2 = 1$. 对于这两部分分别有

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{2},$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = 1.$$

若利用极坐标, 则

$$\begin{aligned} \iint_S (x^2 + y^2) dS &= \iint_{S_1} (x^2 + y^2) dS \\ &\quad + \iint_{S_2} (x^2 + y^2) dS \\ &= \sqrt{2} \int_0^{2\pi} d\varphi \int_0^1 r^3 dr + \int_0^{2\pi} d\varphi \int_0^1 r^3 dr \\ &= \frac{\pi}{2} (1 + \sqrt{2}). \end{aligned}$$

4345. $\iint_S \frac{dS}{(1 + x + y)^2}$, 式中 S 为四面体 $x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0$ 的边界.

解 曲面 S 由四部分组成, 分别为 $S_1: x + y + z = 1, x > 0, y > 0, z > 0; S_2: x = 0, S_3: y = 0, S_4: z = 0$. 于

是,我们有

$$\begin{aligned}
 \iint_S \frac{dS}{(1+x+y)^2} &= \sqrt{3} \int_0^1 dx \int_0^{1-x} \frac{dy}{(1+x+y)^2} \\
 &\quad + \int_0^1 dy \int_0^{1-y} \frac{dz}{(1+y)^2} + \int_0^1 dx \int_0^{1-x} \frac{dz}{(1+x)^2} \\
 &\quad + \int_0^1 dx \int_0^{1-x} \frac{dy}{(1+x+y)^2} \\
 &= (\sqrt{3}+1) \int_0^1 dx \int_0^{1-x} \frac{dy}{(1+x+y)^2} \\
 &\quad + 2 \int_0^1 dx \int_0^{1-x} \frac{dz}{(1+x)^2} \\
 &= (\sqrt{3}+1) \left(\ln 2 - \frac{1}{2} \right) + 2(1 - \ln 2) \\
 &= \frac{3-\sqrt{3}}{2} + (\sqrt{3}-1)\ln 2.
 \end{aligned}$$

4346. $\iint_S |xyz| dS$, 式中 S 为曲面 $z = x^2 + y^2$ 被平面 $z = 1$ 所割下的部分.

解 由于

$$\sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} = \sqrt{1 + 4(x^2 + y^2)},$$

故利用极坐标,并注意对称性,即得

$$\begin{aligned}
 \iint_S |xyz| dS &= 4 \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r^4 \cos\varphi \sin\varphi \sqrt{1+4r^2} r dr \\
 &= 2 \int_0^1 r^5 \sqrt{1+4r^2} dr = \int_0^1 t^2 \sqrt{1+4t} dt \\
 &= \int_1^{\sqrt{5}} \frac{1}{32} (y^2-1)^2 y^2 dy \\
 &= \frac{1}{32} \left(\frac{y^7}{7} - \frac{2y^5}{5} + \frac{y^3}{3} \right) \Big|_1^{\sqrt{5}} = \frac{125\sqrt{5}-1}{420}.
 \end{aligned}$$

*) 作代换 $r^2 = t$.

* *) 作代换 $\sqrt{1+4t} = y$.

4347. $\iint_S \frac{dS}{\rho}$, 式中 S 为椭球表面, ρ 为椭球中心到与椭球表面的元素 dS 相切的平面之间的距离.

解 设椭球面方程为

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

则曲面上任一点 (x, y, z) 的法矢量为 $\left\{ \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\}$.

从而, 由题设知: $\rho = \sqrt{x^2 + y^2 + z^2} \cos(\vec{n}, \vec{r})$, 其中 $\vec{n}$, $\vec{r}$ 分别表示点 (x, y, z) 处的法矢量和矢径, 即

$$\rho = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}},$$

而法线与 Oz 轴夹角的余弦为

$$\frac{\frac{z}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}.$$

于是,

$$\begin{aligned} \iint_S \frac{dS}{\rho} &= \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \frac{c^2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)}{|z|} dx dy \\ &= 2 \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \frac{c \left[\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} \right) + \frac{1}{c^2} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) \right]}{\sqrt{1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)}} dx dy \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^1 dr \int_0^{2\pi} \frac{c}{\sqrt{1-r^2}} \left(\frac{r^2 \cos^2 \theta}{a^2} + \frac{r^2 \sin^2 \theta}{b^2} \right. \\
&\quad \left. + \frac{1}{c^2} - \frac{r^2}{c^2} \right) abrd\theta^*, \\
&= 2\pi abc \int_0^1 \left[\frac{1}{\sqrt{1-r^2}} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) - \sqrt{1-r^2} \right. \\
&\quad \left. \cdot \left(\frac{1}{a^2} + \frac{1}{b^2} \right) + 2 \frac{\sqrt{1-r^2}}{c^2} \right] r dr^*, \\
&= -\pi abc \left[2 \sqrt{1-r^2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) - \frac{2}{3} (1-r^2)^{\frac{3}{2}} \right] \\
&\quad \cdot \left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{4}{3c^2} (1-r^2)^{\frac{3}{2}} \Big|_0^1 \\
&= \frac{4\pi}{3} abc \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right).
\end{aligned}$$

*) 作广义极坐标变换: $x = a \cos \theta, y = b \sin \theta$.

* *) 利用关系式: $\frac{r^2}{\sqrt{1-r^2}} = \frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}$.

4348. $\iint_S x dS$, 式中 S 为螺旋面 $x = u \cos v, y = u \sin v, z = v$ ($0 < u < 1; 0 < v < 2\pi$) 的一部分.

解 由于

$$\begin{aligned}
E &= \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2 \\
&= \cos^2 v + \sin^2 v = 1, \\
G &= \left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 \\
&= u^2 \sin^2 v + u^2 \cos^2 v + 1 = 1 + u^2, \\
F &= \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \cdot \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v} \\
&= -u \sin v \cos v + u \cos v \sin v = 0,
\end{aligned}$$

故得 $\sqrt{EG-F^2} = \sqrt{1+u^2}$. 于是,

$$\begin{aligned}\iint_S z dS &= \int_0^u du \int_0^{2\pi} v \sqrt{1+u^2} dv \\ &= 2\pi^2 \int_0^u \sqrt{1+u^2} du \\ &= 2\pi^2 \left[\frac{u}{2} \sqrt{1+u^2} + \frac{1}{2} \ln(u + \sqrt{1+u^2}) \right] \Big|_0^u \\ &= \pi^2 [a \sqrt{1+a^2} + \ln(a + \sqrt{1+a^2})].\end{aligned}$$

4349. $\iint_S z^2 dS$, 式中 S 为圆锥表面的一部分 $x = r \cos \varphi \sin \alpha$, $y = r \sin \varphi \sin \alpha$, $z = r \cos \alpha$ ($0 \leq r \leq a$; $0 \leq \varphi \leq 2\pi$) 和 α 为常数 ($0 < \alpha < \frac{\pi}{2}$).

解 由于

$$E = \cos^2 \varphi \sin^2 \alpha + \sin^2 \varphi \sin^2 \alpha + \cos^2 \alpha = 1,$$

$$G = r'^2 \cos^2 \varphi \sin^2 \alpha + r^2 \sin^2 \varphi \sin^2 \alpha = r^2 \sin^2 \alpha,$$

$$F = (\cos \varphi \sin \alpha)(-r \sin \varphi \sin \alpha) + \sin \varphi \sin \alpha$$

$$\cdot (r \cos \varphi \sin \alpha) = 0,$$

故得 $\sqrt{EG-F^2} = r \sin \alpha$. 于是,

$$\begin{aligned}\iint_S z^2 dS &= \int_0^{2\pi} d\varphi \int_0^a r^2 \cos^2 \alpha \cdot r \sin \alpha dr \\ &= \frac{\pi a^4}{2} \sin \alpha \cos^2 \alpha.\end{aligned}$$

4350. $\iint_S (xy + yz + zx) dS$, 式中 S 为圆锥曲面 $z = \sqrt{x^2 + y^2}$ 被曲面 $x^2 + y^2 = 2ax$ 所割下的部分.

解 由于

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}}$$

$$= \sqrt{2},$$

又曲面 S 在 Oxy 平面上的射影域为 $x^2 + y^2 \leq 2ax$.

于是, 利用极坐标, 即得

$$\begin{aligned} & \iint_S (xy + yz + zx) dS \\ &= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2a\cos\varphi} [r^2\cos\varphi\sin\varphi + r^2(\cos\varphi + \sin\varphi)] r dr \\ &= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} (2a\cos\varphi)^4 \cos\varphi d\varphi \\ &= 8\sqrt{2}a \int_0^{\frac{\pi}{2}} \cos^5\varphi d\varphi = \frac{64}{15} \sqrt{2} a^4. \end{aligned}$$

4351. 证明 普阿桑公式

$$\begin{aligned} & \iint_S f(ax + by + cz) dS \\ &= 2\pi \int_0^1 f(u \sqrt{a^2 + b^2 + c^2}) du, \end{aligned}$$

式中 S 是球 $x^2 + y^2 + z^2 = 1$ 的表面.

证 取新坐标系 $Ouvw$, 其中原点不变, 平面 $ax + by + cz = 0$ 即为 Ovw 面, u 轴垂直于该面, 则有

$$u = \frac{ax + by + cz}{\sqrt{a^2 + b^2 + c^2}}.$$

在新坐标系下, 公式左端的积分可写为

$$\iint_S f(u \sqrt{a^2 + b^2 + c^2}) dS.$$

显然, 球面 S 的方程为

$$u^2 + v^2 + w^2 = 1 \text{ 或 } v^2 + w^2 = (\sqrt{1 - u^2})^2.$$

若表示成参数式, 则为

$$u = u, \quad v = \sqrt{1-u^2}\cos\omega, \quad \omega = \sqrt{1-u^2}\sin\omega,$$

其中 $-1 \leq u \leq 1$, $0 \leq \omega \leq 2\pi$. 从而

$$\begin{aligned} dS &= \sqrt{EG-F^2} du d\omega \\ &= \sqrt{\frac{1}{1-u^2} \cdot (1-u^2)} du d\omega = du d\omega. \end{aligned}$$

于是,最后得

$$\begin{aligned} \iint_S f(ax+by+cz) dS &= \iint_S f(u \sqrt{a^2+b^2+c^2}) dS \\ &= \int_0^{2\pi} d\omega \int_{-1}^1 f(u \sqrt{a^2+b^2+c^2}) du \\ &= 2\pi \int_{-1}^1 f(u \sqrt{a^2+b^2+c^2}) du. \end{aligned}$$

4352. 求抛物面壳

$$z = \frac{1}{2}(x^2 + y^2) \quad (0 \leq z \leq 1)$$

的质量,此壳的密度按规律 $\rho = z$ 而变更.

解 质量为

$$\begin{aligned} M &= \iint_S \rho dS = \iint_{x^2+y^2 \leq 2} z \sqrt{1+x^2+y^2} dx dy \\ &= \frac{1}{2} \iint_{x^2+y^2 \leq 2} (x^2+y^2) \sqrt{1+x^2+y^2} dx dy \\ &= \frac{1}{2} \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} r^3 \sqrt{1+r^2} dr \\ &= \pi \int_0^{\sqrt{2}} r^3 \sqrt{1+r^2} dr \\ &= \frac{\pi}{2} \int_0^{\sqrt{2}} r^2 \sqrt{1+r^2} d(r^2) \\ &= \frac{\pi}{2} \left(\frac{2}{5} (1+r^2)^{\frac{5}{2}} \Big|_0^{\sqrt{2}} - \frac{2}{3} (1+r^2)^{\frac{3}{2}} \Big|_0^{\sqrt{2}} \right) \end{aligned}$$

$$= \frac{2\pi(1 + 6\sqrt{3})}{15}.$$

4353. 求密度为 ρ_0 的均匀球壳

$$x^2 + y^2 + z^2 = a^2 \quad (z \geq 0)$$

对于 Oz 轴的转动惯量.

解 转动惯量为

$$\begin{aligned} I_x &= \iint_S (x^2 + y^2) \rho_0 dS \\ &= \rho_0 \iint_{x^2 + y^2 \leq a^2} (x^2 + y^2) \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy \\ &= a \rho_0 \int_0^{2\pi} d\varphi \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} dr \\ &= 2\pi a^4 \rho_0 \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta = \frac{4}{3} \pi a^4 \rho_0. \end{aligned}$$

4354. 求密度为 ρ_0 的均匀锥面壳

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 0 \quad (0 \leq z \leq b)$$

对于直线

$$\frac{x}{1} = \frac{y}{0} = \frac{z-b}{0}$$

的转动惯量

解 设 (x, y, z) 为均匀锥面壳上的任一点, 它到直线

$$\frac{x}{1} = \frac{y}{0} = \frac{z-b}{0}$$

的距离为

$$|d| =$$

$$\sqrt{\frac{\begin{vmatrix} y-0 & z-b \\ 0 & 0 \end{vmatrix}^2 + \begin{vmatrix} z-b & x-0 \\ 0 & 1 \end{vmatrix}^2 + \begin{vmatrix} x-0 & y & 0 \\ 1 & 0 \end{vmatrix}^2}{\sqrt{1^2 + 0^2 + 0^2}}} \\ = \sqrt{\left(\frac{b}{a} \sqrt{x^2 + y^2} - b\right)^2 + y^2}.$$

又因

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \frac{\sqrt{a^2 + b^2}}{a}.$$

于是,所求的转动惯量为

$$\begin{aligned} I &= \iint_{x^2+y^2 \leq a^2} \left[\left(\frac{b}{a} \sqrt{x^2+y^2} - b \right)^2 + y^2 \right] \\ &\quad \cdot \rho_0 \frac{\sqrt{a^2+b^2}}{a} dx dy \\ &= \frac{\sqrt{a^2+b^2} \rho_0}{a} \int_0^{2\pi} d\varphi \int_0^a \left[\left(\frac{b}{a} r - b \right)^2 \right. \\ &\quad \left. + r^2 \sin^2 \varphi \right] dr \\ &= \frac{\sqrt{a^2+b^2} \rho_0}{a} \left[2\pi a^2 b^2 \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) + \frac{\pi a^4}{4} \right] \\ &= \frac{\pi a \rho_0 (3a^2 + 2b^2) \sqrt{a^2+b^2}}{12}. \end{aligned}$$

4355. 求均匀的曲面

$$z = \sqrt{x^2 + y^2}$$

被曲面 $x^2 + y^2 = ax$ 所割下部分的重心的坐标.

解 质量为

$$M = \iint_S \rho_0 dS = \sqrt{2} \rho_0 \iint_{x^2+y^2 \leq ax} dx dy$$

$$= \sqrt{2} \rho_0 \left(\frac{a}{2} \right)^2 \pi = \frac{\sqrt{2} \pi a^2 \rho_0}{4}.$$

从而,重心的坐标为

$$\begin{aligned} x_0 &= \frac{1}{M} \cdot \sqrt{2} \rho_0 \iint_{x^2+y^2 \leq ax} x dx dy \\ &= \frac{4}{\pi a^2} \int_0^a x dx \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} dy \\ &= \frac{8}{\pi a^2} \int_0^a x \sqrt{ax-x^2} dx \\ &= \frac{8}{\pi a^2} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\frac{a}{2} + t \right) \sqrt{\left(\frac{a}{2} \right)^2 - t^2} dt^{*)} \\ &= \frac{8}{\pi a} \int_0^{\frac{a}{2}} \sqrt{\left(\frac{a}{2} \right)^2 - t^2} dt = \frac{a}{2}. \end{aligned}$$

$$y_0 = \frac{1}{M} \cdot \sqrt{2} \rho_0 \int_0^a dx \int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} y dy = 0.$$

$$\begin{aligned} z_0 &= \frac{1}{M} \cdot \sqrt{2} \rho_0 \iint_{x^2+y^2 \leq ax} z dx dy \\ &= \frac{4}{\pi a^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a \cos \varphi} r^2 dr \\ &= \frac{8a}{3\pi} \int_0^{\frac{\pi}{2}} \cos^3 \varphi d\varphi = \frac{16a}{9\pi}, \end{aligned}$$

即重心的坐标为 $\left(\frac{a}{2}, 0, \frac{16a}{9\pi} \right)$.

*) 作变换 $t = x - \frac{a}{2}$.

4356. 求均匀曲面

$$z = \sqrt{a^2 - x^2 - y^2} (x \geq 0; y \geq 0; x + y \leq a)$$

的重心的坐标.

解 因为

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}},$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}},$$

所以,

$$\begin{aligned} dS &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy \\ &= \frac{a}{\sqrt{a^2 - x^2 - y^2}} dxdy. \end{aligned}$$

由对称性知,重心的横坐标与纵坐标相等,即

$$x_0 = y_0 = \frac{\iint_S x dS}{\iint_S dS} = \frac{\int_0^a \int_0^{a-y} \frac{ax}{\sqrt{a^2 - x^2 - y^2}} dxdy}{\int_0^a \int_0^{a-x} \frac{a}{\sqrt{a^2 - x^2 - y^2}} dydx}.$$

由于

$$\begin{aligned} &\int_0^a \int_0^{a-y} \frac{ax}{\sqrt{a^2 - x^2 - y^2}} dxdy \\ &= a \int_0^a \left(-\sqrt{a^2 - x^2 - y^2} \right) \Big|_{x=0}^{x=a-y} dy \\ &= a \left[\int_0^a \sqrt{a^2 - y^2} dy - \int_0^a \sqrt{2ay - 2y^2} dy \right] \\ &= a \left[\frac{\pi a^2}{4} - \sqrt{2} \cdot \frac{\pi \left(\frac{a}{2} \right)^2}{2} \right]^{**} \\ &= \frac{\pi a^2}{4} \left(1 - \frac{1}{\sqrt{2}} \right), \\ &\int_0^a \int_0^{a-x} \frac{a}{\sqrt{a^2 - x^2 - y^2}} dydx \end{aligned}$$

$$\begin{aligned}
&= a \int_0^a \arcsin \sqrt{\frac{a-x}{a+x}} dx \\
&= -4a^2 \int_1^0 \frac{u}{(1+u^2)^2} \arcsin u du \\
&= 2a^2 \left(\frac{\arcsin u}{1+u^2} \Big|_1^0 - \int_1^0 \frac{du}{(1+u^2)\sqrt{1-u^2}} \right) \\
&= 2a^2 \left(-\frac{\pi}{4} + \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{u\sqrt{2}}{\sqrt{1-u^2}} \Big|_0^1 \right)^{**)} \\
&= \pi a^2 \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right),
\end{aligned}$$

故有

$$x_0 = y_0 = \frac{\frac{\pi a^3}{4} \left(1 - \frac{1}{\sqrt{2}} \right)}{\pi a^2 \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right)} = \frac{a}{2\sqrt{2}}.$$

又由于

$$\begin{aligned}
\iint_S z dS &= \int_0^a \int_0^{a-x} \sqrt{a^2 - x^2 - y^2} \cdot \frac{a}{\sqrt{a^2 - x^2 - y^2}} dy dx \\
&= a \int_0^a (a-x) dx = \frac{a^3}{2},
\end{aligned}$$

故有

$$\begin{aligned}
z_0 &= \frac{\iint_S z dS}{\iint_S dS} = \frac{\frac{a^3}{2}}{\pi a^2 \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right)} \\
&= \frac{a}{\pi} (\sqrt{2} + 1),
\end{aligned}$$

即重心的坐标为 $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}, \frac{a}{\pi} (\sqrt{2} + 1) \right)$.

*) 由定积分的几何意义知:

$$\int_0^a \sqrt{y(a-y)} dy = \int_0^a \sqrt{\left(\frac{a}{2}\right)^2 - \left(y - \frac{a}{2}\right)^2} dy$$

$$= \frac{\pi a^2}{8}.$$

* *) 利用 1957 题的结果.

4357. 密度为 ρ_0 的均匀截锥面

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = r (0 \leq \varphi \leq 2\pi),$$

$$0 < b \leq r \leq a)$$

以怎样的力吸引质量为 m 位于该曲面顶点的质点?

解 显然曲面顶点为原点 $O(0,0,0)$. 对应于半径 r 处取斜高为 ds 的锥面带, 其面积为

$$dS = 2\pi r ds = 2\sqrt{2}\pi r dr.$$

它与顶点 O 处质量为 m 的质点的引力在 Ox 轴和 Oy 轴上的射影显见为零, 而在 Oz 轴上的射影为

$$dZ = \frac{km \cdot 2\sqrt{2}\pi r dr \rho_0}{r^2 + z^2} \cdot \frac{z}{\sqrt{r^2 + z^2}}$$

$$= \frac{k\pi m \rho_0 dr}{r}.$$

于是, 截圆锥面吸引质量为 m 的质点(在顶点处)的引力在坐标轴上的射影分别为

$$X = 0, \quad Y = 0,$$

$$Z = \int_b^a \frac{k\pi m \rho_0 dr}{r} = k\pi m \rho_0 \ln \frac{a}{b}.$$

4358. 求在点 $M_0(x_0, y_0, z_0)$ 的密度为 ρ_0 的均匀球壳 $x^2 + y^2 + z^2 = a^2$ 的位, 即: 计算积分

$$u = \iint_S \frac{\rho_0 dS}{r},$$

式中

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

解 记 $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$. 由对称性知, 在点 $M_0(x_0, y_0, z_0)$ 的位, 等于在点 $N_0(0, 0, r_0)$ 的位. 由余弦定理知, 球面上任一点 (x, y, z) 到点 N_0 的距离

$$r = \sqrt{a^2 + r_0^2 - 2r_0a\cos\psi} \quad (0 \leq \psi \leq \pi),$$

而球面带 $dS = 2\pi a^2 \sin\psi d\psi$. 于是, 所求的位为

$$u = \iint_S \frac{\rho_0 dS}{r} = 2\pi a^2 \rho_0 \int_0^\pi \frac{\sin\psi d\psi}{\sqrt{a^2 + r_0^2 - 2r_0a\cos\psi}}.$$

令 $u^2 = a^2 + r_0^2 - 2r_0a\cos\psi$, 则

$$2u du = 2r_0 a \sin\psi d\psi,$$

即

$$\sin\psi d\psi = \frac{u}{r_0 a} du.$$

从而, 所求的位为

$$u = \frac{2\pi a \rho_0}{r_0} \int_{|a-r_0|}^{a+r_0} du$$

$$= \begin{cases} 4\pi a \rho_0, & \text{当 } r_0 < a, \\ \frac{4\pi a^2 \rho_0}{r_0}, & \text{当 } r_0 > a, \\ 4\pi a \rho_0, & \text{当 } r_0 = a. \end{cases}$$

也即

$$u = 4\pi \rho_0 \min\left(a, \frac{a^2}{r_0}\right).$$

上述结果表明: 若 M_0 点在球壳内, 则位是个常量; 若 M_0 在球壳外, 则在该点球壳的位等于将球壳质量集中于球心的位; 当 M_0 点从球壳内通过球面时位具有连续性,

从而当 M_0 点在球面上时,位也是个常量,且等于球内任一点的位.

4359. 计算

$$F(t) = \iint_{x+y+z=t} f(x, y, z) dS,$$

式中

$$f(x, y, z) = \begin{cases} 1 - x^2 - y^2 - z^2, & \text{若 } x^2 + y^2 + z^2 \leq 1; \\ 0, & \text{若 } x^2 + y^2 + z^2 > 1. \end{cases}$$

作出函数 $u = F(t)$ 的图形.

解 显然,平面

$$x + y + z = \pm \sqrt{3}$$

是球面 $x^2 + y^2 + z^2 = 1$ 的两个切平面,于是,

$$f(x, y, z) = \begin{cases} 1 - x^2 - y^2 - z^2, & \text{若 } |t| \leq \sqrt{3}, \\ 0 & \text{若 } |t| > \sqrt{3}. \end{cases}$$

由方程组

$$\begin{cases} x + y + z = t, \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

得椭圆方程

$$x^2 + y^2 + [t - (x + y)]^2 = 1,$$

或

$$x^2 + y^2 + xy - t(x + y) = \frac{1 - t^2}{2}, \quad (1)$$

记该椭圆围成的区域为 Ω , 则

$$\begin{aligned} F(t) &= \iint_{\Omega} \{1 - x^2 - y^2 - [t - (x + y)]^2\} \sqrt{3} dx dy \\ &= \sqrt{3} \iint_{\Omega} [1 - t^2 - 2(x^2 + y^2) - 2xy] \end{aligned}$$

$$+ 2t(x + y)]dxdy.$$

作平移变换

$$x = x' + \frac{t}{3}, \quad y = y' + \frac{t}{3},$$

则方程(1)变为

$$x'^2 + y'^2 + x'y' = \frac{1}{2} \left(1 - \frac{t^2}{3} \right), \quad (2)$$

记相应的区域为 Ω' , 而函数为

$$f = 1 - \frac{t^2}{3} - 2(x'^2 + y'^2) - 2x'y'.$$

于是,

$$F(t) = \sqrt{3} \iint_{\Omega'} \left[1 - \frac{t^2}{3} - 2(x'^2 + y'^2) - 2x'y' \right] dx' dy'.$$

再作旋转变换

$$x' = \frac{x'' - y''}{\sqrt{2}}, \quad y' = \frac{x'' + y''}{\sqrt{2}},$$

则方程(2)变为椭圆的标准方程

$$\frac{x''^2}{\left[\frac{1}{\sqrt{3}} \sqrt{1 - \frac{t^2}{3}} \right]^2} + \frac{y''^2}{\left[\sqrt{1 - \frac{t^2}{3}} \right]^2} = 1. \quad (3)$$

记相应的区域为 Ω'' , 而函数为

$$f = 1 - \frac{t^2}{3} - (3x''^2 + y''^2).$$

于是,

$$F(t) = \sqrt{3} \iint_{\Omega''} \left[1 - \frac{t^2}{3} - (3x''^2 + y''^2) \right] dx'' dy''.$$

最后, 作广义的极坐标变换, 即

$$x'' = \frac{1}{\sqrt{3}} \sqrt{1 - \frac{t^2}{3}} r \cos \varphi,$$

$$y'' = \sqrt{1 - \frac{t^2}{3}} r \sin \varphi,$$

则有

$$\begin{aligned} F(t) &= \left(1 - \frac{t^2}{3}\right) \int_0^{2\pi} \int_0^1 \left(1 - \frac{t^2}{3}\right) (r - r^3) dr d\varphi \\ &= \left(1 - \frac{t^2}{3}\right)^2 \int_0^{2\pi} \frac{1}{4} d\varphi = \frac{\pi}{18} (3 - t^2)^2, \end{aligned}$$

其中 $|t| \leq \sqrt{3}$, 而当 $|t| > \sqrt{3}$, 则有

$$F(t) = 0.$$

考虑函数 $u = F(t) (-\infty < t < +\infty)$. 我们有

$$\frac{du}{dt} = -\frac{2\pi}{9} (3 - t^2)t \quad (|t| < \sqrt{3}).$$

当 $t = \sqrt{3}$ 时, u 的左导数 $= -\frac{2\pi}{9} (3 - t^2) \Big|_{t=\sqrt{3}} = 0$, u 的右导数显然为零 (因为 $t \geq \sqrt{3}$ 时, $u \equiv 0$), 故 $t = \sqrt{3}$ 时 u 的导数存在且等于零. 同理可证, $t = -\sqrt{3}$ 时, u 的导数也存在且等于零. 于是, 曲线 $u = F(t)$ 在 $t = 0$ 处以及 $|t| \geq \sqrt{3}$ 的各 t 处切线都平行于 Ot 轴. 又 $t = 0$ 处达极大值 $u = \frac{\pi}{2}$, 且为最大值.

由于

$$\frac{d^2u}{dt^2} = -\frac{2\pi}{3} (1 - t^2),$$

所以当 $t = \pm 1$ 时为拐点. 显然, 图形关于 Ou 轴是对称的. 函数 $u = F(t)$ 的图形, 如图 8.69 所示.

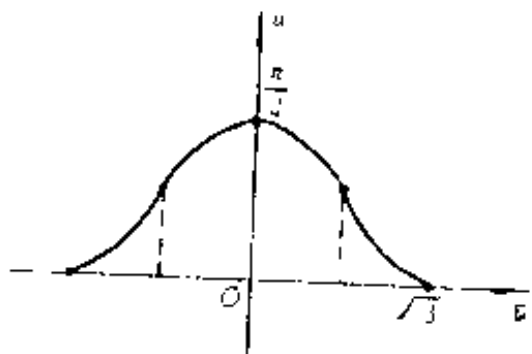


图 8.69

4360. 计算积分

$$F(t) = \iint_{x^2+y^2+z^2=t^2} f(x, y, z) dS,$$

式中

$$f(x, y, z) = \begin{cases} x^2 + y^2, & \text{若 } z \geq \sqrt{x^2 + y^2}; \\ 0, & \text{若 } z < \sqrt{x^2 + y^2}. \end{cases}$$

解 由球面方程 $x^2 + y^2 + z^2 = t^2$ 知

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{t^2 - x^2 - y^2}},$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{t^2 - x^2 - y^2}},$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \frac{|t|}{\sqrt{t^2 - (x^2 + y^2)}},$$

而由

$$\begin{cases} x^2 + y^2 + z^2 = t^2, \\ z^2 = x^2 + y^2 \end{cases}$$

可得

$$x^2 + y^2 = \frac{t^2}{2} = \left(\frac{t}{\sqrt{2}}\right)^2.$$

于是, 积分

$$\begin{aligned}
F(t) &= \iint_{x^2+y^2+z^2=t^2} f(x,y,z) dS \\
&= \iint_{x^2+y^2 \leq \left(\frac{t}{\sqrt{2}}\right)^2} (x^2+y^2) \\
&\quad \cdot \frac{|t|}{\sqrt{t^2-(x^2+y^2)}} dx dy \\
&= |t| \int_0^{2\pi} \int_0^{\frac{|t|}{\sqrt{2}}} \frac{r^3}{\sqrt{t^2-r^2}} dr d\varphi.
\end{aligned}$$

因为

$$\begin{aligned}
\int \frac{r^3}{\sqrt{t^2-r^2}} dr &= \frac{1}{2} \int \frac{t^2-r^2-t^2}{\sqrt{t^2-r^2}} d(t^2-r^2) \\
&= \frac{1}{3} (t^2-r^2)^{\frac{3}{2}} - t^2 \sqrt{t^2-r^2} + C,
\end{aligned}$$

所以

$$\begin{aligned}
\int_0^{\frac{t}{\sqrt{2}}} \frac{r^3}{\sqrt{t^2-r^2}} dr &= \left[\frac{1}{3} (t^2-r^2)^{\frac{3}{2}} \right. \\
&\quad \left. - t^2 \sqrt{t^2-r^2} \right] \Big|_0^{\frac{|t|}{\sqrt{2}}} \\
&= \frac{-5\sqrt{2}}{12} |t|^3 + \frac{2}{3} |t|^3 \\
&= \frac{8-5\sqrt{2}}{12} |t|^3.
\end{aligned}$$

于是,最后得

$$\begin{aligned}
F(t) &= |t| \int_0^{2\pi} \frac{8-5\sqrt{2}}{12} |t|^3 d\varphi \\
&= \frac{(8-5\sqrt{2})\pi}{6} t^4.
\end{aligned}$$

$$F(x, y, z, t) = \iint_S f(\xi, \eta, \zeta) dS,$$

其中 S 是变球

$$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = t^2,$$

且假定 $\sqrt{x^2 + y^2 + z^2} > a > 0$,

$$f(\xi, \eta, \zeta) = \begin{cases} 1, & \text{若 } \xi^2 + \eta^2 + \zeta^2 < a^2; \\ 0, & \text{若 } \xi^2 + \eta^2 + \zeta^2 \geq a^2. \end{cases}$$

解 记 $x^2 + y^2 + z^2 = r^2$. 旋转坐标轴, 使点 $P(x, y, z)$ 位于 Oy 轴的正方向上的点 $P_0(0, 0, r)$, 如图 8.70 所示.

显然, 当 $0 < t \leq r - a$ 及 $t \geq r + a$ 时, 整个球面上的点满足 $\xi^2 + \eta^2 + \zeta^2 \geq a^2$, 此时 $f(\xi, \eta, \zeta) = 0$. 从而, 积分

$$\begin{aligned} F(x, y, z, t) &= \iint_S f(\xi, \eta, \zeta) dS = \\ &= 0. \end{aligned}$$

当 $r - a < t < r + a$ 时, 则 $F(x, y, z, t)$

$$= \iint_{S'} dS',$$

其中 S' 为 S 位于 $\xi^2 + \eta^2 + \zeta^2 = a^2$ 内的部分. 从而, 我们有

$$\begin{aligned} F(x, y, z, t) &= \int_0^{2\pi} d\varphi \int_0^a t^2 \sin\theta d\theta \\ &= 2\pi t^2 (1 - \cos a) \end{aligned}$$

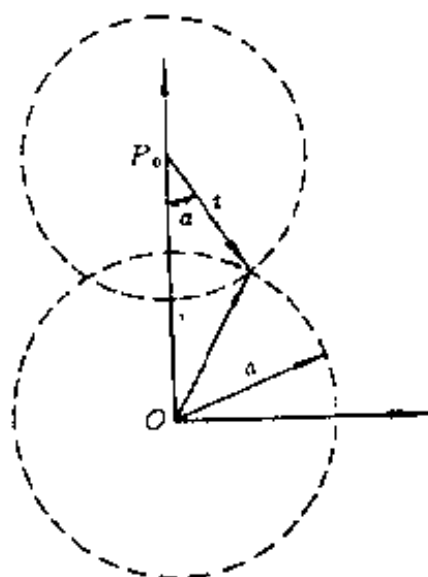


图 8.70

$$\begin{aligned}
&= 2\pi t^2 \left(1 - \frac{t^2 + r^2 - a^2}{2rt} \right) \\
&= \frac{\pi t}{r} [a^2 - (r - t)^2].
\end{aligned}$$

计算下列第二型曲面积分:

4362. $\iint_S xdydz + ydxdz + zdxdy$, 式中 S 为球 $x^2 + y^2 + z^2 = a^2$ 的外表面.

解 根据轮换对称, 只要计算 $\iint_S zdxdy$. 注意到上半球

面 $z = \sqrt{a^2 - x^2 - y^2}$ 应取上侧, 下半球面 $z = -\sqrt{a^2 - x^2 - y^2}$ 应取下侧, 则有

$$\begin{aligned}
\iint_S zdxdy &= \iint_{x^2+y^2 \leq a^2} \sqrt{a^2 - x^2 - y^2} dxdy \\
&\quad - \iint_{x^2+y^2 \leq a^2} (-\sqrt{a^2 - x^2 - y^2}) dxdy \\
&= 2 \iint_{x^2+y^2 \leq a^2} \sqrt{a^2 - x^2 - y^2} dxdy \\
&= 2 \int_0^{2\pi} d\varphi \int_0^a r \sqrt{a^2 - r^2} dr = \frac{4}{3} \pi a^3.
\end{aligned}$$

于是, 积分

$$\iint_S xdydz + ydxdz + zdxdy = 3 \cdot \frac{4}{3} \pi a^3 = 4\pi a^3.$$

4363. $\iint_S f(x)dydz + g(y)dxdz + h(z)dxdy$, 式中 $f(x)$, $g(y)$, $h(z)$ 为连续函数, S 为平行六面体 $0 < x < a$; $0 < y < b$; $0 < z < c$ 的外表面.

解 只要计算任何一个积分, 其它两个可类似地写出

结果. 例如, 下面计算 $\iint_S h(z) dx dy$. 由于六面体有四个面垂直于 Oxy 平面, 故面积分应为零. 从而

$$\begin{aligned}\iint_S h(z) dx dy &= \iint_{\substack{0 \leq x \leq a \\ 0 \leq y \leq a}} h(c) dx dy - \iint_{\substack{0 \leq x \leq a \\ 0 \leq y \leq a}} h(0) dx dy \\ &= abc \cdot \frac{h(c) - h(0)}{c}.\end{aligned}$$

类似地, 可得到 $\iint_S f(x) dx dz$ 及 $\iint_S g(y) dx dz$ 的值. 于是, 所求的积分为

$$\begin{aligned}&\iint_S f(x) dy dz + g(y) dx dz + h(z) dx dy \\ &= abc \left[\frac{f(a) - f(0)}{a} + \frac{g(b) - g(0)}{b} + \frac{h(c) - h(0)}{c} \right].\end{aligned}$$

4364. $\iint_S (y - z) dy dz + (z - x) dx dz + (x - y) dx dy$, 式中 S 为圆锥曲面 $x^2 + y^2 = z^2$ ($0 \leq z \leq h$) 的外表面.

解 方法一:

记 S_1 、 S_2 分别为锥面的底面和侧面, 而 $\cos \alpha$ 、 $\cos \beta$ 、 $\cos \gamma$ 为锥面外法线的方向余弦. 一方面, 我们有

$$\begin{aligned}&\iint_{S_1} (y - z) dy dz + (z - x) dx dz + (x - y) dx dy \\ &= \iint_{x^2 + y^2 \leq h^2} (x - y) dx dy \\ &= \int_0^{2\pi} d\varphi \int_0^h r^2 (\cos \varphi - \sin \varphi) dr\end{aligned}$$

$$= \frac{h^3}{3} \int_0^{2\pi} (\cos\varphi - \sin\varphi) d\varphi = 0.$$

另一方面,在侧面 S_2 上,对于任一点 (x, y, z) , 有

$$\frac{\cos\alpha}{x} = \frac{\cos\beta}{y} = \frac{\cos\gamma}{-z},$$

从而, dS 在各坐标面上的射影分别为

$$\cos\gamma dS = -d\sigma_{xy},$$

$$\cos\alpha dS = -\frac{x}{z}\cos\gamma dS = \frac{x}{z}d\sigma_{xy},$$

$$\cos\beta dS = -\frac{y}{z}\cos\gamma dS = \frac{y}{z}d\sigma_{xy}.$$

于是,

$$\begin{aligned} & \iint_{S_2} (y-z)dydz + (z-x)dx dz + (x-y)dx dy \\ &= \iint_{S_2} \left[(y-z)\cos\alpha + (z-x)\cos\beta + (x-y)\cos\gamma \right] dS \\ &= \iint_{x^2+y^2 \leq h^2} \left[\frac{x}{z}(y-z) + \frac{y}{z}(z-x) - (x-y) \right] d\sigma_{xy} \\ &= -2 \iint_{x^2+y^2 \leq h^2} (x-y)dx dy = 0. \end{aligned}$$

综上所述,我们得

$$\begin{aligned} & \iint_S (y-z)dydz + (z-x)dx dz + (x-y)dx dy \\ &= \iint_{S_1} + \iint_{S_2} = 0. \end{aligned}$$

方法二:

记曲面 S 在各坐标面的射影域分别为 S_{xy}, S_{yz} , 和 S_{xz} . 于是,

$$\begin{aligned}
& \iint_S (y - z) dy dz + (z - x) dx dz + (x - y) dx dy \\
&= \iint_S (y - z) dy dz + \iint_S (z - x) dx dz \\
&\quad + \iint_S (x - y) dx dy \\
&= \left(\iint_{S_{yz}} (y - z) dy dz - \iint_{S_{yz}} (y - z) dy dz \right) \\
&\quad + \left(\iint_{S_{xz}} (z - x) dx dz - \iint_{S_{xz}} (z - x) dx dz \right) \\
&\quad + \left(\iint_{S_{xy}} (x - y) dx dy - \iint_{S_{xy}} (x - y) dx dy \right) \\
&= 0 + 0 + 0 = 0.
\end{aligned}$$

4365. $\iint_S \frac{dy dz}{x} + \frac{dx dz}{y} + \frac{dx dy}{z}$, 式中 S 为椭球 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 的外表面.

解 根据轮换对称, 只要计算一个积分. 例如, 计算

$\iint_S \frac{dx dy}{z}$. 利用广义极坐标, 即得

$$\begin{aligned}
\iint_S \frac{dx dy}{z} &= \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \frac{1}{c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dx dy \\
&\quad - \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \frac{-1}{c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dx dy \\
&= \frac{2}{c} \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \frac{1}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dx dy
\end{aligned}$$

$$\begin{aligned}
&= \frac{2ab}{c} \int_0^{2\pi} d\varphi \int_0^1 \frac{r}{\sqrt{1-r^2}} dr \\
&= \frac{4\pi ab}{c} [-\sqrt{1-r^2}] \Big|_0^1 = 4\pi \cdot \frac{ab}{c}.
\end{aligned}$$

于是,我们有

$$\begin{aligned}
&\iint_S \frac{dydz}{x} + \frac{xdz}{y} + \frac{xdy}{z} \\
&= 4\pi \left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c} \right) \\
&= \frac{4\pi}{abc} (b^2c^2 + a^2c^2 + a^2b^2).
\end{aligned}$$

4366. $\iint_S x^2 dydz + y^2 dx dz + z^2 dx dy$, 式中 S 为球壳

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

的外表面.

解 根据轮换对称, 只要计算 $\iint_S z^2 dx dy$.

注意到

$$z - c = \pm \sqrt{R^2 - (x-a)^2 - (y-b)^2}.$$

并利用极坐标, 即得

$$\begin{aligned}
\iint_S z^2 dx dy &= \iint_{(x-a)^2 + (y-b)^2 \leq R^2} \\
&[c + \sqrt{R^2 - (x-a)^2 - (y-b)^2}]^2 dx dy \\
&- \iint_{(x-a)^2 + (y-b)^2 \leq R^2} \\
&[c - \sqrt{R^2 - (x-a)^2 - (y-b)^2}]^2 dx dy \\
&= 4c \iint_{(x-a)^2 + (y-b)^2 \leq R^2}
\end{aligned}$$

$$\begin{aligned}
& \sqrt{R^2 - (x-a)^2 - (y-b)^2} dx dy \\
&= 4c \int_0^{2\pi} d\varphi \int_0^R \sqrt{R^2 - r^2} r dr \\
&= 8\pi c \left[-\frac{1}{3} (R^2 - r^2)^{\frac{3}{2}} \right] \Big|_0^R = \frac{8}{3} \pi R^3 c.
\end{aligned}$$

于是,我们有

$$\begin{aligned}
& \iint_S x^2 dy dz + y^2 dx dz + z^2 dx dy \\
&= \frac{8}{3} \pi R^3 (a + b + c).
\end{aligned}$$

§ 15. 斯托克斯公式

若 $P = P(x, y, z), Q = Q(x, y, z), R = R(x, y, z)$ 为连续可微分的函数, C 为包围逐片光滑的有界双面曲面 S 的简单封闭逐段光滑的围线, 则有斯托克斯公式:

$$\begin{aligned}
& \oint_C P dx + Q dy + R dz \\
&= \iint_S \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS,
\end{aligned}$$

式中 $\cos \alpha, \cos \beta, \cos \gamma$ 为曲面 S 的法线的方向余弦, 此法线的方向是这样的, 围线 C 环绕着它依反时针方向 (对于右旋坐标系) 而回转。

4367. 应用斯托克斯公式, 计算曲线积分

$$\oint_C y dx + z dy + x dz,$$

式中 C 为圆周 $x^2 + y^2 + z^2 = a^2, x + y + z = 0$, 若从 Ox 轴的正向看去, 这圆周是依反时针方向进行的。

用直接计算法检验结果。

解 平面 $x + y + z = 0$ 的法线的方向余弦为

$$\cos\alpha = \cos\beta = \cos\gamma = -\frac{1}{\sqrt{3}}.$$

于是,

$$\begin{aligned} & \oint_C ydx + zdy + xdz \\ &= \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} dS \\ &= - \iint_S (\cos\alpha + \cos\beta + \cos\gamma) dS \\ &= -\pi a^2(\cos\alpha + \cos\beta + \cos\gamma) = -\sqrt{3}\pi a^2. \end{aligned}$$

下面用直接算法检验结果. 由方程

$$x^2 + y^2 + z^2 = a^2, x + y + z = 0$$

消去 z , 即得曲线 C 在 Oxy 平面上的射影

$$x^2 + y^2 + xy = \frac{a^2}{2}.$$

作旋转变换

$$x = \frac{x' - y'}{\sqrt{2}}, \quad y = \frac{x' + y'}{\sqrt{2}},$$

则方程化为

$$3x'^2 + y'^2 = a^2.$$

因而, 曲线 C 的参数方程可取为

$$\begin{aligned} x &= \frac{a}{\sqrt{2}} \left(\frac{\cos t}{\sqrt{3}} - \sin t \right), \\ y &= \frac{a}{\sqrt{2}} \left(\frac{\cos t}{\sqrt{3}} + \sin t \right), \end{aligned}$$

$$z = \frac{a}{\sqrt{2}} \left(-\frac{2}{\sqrt{3}} \cos t \right) \quad (0 \leq t \leq 2\pi).$$

于是, 曲线积分为

$$\begin{aligned} & \oint_C ydx + zdy + xdz \\ &= \frac{a^2}{2} \int_0^{2\pi} \left[- \left(\frac{\cos t}{\sqrt{3}} + \sin t \right) \left(\frac{\sin t}{\sqrt{3}} + \cos t \right) \right. \\ & \quad \left. - \frac{2}{\sqrt{3}} \cos t \left(-\frac{\sin t}{\sqrt{3}} + \cos t \right) \right. \\ & \quad \left. + \frac{2}{\sqrt{3}} \sin t \left(\frac{\cos t}{\sqrt{3}} - \sin t \right) \right] dt \\ &= \frac{a^2}{2} \int_0^{2\pi} \left(-\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \right) dt \\ &= \frac{a^2}{2} (-\sqrt{3}) \cdot 2\pi = -\sqrt{3} \pi a^2. \end{aligned}$$

可见, 两种算法结果一样.

4368. 计算积分

$$\int_{AmB} (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz,$$

此积分是从点 $A(a, 0, 0)$ 至点 $B(a, 0, h)$ 沿着螺线

$$x = a \cos \varphi, \quad y = a \sin \varphi, \quad z = \frac{h}{2\pi} \varphi$$

上所取的.

解 连接 A, B 两点得线段 AB , 它与 AmB 组成封闭曲线并依正向进行, 则由斯托克斯公式知:

$$\begin{aligned} & \oint_{AmBA} (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz \\ &= \iint_S 0dydz + 0dxdz + 0dxdy = 0. \end{aligned}$$

于是,

$$\begin{aligned} & \int_{AmB} (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz \\ &= \int_{AB} (x^2 - yz)dx + (y^2 - xz)dy + (z^2 - xy)dz \\ &= \int_0^h z^2 dz^{**} = \frac{h^3}{3}. \end{aligned}$$

*) 在线段 AB 上, $x = a, y = 0, dx = dy = 0$, 而 $0 \leq z \leq h$.

4369. 设 C 为位于平面 $x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$ ($\cos\alpha, \cos\beta, \cos\gamma$ 为平面之法线的方向余弦) 上并包围面积为 S 的封闭围线, 求

$$\oint_C \begin{vmatrix} dx & dy & dz \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix},$$

其中围线 C 是依正方向进行的.

解 若记

$$P = \begin{vmatrix} \cos\beta & \cos\gamma \\ y & z \end{vmatrix} = z\cos\beta - y\cos\gamma,$$

$$Q = \begin{vmatrix} \cos\gamma & \cos\alpha \\ z & x \end{vmatrix} = x\cos\gamma - z\cos\alpha$$

$$R = \begin{vmatrix} \cos\alpha & \cos\beta \\ x & y \end{vmatrix} = y\cos\alpha - x\cos\beta,$$

则得

$$\oint_C \begin{vmatrix} dx & dy & dz \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix} = \oint_C Pdx + Qdy + Rdz$$

$$\begin{aligned}
&= \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= 2 \iint_S (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) dS = 2 \iint_S dS = 2S.
\end{aligned}$$

应用斯托克斯公式, 计算积分:

4370. $\oint_C (y+z)dx + (z+x)dy + (x+y)dz$, 式中 C 为依参数 t 增大的方向通过的椭圆 $x = a\sin^2 t, y = 2a\sin t \cdot \cos t, z = a\cos^2 t (0 \leq t \leq 2\pi)$.

解
$$\begin{aligned}
&\oint_C (y+z)dx + (z+x)dy + (x+y)dz \\
&= \iint_S 0dydz + 0dxdz + 0dxdy = 0.
\end{aligned}$$

4371. $\oint_C (y-z)dx + (z-x)dy + (x-y)dz$, 式中 C 为椭圆 $x^2 + y^2 = a^2, \frac{x}{a} + \frac{z}{h} = 1 (a > 0, h > 0)$, 若从 Ox 轴正向看去, 此椭圆是依反时针方向进行的.

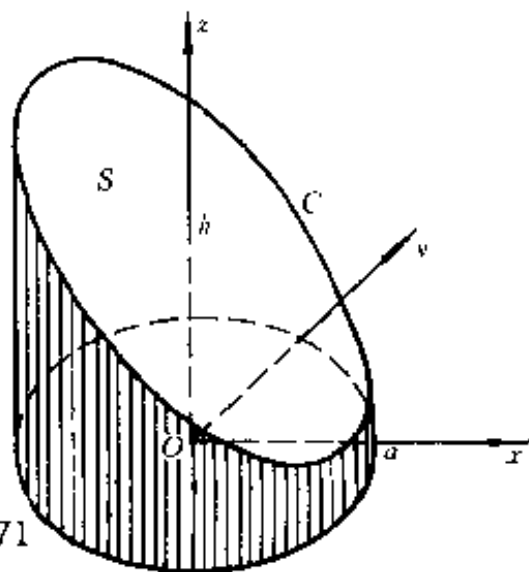


图 8.71

解 椭圆如图 8.71 所示. 把平面 $\frac{x}{a} + \frac{z}{h} = 1$ 上 C 所包围的区域记为 S , 则 S 的法线方向为 $\{h, 0, a\}$. 注意到 S 的法线方向和曲线 C 的方向是正向联系的, 即得

$$\begin{aligned} & \oint_C (y-z)dx + (z-x)dy + (x-y)dz \\ &= -2 \iint_S dydz + dxdz + dxdy \\ &= -2(\cos\alpha - \cos\beta + \cos\gamma) \iint_S dS \\ &= -2 \left(\frac{h}{\sqrt{a^2+h^2}} + 0 + \frac{a}{\sqrt{a^2+h^2}} \right) \\ & \quad \cdot \pi a \sqrt{a^2+h^2} \\ &= -2\pi a(a+h). \end{aligned}$$

4372. $\oint_C (y^2+z^2)dx + (x^2+z^2)dy + (x^2+y^2)dz$, 式中 C 是曲线 $x^2+y^2+z^2=2Rx, x^2+y^2=2rx (0 < r < R, z > 0)$, 此曲线是如下进行的: 由它所包围在球 $x^2+y^2+z^2=2Rr$ 外表面上的最小区域保持在左方.

解 注意到球面的法线的方向余弦为

$$\cos\alpha = \frac{x-R}{R}, \cos\beta = \frac{y}{R}, \cos\gamma = \frac{z}{R},$$

即得

$$\begin{aligned} & \oint_C (y^2+z^2)dx + (x^2+z^2)dy + (x^2+y^2)dz \\ &= 2 \iint_S [(y-z)\cos\alpha + (z-x)\cos\beta \\ & \quad + (x-y)\cos\gamma] dS \\ &= 2 \iint_S \left[(y-z) \left(\frac{x}{R} - 1 \right) + (z-x) \frac{y}{R} \right. \end{aligned}$$

$$+ (x - y) \frac{z}{R} \Big] dS \\ = 2 \iint_S (z - y) dS.$$

由于曲面 S 关于 Oxy 平面对称, 故 $\iint_S y dS = 0$.

又

$$\iint_S z ds = \iint_S R \cos \gamma dS = R \cdot \pi r^2,$$

于是,

$$\oint_C (y^2 + z^2) dx + (z^2 + x^2) dy + (x^2 + y^2) dz \\ = 2\pi R r^2.$$

4373. $\oint_C (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$, 式中 C

为用平面 $x + y + z = \frac{3}{2}a$ 切立方体 $0 < x < a, 0 < y < a, 0 < z < a$ 的表面所得的切痕. 若从 Ox 轴的正向看去, 是依反时针前进的方向的.

解 平面 $x + y + z = \frac{3}{2}a$ 含于立方体内的部分记为 S , 它在 Oxy 平面上的射影域记为 S_{xy} , 其面积显然等于 $\frac{3}{4}a^2$. 当平面 $x + y + z = \frac{3}{2}a$ 取上侧时, 法线方向的单位矢量为 $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$. 于是, 由斯托克斯公式知

$$\oint_C (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz \\ = \iint_S \left[(-2y - 2z) \frac{1}{\sqrt{3}} + (-2z - 2x) \frac{1}{\sqrt{3}} \right.$$

$$\begin{aligned}
& + (-2x - 2y) \frac{1}{\sqrt{3}} \} dS \\
& = -4 \iint_S (x + y + z) \frac{1}{\sqrt{3}} dS \\
& = -6a \iint_S \frac{1}{\sqrt{3}} dS = -6a \iint_{S_{xy}} dx dy \\
& = -6a \cdot \frac{3}{4} a^2 = -\frac{9}{2} a^3.
\end{aligned}$$

4374. $\oint_C y^2 z^2 dx + x^2 z^2 dy + x^2 y^2 dz$, 式中 C 为依参数 t 增大的方向进行的封闭曲线 $x = a \cos t, y = a \cos 2t, z = a \cos 3t$.

解 取 S 为由参数方程

$$\begin{aligned}
x &= u \cos t, y = u \cos 2t, z = u \cos 3t \\
(0 &\leq u \leq a, 0 \leq t \leq 2\pi)
\end{aligned}$$

表示的曲面, 则所给曲线 C 为曲面 S 的边界.

于是, 根据斯托克斯公式, 有

$$\begin{aligned}
& \oint_C y^2 z^2 dx + x^2 z^2 dy + x^2 y^2 dz \\
&= 2 \iint_S [x^2(y-z) dy dz + y^2(z-x) dz dx \\
&\quad + z^2(x-y) dx dy] \\
&= \pm 2 \int_0^{2\pi} \int_0^a [u^2 \cos^2 t (u \cos 2t - u \cos 3t) \\
&\quad \cdot (y'_u z'_t - z'_t y'_u) + u^2 \cos^2 2t (u \cos 3t - u \cos t) \\
&\quad \cdot (z'_u x'_t - x'_t z'_u) + u^2 \cos^2 3t (u \cos t - u \cos 2t) \\
&\quad \cdot (x'_u y'_t - y'_t x'_u)] du dt \\
&= \pm 2 \int_0^a u^4 du \int_0^{2\pi} [\cos^2 t (\cos 2t - \cos 3t) \\
&\quad \cdot (2 \sin 2t \cos 3t - 3 \cos 2t \sin 3t) + \cos^2 2t (\cos 3t - \cos t)
\end{aligned}$$

$$\begin{aligned}
& \cdot (3\sin 3t \cos t - \sin t \cos 3t) + \cos^2 3t (\cos t - \cos 2t) \\
& \cdot (\sin t \cos 2t - 2\sin 2t \cos t) dt \\
& = \pm \frac{2}{5} a^5 \int_{-\pi}^{\pi} [\cos^2 t (\cos 2t - \cos 3t) (2\sin 2t \cos 3t \\
& \quad - 3\cos 2t \sin 3t) + \cos^2 2t (\cos 3t - \cos t) \\
& \quad \cdot (3\sin 3t \cos t - \sin t \cos 3t) + \cos^2 3t (\cos t - \cos 2t) \\
& \quad \cdot (\sin t \cos 2t - 2\sin 2t \cos t)] dt = 0,
\end{aligned}$$

上式中正负号应这样选取,使得 S 的侧正好配合 C 的方向 (t 增大的方向), 积分 $\int_0^{2\pi}$ 可以换为 $\int_{-\pi}^{\pi}$ 是因为被积函数 (t 的函数) 是周期为 2π 的函数, 而 $\int_{-\pi}^{\pi}$ 等于零是因为被积函数为奇函数.

注: 本题若不用斯托克斯公式, 而直接计算线积分, 则较为简单:

$$\begin{aligned}
& \oint_C y^2 z^2 dx + x^2 z^2 dy + x^2 y^2 dz \\
& = - \int_0^{2\pi} a^5 (\cos^2 2t \cos^2 3t \sin t + 2\cos^2 t \cos^2 3t \sin 2t \\
& \quad + 3\cos^2 t \cos^2 2t \sin 3t) dt \\
& = - \int_{-\pi}^{\pi} a^5 (\cos^2 2t \cos^2 3t \sin t + 2\cos^2 t \cos^2 3t \sin 2t \\
& \quad + 3\cos^2 t \cos^2 2t \sin 3t) dt \\
& = 0, \\
& \int_0^{2\pi} \text{可换为} \int_{-\pi}^{\pi} \text{及} \int_{-\pi}^{\pi} = 0 \text{ 的理由同上.}
\end{aligned}$$

4375. 有函数

$$W(x, y, z) = ki \iint_S \frac{\cos(\vec{r}, \vec{n})}{r^2} dS \quad (k = \text{常数}),$$

其中 S 为由围线 C 所界的面积, $\vec{n}$ 为曲面 S 的法线, $\vec{r}$ 为

连接空间的点 $M(x, y, z)$ 与曲面 S 上的动点 $A(\xi, \eta, \zeta)$ 所成之矢径, 证明此函数为通过围线 C 的电流 i 所产生磁场 $\vec{H}$ 的位势(参阅 4340 题)。

证 利用 4340 题指出的定律, 并注意到

$$\frac{\vec{r}}{r^3} = \frac{\partial}{\partial x} \left(\frac{1}{r} \right) \vec{i} + \frac{\partial}{\partial y} \left(\frac{1}{r} \right) \vec{j} + \frac{\partial}{\partial z} \left(\frac{1}{r} \right) \vec{k},$$

其中 $\vec{r} = (\xi - x)\vec{i} + (\eta - y)\vec{j} + (\zeta - z)\vec{k}$, 即得

$$\begin{aligned} \vec{H} &= ki \oint_C \frac{\vec{r} \times d\vec{s}}{r^3} \\ &= ki \left(\left(\oint_C \frac{\partial}{\partial y} \left(\frac{1}{r} \right) d\zeta - \frac{\partial}{\partial z} \left(\frac{1}{r} \right) d\eta \right) \vec{i} \right. \\ &\quad \left. + \left(\oint_C \frac{\partial}{\partial z} \left(\frac{1}{r} \right) d\xi - \frac{\partial}{\partial x} \left(\frac{1}{r} \right) d\zeta \right) \vec{j} \right. \\ &\quad \left. + \left(\oint_C \frac{\partial}{\partial x} \left(\frac{1}{r} \right) d\eta - \frac{\partial}{\partial y} \left(\frac{1}{r} \right) d\xi \right) \vec{k} \right). \end{aligned}$$

利用斯托克斯公式, 并注意到

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{1}{r} \right) &= - \frac{\partial}{\partial \xi} \left(\frac{1}{r} \right), \quad \frac{\partial}{\partial y} \left(\frac{1}{r} \right) = - \frac{\partial}{\partial \eta} \left(\frac{1}{r} \right), \\ \frac{\partial}{\partial z} \left(\frac{1}{r} \right) &= - \frac{\partial}{\partial \zeta} \left(\frac{1}{r} \right) \end{aligned}$$

及 $\Delta \left(\frac{1}{r} \right) = 0$, 从而

$$\begin{aligned} &\frac{\partial^2}{\partial \eta \partial y} \left(\frac{1}{r} \right) + \frac{\partial^2}{\partial \xi \partial z} \left(\frac{1}{r} \right) \\ &= - \frac{\partial^2}{\partial y^2} \left(\frac{1}{r} \right) - \frac{\partial^2}{\partial z^2} \left(\frac{1}{r} \right) = \frac{\partial^2}{\partial x^2} \left(\frac{1}{r} \right), \end{aligned}$$

即得

$$H_x = ki \oint_C \frac{\partial}{\partial y} \left(\frac{1}{r} \right) d\zeta - \frac{\partial}{\partial z} \left(\frac{1}{r} \right) d\eta$$

$$\begin{aligned}
&= ki \iint_S \left[\left(\frac{\partial}{\partial y \partial y} + \frac{\partial}{\partial x \partial z} \right) \frac{1}{r} \right] \vec{i} - \frac{\partial}{\partial x \partial y} \frac{1}{r} \vec{j} \\
&\quad - \frac{\partial}{\partial x \partial z} \frac{1}{r} \vec{k} \cdot \vec{n} dS \\
&= ki \frac{\partial}{\partial x} \iint_S \left(\frac{\partial}{\partial x} \frac{1}{r} \vec{i} + \frac{\partial}{\partial y} \frac{1}{r} \vec{j} + \frac{\partial}{\partial z} \frac{1}{r} \vec{k} \right) \cdot \vec{n} dS \\
&= ki \frac{\partial}{\partial x} \iint_S \frac{\vec{r} \cdot \vec{n}}{r^2} dS = ki \frac{\partial}{\partial x} \iint_S \frac{\cos(\vec{r}, \vec{n})}{r^2} dS.
\end{aligned}$$

同理,

$$H_y = ki \frac{\partial}{\partial y} \iint_S \frac{\cos(\vec{r}, \vec{n})}{r^2} dS,$$

$$H_z = ki \frac{\partial}{\partial z} \iint_S \frac{\cos(\vec{r}, \vec{n})}{r^2} dS.$$

于是,最后得

$$\vec{H} = \frac{\partial W}{\partial x} \vec{i} + \frac{\partial W}{\partial y} \vec{j} + \frac{\partial W}{\partial z} \vec{k},$$

即函数 $W(x, y, z)$ 是磁场 $\vec{H}$ 的位势.

§ 16. 奥斯特洛格拉德斯基公式

若 S 为包含体积 V 的逐片光滑曲面, $P = P(x, y, z)$, $Q = Q(x, y, z)$, $R = R(x, y, z)$ 和它们的一阶偏导函数均为域 $V + S$ 内的连续函数, 则奥斯特洛格拉德斯基公式真确:

$$\iint_S (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz,$$

式中 $\cos \alpha, \cos \beta, \cos \gamma$ 为曲面 S 的外法线的方向余弦.

应用奥斯特洛格拉德斯基公式以变换下列曲面积分, 设光滑的曲

面 S 包围着有界的体积 V , $\cos\alpha, \cos\beta, \cos\gamma$ 为曲面 S 的外法线的方向余弦.

$$4376. \iint_S x^3 dydz + y^3 dx dz + z^3 dx dy.$$

解 由于 $P = x^3, Q = y^3, R = z^3$. 从而

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 3(x^2 + y^2 + z^2).$$

于是,

$$\begin{aligned} \iint_S x^3 dydz + y^3 dx dz + z^3 dx dy \\ = 3 \iiint_V (x^2 + y^2 + z^2) dx dy dz. \end{aligned}$$

$$4377. \iint_S xy dx dy + xz dx dz + yz dy dz.$$

解 由于

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0,$$

故得

$$\begin{aligned} \iint_S xy dx dy + xz dx dz + yz dy dz \\ = \iiint_V 0 dx dy dz = 0. \end{aligned}$$

$$4378. \iint_S \frac{x \cos\alpha + y \cos\beta + z \cos\gamma}{\sqrt{x^2 + y^2 + z^2}} dS.$$

解 由于

$$P = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad Q = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

$$R = \frac{z}{\sqrt{x^2 + y^2 + z^2}},$$

从而

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{2}{\sqrt{x^2 + y^2 + z^2}}.$$

于是,

$$\begin{aligned} & \iint_S \frac{xcos\alpha + ycos\beta + zcos\gamma}{\sqrt{x^2 + y^2 + z^2}} dS \\ &= 2 \iiint_V \frac{dxdydz}{\sqrt{x^2 + y^2 + z^2}}. \end{aligned}$$

$$4379. \iint_S \left(\frac{\partial u}{\partial x} \cos\alpha + \frac{\partial u}{\partial y} \cos\beta + \frac{\partial u}{\partial z} \cos\gamma \right) dS.$$

解 由于

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \Delta u,$$

故得

$$\begin{aligned} & \iint_S \left(\frac{\partial u}{\partial x} \cos\alpha + \frac{\partial u}{\partial y} \cos\beta + \frac{\partial u}{\partial z} \cos\gamma \right) dS \\ &= \iiint_V \Delta u dx dy dz. \end{aligned}$$

$$\begin{aligned} 4380. \iint_S & \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos\beta \right. \\ & \left. + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos\gamma \right] dS. \end{aligned}$$

解 记

$$\begin{aligned} P^* &= \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, & Q^* &= \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \\ R^* &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \end{aligned}$$

则易知:

$$\frac{\partial P^*}{\partial x} + \frac{\partial Q^*}{\partial y} + \frac{\partial R^*}{\partial z} = 0.$$

于是,原面积分等于零.

4381. 证明:若 S 为封闭的简单曲面而 $\vec{l}$ 为任何的固定方向,则

$$\iint_S \cos(\vec{n}, \vec{l}) dS = 0,$$

式中 $\vec{n}$ 为曲面 S 的外法线.

证 因为

$$\cos(\vec{n}, \vec{l}) = \cos\alpha \cos(\vec{l}, x) + \cos\beta \cos(\vec{l}, y) + \cos\gamma \cos(\vec{l}, z),$$

其中 $\cos\alpha, \cos\beta, \cos\gamma$ 为 $\vec{n}$ 的方向余弦,故有

$$\begin{aligned} \iint_S \cos(\vec{n}, \vec{l}) dS &= \iint_S \cos(\vec{l}, x) dydz \\ &+ \cos(\vec{l}, y) dxdz + \cos(\vec{l}, z) dxdy. \end{aligned}$$

由于 $\vec{l}$ 为固定方向,从而 $\cos(\vec{l}, x), \cos(\vec{l}, y), \cos(\vec{l}, z)$ 均为常数.于是,

$$\begin{aligned} \iint_S \cos(\vec{n}, \vec{l}) dS &= \iiint_V \left\{ \frac{\partial \cos(\vec{l}, x)}{\partial x} \right. \\ &+ \frac{\partial \cos(\vec{l}, y)}{\partial y} + \frac{\partial \cos(\vec{l}, z)}{\partial z} \Big\} dxdydz \\ &= \iiint_V 0 dxdydz = 0. \end{aligned}$$

4382. 证明:由曲面 S 所包围的体积等于

$$V = \frac{1}{3} \iint_S (x \cos\alpha + y \cos\beta + z \cos\gamma) dS,$$

式中 $\cos\alpha, \cos\beta, \cos\gamma$ 为曲面 S 的外法线的方向余弦.

证 由奥氏公式,有

$$\begin{aligned}
& \iint_C (x \cos \alpha + y \cos \beta + z \cos \gamma) dS \\
&= \iint_S x dy dz + y dz dx + z dx dy \\
&= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dx dy dz \\
&= \iiint_V 3 dx dy dz = 3V,
\end{aligned}$$

由此可知

$$V = \frac{1}{3} \iint_C (x \cos \alpha + y \cos \beta + z \cos \gamma) dS.$$

证毕.

4383. 证明, 由平滑的圆锥曲面 $F(x, y, z) = 0$ 和平面 $Ax + By + Cz + D = 0$ 所包围的锥体体积等于

$$V = \frac{1}{3}SH,$$

式中 S 为位于已知平面上的锥底之面积, H 为锥的高.

证 方法一:

不失一般性, 设坐标原点位于圆锥曲面 $F(x, y, z) = 0$ 的顶点. 于是 $F(x, y, z)$ 是 x, y, z 的二次齐次函数. 因此, 根据尤拉定理知

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = 2F(x, y, z) \quad (1)$$

由 4382 题的结果, 有

$$\begin{aligned}
V &= \frac{1}{3} \iint_{S+S_1} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS \\
&= \frac{1}{3} \iint_S (x \cos \alpha + y \cos \beta + z \cos \gamma) dS
\end{aligned}$$

$$+ \frac{1}{3} \iint_{S_1} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS, \quad (2)$$

其中 S 为锥底(位于平面 $Ax + By + Cz + D = 0$ 上), 而 S_1 是圆锥的侧面. 在锥面 S_1 (即 $F(x, y, z) = 0$) 上, 有

$$\begin{aligned} \cos \alpha &= \frac{F'_x}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}}, \\ \cos \beta &= \frac{F'_y}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}}, \\ \cos \gamma &= \frac{F'_z}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}}. \end{aligned}$$

于是, 注意到(1)式, 即知在 S_1 上有

$$\begin{aligned} & x \cos \alpha + y \cos \beta + z \cos \gamma \\ &= \frac{x F'_x + y F'_y + z F'_z}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}} \\ &= \frac{2F(x, y, z)}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}} = 0, \end{aligned}$$

从而

$$\iint_{S_1} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS = 0. \quad (3)$$

又在平面 $Ax + By + Cz + D = 0$ 上, 有

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \vec{r} \cdot \vec{n} = H,$$

其中 $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ 是从原点 $(0, 0, 0)$ 到点 (x, y, z) 的矢径, $\vec{n}$ 为平面(锥底)的外法线单位向量, H 为从原点到平面的距离(即锥的高). 于是,

$$\begin{aligned} & \iint_S (x \cos \alpha + y \cos \beta + z \cos \gamma) dS \\ &= H \iint_S dS = HS. \end{aligned}$$

由此,再注意到(2)式与(3)式,即得 $V = \frac{1}{3}SH$.

方法二:

取坐标系 $Ox'y'z'$, 使圆锥的顶点在坐标原点, $Ox'y'$ 平面平行于圆锥的底面, 由于在 z 处的圆锥的截面面积

$$S(z) = \frac{S z'^2}{H^2}$$

故所求的体积为

$$\begin{aligned} V &= \int_0^H S(z') dz' \\ &= \int_0^H \frac{S}{H^2} z' dz' = \frac{1}{3}SH. \end{aligned}$$

4384. 求由曲面: $z = \pm c$ 及

$$\begin{aligned} x &= a \cos u \cos v + b \sin u \sin v, \\ y &= a \cos u \sin v - b \sin u \cos v, \\ z &= c \sin u \end{aligned}$$

所界物体的体积.

解 方法一:

我们有

$$x^2 + y^2 = a^2 \cos^2 u + b^2 \sin^2 u, \quad (1)$$

以 $z = c \sin u$ 代入得

$$x^2 + y^2 + \frac{a^2 - b^2}{c^2} z^2 = a^2 \quad (2)$$

故所界物体由平面 $z = c, z = -c$ 及曲面(2)围成. 利用

4382 题的结果,即知所求的体积为

$$V = \frac{1}{3} \iint_{S_1 + S_2 + S_3} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS, \quad (3)$$

其中 S_1, S_2 分别是平面 $z = c, z = -c$ 上的部分(此时 $u = \frac{\pi}{2}, u = -\frac{\pi}{2}$, 从而 $x^2 + y^2 = b^2$, 故 S_1, S_2 为圆盘 $x^2 + y^2 \leq b^2$), S_3 表曲面(2)的部分, $\cos \alpha, \cos \beta, \cos \gamma$ 表外法线的方向余弦. 显然, 在 S_1 上, $\cos \alpha = \cos \beta = 0, \cos \gamma = \frac{c}{|c|}$. 于是,

$$\begin{aligned} \iint_{S_1} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS &= \iint_{S_1} \frac{c^2}{|c|} dS \\ &= |c| \pi b^2. \end{aligned}$$

同理可得

$$\iint_{S_2} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS = |c| \pi b^2.$$

此外

$$\begin{aligned} &\iint_{S_3} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS \\ &= \iint_{S_3} x dy dz + y dz dx + z dx dy \\ &= \pm \int_0^{2\pi} d\nu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [(a \cos u \cos \nu + b \sin u \sin \nu) \\ &\quad \cdot (y'_u z'_\nu - y'_\nu z'_u) \\ &\quad - (a \cos u \sin \nu - b \sin u \cos \nu) (z'_u x'_\nu - z'_\nu x'_u) \\ &\quad + c \sin u (x'_u y'_\nu - x'_\nu y'_u)] du \end{aligned}$$

$$-\pm \int_0^{2\pi} dv \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ca^2 \cos u du = \pm 4\pi ca^2, \quad (4)$$

其中的正负号应这样选取,使对应于 S_3 的外侧. 下面确定此正负号. 由(2), S_3 的方程可写为 $F(x, y, z) = a^2$, 其中 $F(x, y, z) = x^2 + y^2 + \frac{a^2 - b^2}{c^2} z^2$ 是二次齐次函数. 于是, 在 S_3 上, 有

$$\begin{aligned} \cos \alpha &= \frac{F'_x}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}}, \\ \cos \beta &= \frac{F'_y}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}}, \\ \cos \gamma &= \frac{F'_z}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}}, \end{aligned}$$

其中正号对应于 S_3 的一侧, 负号对应于 S_3 的另一侧. 于是, 根据齐次函数的尤拉定理, 在 S_3 (外侧) 上有

$$\begin{aligned} & x \cos \alpha + y \cos \beta + z \cos \gamma \\ &= \frac{x F'_x + y F'_y + z F'_z}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}} \\ &= \frac{2F}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}} \\ &= \frac{2a^2}{\pm \sqrt{F_x'^2 + F_y'^2 + F_z'^2}}. \end{aligned} \quad (5)$$

但在 S_3 与 Oxy 平面的交线 (即 $x^2 + y^2 = a^2, z = 0$) 的各点上, 对 S_3 的外侧, 显然有 (注意到曲面(2)关于 Oxy 坐标平面对称)

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \vec{r} \cdot \vec{n} > 0,$$

(这是因为此时向径 $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ 与外法线单位向量 $\vec{n}$ 的方向一致), 由此可知, 在(5)式中应取正号, 于是

$$\begin{aligned} & \iint_{S_3} (x\cos\alpha + y\cos\beta + z\cos\gamma) dS \\ &= \iint_{S_3} \frac{2a^2}{\sqrt{F'^2_x + F'^2_y + F'^2_z}} dS > 0. \end{aligned}$$

从而, 由(4)式知

$$\iint_{S_3} (x\cos\alpha + y\cos\beta + z\cos\gamma) dS = 4\pi|c|a^2.$$

综上所述, 最后得(注意(3)式)

$$\begin{aligned} V &= \frac{1}{3} (4\pi|c|a^2 + |c|\pi b^2 + |c|\pi b^2) \\ &= \frac{4\pi}{3} (a^2 + \frac{b^2}{2}) |c|. \end{aligned}$$

方法二:

不用面积分求体积的公式(3), 而直接计算体积较为简单. 由(1)式知, 平面 $z = \text{常数}$ (即 $u = \text{常数}$) 与曲面(2)的截面 $S(z)$ 是圆, 故所求的体积为

$$\begin{aligned} V &= \int_{-c}^c dz \iint_{S(z)} dxdy = \int_{-c}^c S(z) dz \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi(a^2\cos^2 u + b^2\sin^2 u) |c| d(\sin u) \\ &= |c|\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [a^2 + (b^2 - a^2)\sin^2 u] d(\sin u) \\ &= \pi|c| [2a^2 + \frac{2}{3}(b^2 - a^2)] \\ &= \frac{4\pi}{3} (a^2 + \frac{b^2}{2}) |c|. \end{aligned}$$

4385. 求由曲面

$$x = u \cos v, y = u \sin v, z = -u + a \cos v \quad (u \geq 0)$$

及平面 $x = 0, z = 0 (a > 0)$ 所界物体的体积.

解 方法一:

用 S_1 表物体表面位于平面 $z = 0$ 上的那一部分, S_2 为物体表面由所给参数方程给出的曲面上那一部分, 此外, 物体表面在平面 $x = 0$ 上的那部分显然是一线段 $x = 0, y = 0, 0 \leq z \leq a$. 于是, 利用 4382 题的结果, 即知所求体积为

$$V = \frac{1}{3} \iint_{S_1 + S_2} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS, \quad (1)$$

其中 $\cos \alpha, \cos \beta, \cos \gamma$ 是向外法线的方向余弦. 显然, 在 S_1 上, $\cos \alpha = 0, \cos \beta = 0, \cos \gamma = -1, z = 0$, 故

$$\iint_{S_1} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS = 0. \quad (2)$$

另外, 我们有

$$\begin{aligned} & \iint_{S_2} (x \cos \alpha + y \cos \beta + z \cos \gamma) dS \\ &= \iint_{S_2} x dy dz + y dz dx + z dx dy \\ &= \pm \iint_D [-u \cos v (y'_u z'_v - y'_v z'_u) \\ & \quad + u \sin v (z'_u x'_v - z'_v x'_u) \\ & \quad + (-u + a \cos v) (x'_u y'_v - x'_v y'_u)] du dv \\ &= \pm \iint_D [u \cos v (u \cos v - a \sin^2 v) \end{aligned}$$

$$\begin{aligned}
& + u \sin v (a \sin v \cos v + u \sin v) \\
& + (-u + a \cos v) u] du dv \\
= & \pm \iint_D a u \cos v du dv = \pm \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dv \int_0^{a \cos v} a u \cos v du \\
= & \pm \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} a^3 \cos^3 v \right) dv = \pm \int_0^{\frac{\pi}{2}} a^3 (1 - \sin^2 v) d(\sin v) \\
= & \pm a^3 \left(\sin v - \frac{\sin^3 v}{3} \right) \Big|_0^{\frac{\pi}{2}} = \pm \frac{2}{3} a^3,
\end{aligned}$$

其中的正负号应这样选取,使对应于 S_2 的外侧, D 为 u, v 的变化区域(对应于 S_2). 由此,再注意到(1)式与(2)式,即得 $V = \pm \frac{2}{9} a^3$. 但体积恒为正($V > 0$)故必有 $V = \frac{2}{9} a^3$.

方法二:

本题若不利用面积分计算体积的公式(1),而直接计算体积,则较为简单(下面 Ω 表物体在 Oxy 平面上的投影):

$$\begin{aligned}
V &= \iint_{\Omega} z dx dy = \iint_D (-u + a \cos v) \\
&\quad \cdot \left| \frac{D(x, y)}{D(u, v)} \right| du dv \\
&= \iint_D (-u + a \cos v) u du dv \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dv \int_0^{a \cos v} (-u + a \cos v) u du \\
&= \frac{a^3}{6} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 v dv
\end{aligned}$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^2 v) d(\sin v) = \frac{2}{9} a^3.$$

4386. 证明公式

$$\begin{aligned} & \frac{d}{dt} \left\{ \iiint_{x^2+y^2+z^2 \leq t^2} f(x, y, z, t) dx dy dz \right\} \\ &= \iint_{x^2+y^2+z^2=t^2} f(x, y, z, t) dS \\ &+ \iiint_{x^2+y^2+z^2 \leq t^2} \frac{\partial f}{\partial t} dx dy dz \quad (t > 0). \end{aligned}$$

证 证法一

作变量代换 $x = tu, y = tv, z = t\omega$ ($t > 0$ 固定), 则
(利用奥氏公式)

$$\begin{aligned} & \frac{d}{dt} \left\{ \iiint_{x^2+y^2+z^2 \leq t^2} f(x, y, z, t) dx dy dz \right\} \\ &= \frac{d}{dt} \left\{ \iiint_{u^2+v^2+\omega^2 \leq 1} t^3 f(tu, tv, t\omega, t) du dv d\omega \right\} \\ &= \iiint_{u^2+v^2+\omega^2 \leq 1} \left[t^3 \left(\frac{\partial f}{\partial x} u + \frac{\partial f}{\partial y} v + \frac{\partial f}{\partial z} \right. \right. \\ &+ \left. \left. \frac{\partial f}{\partial t} \right) + 3t^2 f \right] du dv d\omega \\ &= \iiint_{u^2+v^2+\omega^2 \leq 1} t^3 \left\{ \frac{1}{t} \left[\frac{\partial}{\partial x}(fx) + \frac{\partial}{\partial y}(fy) \right. \right. \\ &+ \left. \left. \frac{\partial}{\partial z}(fz) \right] \right\} du dv d\omega \\ &+ \iiint_{u^2+v^2+\omega^2 \leq 1} t^3 \frac{\partial f}{\partial t} du dv d\omega \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{t} \iiint_{x^2+y^2+z^2 \leq t^2} \left(\frac{\partial}{\partial x}(fx) + \frac{\partial}{\partial y}(fy) \right. \\
&\quad \left. + \frac{\partial}{\partial z}(fz) \right) dx dy dz \\
&\quad + \iiint_{x^2+y^2+z^2 \leq t^2} \frac{\partial f}{\partial x} dx dy dz \\
&= \frac{1}{t} \iint_{x^2+y^2+z^2=t^2} (fx \cos \alpha + fy \cos \beta \\
&\quad + fz \cos \gamma) dS \\
&\quad + \iiint_{x^2+y^2+z^2 \leq t^2} \frac{\partial f}{\partial x} dx dy dz \quad (t > 0),
\end{aligned}$$

其中 $\cos \alpha, \cos \beta, \cos \gamma$ 为球面 $x^2 + y^2 + z^2 = t^2$ 上向外法线的方向余弦. 显然

$$\cos \alpha = \frac{x}{t}, \cos \beta = \frac{y}{t}, \cos \gamma = \frac{z}{t},$$

故

$$\begin{aligned}
&\iint_{x^2+y^2+z^2=t^2} (fx \cos \alpha + fy \cos \beta + fz \cos \gamma) dS \\
&= \iint_{x^2+y^2+z^2=t^2} f \cdot \frac{x^2 + y^2 + z^2}{t} dS \\
&= t \iint_{x^2+y^2+z^2=t^2} f dS.
\end{aligned}$$

于是,最后得

$$\begin{aligned}
&\frac{d}{dt} \left\{ \iiint_{x^2+y^2+z^2 \leq t^2} f(x, y, z, t) dx dy dz \right\} \\
&= \iint_{x^2+y^2+z^2=t^2} f dS
\end{aligned}$$

$$+ \iiint_{x^2+y^2+z^2 \leq t^2} \frac{\partial f}{\partial t} dx dy dz \quad (t > 0).$$

证法二

不利用奥氏公式更简单些. 采用球坐标, 我们有

$$\begin{aligned} & \frac{d}{dt} \left\{ \iiint_{x^2+y^2+z^2 \leq t^2} f(x, y, z, t) dx dy dz \right\} \\ &= \frac{d}{dt} \left\{ \int_0^t \left[\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(r \cos \varphi \cos \psi, r \sin \varphi \cos \psi, \right. \right. \\ & \quad \left. \left. r \sin \psi, t) r^2 \cos \psi d\psi d\varphi \right] dr \right\} \\ &= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t \cos \varphi \cos \psi, t \sin \varphi \cos \psi, \\ & \quad t \sin \psi, t) t^2 \cos \psi d\psi d\varphi \\ & \quad + \int_0^t \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial t} f(r \cos \varphi \cos \psi, r \sin \varphi \cos \psi, \\ & \quad r \sin \psi, t) r^2 \cos \psi d\psi d\varphi dr \\ &= \iint_{x^2+y^2+z^2=t^2} f(x, y, z, t) dS \\ & \quad + \iiint_{x^2+y^2+z^2 < t^2} \frac{\partial f}{\partial t} dx dy dz. \end{aligned}$$

利用奥斯特洛格拉德斯基公式计算下列面积分:

4387. $\iint_S x^2 dy dz + y^2 dx dz + z^2 dx dy$, 式中 S 为立方体 $0 < x < a, 0 < y < a, 0 < z < a$ 的边界的外表面.

解
$$\begin{aligned} & \iint_S x^2 dy dz + y^2 dx dz + z^2 dx dy \\ &= 2 \iiint_V (x + y + z) dx dy dz \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^a dx \int_0^a dy \int_0^a (x + y + z) dz \\
&= 6 \int_0^a dx \int_0^a dy \int_0^a z dz = 3a^3.
\end{aligned}$$

4388. $\iint_S x^3 dydz + y^3 dx dz + z^3 dx dy$, 式中 S 为球 $x^2 + y^2 + z^2 = a^2$ 的外表面.

解
$$\begin{aligned}
&\iint_S x^3 dydz + y^3 dx dz + z^3 dx dy \\
&= 3 \iiint_V (x^2 + y^2 + z^2) dx dy dz \\
&= 3 \int_0^{2\pi} d\varphi \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^a r^4 \cos\psi dr \\
&= 6\pi \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\psi d\psi \right] \left(\int_0^a r^4 dr \right) = \frac{12}{5} \pi a^5.
\end{aligned}$$

4389. $\iint_S (x - y + z) dydz + (y - z + x) dx dz + (z - x + y) \cdot dx dy$, 式中 S 为曲面

$|x - y + z| + |y - z + x| + |z - x + y| = 1$ 的外表面.

解
$$\begin{aligned}
&\iint_S (x - y + z) dydz + (y - z + x) dx dz \\
&\quad + (z - x + y) dx dy \\
&= \iiint_V 3 dx dy dz,
\end{aligned}$$

其中 V 为由曲面 $|x - y + z| + |y - z + x| + |z - x + y| = 1$ 围成的体积. 作变换

$$u = x - y + z, v = y - z + x, w = z - x + y,$$

则

$$\frac{D(u, v, \omega)}{D(x, y, z)} = 4,$$

且由 $|u| + |v| + |\omega| = 1$ 围成的体积等于 $\frac{4}{3}$. 于是, 所求的积分

$$\begin{aligned} & \iint_S (x - y + z) dy dz + (y - z + x) dx dz \\ & + (z - x + y) dx dy \\ & = \iiint_{|u| + |v| + |\omega| \leq 1} 3 \cdot \frac{1}{4} du dv d\omega \\ & = \frac{3}{4} \cdot \frac{4}{3} = 1. \end{aligned}$$

*) 由 $|u| + |v| + |\omega| = 1$ 围成的体积是对称于坐标原点的正八面体的体积, 其大小等于由平面 $u + v + \omega = 1, u = 1, v = 0, \omega = 0$ 所围成的四面体体积的 8 倍, 即为 $8 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{4}{3}$.

4390. 计算

$$\iint_S (x^2 \cos \alpha + y^2 \cos \beta + z^2 \cos \gamma) dS,$$

式中 S 为圆锥曲面 $x^2 + y^2 = z^2 (0 \leq z \leq h)$ 的一部分, $\cos \alpha, \cos \beta, \cos \gamma$ 为此曲面外法线的方向余弦.

解 并合平面 $S_1: z = h, x^2 + y^2 \leq h^2$ 的部分得一立体 V , 则(利用奥氏公式)

$$\begin{aligned} & \iint_{S+S_1} (x^2 \cos \alpha + y^2 \cos \beta + z^2 \cos \gamma) dS \\ & = 2 \iiint_V (x + y + z) dx dy dz \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^{2\pi} d\varphi \int_0^h r dr \int_0^h [r(\cos\varphi + \sin\varphi) + z] dz \\
&= 2\pi \int_0^h (rh^2 - r^3) dr = \frac{\pi h^4}{2}.
\end{aligned}$$

又因

$$\begin{aligned}
&\iint_{S_1} (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS \\
&= h^2 \iint_{x^2+y^2 \leq h^2} dx dy = \pi h^4,
\end{aligned}$$

于是,

$$\begin{aligned}
&\iint_S (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS \\
&= \frac{\pi h^4}{2} - \pi h^4 = -\frac{\pi h^4}{2}.
\end{aligned}$$

4391. 证明公式

$$\iiint_V \frac{d\xi d\eta d\zeta}{r} = \frac{1}{2} \iint_S \cos(\vec{r}, \vec{n}) dS,$$

其中 S 为包围体积 V 的封闭曲面, $\vec{n}$ 为封闭曲面 S 上的动点 (ξ, η, ζ) 处的外法线, 而

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2},$$

$\vec{r}$ 为从点 (x, y, z) 到点 (ξ, η, ζ) 的矢径.

证 先设曲面 S 不包围点 (x, y, z) (即点 (x, y, z) 在 V 之外), 我们有

$$\begin{aligned}
\cos(\vec{r}, \vec{n}) &= \cos(\vec{r}, x) \cos\alpha + \cos(\vec{r}, y) \cos\beta \\
&\quad + \cos(\vec{r}, z) \cos\gamma,
\end{aligned}$$

其中 $\cos\alpha, \cos\beta, \cos\gamma$ 为 $\vec{n}$ 的方向余弦. 由于

$$\cos(\vec{r}, x) = \frac{\xi - x}{r}, \cos(\vec{r}, y) = \frac{\eta - y}{r},$$

$$\cos(\vec{r}, z) = \frac{\xi - z}{r},$$

故

$$\begin{aligned}\cos(\vec{r}, \vec{n}) &= \frac{\xi - x}{r} \cos \alpha + \frac{\eta - y}{r} \\ &\quad \cdot \cos \beta + \frac{\xi - z}{r} \cos \gamma.\end{aligned}$$

于是, 利用奥氏公式, 得

$$\begin{aligned}\iint_S \cos(\vec{r}, \vec{n}) dS &= \iint_S \left(\frac{\xi - x}{r} \cos \alpha + \frac{\eta - y}{r} \right. \\ &\quad \left. \cdot \cos \beta + \frac{\xi - z}{r} \cos \gamma \right) dS \\ &= \iiint_V \left[\frac{\partial}{\partial \xi} \left(\frac{\xi - x}{r} \right) + \frac{\partial}{\partial \eta} \left(\frac{\eta - y}{r} \right) \right. \\ &\quad \left. + \frac{\partial}{\partial \xi} \left(\frac{\xi - z}{r} \right) \right] d\xi d\eta d\xi \\ &= \iiint_V \frac{2}{r} d\xi d\eta d\xi,\end{aligned}$$

故

$$\iiint_V \frac{d\xi d\eta d\xi}{r} = \frac{1}{2} \iint_S \cos(\vec{r}, \vec{n}) dS.$$

次设曲面 S 包围点 (x, y, z) . 这时, 不能对 V 应用奥氏公式, 必须用一小区域将点 (x, y, z) 挖掉, 即以点 (x, y, z) 为中心, ϵ 为半径作一开球域 V_ϵ (ϵ 充分小), 其边界 (球面) 以 S_ϵ 表示. 对闭域 $V - V_\epsilon$ 应用奥氏公式, 仿上可得

$$\iint_S \cos(\vec{r}, \vec{n}) dS + \iint_{S_\epsilon} \cos(\vec{r}, \vec{n}) dS$$

$$\begin{aligned}
&= \iiint_{V-V_\epsilon} \left[\frac{\partial}{\partial \xi} \left(\frac{\xi - x}{r} \right) + \frac{\partial}{\partial \eta} \left(\frac{\eta - y}{r} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial \zeta} \left(\frac{\zeta - z}{r} \right) \right] d\xi d\eta d\zeta \\
&= 2 \iiint_{V-V_\epsilon} \frac{d\xi d\eta d\zeta}{r}.
\end{aligned}$$

但在 S_ϵ 上, $\vec{n}$ 的方向与 $\vec{r}$ 的方向相反, 故 $\cos(\vec{r}, \vec{n}) = -1$. 于是,

$$\iint_{S_\epsilon} \cos(\vec{r}, \vec{n}) dS = -4\pi\epsilon^2.$$

由此可知, 在前式中令 $\epsilon \rightarrow +0$ 取极限, 即得

$$\begin{aligned}
\iiint_V \frac{d\xi d\eta d\zeta}{r} &= \lim_{\epsilon \rightarrow +0} \iiint_{V-V_\epsilon} \frac{d\xi d\eta d\zeta}{r} \\
&= \frac{1}{2} \iint_S \cos(\vec{r}, \vec{n}) dS.
\end{aligned}$$

证毕.

4392. 计算高斯积分

$$I(x, y, z) = \iint_S \frac{\cos(\vec{r}, \vec{n})}{r^2} dS,$$

式中 S 为包含体积 V 的简单封闭平滑曲面, $\vec{n}$ 为曲面 S 上在点 (ξ, η, ζ) 处的外法线, $\vec{r}$ 为连接点 (x, y, z) 和点 (ξ, η, ζ) 的矢径,

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}.$$

研究两种情形: (a) 当曲面 S 不包围点 (x, y, z) ; (b) 当曲面 S 包围点 (x, y, z) .

解 设法线 $\vec{n}$ 的方向余弦为 $\cos\alpha, \cos\beta, \cos\gamma$, 则

$$\begin{aligned}\cos(\vec{r}, \vec{n}) &= \cos(\vec{r}, x)\cos\alpha + \cos(\vec{r}, y)\cos\beta \\ &+ \cos(\vec{r}, z)\cos\gamma \\ &= \frac{\xi - x}{r}\cos\alpha + \frac{\eta - y}{r}\cos\beta + \frac{\zeta - z}{r}\cos\gamma.\end{aligned}$$

因此, 高斯积分

$$\begin{aligned}I(x, y, z) &= \iint_S \frac{\xi - x}{r^3} d\eta d\zeta \\ &+ \frac{\eta - y}{r^3} d\zeta d\xi + \frac{\zeta - z}{r^3} d\xi d\eta,\end{aligned}$$

这里 $P = \frac{\xi - x}{r^3}$, $Q = \frac{\eta - y}{r^3}$, $R = \frac{\zeta - z}{r^3}$. 于是,

$$\begin{aligned}\frac{\partial P}{\partial \xi} &= \frac{1}{r^3} - \frac{3(\xi - x)^2}{r^5}, \\ \frac{\partial Q}{\partial \eta} &= \frac{1}{r^3} - \frac{3(\eta - y)^2}{r^5}, \\ \frac{\partial R}{\partial \zeta} &= \frac{1}{r^3} - \frac{3(\zeta - z)^2}{r^5}\end{aligned}$$

它们仅在点 (x, y, z) 处不连续. 因此

(a) 当曲面 S 不包围点 (x, y, z) 时, 则

$$\frac{\partial P}{\partial \xi} + \frac{\partial Q}{\partial \eta} + \frac{\partial R}{\partial \zeta} = 0.$$

于是, 利用奥氏公式有

$$I(x, y, z) = \iint_S \frac{\cos(\vec{r}, \vec{n})}{r^2} dS = 0.$$

(b) 当曲面 S 包围点 (x, y, z) 时, 则我们以点 (x, y, z) 为中心, ϵ 为半径作一球 V_ϵ 包围在 S 内, 此球面记以 S_ϵ . 将奥氏公式用于 $V - V_\epsilon$ 上,

即得

$$\iint_{S+S_\epsilon} \frac{\cos(\vec{r}, \vec{n})}{r^2} dS = 0.$$

但因

$$\iint_{S_\epsilon} \frac{\cos(\vec{r}, \vec{n})}{r^2} dS = \iint_{S_\epsilon} \left(-\frac{1}{\epsilon^2} \right) dS = -4\pi,$$

故得

$$I(x, y, z) = \iint_S \frac{\cos(\vec{r}, \vec{n})}{r^2} dS = 4\pi.$$

4393. 证明: 若

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

及 S 为包围有界体积 V 的光滑曲面, 则下列公式正确:

$$(a) \iint_S \frac{\partial u}{\partial n} dS = \iiint_V \Delta u dx dy dz;$$

$$(6) \iint_S u \frac{\partial u}{\partial n} dS \\ = \iiint_V \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dx dy dz \\ + \iiint_V u \Delta u dx dy dz,$$

式中 u 和它的直到二阶的偏导函数是在域 $V + S$ 内连续的函数, $\frac{\partial u}{\partial n}$ 为沿曲面 S 的外法线的导函数.

证 由于

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma,$$

因此, 利用奥氏公式即得

$$\iint_S \frac{\partial u}{\partial n} dS = \iint_S \left(\frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta \right. \\ \left. + \frac{\partial u}{\partial z} \cos \gamma \right) dS$$

$$\begin{aligned}
&= \iiint_V \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dx dy dz \\
&= \iiint_V \Delta u dx dy dz.
\end{aligned}$$

$$\begin{aligned}
(6) \quad \iint_S u \frac{\partial u}{\partial n} dS &= \iint_S \left(u \frac{\partial u}{\partial x} \cos \alpha \right. \\
&\quad \left. + u \frac{\partial u}{\partial y} \cos \beta + u \frac{\partial u}{\partial z} \cos \gamma \right) dS \\
&= \iiint_V \left[\frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(u \frac{\partial u}{\partial y} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial z} \left(u \frac{\partial u}{\partial z} \right) \right] dx dy dz \\
&= \iiint_V u \Delta u dx dy dz + \iiint_V \left[\left(\frac{\partial u}{\partial x} \right)^2 \right. \\
&\quad \left. + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dx dy dz.
\end{aligned}$$

4394. 证明空间的格林第二公式

$$\iiint_V \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx dy dz = \iint_S \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial v}{\partial n} \\ u & v \end{vmatrix} dS,$$

式中体积 V 是由曲面 S 所包围的, $\vec{n}$ 是曲面 S 的外法线方向, 而函数 $u = u(x, y, z), v = v(x, y, z)$ 为域 $V + S$ 内可微分两次的函数.

$$\begin{aligned}
\text{证} \quad \iint_S \begin{vmatrix} \frac{\partial u}{\partial n} & \frac{\partial v}{\partial n} \\ u & v \end{vmatrix} dS \\
= \iint_S \left[\left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) \cos \alpha + \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} \right) \cos \beta \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) \cos \gamma \Big) dS \\
& = \iiint_V \left[\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} \right) \right. \\
& \quad \left. + \frac{\partial}{\partial z} \left(v \frac{\partial u}{\partial z} - u \frac{\partial v}{\partial z} \right) \right] dx dy dz \\
& = \iiint_V \left[v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \right. \\
& \quad \left. - u \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \right] dx dy dz \\
& = \iiint_V \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx dy dz.
\end{aligned}$$

4395. 函数 $u = u(x, y, z)$ 在某一域内具有直到二阶的连续导函数, 若

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

则 $u(x, y, z)$ 在这个域内称为调和函数.

证明: 若 u 是被平滑曲面 S 所包围的有界闭域 V 内的调和函数, 则下列公式是正确的.

$$(a) \iint_S \frac{\partial u}{\partial n} dS = 0;$$

$$\begin{aligned}
(b) \iiint_V \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dx dy dz \\
= \iint_S u \frac{\partial u}{\partial n} dS,
\end{aligned}$$

式中 $\vec{n}$ 为曲面 S 的外法线.

试用公式 (a), 证明在域 V 内的调和函数由它在界限 S 上的值唯一地确定.

证 (a) 由于 $\Delta u = 0$, 故利用 4393 题(a) 的结果, 即得

$$\int_S \frac{\partial u}{\partial n} dS = \iiint_V 0 dx dy dz = 0.$$

(6) 利用 4393 题(σ) 的结果, 即得

$$\begin{aligned} \int_S u \frac{\partial u}{\partial n} dS &= \iiint_V u \cdot 0 dx dy dz \\ &+ \iiint_V \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dx dy dz \\ &= \iiint_V \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dx dy dz. \end{aligned}$$

与 4333 题一样, 只要证明: 若在界限 S 上调和函数 $u = 0$, 则它在域 V 上也恒有 $u = 0$. 事实上, 利用本题 (6), 得

$$\iiint_V \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dx dy dz = 0.$$

因此,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0,$$

即在域 V 上 $u \equiv$ 常数. 但在 S 上 $u = 0$, 故在域 V 上 $u = 0$. 这就是证明: 在域 V 内的调和函数由它在界限 S 上的值唯一地确定.

4396. 证明: 若函数 $u = u(x, y, z)$ 是在由光滑曲面 S 所包围着的有界闭域 V 内的调和函数, 则

$$u(x, y, z) = \frac{1}{4\pi} \int_S \left(u \frac{\cos(\vec{r}, \vec{n})}{r^2} + \frac{1}{r} \frac{\partial u}{\partial n} \right) dS,$$

式中 $\vec{r}$ 是从域 V 的内面的点 (x, y, z) 引至曲面 S 上的动点 (ξ, η, ζ) 的矢径, 而

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2},$$

$\vec{n}$ 为曲面 S 上在点 (ξ, η, ζ) 的外法线向量.

证 在 4394 题中令 $v = \frac{1}{r}$, 则当 $(\xi, \eta, \zeta) \neq (x, y, z)$ 时, 有 $\Delta v = 0$. 现以点 $P(x, y, z)$ 为中心, ρ 为半径作一球面 S_ρ 含于曲面 S 内, 再将 4394 题应用到由曲面 $S + S_\rho$ 所包围的体积 V 内, 即得

$$\iint_{S+S_\rho} \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial(\frac{1}{r})}{\partial n} \right\} dS = 0,$$

或

$$\begin{aligned} & \iint_{S+S_\rho} \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial(\frac{1}{r})}{\partial n} \right\} dS \\ &= - \iint_{S_\rho} \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial(\frac{1}{r})}{\partial n} \right\} dS. \end{aligned}$$

显然, S 上的法线是向外的, 而 S_ρ 上的法线是指向球心的, 即指向半径减少一方. 因此,

$$\frac{\partial(\frac{1}{r})}{\partial n} = - \frac{\partial(\frac{1}{r})}{\partial r} \Big|_{r=\rho} = \frac{1}{\rho^2}.$$

于是, 我们有

$$\iint_{S_\rho} \left(\frac{1}{\rho} \frac{\partial u}{\partial n} - \frac{u}{\rho^2} \right) dS = - \iint_{S_\rho} \left\{ \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial(\frac{1}{r})}{\partial n} \right\} dS.$$

但

$$\iint_{S_\rho} \frac{1}{\rho} \frac{\partial u}{\partial n} dS = \frac{1}{\rho} \iint_{S_\rho} \frac{\partial u}{\partial n} dS = 0,$$

又利用中值定理,得

$$\begin{aligned}\iint_{S_\rho} \frac{u}{\rho^2} dS &= \frac{1}{\rho^2} u(x', y', z') \cdot 4\pi\rho^2 \\ &= 4\pi u(x', y', z'),\end{aligned}$$

其中 $u(x', y', z')$ 为函数 u 在球面 S_ρ 上某点之值. 从而

$$u(x', y', z') = \frac{1}{4\pi} \iint_S \left[\frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial(\frac{1}{r})}{\partial n} \right] dS.$$

上式右端与 ρ 无关. 而 $\lim_{\rho \rightarrow +0} u(x', y', z') = u(x, y, z)$.

因而, 令 $\rho \rightarrow +0$, 即得

$$u(x, y, z) = \frac{1}{4\pi} \iint_S \left[\frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial(\frac{1}{r})}{\partial n} \right] dS.$$

又由于

$$\begin{aligned}\frac{\partial(\frac{1}{r})}{\partial n} &= \frac{\partial(\frac{1}{r})}{\partial r} \frac{\partial r}{\partial n} = -\frac{1}{r^2} \left(\frac{\partial r}{\partial \xi} \cos \alpha \right. \\ &\quad \left. + \frac{\partial r}{\partial \eta} \cos \beta + \frac{\partial r}{\partial \zeta} \cos \gamma \right) \\ &= -\frac{1}{r^2} \left(\frac{\xi - x}{r} \cos \alpha + \frac{\eta - y}{r} \cos \beta + \frac{\zeta - z}{r} \cos \gamma \right) \\ &= -\frac{\cos(\vec{r}, \vec{n})}{r^2}.\end{aligned}$$

故最后得

$$u(x, y, z) = \frac{1}{4\pi} \iint_S \left(\frac{u \cos(\vec{r}, \vec{n})}{r^2} + \frac{1}{r} \frac{\partial u}{\partial n} \right) dS.$$

4397. 证明, 若 $u = u(x, y, z)$ 为在以 R 为半径, 以点 (x_0, y_0, z_0) 为球心的球 S 内的调和函数, 则

$$u(x_0, y_0, z_0) = \frac{1}{4\pi R^2} \iint_S u(x, y, z) dS$$

(中值定理).

证 在球 S 上应用 4396 题的结果, 即得

$$\begin{aligned} u(x_0, y_0, z_0) &= \iint_S \left(\frac{u \cos(\vec{r}, \vec{n})}{r^2} + \frac{1}{r} \frac{\partial u}{\partial n} \right) dS \\ &= \frac{1}{4\pi} \iint_S \left(\frac{u}{R^2} + \frac{1}{R} \frac{\partial u}{\partial n} \right) dS \\ &= \frac{1}{4\pi R^2} \int_S u(x, y, z) dS. \end{aligned}$$

*) 利用 4395 题的结果, 有

$$\frac{1}{4\pi R} \iint_S \frac{\partial u}{\partial n} dS = 0.$$

4398. 证明, 在有界闭域 V 内连续且在其内部是调和的函数

$u = u(x, y, z)$, 若它不是常数, 则在域内的点函数不能达到最大和最小的值(极大值原则).

证 证明与 4337 题(平面情形)完全类似. 设有界闭域为 $\bar{\Omega}$, 它是由有界开域 Ω 及其边界 $\partial\Omega$ 构成. 我们要证明: 如果 $u(x, y, z)$ 在 $\bar{\Omega}$ 的某内点 $P_0(x_0, y_0, z_0)$ 达到其最大值或最小值(例如, 设达到最大值), 则 $u(x, y, z)$ 在 $\bar{\Omega}$ 上必为常数, 下分三步证明:

(1) 先证: 若球域 $V_\rho = \{(x, y, z) | (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq \rho^2\}$ 完全属于 Ω , 则 $u(x, y, z)$ 在 V_ρ 上必为常数.

对任何的 $0 < r \leq \rho$, 用 S_r 表球面 $\{(x, y, z) | (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2\}$. 由 4397 题的结果知

$$u(x_0, y_0, z_0) = \frac{1}{4\pi r^2} \iint_{S_r} u(x, y, z) dS,$$

故

$$\frac{1}{4\pi r^2} \iint_{S_r} [u(x_0, y_0, z_0) - u(x, y, z)] dS = 0, \quad (1)$$

但因 $u(x_0, y_0, z_0)$ 是最大值, 故在 S_r 上恒有

$$u(x_0, y_0, z_0) - u(x, y, z) \geq 0.$$

由此, 根据(1), 即易知在 S_r 上 $u(x_0, y_0, z_0) - u(x, y, z) \equiv 0$. 因为, 若有某点 $(x_1, y_1, z_1) \in S_r$ 使

$$u(x_0, y_0, z_0) - u(x_1, y_1, z_1) = \tau > 0,$$

则由 $u(x, y, z)$ 的连续性知, 必有以 (x_1, y_1, z_1) 为中心的某小球域 σ 存在, 使当 $(x, y, z) \in \sigma$ 时, 恒有

$$u(x_0, y_0, z_0) - u(x, y, z) \geq \frac{\tau}{2}.$$

用 S' 表 S_r 含于 σ 内的部分, 则

$$\begin{aligned} & \iint_{S_r} [u(x_0, y_0, z_0) - u(x, y, z)] dS \\ & \geq \iint_{S'} [u(x_0, y_0, z_0) - u(x, y, z)] dS \\ & \geq \iint_{S'} \frac{\tau}{2} dS = \frac{1}{2} \tau D_r > 0, \end{aligned}$$

其中 D_r 表 S' 的面积, 此显然与(1)式矛盾. 于是, 在 S_r 上有

$$u(x_0, y_0, z_0) - u(x, y, z) \equiv 0.$$

再根据 r 的任意性 ($0 < r \leq \rho$), 即知对任何 $(x, y, z) \in V_\rho$, 都有 $u(x, y, z) = u(x_0, y_0, z_0)$; 换句话说, $u(x, y, z)$ 在 V_ρ 上是常数.

(2) 次证: 设 $P^*(x^*, y^*, z^*)$ 为 Ω 的任一内点 (即 $P^* \in \Omega$), 则必有

$$u(x^*, y^*, z^*) = u(x_0, y_0, z_0).$$

用完全含于 Ω 内的折线 l 将点 $P_0(x_0, y_0, z_0)$ 与点 $P^*(x^*, y^*, z^*)$ 连接起来, 用 δ 表 $\partial\Omega$ 与 l 之间的距离, 即

$$\delta = \min \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

其中的 $\min$ 是对一切 $(x, y, z) \in \partial\Omega, (x', y', z') \in l$ 来取的 (由于 $\partial\Omega, l$ 是互不相交的有界闭集, 可证 $\min$ 一定能达到, 从而 $\delta > 0$). 取 $0 < \delta' < \delta$, 以点 P_0 为中心, δ' 为半径作一球, 得球域 $V_0 = \{(x, y, z) | (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq \delta'^2\}$. 此球域完全含于 Ω 内, 则 (1) 段已证的结果知, $u(x, y, z)$ 在 V_0 中为常数. 特别是 $u(x_1, y_1, z_1) = u(x_0, y_0, z_0)$. 这里点 $P_1(x_1, y_1, z_1)$ 代表球面 $S_0 = \{(x, y, z) | (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \delta'^2\}$ 与折线 l 的交点 (参看 4337 题的图). 又以点 P_1 为中心, δ' 为半径作一球, 得球域 $V_1 = \{(x, y, z) | (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \leq \delta'^2\}$. 于是, V_1 也完全含于 Ω 内. 由于 $u(x, y, z)$ 也在点 $P_1(x_1, y_1, z_1)$ 达到最大值, 故将 (1) 段的结果用于 V_1 , 可知 $u(x, y, z)$ 在 V_1 上是常数. 特别是 $u(x_2, y_2, z_2) = u(x_1, y_1, z_1)$. 这里点 $P_2(x_2, y_2, z_2)$ 为球面 $S_1 = \{(x, y, z) | (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = \delta'^2\}$ 与 l 的交点 (除 P_0 外的另一交点). 再以点 P_2 为中心, δ' 为半径作一球域 $V_2, \dots$ 这样继续作下去. 显然, 至多经过 n 次 (n 为大于 $\frac{5}{\delta'}$ 的最小正

整数, s 表折线 l 的长), 点 $P^*(x^*, y^*, z^*)$ 必属于 V_{n-1} . 从而

$$\begin{aligned} u(x^*, y^*, z^*) &= u(x_{n-1}, y_{n-1}, z_{n-1}) = \cdots \\ &= u(x_1, y_1, z_1) = u(x_0, y_0, z_0). \end{aligned}$$

(3) 由(2)段的结果知, $u(x, y, z)$ 在 Ω 上是常数, 根据 $u(x, y, z)$ 在 $\bar{\Omega}$ 上的连续性, 通过由 Ω 的点趋向 $\partial\Omega$ 的点取极限, 即知 $u(x, y, z)$ 在 $\bar{\Omega}$ 上是常数. 证毕.

注: 从证明过程中看出, 需假定区域 Ω (从而 $\bar{\Omega}$) 是连通的. 事实上, 若 Ω 不连通, 则结论不一定成立. 例如, 设 $\bar{\Omega} = V_1 + V_2$, 其中 V_1 与 V_2 是两个互无公共点的闭球域, 而令

$$u(x, y, z) = \begin{cases} C_1, & (x, y, z) \in V_1, \\ C_2, & (x, y, z) \in V_2, \end{cases}$$

其中 $C_1 \neq C_2$ 是两个常数, 则 $u(x, y, z)$ 显然是 $\bar{\Omega}$ 上的调和函数且在 Ω 上不是常数, 但它却在其内点达到最大值与最小值.

4399. 物体 V 全部浸溺于液体中, 从巴斯葛耳定律出发, 证明液体的浮力等于物体同体积液体之重而方向垂直向上 (阿基米德定律).

证 将 Oxy 坐标面取在液面上, 而 Oz 轴垂直液面向下. 设液体比重为 ρ , 浸入液体的物体 V 的表面积为 S . 若对应于面积元素 dS 液体的深度为 z , 则在 dS 上所受的壓力为 $\rho z dS$. 由于此压力总是垂直于 dS 面的, 故压力在各坐标轴上的射影为

$$-\rho z \cos \alpha dS, -\rho z \cos \beta dS, -\rho z \cos \gamma dS.$$

利用奥氏公式, 即得作用于物体整个表面的总压力在各

坐标轴上的射影

$$P_x = -\rho \iint_S z \cos \alpha dS = -\rho \iiint_V 0 dx dy dz = 0,$$

$$P_y = -\rho \iint_S z \cos \beta dS = -\rho \iiint_V 0 dx dy dz = 0,$$

$$P_z = -\rho \iint_S z \cos \gamma dS = -\rho \iiint_V 1 dx dy dz = -\rho V.$$

因此,压力的主向量即合力,朝着垂直向上的方向,其大小等于被物体排出的液体的重量. 这就是阿基米德定律.

4400. 设 S_t 是变动的球 $(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = t^2$, 而函数 $f(\xi, \eta, \zeta)$ 是连续的, 证明函数

$$u(x, y, z, t) = \frac{1}{4\pi} \iint_{S_t} \frac{f(\xi, \eta, \zeta)}{t} dS_t$$

满足波动方程式

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$

和初值条件 $u|_{t=0} = 0, \frac{\partial u}{\partial t} \Big|_{t=0} = f(x, y, z)$.

证 首先指出, 本题应设 $f(\xi, \eta, \zeta)$ 具有连续的二阶偏导函数. 先验证函数 u 满足初值条件 $u|_{t=0} = 0$ (意即 $\lim_{t \rightarrow +0} u = 0$) 及 $\frac{\partial u}{\partial t} \Big|_{t=\epsilon} = f(x, y, z)$ (意即 $\lim_{t \rightarrow +0} \frac{\partial u}{\partial t} = f(x, y, z)$). 今固定 (x, y, z) . 由连续性知, 存在常数 $M > 0$, 使当 $(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 \leq 1$ 时恒有

$$|f(\xi, \eta, \zeta)| \leq M, |f'_\xi(\xi, \eta, \zeta)| \leq M,$$

$$|f'_\eta(\xi, \eta, \zeta)| \leq M, |f'_\zeta(\xi, \eta, \zeta)| \leq M.$$

当 $0 < t < 1$ 时, 我们有

$$\begin{aligned}
|u(x, y, z, t)| &\leq \frac{1}{4\pi t} \iint_{S_t} |f(\xi, \eta, \zeta)| dS_t \\
&\leq \frac{1}{4\pi t} \iint_S M dS_t = \frac{1}{4\pi t} \cdot M 4\pi t^2 \\
&= Mt,
\end{aligned}$$

由此可知, $\lim_{t \rightarrow +0} u(x, y, z, t) = 0$.

又作变量代换 $\xi = x + ut, \eta = y + vt, \zeta = z + \omega t$, 则有

$$\begin{aligned}
u(x, y, z, t) &= \frac{1}{4\pi} \iint_S f(x + ut, y + vt, z + \omega t) \\
&\quad \cdot t dS,
\end{aligned} \tag{1}$$

其中 S 是单位球面 $u^2 + v^2 + \omega^2 = 1$. 于是,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{4\pi} \frac{\partial}{\partial t} \iint_S f(x + ut, y + vt, z + \omega t) t dS \\
&= \frac{1}{4\pi} \iint_S \left(\frac{\partial f}{\partial \xi} u + \frac{\partial f}{\partial \eta} v + \frac{\partial f}{\partial \zeta} \omega \right) t dS \\
&\quad + \frac{1}{4\pi} \iint_S f(x + ut, y + vt, z + \omega t) dS \\
&= I_1 + I_2.
\end{aligned} \tag{2}$$

显然, 当 $0 < t < 1$ 时,

$$|I_1| \leq \frac{t}{4\pi} \iint_S 3M dS = 3Mt,$$

故 $\lim_{t \rightarrow +0} I_1 = 0$, 又显然(由于连续性)

$$\begin{aligned}
\lim_{t \rightarrow +0} I_2 &= \frac{1}{4\pi} \iint_S f(x, y, z) dS \\
&= \frac{f(x, y, z)}{4\pi} \iint_S dS = f(x, y, z).
\end{aligned}$$

因此,得

$$\lim_{t \rightarrow +0} \frac{\partial u}{\partial t} = f(x, y, z).$$

下面再验证 u 满足波动方程. 由(2)式,利用奥氏公式,有(V 为球体 $u^2 + v^2 + \omega^2 \leq 1$, V_t 为球体 $u_1^2 + v_1^2 + \omega_1^2 \leq t^2$)

$$\begin{aligned} I_1 &= \frac{t}{4\pi} \iint_S \left(\frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta + \frac{\partial f}{\partial z} \cos \gamma \right) dS \\ &= \frac{t^2}{4\pi} \iiint_V \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f(x + ut, y + vt, \\ &\quad z + \omega t) du dv d\omega \\ &= \frac{1}{4\pi t} \iiint_{V_t} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f(x + u_1, y + v_1, \\ &\quad z + \omega_1) du_1 dv_1 d\omega_1 \\ &= \frac{1}{4\pi t} \iiint_{V_t} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f(x + u_1, y + v_1, \\ &\quad z + \omega_1) du_1 dv_1 d\omega_1 \\ &= \frac{1}{4\pi t} \triangle \left(\iiint_{V_t} f(x + u_1, y + v_1, \right. \\ &\quad \left. z + \omega_1) du_1 dv_1 d\omega_1 \right) \\ &= \frac{1}{4\pi t} \triangle \left(\int_0^t \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x + r \cos \phi \cos \varphi, y \right. \\ &\quad \left. + r \cos \phi \sin \varphi, z + r \sin \phi) r^2 \cos \phi d\phi d\varphi dr \right) \\ &= \frac{I_3}{4\pi t}, \end{aligned}$$

其中 $\cos \alpha = u, \cos \beta = v, \cos \gamma = \omega$ 为 S 的外法线的方向余弦, 又由(2)式及(1)式, 有

$$\begin{aligned}
 I_2 &= \frac{1}{4\pi t} \iint_S f(x + ut, y + vt, z + wt) t dS \\
 &= \frac{u}{t},
 \end{aligned}$$

从而

$$\frac{\partial u}{\partial t} = \frac{I_3}{4\pi t} + \frac{u}{t} \quad (t > 0).$$

于是,

$$\begin{aligned}
 \frac{\partial^2 u}{\partial t^2} &= \frac{1}{4\pi t} \frac{\partial I_3}{\partial t} - \frac{I_3}{4\pi t^2} - \frac{u}{t^2} + \frac{1}{t} \frac{\partial u}{\partial t} \\
 &= \frac{1}{4\pi t} \frac{\partial I_3}{\partial t} - \frac{I_3}{4\pi t^2} - \frac{u}{t^2} + \frac{1}{t} \left(\frac{I_3}{4\pi t} + \frac{u}{t} \right) \\
 &= \frac{1}{4\pi t} \frac{\partial I_3}{\partial t} \quad (t > 0).
 \end{aligned} \tag{3}$$

但

$$\begin{aligned}
 \frac{\partial I_3}{\partial t} &= \frac{\partial}{\partial t} \Delta \left(\int_0^t \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x + r \cos \psi \cos \varphi, \right. \\
 &\quad \left. y + r \cos \psi \sin \varphi, z + r \sin \psi) r^2 \cos \psi d\psi d\varphi dr \right) \\
 &= \Delta \left[\frac{\partial}{\partial t} \int_0^t \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x + r \cos \psi \cos \varphi, \right. \\
 &\quad \left. y + r \cos \psi \sin \varphi, z + r \sin \psi) r^2 \cos \psi d\psi d\varphi dr \right] \\
 &= \Delta \left[\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x + t \cos \psi \cos \varphi, \right. \\
 &\quad \left. y + t \cos \psi \sin \varphi, z + t \sin \psi) t^2 \cos \psi d\psi d\varphi \right] \\
 &= \Delta \left(\iint_{S_t} f(\xi, \eta, \zeta) dS_t \right)
 \end{aligned}$$

故

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4\pi t} \Delta \left(\iint_{S_t} f(\xi, \eta, \zeta) dS_t \right)$$

$$\begin{aligned}
&= \Delta \left(\frac{1}{4\pi} \iint_{S_t} \frac{f(\xi, \eta, \zeta)}{t} dS_t \right) \\
&= \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} (t > 0).
\end{aligned}$$

证毕.

§ 17. 场论初步

1° 梯度 若 $\vec{u}(r) = u(x, y, z)$ (其中 $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$) 是连续可微分的数量场, 则称向量

$$\text{grad} u = \frac{\partial u}{\partial x} \vec{i} + \frac{\partial u}{\partial y} \vec{j} + \frac{\partial u}{\partial z} \vec{k}$$

为 $u(r)$ 的梯度, 或简记为 $\text{grad} u = \nabla u$, 其中 $\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$.
于已知点 (x, y, z) 场 u 的梯度的方向是与过此点的等位面 $u(x, y, z) = C$ 的法线方向一致. 对于场的每一点此向量给出函数 u 变化之最大速度的大小

$$|\text{grad} u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}$$

与方向.

在某方向 $\vec{l} \{\cos \alpha, \cos \beta, \cos \gamma\}$ 上场 u 的导数等于

$$\frac{\partial u}{\partial l} = \text{grad} u \cdot \vec{l} = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma.$$

2° 场的散度与场的旋度 若

$$\vec{a}(\vec{r}) = a_x(x, y, z)\vec{i} + a_y(x, y, z)\vec{j} + a_z(x, y, z)\vec{k}$$

是连续可微分的向量场, 则称数量

$$\text{div} \vec{a} = \nabla \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

为这个场的散度.

向量

$$\operatorname{rot} \vec{a} = \nabla \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

名为场的旋度.

3° 穿过曲面的流量 若向量 $\vec{a}(\vec{r})$ 在域 Ω 内产生向量场, 则称积分

$$\iint_S \vec{a} n dS = \iint_S (a_x \cos \alpha + a_y \cos \beta + a_z \cos \gamma) dS$$

为穿过位于域 Ω 内的已知曲面 S 的流向法线上单位向量 $\vec{n} \{ \cos \alpha, \cos \beta, \cos \gamma \}$ 所指的那一面的流量. 在向量的论述中 奥斯特洛格拉德斯基公式 具有下面的形状:

$$\iint_S \vec{a} n dS = \iiint_V \operatorname{div} \vec{a} dx dy dz,$$

式中 S 为包围体积 V 的曲面, $\vec{n}$ 为曲面 S 的外法线单位向量.

4° 向量的环流 数

$$\int_C \vec{a} d\vec{r} = \int_C a_x dx + a_y dy + a_z dz$$

称为向量 $\vec{a}(\vec{r})$ 沿某曲线 C 所取的线积分(场作的功).

若围线 C 是封闭的, 则称线积分为向量 $\vec{a}$ 沿围线 C 的环流.

在向量的形式上斯托克斯公式为

$$\oint_C \vec{a} d\vec{r} = \iint_S \operatorname{rot} \vec{a} n dS,$$

式中 C 为包围曲面 S 的封闭围线, 并且对曲面 S 的法线 $\vec{n}$ 之方向应当这样来选择: 使得立于曲面 S 上的观察者, 以头向着法线的方向, 围线 C 的回绕是依反时针前进的方向作成的(对于右旋坐标系).

5° 有势场 向量场 $\vec{a}(\vec{r})$ 是某数量 u 的梯度即 $\operatorname{grad} u = \vec{a}$, $\vec{a}$ 名为有势场, 而数量 u 名为场的势.

若势 u 为单值函数, 则

$$\int_{AB} \vec{a} d\vec{r} = u(B) - u(A).$$

特别是, 在这个情形向量 $\vec{a}$ 的环流等于零.

给定在单联通域内的场 $\vec{a}$ 为有势场的充要条件, 是条件 $\text{rot} \vec{a} = \vec{0}$ 满足, 就是说, 这样的场应当是无旋场.

4401. 求场

$$u = x^2 + 2y^2 + 3z^2 + xy + 3x - 2y - 6z$$

在: (a) $O(0, 0, 0)$; (б) $A(1, 1, 1)$; (в) $B(2, 1, 1)$ 诸点梯度的大小和方向. 在场的怎样的点, 梯度等于零?

解 $\frac{\partial u}{\partial x} = 2x + y + 3, \frac{\partial u}{\partial y} = 4y + x - 2, \frac{\partial u}{\partial z} = 6z - 6.$

(a) 在 O 点, 有

$$\text{grad} u = 3\vec{i} - 2\vec{j} - 6\vec{k}, |\text{grad} u| = 7,$$

方向: $\cos \alpha = \frac{3}{7}, \cos \beta = -\frac{2}{7}, \cos \gamma = -\frac{6}{7};$

(б) 在 A 点, 有

$$\text{grad} u = 6\vec{i} + 3\vec{j}, |\text{grad} u| = 3\sqrt{5},$$

方向: $\cos \alpha = \frac{2}{\sqrt{5}}, \cos \beta = \frac{1}{\sqrt{5}}, \cos \gamma = 0;$

(в) 在 B 点, 有

$$\text{grad} u = 7\vec{i}, |\text{grad} u| = 7,$$

方向: $\cos \alpha = 1, \cos \beta = \cos \gamma = 0.$

一般地说, 我们有

$$|\text{grad} u| = \sqrt{(2x + y + 3)^2 + (4y + x - 2)^2 + (6z - 6)^2}.$$

要 $|\text{grad} u| = 0$, 只要

$$\begin{cases} 2x + y + 3 = 0, \\ 4y + x - 2 = 0, \\ 6z - 6 = 0. \end{cases}$$

解之,得 $x = -2, y = 1, z = 1$, 即在点 $(-2, 1, 1)$ 梯度为零.

4402. 在空间 $Oxyz$ 的那些点, 场

$$u = x^3 + y^3 + z^3 - 3xyz$$

的梯度 (a) 垂直于 Oz 轴; (b) 平行于 Oz 轴; (c) 等于零?

解 $\text{grad} u = (3x^2 - 3yz)\vec{i} + (3y^2 - 3xz)\vec{j}$
 $+ (3z^2 - 3xy)\vec{k}.$

(a) 要 $\text{grad} u \perp Oz$, 只要 $\text{grad} u \cdot \vec{k} = 0$, 即 $3z^2 - 3xy = 0$ 或 $z^2 = xy$. 因此, 在满足 $z^2 = xy$ 的点 (x, y, z) , 其梯度垂直于 Oz 轴.

(b) 要 $\text{grad} u \parallel Oz$, 只要

$$\begin{cases} 3x^2 - 3yz = 0, \\ 3y^2 - 3xz = 0. \end{cases}$$

解之得 $x = y = 0$ 及 $x = y = z$. 因此, 在点 $(0, 0, z)$ 及 (x, y, z) (其中 $x = y = z$), 其梯度平行于 Oz 轴.

(c) 要 $|\text{grad} u| = 0$, 只要

$$\begin{cases} 3x^2 - 3yz = 0, \\ 3y^2 - 3xz = 0, \\ 3z^2 - 3xy = 0. \end{cases}$$

解之, 得 $x = y = z$. 因此, 在满足 $x = y = z$ 的点 (x, y, z) , 其梯度等于零.

4403. 已给数量场

$$u = \ln \frac{1}{r},$$

其中 $r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, 在空间 $Oxyz$ 的哪些点下面等式成立

$$|\operatorname{grad} u| = 1?$$

解 $\frac{\partial u}{\partial x} = -\frac{x-a}{r^2}, \frac{\partial u}{\partial y} = -\frac{y-b}{r^2}, \frac{\partial u}{\partial z} = -\frac{z-c}{r^2}.$

于是,

$$\begin{aligned} |\operatorname{grad} u| &= \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2} \\ &= \sqrt{\frac{1}{r^4}[(x-a)^2 + (y-b)^2 + (z-c)^2]} \\ &= \frac{1}{r}. \end{aligned}$$

要 $|\operatorname{grad} u| = 1$, 只要 $r = 1$ 即在以点 (a, b, c) 为中心, 1 为半径的球面上, 均有

$$\left| \operatorname{grad} \left(\ln \frac{1}{r} \right) \right| = 1,$$

其中 $r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.

4404. 作数量场

$$u = \sqrt{x^2 + y^2 + (z+8)^2} + \sqrt{x^2 + y^2 + (z-8)^2}$$

的等位面. 求通过点 $M(9, 12, 28)$ 的等位面. 在域 $x^2 + y^2 + z^2 \leq 36$ 内 $\max u$ 等于什么?

解 等位面可由

$$u = \sqrt{x^2 + y^2 + (z+8)^2} + \sqrt{x^2 + y^2 + (z-8)^2}$$

化简得到. 显然有

$$u \geq \sqrt{(z+8)^2} + \sqrt{(z-8)^2} \geq z+8 - (z-8) = 16.$$

于是,当 $u \geq 16$ 时,有

$$\begin{aligned} u - \sqrt{x^2 + y^2 + (z-8)^2} \\ = \sqrt{x^2 + y^2 + (z+8)^2}. \end{aligned}$$

平方化简可得

$$u^2 - 32z = 2u \sqrt{x^2 + y^2 + (z-8)^2},$$

再平方化简,即得等位面方程

$$\frac{4(x^2 + y^2)}{u^2 - 256} + \frac{4z^2}{u^2} = 1 \quad (u \geq 16),$$

这是绕 Oz 轴旋转的一个旋转面. 图形省略.

当 $x=9, y=12, z=28$ 时, $u=64$. 因此,等位面方程为

$$\frac{x^2 + y^2}{960} + \frac{z^2}{1024} = 1.$$

在域 $x^2 + y^2 + z^2 \leq 36$ 内,由于

$$\begin{aligned} u &= \sqrt{x^2 + y^2 + z^2 + 16z + 64} \\ &+ \sqrt{x^2 + y^2 + z^2 - 16z + 64} \\ &\leq \sqrt{100 + 16z} + \sqrt{100 - 16z} \quad (0 \leq z \leq 6), \end{aligned}$$

故函数 $f(z) = \sqrt{100 + 16z} + \sqrt{100 - 16z}$ 在 $[0, 6]$ 上

的最大值即 u 的最大值. 但是,

$$\begin{aligned} f(z) &= 8 \left(\frac{1}{\sqrt{100 + 16z}} - \frac{1}{\sqrt{100 - 16z}} \right) < 0 \\ &\quad (0 < z \leq 6), \end{aligned}$$

故 $f(z)$ 在 $[0, 6]$ 上是严格减函数,从而

$$\max_{0 \leq z \leq 5} f(z) = f(0) = 20.$$

因此,有

$$\max u = 20.$$

4405. 求场

$$u = \frac{x}{x^2 + y^2 + z^2}$$

在点 $A(1, 2, 2)$ 及 $B(-3, 1, 0)$ 的梯度之间

$$\text{解} \quad \frac{\partial u}{\partial x} = \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2},$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2 + z^2)^2},$$

$$\frac{\partial u}{\partial z} = -\frac{2xz}{(x^2 + y^2 + z^2)^2}.$$

在 A, B 点的梯度分别为

$$\text{grad}u(A) = \frac{7}{81} i - \frac{4}{81} j - \frac{4}{81} k,$$

$$\text{grad}u(B) = -\frac{2}{25} i + \frac{3}{50} j.$$

于是,

$$\begin{aligned} \cos\theta &= \frac{\text{grad}u(A) \cdot \text{grad}u(B)}{|\text{grad}u(A)| \cdot |\text{grad}u(B)|} \\ &= \frac{-\frac{4}{405}}{\frac{1}{9} \cdot \frac{1}{10}} = -\frac{8}{9} \end{aligned}$$

4406. 设已知数量场

$$u = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

作出场的等位面 and 梯度的等模面.

在域 $1 < z < 2$ 内求 $\inf u, \sup u, \inf |\text{grad}u|$,

$\sup |\operatorname{grad} u|$.

解 将 $u = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$ 化简整理, 即得

$$x^2 + y^2 + \frac{u^2 - 1}{u^2} z^2 = 0.$$

其中显然有 $0 < |u| < 1$. 由此可知, 等位面是一个以原点为顶点, Oz 轴为旋转轴的圆锥, 但要去掉原点 $O(0, 0, 0)$. 因此, 它是一个圆锥孔, 又

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{xz}{\sqrt{(x^2 + y^2 + z^2)^3}}, \\ \frac{\partial u}{\partial y} &= -\frac{yz}{\sqrt{(x^2 + y^2 + z^2)^3}}, \\ \frac{\partial u}{\partial z} &= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \\ &\quad - \frac{z^2}{\sqrt{(x^2 + y^2 + z^2)^3}} = -\frac{x^2 + y^2}{\sqrt{(x^2 + y^2 + z^2)^3}},\end{aligned}$$

故有

$$|\operatorname{grad} u| = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2}. \text{ 令 } \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} = c.$$

显见此等模面是一个以 Oz 轴为旋转轴的旋转面. 现在令 $y = 0$, 得

$$x = cx^2 + cz^2 \text{ 或 } \left(x - \frac{1}{2c}\right)^2 + z^2 = \frac{1}{4c^2} (c \neq 0),$$

它是 Oxz 面上的圆. 因此, 梯度的等模面是一个旋转环面.

当 $1 < z < 2$ 时, 显然有 $0 < u \leq 1$; 且当 $x = y = 0$ 时, $u = 1$, 而当 $x^2 + y^2$ 充分大时 u 可任意小, 故

$$\inf_{1 < z < 2} u = 0, \sup_{1 < z < 2} u = 1.$$

另外,显然

$$\inf_{1 \leq z \leq 2} |\operatorname{grad} u| = \inf_{1 \leq z \leq 2} \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} = 0.$$

由于对于常数 $a > 0$, 函数 $f(t) = \frac{\sqrt{t}}{t+a}$ ($0 \leq t < +\infty$)

当 $t = a$ 时达最大值 $f(a) = \frac{1}{2\sqrt{a}}$ (这可从讨论 $f(t)$ 简

单地得知), 故对于固定的 z , $\frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2}$ 的最大值是

$$\frac{1}{2\sqrt{z^2}} = \frac{1}{2z} \quad (z > 0 \text{ 时}), \text{ 由此可知}$$

$$\sup_{1 \leq z \leq 2} |\operatorname{grad} u| = \sup_{1 \leq z \leq 2} \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} = \frac{1}{2}.$$

4407. 求在点 $M_0(x_0, y_0, z_0)$ 处之二无限接近的等位面

$$u(x, y, z) = c \text{ 及 } u(x, y, z) = c + \Delta c$$

之间的距离准确到高阶无穷小, 其中 $u(x_0, y_0, z_0) = c$,

解 过点 M_0 作等位面 $u(x, y, z) = c$ 的垂线, 交等位面 $u(x, y, z) = c + \Delta c$ 于点 $M_1(x_1, y_1, z_1)$, 则显然二等位面 $u(x, y, z) = c$ 和 $u(x, y, z) = c + \Delta c$ 之间的距离 $d = |\overrightarrow{M_0 M_1}|$. 由于梯度垂直于等位面, 因此 $\operatorname{grad} u(x_0, y_0, z_0)$ 的方向与 $\overrightarrow{M_0 M_1}$ 的方向或者重合, 或者相反. 于是, 注意到 $u(x_0, y_0, z_0) = c, u(x_1, y_1, z_1) = c + \Delta c$, 知

$$\Delta c = u(x_1, y_1, z_1) - u(x_0, y_0, z_0)$$

$$= \frac{\partial u}{\partial x} \Big|_{(x_0, y_0, z_0)} (x_1 - x_0) + \frac{\partial u}{\partial y} \Big|_{(x_0, y_0, z_0)}$$

$$\cdot (y_1 - y_0)$$

$$+ \frac{\partial u}{\partial z} \Big|_{(x_0, y_0, z_0)} (z_1 - z_0)$$

$$\begin{aligned}
&= [\text{grad } u(x_0, y_0, z_0)] \cdot \overrightarrow{M_0 M_1} \\
&= \pm |\text{grad } u(x_0, y_0, z_0)| \cdot |\overrightarrow{M_0 M_1}| \\
&= \pm |\text{grad } u(x_0, y_0, z_0)| d.
\end{aligned}$$

由此可知(准确到高阶无穷小)

$$d = \frac{|\Delta c|}{|\text{grad } u(x_0, y_0, z_0)|}.$$

4408. 证明公式

$$(a) \text{grad}(u + c) = \text{grad} u \quad (c \text{ 为常数});$$

$$(b) \text{grad} cu = c \text{grad} u \quad (c \text{ 为常数});$$

$$(c) \text{grad}(u + v) = \text{grad} u + \text{grad} v;$$

$$(d) \text{grad} uv = v \text{grad} u + u \text{grad} v;$$

$$(e) \text{grad}(u^2) = 2u \text{grad} u;$$

$$(f) \text{grad} f(u) = f'(u) \text{grad} u.$$

证 (a) 由于 $\frac{\partial(u+c)}{\partial x} = \frac{\partial u}{\partial x}, \quad \frac{\partial(u+c)}{\partial y} = \frac{\partial u}{\partial y},$

$$\frac{\partial(u+c)}{\partial z} = \frac{\partial u}{\partial z}, \text{ 故得}$$

$$\text{grad}(u+c) = \text{grad} u.$$

$$(b) \text{ 由于 } \frac{\partial(cu)}{\partial x} = c \frac{\partial u}{\partial x}, \frac{\partial(cu)}{\partial y}$$

$$= c \frac{\partial u}{\partial y}, \frac{\partial(cu)}{\partial z} = c \frac{\partial u}{\partial z}, \text{ 故得}$$

$$\text{grad} cu = c \text{grad} u.$$

$$(c) \text{ 由于 } \frac{\partial(u+v)}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, \frac{\partial(u+v)}{\partial y}$$

$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \frac{\partial(u+v)}{\partial z} = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z}, \text{ 故得}$$

$$\text{grad}(u+v) = \text{grad} u + \text{grad} v.$$

$$(d) \text{ 由于 } \frac{\partial(uv)}{\partial x} = u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x}, \frac{\partial(uv)}{\partial y} = u \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y}, \frac{\partial(uv)}{\partial z}$$

$$= u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x}, \text{故得}$$

$$\operatorname{grad} uv = u \operatorname{grad} v + v \operatorname{grad} u.$$

(1) 在(1)中令 $v = u$, 即得

$$\operatorname{grad}(u^2) = 2u \operatorname{grad} u.$$

$$\begin{aligned} \text{(c) 由于 } \frac{\partial f(u)}{\partial x} &= f'(u) \frac{\partial u}{\partial x}, \frac{\partial f(u)}{\partial y} = f'(u) \cdot \frac{\partial u}{\partial y}, \frac{\partial f(u)}{\partial z} \\ &= f'(u) \frac{\partial u}{\partial z}, \text{故得} \end{aligned}$$

$$\operatorname{grad} f(u) = f'(u) \operatorname{grad} u.$$

4409. 计算: (a) $\operatorname{grad} r$, (b) $\operatorname{grad} r^2$, (c) $\operatorname{grad} \frac{1}{r}$, 其中 $r = \sqrt{x^2 + y^2 + z^2}$.

$$\text{解 (a) } \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}. \text{ 于是,}$$

$$\operatorname{grad} r = \frac{\vec{r}}{r}, \text{ 其中 } \vec{r} = x \vec{i} + y \vec{j} + z \vec{k},$$

$$\text{(b) } \operatorname{grad}(r^2) = 2r \operatorname{grad} r = 2r \cdot \frac{\vec{r}}{r} = 2 \vec{r},$$

$$\text{(c) } \operatorname{grad} \frac{1}{r} = -\frac{1}{r^2} \operatorname{grad} r = -\frac{\vec{r}}{r^3}.$$

4410. 求 $\operatorname{grad} f(r)$, 其中 $r = \sqrt{x^2 + y^2 + z^2}$.

$$\text{解 } \operatorname{grad} f(r) = f'(r) \operatorname{grad} r = f'(r) \cdot \frac{\vec{r}}{r}.$$

*) 利用 4408 题的结果.

* *) 利用 4409 题的结果.

4411. 求 $\operatorname{grad}(\vec{c} \cdot \vec{r})$, 其中 $\vec{c}$ 为常向量, $\vec{r}$ 为从坐标原点起的向径.

解 设 $\vec{c} = c_x \vec{i} + c_y \vec{j} + c_z \vec{k}$, 其中 c_x, c_y, c_z 为常数.

由于

$$\vec{c} \cdot \vec{r} = c_x x + c_y y + c_z z$$

$$\text{及 } \frac{\partial(\vec{c} \cdot \vec{r})}{\partial x} = c_x, \frac{\partial(\vec{c} \cdot \vec{r})}{\partial y} = c_y, \frac{\partial(\vec{c} \cdot \vec{r})}{\partial z} = c_z,$$

$$\text{故 } \operatorname{grad}(\vec{c} \cdot \vec{r}) = \vec{c}.$$

4412. 求 $\operatorname{grad}\{|\vec{c} \times \vec{r}|^2\}$ ($\vec{c}$ 为常向量).

解 $|\vec{c} \times \vec{r}|^2 = (c_y z - c_z y)^2 + (c_z x - c_x z)^2 + (c_x y - c_y x)^2$. 于是,

$$\begin{aligned} \operatorname{grad}\{|\vec{c} \times \vec{r}|^2\} &= [2c_z(c_y x - c_x z) - 2c_y(c_x y - c_y x) \\ &\quad + 2c_x(c_y z - c_z y)] \vec{i} + [-2c_z(c_y z - c_z y) \\ &\quad + 2c_x(c_x y - c_y x)] \vec{j} \\ &\quad + [2c_y(c_y z - c_x y) - 2c_x(c_z x - c_x z)] \vec{k} \\ &= 2[x(c_x^2 + c_y^2 + c_z^2) - c_x(c_x x + c_y y + c_z z)] \vec{i} \\ &\quad + 2[y(c_x^2 + c_y^2 + c_z^2) - c_y(c_x x + c_y y + c_z z)] \vec{j} \\ &\quad + 2[z(c_x^2 + c_y^2 + c_z^2) - c_z(c_x x + c_y y + c_z z)] \vec{k} \\ &= 2\vec{r}(\vec{c} \cdot \vec{c}) - 2\vec{c}(\vec{r} \cdot \vec{c}). \end{aligned}$$

4413. 证明公式

$$\operatorname{grad} f(u, v) = \frac{\partial f}{\partial u} \operatorname{grad} u + \frac{\partial f}{\partial v} \operatorname{grad} v.$$

证 由于

$$\frac{\partial f(u, v)}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x},$$

$$\frac{\partial f(u, v)}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y},$$

$$\frac{\partial f(u, v)}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z},$$

故有

$$\begin{aligned}\operatorname{grad} f(u, v) &= \frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} \vec{i} + \frac{\partial u}{\partial y} \vec{j} + \frac{\partial u}{\partial z} \vec{k} \right) \\ &\quad + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} \vec{i} + \frac{\partial v}{\partial y} \vec{j} + \frac{\partial v}{\partial z} \vec{k} \right) \\ &= \frac{\partial f}{\partial u} \operatorname{grad} u + \frac{\partial f}{\partial v} \operatorname{grad} v.\end{aligned}$$

4414. 证明公式

$$\nabla^2(uv) = u\nabla^2v + v\nabla^2u + 2\nabla u \nabla v,$$

$$\text{其中} \quad \nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z},$$

$$\nabla^2 = \nabla \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

证 由于 $\nabla(uv) = u\nabla v + v\nabla u$, 故

$$\begin{aligned}\nabla^2(uv) &= \nabla(\nabla(uv)) = \nabla(u\nabla v + v\nabla u) \\ &= \nabla(u\nabla v) + \nabla(v\nabla u) \\ &= (u\nabla^2v + \nabla u \nabla v) + (v\nabla^2u + \nabla u \nabla v) \\ &= u\nabla^2v + v\nabla^2u + 2\nabla u \nabla v.\end{aligned}$$

4415. 证明: 若函数 $u = u(x, y, z)$ 在凸形域 Ω 内可微分且 $|\operatorname{grad} u| \leq M$, 其中 M 为常数, 则对于 Ω 中任意两点 A, B 有:

$$|u(A) - u(B)| \leq M\rho(A, B),$$

式中 $\rho(A, B)$ 为 A 与 B 两点间之距离.

证 由于 Ω 为凸形域, 故线段 $\overline{AB}$ 整个属于 Ω . 设 B 的坐标为 (x_0, y_0, z_0) , A 的坐标为 (x_1, y_1, z_1) , 且令 $x_1 - x_0 = \Delta x, y_1 - y_0 = \Delta y, z_1 - z_0 = \Delta z$. 并考虑一元函数 $f(t) = u(x_0 + t\Delta x, y_0 + t\Delta y, z_0 + t\Delta z)$ ($0 \leq t \leq 1$), 显然 $f(0) = u(B), f(1) = u(A)$, 且 $f(t)$ 在 $[0, 1]$ 可微, 并且

$$\begin{aligned}
 f'(t) &= u'_x(x_0 + t\Delta x, y_0 + t\Delta y, z_0 + t\Delta z) \Delta x \\
 &\quad + u'_y(x_0 + t\Delta x, y_0 + t\Delta y, z_0 + t\Delta z) \Delta y \\
 &\quad + u'_z(x_0 + t\Delta x, y_0 + t\Delta y, z_0 + t\Delta z) \Delta z.
 \end{aligned}$$

于是,由微分学中值定理知

$$\begin{aligned}
 u(A) - u(B) &= f(1) - f(0) = f(\xi) \\
 &= u'_x(x_0 + \xi\Delta x, y_0 + \xi\Delta y, z_0 + \xi\Delta z) \Delta x \\
 &\quad + u'_y(x_0 + \xi\Delta x, y_0 + \xi\Delta y, z_0 + \xi\Delta z) \Delta y \\
 &\quad + u'_z(x_0 + \xi\Delta x, y_0 + \xi\Delta y, z_0 + \xi\Delta z) \Delta z \\
 &= [\text{gradu}(x_0 + \xi\Delta x, y_0 + \xi\Delta y, z_0 + \xi\Delta z)] \cdot \overrightarrow{BA},
 \end{aligned}$$

由此可知

$$\begin{aligned}
 |u(A) - u(B)| &= |[\text{gradu}(x_0 + \xi\Delta x, y_0 + \xi\Delta y, z_0 + \xi\Delta z)] \cdot \overrightarrow{BA}| \\
 &\leq |\text{gradu}(x_0 + \xi\Delta x, y_0 + \xi\Delta y, z_0 + \xi\Delta z)| \cdot |\overrightarrow{BA}| \\
 &\leq M\rho(A, B).
 \end{aligned}$$

4416. 求场 $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ 在已知点 $M(x, y, z)$ 沿此点的向径 $\vec{r}$ 之方向的导数.

在什么情况下,此导数将等于梯度的大小?

$$\begin{aligned}
 \text{解} \quad \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma, \text{ 其中 } \cos \alpha \\
 &= \frac{x}{r}, \cos \beta = \frac{y}{r}, \cos \gamma = \frac{z}{r}, r = \sqrt{x^2 + y^2 + z^2}. \text{ 于是} \\
 \frac{\partial u}{\partial r} &= \frac{2x}{a^2} \cdot \frac{x}{r} + \frac{2y}{b^2} \cdot \frac{y}{r} + \frac{2z}{c^2} \cdot \frac{z}{r} = \frac{2u}{r}.
 \end{aligned}$$

$$\text{又 } |\text{gradu}| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}.$$

要 $|\operatorname{grad} u| = \frac{\partial u}{\partial r}$, 只要 $\frac{u}{r} = \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$, 即只要 $a = b = c$, 此即所求之解.

4417. 求场 $u = \frac{1}{r}$ (其中 $r = \sqrt{x^2 + y^2 + z^2}$) 在方向 $\vec{l} \{\cos \alpha, \cos \beta, \cos \gamma\}$ 上的导数.

在什么情况下, 此导数等于零?

解 $\frac{\partial u}{\partial x} = -\frac{x}{r^3}, \frac{\partial u}{\partial y} = -\frac{y}{r^3}, \frac{\partial u}{\partial z} = -\frac{z}{r^3}$. 于是,

$$\begin{aligned} \frac{\partial u}{\partial l} &= -\frac{x}{r^3} \cos \alpha - \frac{y}{r^3} \cos \beta - \frac{z}{r^3} \cos \gamma \\ &= -\frac{1}{r^2} [\cos(\vec{r}, x) \cos \alpha + \cos(\vec{r}, y) \cos \beta \\ &\quad + \cos(\vec{r}, z) \cos \gamma] \\ &= -\frac{\cos(\vec{l}, \vec{r})}{r^2}. \end{aligned}$$

要 $\frac{\partial u}{\partial l} = 0$, 只要 $\cos(\vec{l}, \vec{r}) = 0$, 即 $\vec{l} \perp \vec{r}$, 此即所求之解.

4418. 求场 $u = u(x, y, z)$ 在场 $v = v(x, y, z)$ 的梯度方向的导数.

在什么情况下, 此导数等于零?

解 $\vec{l} = \operatorname{grad} v, \vec{l}_0 = \frac{\operatorname{grad} v}{|\operatorname{grad} v|}$. 于是,

$$\frac{\partial u}{\partial l} = \operatorname{grad} u \cdot \vec{l}_0 = \frac{\operatorname{grad} u \cdot \operatorname{grad} v}{|\operatorname{grad} v|}.$$

要 $\frac{\partial u}{\partial l} = 0$, 只要 $\operatorname{grad} u \perp \operatorname{grad} v$, 此即所求之解.

4419⁺. 设:

$$u = \operatorname{arctg} \frac{z}{\sqrt{x^2 + y^2}} \text{ 及 } \vec{c} = \vec{i} + \vec{j} + \vec{k},$$

计算 $\vec{a} = \vec{c} \times \operatorname{grad} u.$

$$\begin{aligned}
 \text{解} \quad \frac{\partial u}{\partial x} &= \frac{1}{1 + \frac{z^2}{x^2 + y^2}} \left(-\frac{xz}{(x^2 + y^2)^{\frac{3}{2}}} \right) \\
 &= -\frac{xz}{(x^2 + y^2 + z^2)(x^2 + y^2)^{\frac{1}{2}}}, \\
 \frac{\partial u}{\partial y} &= -\frac{yz}{(x^2 + y^2 + z^2)(x^2 + y^2)^{\frac{1}{2}}}, \\
 \frac{\partial u}{\partial z} &= \frac{(x^2 + y^2)^{\frac{1}{2}}}{x^2 + y^2 + z^2}.
 \end{aligned}$$

于是,

$$\begin{aligned}
 \vec{a} &= \vec{c} \times \text{grad} u = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} \\
 &= \frac{1}{(x^2 + y^2 + z^2)(x^2 + y^2)^{\frac{1}{2}}} [(x^2 + y^2 + yz) \vec{i} \\
 &\quad - (x^2 + y^2 + xz) \vec{j} + (x - y)z \vec{k}].
 \end{aligned}$$

4420. 确定向量场

$$\vec{a} = x \vec{i} + y \vec{j} + 2z \vec{k}$$

的力线.

解 力线系这样的一条曲线 C , 在 C 上每一点的切线与向量场在该点的方向重合. 因此, 有 $d\vec{r} \parallel \vec{a}$, 即

$$\frac{dx}{a_x} = \frac{dy}{a_y} = \frac{dz}{a_z},$$

其中 $\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$.

今有 $a_x = x, a_y = y, a_z = 2z$, 故得

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{2z}.$$

解之,得 $y = c_1 x, z = c_2 x^2$.

4421. 用直接计算的方向证明向量 $\vec{a}$ 的散度与直角坐标系的选择无关.

证 设除直角坐标系 $Oxyz$ (坐标轴方向的单位向量为 $\vec{i}, \vec{j}, \vec{k}$) 外, 另有直角坐标系 $O'x'y'z'$ (坐标轴方向的单位向量为 $\vec{i}', \vec{j}', \vec{k}'$). 我们要证

$$\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = \frac{\partial a_{x'}}{\partial x'} + \frac{\partial a_{y'}}{\partial y'} + \frac{\partial a_{z'}}{\partial z'}.$$

设

$$\begin{cases} \vec{i}' = \cos\alpha_1 \vec{i} + \cos\beta_1 \vec{j} + \cos\gamma_1 \vec{k}, \\ \vec{j}' = \cos\alpha_2 \vec{i} + \cos\beta_2 \vec{j} + \cos\gamma_2 \vec{k}, \\ \vec{k}' = \cos\alpha_3 \vec{i} + \cos\beta_3 \vec{j} + \cos\gamma_3 \vec{k}. \end{cases}$$

又设 $\vec{r}_0 = O\vec{O}' = a\vec{i} + b\vec{j} + c\vec{k}$. 于是, 空间一点 P 在两个坐标系中的坐标 (x, y, z) 与 (x', y', z') 之间的关系为 (令 $\vec{r} = \overrightarrow{OP}, \vec{r}' = \overrightarrow{O'P}$):

$$\begin{aligned} x' &= \vec{r}' \cdot \vec{i}' = (\vec{r} - \vec{r}_0) \cdot \vec{i}' \\ &= (x - a)\cos\alpha_1 + (y - b)\cos\beta_1 + (z - c)\cos\gamma_1, \\ y' &= \vec{r}' \cdot \vec{j}' = (\vec{r} - \vec{r}_0) \cdot \vec{j}' \\ &= (x - a)\cos\alpha_2 + (y - b)\cos\beta_2 + (z - c)\cos\gamma_2, \\ z' &= \vec{r}' \cdot \vec{k}' = (\vec{r} - \vec{r}_0) \cdot \vec{k}' \\ &= (x - a)\cos\alpha_3 + (y - b)\cos\beta_3 + (z - c)\cos\gamma_3. \end{aligned}$$

我们有

$$\begin{aligned} \vec{a} &= a_x \vec{i}' + a_y \vec{j}' + a_z \vec{k}' \\ &= a_x (\cos\alpha_1 \vec{i} + \cos\beta_1 \vec{j} + \cos\gamma_1 \vec{k}) \\ &\quad + a_y (\cos\alpha_2 \vec{i} + \cos\beta_2 \vec{j} + \cos\gamma_2 \vec{k}) \\ &\quad + a_z (\cos\alpha_3 \vec{i} + \cos\beta_3 \vec{j} + \cos\gamma_3 \vec{k}) \end{aligned}$$

$$+ a_{z'} (\cos\alpha_3 \vec{i} + \cos\beta_3 \vec{j} + \cos\gamma_3 \vec{k}).$$

由此可知

$$a_x = a_{x'} \cos\alpha_1 + a_{y'} \cos\alpha_2 + a_{z'} \cos\alpha_3$$

$$a_y = a_{x'} \cos\beta_1 + a_{y'} \cos\beta_2 + a_{z'} \cos\beta_3$$

$$a_z = a_{x'} \cos\gamma_1 + a_{y'} \cos\gamma_2 + a_{z'} \cos\gamma_3.$$

于是,

$$\begin{aligned} \frac{\partial a_x}{\partial x} &= \frac{\partial a_x}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial a_x}{\partial y'} \frac{\partial y'}{\partial x} + \frac{\partial a_x}{\partial z'} \frac{\partial z'}{\partial x} \\ &= \left(\cos\alpha_1 \frac{\partial a_{x'}}{\partial x} + \cos\alpha_2 \frac{\partial a_{y'}}{\partial x} + \cos\alpha_3 \frac{\partial a_{z'}}{\partial x} \right) \cos\alpha_1 \\ &\quad + \left(\cos\alpha_1 \frac{\partial a_{x'}}{\partial y'} + \cos\alpha_2 \frac{\partial a_{y'}}{\partial y'} + \cos\alpha_3 \frac{\partial a_{z'}}{\partial y'} \right) \cos\alpha_2 \\ &\quad + \left(\cos\alpha_1 \frac{\partial a_{x'}}{\partial z'} + \cos\alpha_2 \frac{\partial a_{y'}}{\partial z'} + \cos\alpha_3 \frac{\partial a_{z'}}{\partial z'} \right) \cos\alpha_3. \end{aligned}$$

同理,可得

$$\begin{aligned} \frac{\partial a_y}{\partial y} &= \left(\cos\beta_1 \frac{\partial a_{x'}}{\partial x'} + \cos\beta_2 \frac{\partial a_{y'}}{\partial x'} + \cos\beta_3 \frac{\partial a_{z'}}{\partial x'} \right) \cos\beta_1 \\ &\quad + \left(\cos\beta_1 \frac{\partial a_{x'}}{\partial y'} + \cos\beta_2 \frac{\partial a_{y'}}{\partial y'} + \cos\beta_3 \frac{\partial a_{z'}}{\partial y'} \right) \cos\beta_2 \\ &\quad + \left(\cos\beta_1 \frac{\partial a_{x'}}{\partial z'} + \cos\beta_2 \frac{\partial a_{y'}}{\partial z'} + \cos\beta_3 \frac{\partial a_{z'}}{\partial z'} \right) \cos\beta_3, \\ \frac{\partial a_z}{\partial z} &= \left(\cos\gamma_1 \frac{\partial a_{x'}}{\partial x'} + \cos\gamma_2 \frac{\partial a_{y'}}{\partial x'} + \cos\gamma_3 \frac{\partial a_{z'}}{\partial x'} \right) \cos\gamma_1 \\ &\quad + \left(\cos\gamma_1 \frac{\partial a_{x'}}{\partial y'} + \cos\gamma_2 \frac{\partial a_{y'}}{\partial y'} + \cos\gamma_3 \frac{\partial a_{z'}}{\partial y'} \right) \cos\gamma_2 \\ &\quad + \left(\cos\gamma_1 \frac{\partial a_{x'}}{\partial z'} + \cos\gamma_2 \frac{\partial a_{y'}}{\partial z'} + \cos\gamma_3 \frac{\partial a_{z'}}{\partial z'} \right) \cos\gamma_3. \end{aligned}$$

将这三式相加,得

$$\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = (\vec{i}' \cdot \vec{i}') \frac{\partial a_{x'}}{\partial x'}$$

$$\begin{aligned}
& + (j' \cdot i') \frac{\partial a_{x'}}{\partial x'} + (k' \cdot i') \frac{\partial a_{x'}}{\partial x'} \\
& + (i' \cdot j') \frac{\partial a_{x'}}{\partial y'} + (j' \cdot j') \frac{\partial a_{y'}}{\partial y'} + (k' \cdot j') \frac{\partial a_{x'}}{\partial y'} \\
& + (i' \cdot k') \frac{\partial a_{x'}}{\partial z'} + (j' \cdot k') \frac{\partial a_{y'}}{\partial z'} + (k' \cdot k') \frac{\partial a_{x'}}{\partial z'} \\
& = \frac{\partial a_{x'}}{\partial x'} + \frac{\partial a_{y'}}{\partial y'} + \frac{\partial a_{z'}}{\partial z'}.
\end{aligned}$$

证毕.

4422. 证明

$$\operatorname{div} \vec{a}(M) = \lim_{d(S) \rightarrow 0} \frac{1}{V} \iint_S \vec{a} \cdot \vec{n} dS,$$

其中 S 为围绕着点 M 和界有体积 V 的封闭曲面, $\vec{n}$ 为曲面 S 之外法线, $d(S)$ 为曲面 S 的直径.

证 由于

$$\vec{a} \cdot \vec{n} = a_x \cos \alpha + a_y \cos \beta + a_z \cos \gamma,$$

其中 $\cos \alpha, \cos \beta, \cos \gamma$ 是 $\vec{n}$ 之方向余弦. 应用奥氏公式以及积分中值定理, 得

$$\begin{aligned}
\iint_S \vec{a} \cdot \vec{n} dS &= \iint_S (a_x \cos \alpha + a_y \cos \beta + a_z \cos \gamma) dS \\
&= \iiint_V \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) dx dy dz \\
&= \iiint_V (\operatorname{div} \vec{a}) dx dy dz \\
&= \operatorname{div} \vec{a}(M_1) \cdot V,
\end{aligned}$$

其中 M_1 是 V 中某点, 即

$$\operatorname{div} \vec{a}(M_1) = \frac{1}{V} \iint_S \vec{a} \cdot \vec{n} dS.$$

令 $d(S) \rightarrow 0$, 这时 V 缩向点 M , 从而点 $M_1 \rightarrow M$, 取极限, 即得

$$\operatorname{div} \vec{a}(M) = \lim_{d(S) \rightarrow 0} \frac{1}{V} \iint_S \vec{a} \cdot \vec{n} dS.$$

证毕.

4423. 求:

$$\operatorname{div} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_x & \omega_y & \omega_z \end{vmatrix}.$$

解

$$\begin{aligned} & \operatorname{div} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_x & \omega_y & \omega_z \end{vmatrix} \\ &= \operatorname{div} \left[\left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z} \right) \vec{i} + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x} \right) \vec{j} \right. \\ & \quad \left. + \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) \vec{k} \right] \\ &= \frac{\partial}{\partial x} \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x} \right) \\ & \quad + \frac{\partial}{\partial z} \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) \\ &= 0 \end{aligned}$$

4424. 证明:

$$(a) \operatorname{div}(\vec{a} + \vec{b}) = \operatorname{div} \vec{a} + \operatorname{div} \vec{b};$$

$$(\sigma) \operatorname{div}(u \vec{c}) = \vec{c} \cdot \operatorname{grad} u \quad (\vec{c} \text{ 为常量, } u \text{ 为数量});$$

$$(b) \operatorname{div}(u \vec{a}) = u \operatorname{div} \vec{a} + \vec{a} \cdot \operatorname{grad} u.$$

证 (a) 设 $\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$, $\vec{b} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}$.

$$\begin{aligned} \text{由于 } \frac{\partial(a_x + b_x)}{\partial x} &= \frac{\partial a_x}{\partial x} + \frac{\partial b_x}{\partial x}, \frac{\partial(a_y + b_y)}{\partial y} \\ &= \frac{\partial a_y}{\partial y} + \frac{\partial b_y}{\partial y} \text{ 及 } \frac{\partial(a_z + b_z)}{\partial z} = \frac{\partial a_z}{\partial z} + \frac{\partial b_z}{\partial z}, \text{ 故得} \\ \operatorname{div}(\vec{a} + \vec{b}) &= \operatorname{div} \vec{a} + \operatorname{div} \vec{b}. \end{aligned}$$

(σ) 设 $\vec{c} = c_x \vec{i} + c_y \vec{j} + c_z \vec{k}$, 其中 c_x, c_y, c_z 为常数.

$$\begin{aligned} \text{由于 } \frac{\partial(uc_x)}{\partial x} &= c_x \frac{\partial u}{\partial x}, \frac{\partial(uc_y)}{\partial y} = c_y \frac{\partial u}{\partial y} \text{ 及 } \frac{\partial(uc_z)}{\partial z} = c_z \frac{\partial u}{\partial z}, \\ \text{故得} \end{aligned}$$

$$\operatorname{div}(u \vec{c}) = \vec{c} \cdot \operatorname{grad} u.$$

$$\begin{aligned} \text{(b) 由于 } \frac{\partial(ua_x)}{\partial x} &= u \frac{\partial a_x}{\partial x} + a_x \frac{\partial u}{\partial x}, \frac{\partial(ua_y)}{\partial y} \\ &= u \frac{\partial a_y}{\partial y} + a_y \frac{\partial u}{\partial y} \text{ 及 } \frac{\partial(ua_z)}{\partial z} = u \frac{\partial a_z}{\partial z} + a_z \frac{\partial u}{\partial z}, \text{ 故得} \\ \operatorname{div}(u \vec{a}) &= u \operatorname{div} \vec{a} + \vec{a} \cdot \operatorname{grad} u. \end{aligned}$$

4425. 求 $\operatorname{div}(\operatorname{grad} u)$.

$$\begin{aligned} \text{解 } \operatorname{div}(\operatorname{grad} u) &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial z} \right) \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ &= \Delta u \text{ (或记成 } \nabla^2 u \text{)}. \end{aligned}$$

4426. 求 $\operatorname{div}[\operatorname{grad} f(r)]$, 其中 $r = \sqrt{x^2 + y^2 + z^2}$. 在什么情况下 $\operatorname{div}[\operatorname{grad} f(r)] = 0$?

解 由 4410 题的结果知,

$$\operatorname{grad} f(r) = f(r) \cdot \frac{\vec{r}}{r}.$$

于是,

$$\begin{aligned}
\operatorname{div}[\operatorname{grad} f(r)] &= \frac{\partial}{\partial x} \left[f'(r) \frac{x}{r} \right] \\
&\quad + \frac{\partial}{\partial y} \left[f'(r) \frac{y}{r} \right] + \frac{\partial}{\partial z} \left[f'(r) \frac{z}{r} \right] \\
&= f''(r) \left(\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} \right) + f'(r) \\
&\quad \left(\frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} + \frac{r^2 - z^2}{r^3} \right) \\
&= f''(r) + \frac{2}{r} f'(r).
\end{aligned}$$

要 $\operatorname{div}[\operatorname{grad} f(r)] = 0$, 只要 $f''(r) + \frac{2}{r} f'(r) = 0$.

将上述方程写成下述形式:

$$r f''(r) + 2 f'(r) = 0.$$

或 $(r f'(r) + f(r))' + f'(r) = 0$.

积分之, 即得

$$r f(r) + f(r) = C \quad (C \text{ 为常数}).$$

再积分之, 得

$$r f(r) = C r + C_1 \quad (C_1 \text{ 为常数}).$$

于是, 最后得

$$f(r) = C + \frac{C_1}{r},$$

此即所求之解.

4427. 计算: (a) $\operatorname{div} \vec{r}$; (b) $\operatorname{div} \frac{\vec{r}}{r}$.

解 (a) $\operatorname{div} \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$

$$\begin{aligned}
(b) \operatorname{div} \frac{\vec{r}}{r} &= \frac{\partial}{\partial x} \left(\frac{x}{r} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r} \right) \\
&= \left(\frac{1}{r} - \frac{x^2}{r^3} \right) + \left(\frac{1}{r} - \frac{y^2}{r^3} \right) + \left(\frac{1}{r} - \frac{z^2}{r^3} \right)
\end{aligned}$$

$$= \frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

4428. 计算 $\operatorname{div}[f(r) \vec{c}]$, 式中 $\vec{c}$ 为常向量.

$$\begin{aligned}\text{解} \quad \operatorname{div}[f(r) \vec{c}] &= \vec{c} \cdot \operatorname{grad} f(r)^{*)} \\ &= \vec{c} \cdot f'(r) \frac{\vec{r}^{**})}{r} = \frac{f'(r)}{r} (\vec{c} \cdot \vec{r}).\end{aligned}$$

*) 利用 4424 题(6) 的结果.

**) 利用 4410 题的结果.

4429. 求 $\operatorname{div}[f(r) \vec{r}]$. 在什么情况下此向量的散度等于零?

$$\begin{aligned}\text{解} \quad \operatorname{div}[f(r) \vec{r}] &= f(r) \operatorname{div} \vec{r} + \vec{r} \cdot \operatorname{grad} f(r)^{*)} \\ &= 3f(r) + \vec{r} \cdot f'(r) \frac{\vec{r}^{**})}{r} \\ &= 3f(r) + rf'(r). \\ \text{要 } \operatorname{div}[f(r) \vec{r}] &= 0, \text{ 只要 } 3f(r) + rf'(r) = 0, \text{ 即}\end{aligned}$$

$$\frac{f'(r)}{f(r)} = -\frac{3}{r}.$$

积分之, 即得

$$f(r) = \frac{C}{r^3} \quad (C \text{ 为常数}),$$

此即所求之解.

*) 利用 4424 题(n) 的结果.

**) 利用 4410 题的结果.

4430. 求: (a) $\operatorname{div}(u \operatorname{grad} u)$; (6) $\operatorname{div}(u \operatorname{grad} v)$.

$$\begin{aligned}\text{解} \quad \text{(a)} \operatorname{div}(u \operatorname{grad} u) &= u \operatorname{div}(\operatorname{grad} u) + \operatorname{grad} u \cdot \operatorname{grad} u^{*)} \\ &= u \Delta u + (\operatorname{grad} u)^2^{**}) \\ \text{(6)} \operatorname{div}(u \operatorname{grad} v) &= u \operatorname{div}(\operatorname{grad} v) + \operatorname{grad} v \cdot \operatorname{grad} u^{*)} \\ &= u \Delta v + \operatorname{grad} u \cdot \operatorname{grad} v^{**}).\end{aligned}$$

*) 利用 4424 题(n) 的结果.

* *) 利用 4425 题的结果.

4437. 物体以一定的角速度 ω 依逆时针方向绕 Oz 轴旋转. 求速度向量 $\vec{v}$ 和加速度向量 $\vec{w}$ 在空间的点 $M(x, y, z)$ 和在已知时刻的散度.

解 $\vec{v} = \vec{v}_0 + \vec{\omega} \times \vec{r}$. 微分之, 即得

$$\begin{aligned}\vec{w} &= \vec{w}_0 + \vec{\omega} \times \dot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} \\ &= \vec{w}_0 + \vec{\omega} \times \dot{\vec{r}} + \vec{\omega} \times \vec{v} \\ &= \vec{w}_0 + \vec{\omega} \times \vec{r} + \vec{\omega} \times (\vec{v}_0 + \vec{\omega} \times \vec{r}) \\ &= \vec{w}_0 + \vec{\omega} \times \vec{r} + \vec{\omega} \times \vec{v}_0 + (\vec{\omega} \cdot \vec{r}) \vec{\omega} - (\vec{\omega} \cdot \vec{\omega}) \vec{r}^{**}.\end{aligned}$$

为了计算 $\operatorname{div} \vec{v}$ 和 $\operatorname{div} \vec{w}$, 先计算 $\operatorname{div}(\vec{a} \times \vec{r})$, 此处 $\vec{a}$ 为常向量. 由于

$$\begin{aligned}(\vec{a} \times \vec{r})_x &= a_y z - a_z y, & (\vec{a} \times \vec{r})_y &= a_z x - a_x z, \\ (\vec{a} \times \vec{r})_z &= a_x y - a_y x,\end{aligned}$$

故得

$$\begin{aligned}\operatorname{div}(\vec{a} \times \vec{r}) &= \frac{\partial}{\partial x}(a_y z - a_z y) + \frac{\partial}{\partial y}(a_z x - a_x z) \\ &+ \frac{\partial}{\partial z}(a_x y - a_y x) = 0.\end{aligned}$$

于是, 即得

$$\operatorname{div} \vec{v} = \operatorname{div} \vec{v}_0 + \operatorname{div}(\vec{\omega} \times \vec{r}) = 0.$$

$$\begin{aligned}\operatorname{div} \vec{w} &= \operatorname{div} \vec{w}_0 + \operatorname{div}(\vec{\omega} \times \vec{r}) + \operatorname{div}(\vec{\omega} \times \vec{v}_0) \\ &+ \operatorname{div}[(\vec{\omega} \cdot \vec{r}) \vec{\omega}] - \operatorname{div}[(\vec{\omega} \cdot \vec{\omega}) \vec{r}],\end{aligned}$$

$$\begin{aligned}\text{而 } \operatorname{div}[(\vec{\omega} \cdot \vec{r}) \vec{\omega}] &= \vec{\omega} \cdot \vec{r} \operatorname{div} \vec{\omega} + \vec{\omega} \cdot \operatorname{grad}(\vec{\omega} \cdot \vec{r})^{***} \\ &= \vec{\omega} \cdot \vec{\omega}^{***} = \omega^2\end{aligned}$$

$$\text{及 } \operatorname{div}[(\vec{\omega} \cdot \vec{\omega}) \vec{r}] = \vec{\omega} \cdot \vec{\omega} \operatorname{div} \vec{r} + \vec{r} \cdot \operatorname{grad}(\vec{\omega} \cdot \vec{\omega})$$

$$= 3\omega^2,$$

从而,最后得

$$\operatorname{div} \hat{\omega} = \omega^2 - 3\omega^2 = -2\omega^2.$$

*) 利用向量代数中的公式(二重外积展开式):

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}.$$

* *) 利用 4424 题(B) 的结果.

* * *) 利用 4411 题的结果.

4432. 求由引力中心的有限系统所产生的动力场之散度.

解 引力 $\vec{F} = \frac{k \vec{r}}{r^3}$ (k 为常数). 于是,

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial}{\partial x} \left(\frac{kx}{r^3} \right) + \frac{\partial}{\partial y} \left(\frac{ky}{r^3} \right) + \frac{\partial}{\partial z} \left(\frac{kz}{r^3} \right) \\ &= k \left[\left(\frac{1}{r^3} - \frac{3x^2}{r^5} \right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5} \right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right) \right] \\ &= k \left[\frac{3}{r^3} - \frac{3(x^2 + y^2 + z^2)}{r^5} \right] \\ &= k \left(\frac{3}{r^3} - \frac{3}{r^3} \right) = 0. \end{aligned}$$

4433⁺. 求由极坐标 r 与 φ 所表的平面向量 $\vec{a} = \vec{a}(r, \varphi)$ 之散度的表示式.

解 设极坐标的 r 线与 φ 线的单位矢量为 $\vec{e}_r$ 与 $\vec{e}_\varphi$, 且

$$\vec{a}(r, \varphi) = a_r(r, \varphi) \vec{e}_r + a_\varphi(r, \varphi) \vec{e}_\varphi.$$

这里自然假定 a_r, a_φ 都具有连续的偏导函数. 取面积元素 $\Delta S = r \Delta \varphi \Delta r$, 记其界线为 ΔC . 首先, 推导矢量 $\vec{a}$ 经过界线 ΔC 的通量, 即矢流. 通量可分两部分: 一部分是经过 r 线的; 另一部分是经过 φ 线的. 它们分别是

$$\int_r^{r+\Delta r} a_\varphi(r, \varphi + \Delta \varphi) dr - \int_r^{r+\Delta r} a_\varphi(r, \varphi) dr$$

$$\begin{aligned}
&= \int_r^{r+\Delta r} [a_\varphi(r, \varphi + \Delta\varphi) - a_\varphi(r, \varphi)] dr \\
&= \int_r^{r+\Delta r} \frac{\partial a_\varphi(r, \varphi)}{\partial \varphi} \Delta\varphi dr \\
&= \frac{\partial a_\varphi(r, \varphi)}{\partial \varphi} \Delta\varphi \Delta r, \\
\int_\varphi^{\varphi+\Delta\varphi} a_r(r + \Delta r, \varphi)(r + \Delta r) d\varphi - \int_\varphi^{\varphi+\Delta\varphi} a_r(r, \varphi) r d\varphi \\
&= \int_\varphi^{\varphi+\Delta\varphi} [a_r(r + \Delta r, \varphi)(r + \Delta r) \\
&\quad - a_r(r, \varphi)r] d\varphi \\
&= \int_\varphi^{\varphi+\Delta\varphi} \frac{\partial [a_r(r, \varphi)r]}{\partial r} \Delta r d\varphi \\
&= \frac{\partial [a_r(r, \varphi)r]}{\partial r} \Delta r \Delta\varphi,
\end{aligned}$$

且由于 a_r, a_φ 的偏导函数的连续性, 当 $\Delta r, \Delta\varphi$ 取得愈小时, 上述近似等式愈精确. 于是, 矢量 $\vec{a}$ 经过 ΔC 的通量

$$\oint_{\Delta C} \vec{a} \cdot \vec{n} ds = \left\{ \frac{\partial a_\varphi(r, \varphi)}{\partial \varphi} + \frac{\partial [a_r(r, \varphi)r]}{\partial r} \right\} \Delta\varphi \Delta r,$$

其中 $\vec{n}$ 为曲线 ΔC 的外法线方向, 而且当 $\Delta r, \Delta\varphi$ 愈小时此近似等式愈精确.

于是, 根据散度的定义, 并注意到 ΔS 收缩为一点 (r, φ) 与 $\Delta r \rightarrow 0, \Delta\varphi \rightarrow 0$ 等价, 从而即得

$$\begin{aligned}
\operatorname{div} \vec{a} &= \lim_{\Delta S \rightarrow 0} \frac{\oint_{\Delta C} \vec{a} \cdot \vec{n} ds}{\Delta S} \\
&= \lim_{\substack{\Delta r \rightarrow 0 \\ \Delta\varphi \rightarrow 0}} \frac{\left\{ \frac{\partial a_\varphi(r, \varphi)}{\partial \varphi} + \frac{\partial [a_r(r, \varphi)r]}{\partial r} \right\} \Delta r \Delta\varphi}{r \Delta r \Delta\varphi}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{r} \left\{ \frac{\partial a_\varphi(r, \varphi)}{\partial \varphi} + \frac{\partial [a_r(r, \varphi)r]}{\partial r} \right\} \\
&= \frac{1}{r} \left[\frac{\partial (r a_r)}{\partial r} + \frac{\partial a_\varphi}{\partial \varphi} \right].
\end{aligned}$$

4434. 设

$$x = f(u, v, w), y = g(u, v, w), z = h(u, v, w),$$

用直交曲线坐标 u, v, w 表示 $\operatorname{div} \vec{a}(x, y, z)$.

作为特殊的情形, 求

用柱坐标和球坐标

表示 $\operatorname{div} \vec{a}$ 的表示式.

解 考虑向量 $\vec{a}$ 通过由曲面 $u =$ 常数, $v =$ 常数, $w =$ 常数所界的小立体 (接近于长方体) V 的表面 S 的流量 (图

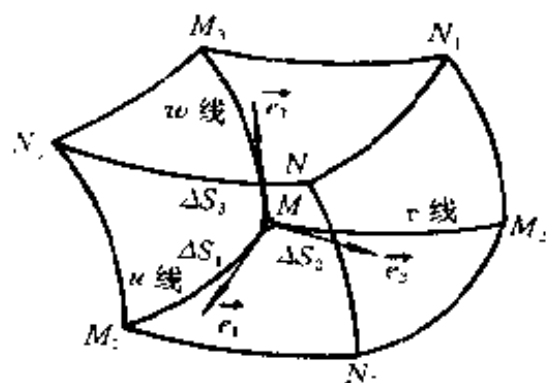


图 8.72

8.72), 我们有 $\vec{a} = a_u \vec{e}_1 + a_v \vec{e}_2 + a_w \vec{e}_3$.

在 u 曲线上, 只有 u 变化 (v 和 w 都是常数), 故

$$d\vec{r} = \frac{\partial x}{\partial u} du \vec{i} + \frac{\partial y}{\partial u} du \vec{j} + \frac{\partial z}{\partial u} du \vec{k},$$

从而

$$ds = |d\vec{r}| = L du,$$

$$\begin{aligned}
\text{其中} \quad L &= \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} \\
&= \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial h}{\partial u}\right)^2},
\end{aligned}$$

ds_1 为 u 曲线上的弧元素. 同理可得

$$dx_2 = Mdv, ds_3 = Nd\omega,$$

其中 dx_2, ds_3 分别为 v, ω 曲线上的弧元素, 而

$$M = \sqrt{\left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2},$$

$$N = \sqrt{\left(\frac{\partial f}{\partial \omega}\right)^2 + \left(\frac{\partial g}{\partial \omega}\right)^2 + \left(\frac{\partial h}{\partial \omega}\right)^2}.$$

由于坐标曲线互相垂直, $\triangle s_1, \triangle s_2, \triangle s_3$ 都很小, 故 V 接近于长方体. 因此, 其体积为

$$\begin{aligned} V &\doteq \triangle s_1 \triangle s_2 \triangle s_3 \doteq ds_1 ds_2 ds_3 \\ &= LMNdudvd\omega. \end{aligned}$$

现计算 $\vec{a}$ 通过 V 的表面 S 向外的流量 $\iint_S a_n dS$, S 共包括

六块小曲面(图 8.72), 记垂直于 $\vec{e}_1$ 方向的两块为 S_1 与 S_2 (即图中的 $MM_2N_1M_3$ 与 $M_1N_3NN_2$), 垂直于 $\vec{e}_2$ 方向的两块为 S_3 与 S_4 , 垂直于 $\vec{e}_3$ 方向的两块为 S_5 与 S_6 . 显然, 由于曲面很小, 有

$$\begin{aligned} &\iint_{S_2} a_n dS + \iint_{S_1} a_n dS \\ &\doteq a_u \triangle S_2 \triangle S_3 | (u + \triangle u, v, \omega) \\ &\quad - a_u \triangle S_2 \triangle S_3 | (u, v, \omega) \\ &\doteq a_u MNdv d\omega | (u + \triangle u, v, \omega) \\ &\quad - a_u MNdv d\omega | (u, v, \omega) \\ &\doteq \frac{\partial(a_u MNdv d\omega)}{\partial u} du \\ &= \frac{\partial(MNa_u)}{\partial u} dudvd\omega. \end{aligned}$$

同理可得

$$\iint_{S_1} a_n dS + \iint_{S_3} a_n dS = -\frac{\partial(NLa_v)}{\partial u} dudvd\omega,$$

$$\iint_{S_5} a_n dS + \iint_{S_7} a_n dS = -\frac{\partial(LMa_w)}{\partial w} dudvd\omega.$$

相加即得

$$\iint_S a_n dS = \left[\frac{\partial(MNa_u)}{\partial u} + \frac{\partial(NLa_v)}{\partial v} + \frac{\partial(LMa_w)}{\partial w} \right] dudvd\omega.$$

于是,

$$\frac{\iint_S a_n dS}{V} = \frac{1}{LMN} \left[\frac{\partial}{\partial u}(MNa_u) + \frac{\partial}{\partial v}(NLa_v) + \frac{\partial}{\partial w}(LMa_w) \right].$$

显然,当小立体 V 愈缩向点 M (V 愈小) 时,上述各近似等式都愈精确. 于是,令 V 缩向 M (即 S 的直径 $d(S)$ 趋于零) 取极限,利用 4422 题的结果,得

$$\begin{aligned} \operatorname{div} \vec{a} &= \lim_{d(S) \rightarrow 0} \frac{\iint_S a_n dS}{V} \\ &= \frac{1}{LMN} \left[\frac{\partial}{\partial u}(MNa_u) + \frac{\partial}{\partial v}(NLa_v) + \frac{\partial}{\partial w}(LMa_w) \right]. \end{aligned}$$

特别是在柱坐标情形下,有

$$\begin{aligned} x &= r \cos \varphi, y = r \sin \varphi, z = z \\ (u &= r, v = \varphi, w = z). \end{aligned}$$

从而

$$L = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1,$$

$$M = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = r,$$

$$N = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = 1.$$

于是,

$$\operatorname{div} \vec{a} = \frac{1}{r} \left[\frac{\partial}{\partial r} (r a_r) + \frac{\partial a_\varphi}{\partial \varphi} + r \frac{\partial a_z}{\partial z} \right].$$

在球坐标情形下,有

$$x = \rho \sin \theta \cos \varphi, y = \rho \sin \theta \sin \varphi,$$

$$z = \rho \cos \theta (u = \rho, v = \theta, \omega = \varphi).$$

于是,

$$L = \sqrt{\left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2} = 1,$$

$$M = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = \rho$$

$$N = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = \rho \sin \theta.$$

由此可知

$$\begin{aligned} \operatorname{div} \vec{a} &= \frac{1}{\rho^2 \sin \theta} \left[\frac{\partial}{\partial \rho} (a_\rho \rho^2 \sin \theta) \right. \\ &\quad \left. + \frac{\partial}{\partial \theta} (a_\theta \rho \sin \theta) + \frac{\partial}{\partial \varphi} (a_\varphi \rho) \right] \\ &= \frac{1}{\rho^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial \rho} (a_\rho \rho^2) \right. \\ &\quad \left. + \rho \frac{\partial}{\partial \theta} (a_\theta \sin \theta) + \rho \frac{\partial a_\varphi}{\partial \varphi} \right]. \end{aligned}$$

4435. 证明:

$$(a) \operatorname{rot}(\vec{a} + \vec{b}) = \operatorname{rot} \vec{a} + \operatorname{rot} \vec{b};$$

$$(6) \operatorname{rot}(u \vec{a}) = u \operatorname{rot} \vec{a} + \operatorname{grad} u \times \vec{a}.$$

证 (a) 设 $\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$, $\vec{b} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}$, 则有

$$\begin{aligned} \operatorname{rot}(\vec{a} + \vec{b}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x + b_x & a_y + b_y & a_z + b_z \end{vmatrix} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_x & b_y & b_z \end{vmatrix} \\ &= \operatorname{rot} \vec{a} + \operatorname{rot} \vec{b}. \end{aligned}$$

$$\begin{aligned} (6) \operatorname{rot}_y(u \vec{a}) &= \frac{\partial}{\partial y}(ua_z) - \frac{\partial}{\partial z}(ua_y) \\ &= u \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + \left(a_z \frac{\partial u}{\partial y} - a_y \frac{\partial u}{\partial z} \right) \\ &= u \operatorname{rot}_y \vec{a} + (\operatorname{grad} u \times \vec{a})_y, \end{aligned}$$

同法可得

$$\operatorname{rot}_y(u \vec{a}) = u \operatorname{rot}_y \vec{a} + (\operatorname{grad} u \times \vec{a})_y,$$

$$\operatorname{rot}_z(u \vec{a}) = u \operatorname{rot}_z \vec{a} + (\operatorname{grad} u \times \vec{a})_z.$$

于是,

$$\operatorname{rot}(u \vec{a}) = u \operatorname{rot} \vec{a} + \operatorname{grad} u \times \vec{a}.$$

4436. 求: (a) $\operatorname{rot} \vec{r}$; (6) $\operatorname{rot}[f(r) \vec{r}]$.

$$\begin{aligned} \text{解} \quad (a) \operatorname{rot} \vec{r} &= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) \vec{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) \vec{j} \\ &\quad + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \vec{k} = \vec{0}. \end{aligned}$$

$$\begin{aligned}
 (5) \operatorname{rot}[f(r) \vec{r}] &= f(r) \operatorname{rot} \vec{r} + \operatorname{grad} f(r) \times \vec{r} \\
 &= 0 + f'(r) \frac{\vec{r}}{r} \times \vec{r} \\
 &= 0.
 \end{aligned}$$

*) 利用 4435 题(6) 的结果.

* *) 利用 4410 题的结果.

4437. 求: (a) $\operatorname{rot} \vec{c} f(r)$, (6) $\operatorname{rot}[\vec{c} \times f(r) \vec{r}]$ ($\vec{c}$ 为定向量).

解 (a) $\operatorname{rot} \vec{c} f(r) = f(r) \operatorname{rot} \vec{c} + \operatorname{grad} f(r) \times \vec{c}$
 $= \frac{f'(r)}{r} (\vec{r} \times \vec{c}).$

$$\begin{aligned}
 (6) \operatorname{rot}[\vec{c} \times f(r) \vec{r}] &= f(r) \operatorname{rot}(\vec{c} \times \vec{r}) \\
 &+ \operatorname{grad} f(r) \times (\vec{c} \times \vec{r}). \text{ 但是,}
 \end{aligned}$$

$$\operatorname{rot}(\vec{c} \times \vec{r}) =$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ c_y z - c_z y & c_z x - c_x z & c_x y - c_y x \end{vmatrix} = 2 \vec{c}$$

$$\begin{aligned}
 \operatorname{grad} f(r) \times (\vec{c} \times \vec{r}) &= \frac{f'(r)}{r} \vec{r} \times (\vec{c} \times \vec{r}) \\
 &= \frac{f'(r)}{r} [(\vec{r} \cdot \vec{r}) \vec{c} - (\vec{r} \cdot \vec{c}) \vec{r}],
 \end{aligned}$$

故最后得

$$\begin{aligned}
 \operatorname{rot}[\vec{c} \times f(r) \vec{r}] &= 2f(r) \vec{c} \\
 &+ \frac{f'(r)}{r} [(\vec{r} \cdot \vec{r}) \vec{c} - (\vec{r} \cdot \vec{c}) \vec{r}]
 \end{aligned}$$

4438. 证明 $\operatorname{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot \operatorname{rot} \vec{a} - \vec{a} \cdot \operatorname{rot} \vec{b}$.

证 $\operatorname{div}(\vec{a} \times \vec{b}) = \frac{\partial}{\partial x}(a_z b_y - a_y b_z) + \frac{\partial}{\partial y}(a_x b_z - a_z b_x)$
 $+ \frac{\partial}{\partial z}(a_y b_x - a_x b_y)$

$$\begin{aligned}
&= b_1 \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) + b_2 \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \\
&+ b_3 \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) - a_1 \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \\
&- a_2 \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) - a_3 \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\
&= \vec{b} \cdot \operatorname{rot} \vec{a} - \vec{a} \cdot \operatorname{rot} \vec{b}.
\end{aligned}$$

4439. 求: (a) $\operatorname{rot}(\operatorname{grad} u)$; (b) $\operatorname{div}(\operatorname{rot} \vec{a})$.

解 (a) $\operatorname{rot}(\operatorname{grad} u) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} = \vec{0}.$

$$\begin{aligned}
(b) \operatorname{div}(\operatorname{rot} \vec{a}) &= \frac{\partial}{\partial x} \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) \\
&+ \frac{\partial}{\partial y} \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) \\
&+ \frac{\partial}{\partial z} \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) = 0.
\end{aligned}$$

4440. 物体以一定的角速度 ω 围绕轴 $l\{\cos\alpha, \cos\beta, \cos\gamma\}$ 旋转. 求速度向量 $\vec{v}$ 在空间的点 $M(x, y, z)$ 和在已知时刻的旋度.

解 $\vec{v} = \vec{v}_0 + \vec{\omega} \times \vec{r}$. 从而有

$$\begin{aligned}
v_x &= v_{0x} + \omega_y z - \omega_z y, & v_y &= v_{0y} + \omega_z x - \omega_x z, \\
v_z &= v_{0z} + \omega_x y - \omega_y x.
\end{aligned}$$

由于 $\operatorname{rot}_x \vec{v} = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 2\omega_x$, $\operatorname{rot}_y \vec{v} = 2\omega_y$ 及 $\operatorname{rot}_z \vec{v} = 2\omega_z$, 故 $\operatorname{rot} \vec{v} = 2\vec{\omega}$.

4441. 求向量 $\vec{r}$ 的流量: (a) 穿过圆锥形 $x^2 + y^2 \leq z^2$ ($0 \leq z \leq$

h 的侧表面; (6) 穿过此圆锥形的底.

解 (a) 在侧面上, 点的向径的方向与圆锥的母线重合. 因此, 点的向径与圆锥在该点的法线互相垂直, 即 $\vec{r}$ 在法线方向上的射影 $\vec{r}^n = 0$. 于是, 向量 r 穿过侧面 D 的流量为

$$\iint_D r_n dS = 0.$$

(6) 在圆锥形的底面上, $r_n = h$. 于是, 所求的流量为

$$\iint_{x^2+y^2 \leq h^2} r_n dS = h \cdot \pi h^2 = \pi h^3.$$

4442. 求向量 $\vec{a} = yz \vec{i} + zx \vec{j} + xy \vec{k}$ 的流量: (a) 穿过圆柱 $x^2 + y^2 \leq a^2$ ($0 \leq z \leq h$) 的侧表面; (6) 穿过此圆柱的全表面.

解 先求 (6), 由于

$$\iint_S a_n dS = \iiint_V \operatorname{div} \vec{a} dV = \iiint_V 0 dV = 0,$$

故向量 $\vec{a}$ 穿过圆柱的全表面的流量为零.

再求 (a), 又由于 $S = S_{\text{侧}} + S_{\text{上、下底}}$ 及在上、下底上 $a_n = xy$, 故有

$$\begin{aligned} \iint_{S_{\text{上、下底}}} a_n dS &= \iint_{x^2+y^2 \leq a^2} xy dx dy \\ &= 2 \int_0^{2\pi} d\varphi \int_0^a r^3 \sin\varphi \cos\varphi dr = 0. \end{aligned}$$

于是,

$$\iint_{S_{\text{侧}}} a_n dS = 0,$$

即向量 $\vec{a}$ 穿过侧面的流量也为零.

4443. 求向径 $\vec{r}$ 穿过曲面

$$z = 1 - \sqrt{x^2 + y^2} \quad (0 \leq z \leq 1)$$

的流量

解 设 S 为所给的曲面(锥), D 为锥的底面(即 Oxy 平面上的圆域 $x^2 + y^2 \leq 1$). 由于

$$\begin{aligned} \iint_S \vec{r}_n dS + \iint_D \vec{r}_n dS &= \iiint_V \operatorname{div} \vec{r} dV \\ &= 3 \int_0^{2\pi} d\varphi \int_0^1 r dr \int_0^1 dz = \pi \end{aligned}$$

及在 D 上, $\vec{r} \perp \vec{n}$, 故 $r_n = 0$, $\iint_D \vec{r}_n dS = 0$, 从而, 得

$$\iint_S \vec{r}_n dS = \pi.$$

4444. 求向量 $\vec{a} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$ 穿过球 $x^2 + y^2 + z^2 = 1$, $x > 0, y > 0, z > 0$ 的正八分之一的流量.

解 设 S 为所给的曲面, S_1, S_2 及 S_3 为球内三个坐标平面上的部分, 则有

$$\begin{aligned} & \iint_S \vec{a}_n dS + \iint_{S_1} \vec{a}_n dS + \iint_{S_2} \vec{a}_n dS + \iint_{S_3} \vec{a}_n dS \\ &= \iiint_{\substack{x^2+y^2+z^2 \leq 1 \\ x>0, y>0, z>0}} \operatorname{div} \vec{a} dV = 2 \iiint_{\substack{x^2+y^2+z^2 \leq 1 \\ x>0, y>0, z>0}} (x+y+z) dx dy dz \\ &= 2 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^1 r^2 \cos\psi \cdot r (\cos\varphi \cos\psi \\ & \quad + \sin\varphi \cos\psi + \sin\psi) dr \end{aligned}$$

$$\begin{aligned}
&= 2 \cdot \frac{1}{4} \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} [\cos\psi \sin\psi + \cos^2\psi (\cos\varphi + \sin\varphi)] d\psi \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} + \frac{\pi}{4} (\cos\varphi + \sin\varphi) \right] d\varphi \\
&= \frac{3}{8} \pi.
\end{aligned}$$

但在 $S_i (i = 1, 2, 3)$ 上, 显然有 $\vec{a} \perp \vec{n}$, 故 $a_n = 0$, 从而 $\iint_{S_i} a_n dS = 0 (i = 1, 2, 3)$. 于是, 所求的流量为

$$\iint_S a_n dS = \frac{3}{8} \pi.$$

4445. 求向量 $\vec{a} = y\vec{i} + z\vec{j} + x\vec{k}$ 穿过由诸平面 $x = 0, y = 0, z = 0, x + y + z = a (a > 0)$ 所包围角锥的全表面的流量.

利用奥斯特洛格拉德斯基公式, 验证结果.

解 方法一:

由于 $\operatorname{div} \vec{a} = 0$,

故所求的流量为

$$\iint_S a_n dS = \iiint_V \operatorname{div} \vec{a} dV = 0.$$

方法二:

如图 8.73 所示.

在平面 $z = 0 (S_1)$ 上.

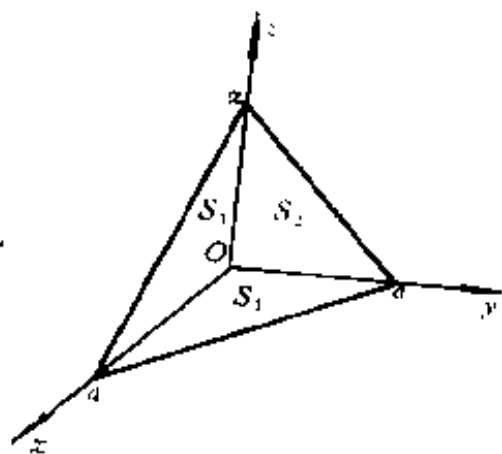


图 8.73

$\vec{n} = \{0, 0, -1\}$; 在平面 $y = 0 (S_2)$ 上, $\vec{n} = \{0, -1, 0\}$;

在平面 $x = 0 (S_3)$ 上, $\vec{n} = \{-1, 0, 0\}$.

于是, 向量 $\vec{a}$ 穿过曲面 S_1 的流量为

$$\begin{aligned}\iint_{S_1} \vec{a}_n dS &= \iint_{S_1} \vec{a} \cdot \vec{n} dS = \iint_{\substack{x^2+y^2 \leq a^2 \\ z=0}} (-x) dx dy \\ &= -\frac{a^3}{6}.\end{aligned}$$

同法可求得向量 $\vec{a}$ 穿过 S_2 及 S_3 面的流量也为 $-\frac{a^3}{6}$.

对于平面 $x+y+z=a(S_4)$, 其法向量为 $\vec{n} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, 故流量为

$$\begin{aligned}\iint_{S_4} \vec{a}_n dS &= \frac{1}{\sqrt{3}} \iint_{S_4} (y+z+x) dS \\ &= \frac{1}{\sqrt{3}} \iint_{\substack{x^2+y^2 \leq a^2 \\ x+y+z=a}} \vec{a} \cdot \sqrt{1^2+1^2+1^2} dx dy = \frac{a^3}{2}.\end{aligned}$$

因此, 最后得向量 $\vec{a}$ 穿过角锥全表面的流量为

$$\sum_{i=1}^4 \iint_{S_i} \vec{a}_n dS = \frac{a^3}{2} + 3\left(-\frac{a^3}{6}\right) = 0.$$

4446. 证明: 向量 $\vec{a}$ 穿过由方程式 $\vec{r} = \vec{r}(u, v) ((u, v) \in \Omega)$ 所给出的曲面 S 的流量等于

$$\iint_S \vec{a} \cdot \vec{n} dS = \iint_{\Omega} \left(\vec{a} \cdot \frac{\partial \vec{r}}{\partial u} \frac{\partial \vec{r}}{\partial v} \right) du dv,$$

式中 $\vec{n}$ 为曲面 S 的法线之单位向量.

证 设曲面 S 的方程为

$$\vec{r} = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k},$$

则有

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u} \vec{i} + \frac{\partial y}{\partial u} \vec{j} + \frac{\partial z}{\partial u} \vec{k},$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v} \vec{i} + \frac{\partial y}{\partial v} \vec{j} + \frac{\partial z}{\partial v} \vec{k}.$$

从而

$$\begin{aligned} \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \\ &= \left(\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right) \vec{i} + \left(\frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \right) \vec{j} \\ &\quad + \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) \vec{k}. \end{aligned}$$

因此, 易得

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{EG - F^2}$$

又 $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ 的方向显然是法线 $\vec{n}$ 的方向. 于是, 我们有

$$\begin{aligned} \iint_S \vec{a} \cdot \vec{n} dS &= \iint_{\vec{n}} \vec{a} \cdot \sqrt{EG - F^2} \vec{n} dudv \\ &= \iint_{\vec{n}} \vec{a} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) dudv \\ &= \iint_{\vec{n}} \left(\vec{a} \cdot \frac{\partial \vec{r}}{\partial u} \frac{\partial \vec{r}}{\partial v} \right) dudv. \end{aligned}$$

4447. 求向量 $\vec{a} = m \frac{\vec{r}}{r^3}$ (m 为常数) 穿过围绕坐标原点的封闭

曲面 S 的流量.

解 所求的流量为

$$\begin{aligned}\iint_S a_n dS &= m \iint_S \frac{1}{r^3} \vec{r} \cdot \vec{n} dS = m \iint_S \frac{\cos(\vec{r}, \vec{n})}{r^2} dS \\ &= m \cdot 4\pi^{*}) = 4\pi m.\end{aligned}$$

*) 利用 4392 题(σ) 的结果.

4448. 已知向量

$$\vec{a}(\vec{r}) = \vec{a} \sum_{i=1}^n \text{grad} \left(-\frac{e_i}{4\pi r_i} \right),$$

其中 e_i 为常数, r_i 为点 M_i (起点) 距动点 $M(\vec{r})$ 的距离. 求此向量穿过围绕点 $M_i (i = 1, 2, \dots, n)$ 的封闭曲面 S 的流量.

解 首先, 我们有

$$\vec{a} = \sum_{i=1}^n \text{grad} \left(-\frac{e_i}{4\pi r_i} \right) = \sum_{i=1}^n \frac{e_i \vec{r}_i}{4\pi r_i^3}.$$

其次, 我们考虑这样一个立体(V), 它由曲面 S 及包围点 $M_i (i = 1, 2, \dots, n)$ 的 n 个小球所围成(这些小球的球心在点 M_i , 半径为 ρ_i). 由于 $\text{div} \vec{a}$ 在 V 内为零, 故

$$\iint_S a_n dS = \sum_{j=1}^n \iint_{S_j} a_n dS,$$

其中 S_j 为第 j 个小球面. 但是

$$\iint_{S_j} a_n dS = \iint_{S_j} \left(\sum_{i=1}^n \frac{e_i \vec{r}_i}{4\pi r_i^3} \right) \cdot \vec{n} dS.$$

由于

$$\begin{aligned}\iint_{S_j} \frac{1}{r_{ij}^3} (\vec{r}_{ij} \cdot \vec{n}) dS &= \iint_{S_j} \frac{\cos(\vec{r}_{ij}, \vec{n})}{r_{ij}^2} dS \\ &= \begin{cases} 0, & \text{当 } j \neq i \text{ 时;} \\ 4\pi, & \text{当 } j = i \text{ 时,} \end{cases}\end{aligned}$$

故得

$$\iint_{S_j} a_n dS = e_j.$$

从而

$$\iint_S a_n dS = \sum_{j=1}^n e_j.$$

*) 利用 4392 题的结果.

4449. 证明:

$$\iint_S \frac{\partial u}{\partial n} dS = \iiint_V \nabla^2 u dx dy dz,$$

其中曲面 S 包围体积 V .

证 参看 4393 题(a).

4450. 在单位时间内经过曲面元素 dS 而进入温度场 u 的热量等于

$$dQ = -k \vec{n} \cdot \text{grad} u dS,$$

其中 k 为内热的传导系数, $\vec{n}$ 为曲面 S 的法线之单位向量. 求在单位时间内物体 V 所积累的热量. 研究温度上升的速度以推出为物体温度所满足的方程式(热传导方程式).

解 由于

$$dQ = -k \vec{n} \cdot \text{grad} u dS = -k \text{grad}_n u dS,$$

故由奥氏公式, 即得

$$Q = - \iint_S k \operatorname{grad}_n u dS = \iiint_V k \operatorname{div}(\operatorname{grad} u) dV.$$

因此,每单位时间内向立体内部流入的热量为

$$\iiint_V \operatorname{div}(k \operatorname{grad} u) dV. \quad (1)$$

这一热量引起立体内部温度的增加,现在我们从另一方面再来计算此热量.在时间 dt 内温度 u 增加

$$du = \frac{\partial u}{\partial t} dt.$$

需要对体积元素 dV 输入热量

$$c du \rho dV = c \frac{\partial u}{\partial t} \rho dt dV,$$

其中 c 为物体在所考察的点处的热容量.于是,在时间 dt 内整个立体就要吸收热量

$$dt \iiint_V c \rho \frac{\partial u}{\partial t} dV,$$

而在每单位时间内所吸收的热量即为

$$\iiint_V c \rho \frac{\partial u}{\partial t} dV. \quad (2)$$

比较(1)式及(2)式,便得等式

$$\iiint_V \left\{ c \rho \frac{\partial u}{\partial t} - \operatorname{div}(k \operatorname{grad} u) \right\} dV = 0.$$

由于上式对取在所考察境域内的任何立体 V 都适合,且被积函数显见连续,故根据 4097 题的结果,当点属于所考察的境域时,恒有

$$c \rho \frac{\partial u}{\partial t} = \operatorname{div}(k \operatorname{grad} u),$$

此即所求的热传导方程.

4451. 在运动中不可压缩的液体占有体积 V . 假定在域 V 内源

泉和漏孔都不存在,试推出连续性的方程:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0,$$

式中 $\rho = \rho(x, y, z, t)$ 为液体密度, $\vec{v}$ 为速度向量, t 为时间.

解 首先,我们已知:在每单位时间内自 V 中的任一立体 V' 的表面 S' 向外流出的流量 Q 为

$$Q = \iint_{S'} \rho v_n dS = \iiint_{V'} \operatorname{div}(\rho \vec{v}) dV. \quad (1)$$

现在我们用另一法来计算 Q ,如考虑到在时间 dt 内密度 ρ 增加 $\frac{\partial \rho}{\partial t} dt$,则立体元素 dV 的质量就增加 $\frac{\partial \rho}{\partial t} dt dV$,而整个所考察的立体 V' 的质量就增加

$$dt \iiint_{V'} \frac{\partial \rho}{\partial t} dV.$$

因此,每单位时间内 V' 中质量减少

$$- \iiint_{V'} \frac{\partial \rho}{\partial t} dV.$$

由于 V 内无源泉和漏孔,故这个减少的质量正好就是从 V' 的表面 S' 流出的质量流量 Q ,即

$$Q = - \iiint_{V'} \frac{\partial \rho}{\partial t} dV. \quad (2)$$

比较(1)式和(2)式,便得等式

$$\iiint_{V'} \left\{ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) \right\} dV = 0.$$

由于上式对 V 中任一立体 V' 均成立,且被积函数连续,故根据 4097 题的结果,当 $(x, y, z) \in V$ 时,恒有

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0.$$

4452. 求向量 $\vec{a} = \vec{r}$ 沿着螺线

$$\vec{r} = a \cos t \vec{i} + a \sin t \vec{j} + b t \vec{k} \quad (0 \leq t \leq 2\pi)$$

的一段功.

解 由于

$$d\vec{r} = (-a \sin t \vec{i} + a \cos t \vec{j} + b \vec{k}) dt,$$

$$\vec{a} \cdot d\vec{r} = b^2 t dt,$$

故所求的功为

$$W = \int_0^{2\pi} b^2 t dt = 2\pi^2 b^2.$$

4453. 求向量 $\vec{a} = f(r)\vec{r}$ (其中 f 是连续函数) 沿着弧 AB 的功.

解 所求的功为

$$\begin{aligned} W &= \int_{r_A}^{r_B} f(r) \vec{r} \cdot d\vec{r} = \int_{r_A}^{r_B} f(r) \vec{r} \cdot \vec{t}^0 ds \\ &= \int_{r_A}^{r_B} f(r) r dr, \end{aligned}$$

其中 $\vec{t}^0$ 是单位切向量.

4454. 求向量

$$\vec{a} = -y\vec{i} + x\vec{j} + c\vec{k}$$

(c 为常数) 的环流: (a) 沿着圆周 $x^2 + y^2 = 1, z = 0$.

(σ) 沿着圆周 $(x-2)^2 + y^2 = 1, z = 0$.

解 (a) 圆 $x^2 + y^2 = 1, z = 0$ 的向径 $\vec{r}$ 适合方程

$$\vec{r} = \cos t \vec{i} + \sin t \vec{j} + 0 \vec{k} \quad (0 \leq t \leq 2\pi).$$

由于

$$\begin{aligned}\vec{a} \cdot d\vec{r} &= (-\sin t \vec{i} + \cos t \vec{j} + c \vec{k}) \cdot \\ &\quad (-\sin t \vec{i} + \cos t \vec{j} + 0 \vec{k}) dt \\ &= dt,\end{aligned}$$

故所求的环流为

$$\int_0^{2\pi} dt = 2\pi.$$

(σ) 对于圆 $(x-2)^2 + y^2 = 1, z=0$, 有

$$\vec{r} = (2 + \cos t) \vec{i} + \sin t \vec{j} + 0 \vec{k} \quad (0 \leq t \leq 2\pi).$$

由于

$$\vec{a} \cdot d\vec{r} = (2\cos t + 1) dt,$$

故所求的环流为

$$\int_0^{2\pi} (2\cos t + 1) dt = 2\pi.$$

4455. 求向量 $\vec{a} = \text{grad} \left(\arctg \frac{y}{x} \right)$ 沿着围线 C 的环流 Γ ; (a) C 不围绕 Oz 轴; (σ) C 围绕 Oz 轴.

解 我们有

$$\vec{a} = -\frac{y}{x^2 + y^2} \vec{i} + \frac{x}{x^2 + y^2} \vec{j}.$$

于是, 易知

$$\text{rot} \vec{a} = \vec{0} \quad (\text{除 } x = y = 0, \text{ 即 } Oz \text{ 轴上的点}).$$

(a) 若 C 不围绕 Oz 轴, 则可于 C 上张一曲面 S , 使 S 与 Oz 轴不相交, 于是, 根据斯托克斯公式, 得

$$\Gamma = \oint_C \vec{a} \cdot d\vec{r} = \iint_S \vec{n} \cdot \text{rot} \vec{a} \, dS = 0.$$

(σ) 若 C 围绕 Oz 轴. 先设 C 正好围绕 Oz 轴旋转一周, 取常数 $\tau < 0$ 充分小, 使 C 位于平面 $z = \tau$ 的上方. 在

平面 $z = \tau$ 上围绕 Oz 轴取一圆周 $C_r(x^2 + y^2 = r^2, z = \tau)$ 充分小, 使半径 r 小于 C 到 Oz 轴的距离. 以 C 与 C_r 为边界张上一曲面 S , 使 S 与 Oz 轴不相交. 由斯托克斯公式, 得

$$\oint_C \vec{a} \cdot d\vec{r} + \oint_{C_r} \vec{a} \cdot d\vec{r} = \iint_S \vec{n} \cdot \text{rot} \vec{a} \, dS = 0,$$

其中 $-C_r$ 表示沿顺时针方向取向. 于是

$$\Gamma = \oint_C \vec{a} \cdot d\vec{r} = \oint_{C_r} \vec{a} \cdot d\vec{r}.$$

但取 C_r 的参数方程 $x = r \cos \theta, y = r \sin \theta, z = \tau$ 后, 得

$$\begin{aligned} \oint_{C_r} \vec{a} \cdot d\vec{r} &= \int_0^{2\pi} \left[\left(-\frac{r \sin \theta}{r^2} \right) (-r \sin \theta) \right. \\ &\quad \left. + \left(\frac{r \cos \theta}{r^2} \right) (r \cos \theta) \right] d\theta \\ &= \int_0^{2\pi} d\theta = 2\pi. \end{aligned}$$

从而

$$\Gamma = \oint_C \vec{a} \cdot d\vec{r} = 2\pi.$$

现设 C 围绕 Oz 轴旋转了 n 圈. 为叙述简单起见, 假定 $n = 2$. 在平面

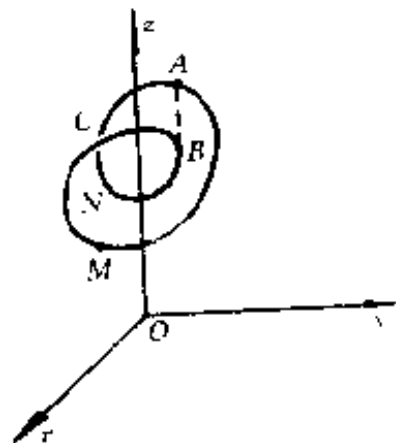


图 8.74

Ozx 上引辅助线(直线段) AB , 将 C 分解成两个只绕 Oz 轴转一周的闭曲线 $C_1 = ABMA$ 与 $C_2 = ANBA$ (图 8.74). 根据前面已证的结果可知

$$\oint_{C_1} \vec{a} \cdot d\vec{r} = 2\pi, \quad \oint_{C_2} \vec{a} \cdot d\vec{r} = 2\pi.$$

于是,注意到 $\overline{AB}$ 上的线积分(第二型)与 $\overline{BA}$ 上的线积分相消,即得

$$\Gamma = \oint_C \vec{a} \cdot d\vec{r} = \oint_{C_1} \vec{a} \cdot d\vec{r} + \oint_{C_2} \vec{a} \cdot d\vec{r} = 4\pi.$$

完全类似地,可得一般情况(C 围绕 Oz 轴转 n 圈)时,有

$$\Gamma = \oint_C \vec{a} \cdot d\vec{r} = 2\pi n.$$

4456^{*}. 平面的不可压缩稳流由速度向量

$$\vec{\omega} = u(x, y)\vec{i} + v(x, y)\vec{j}$$

描写出来,求出:(1)经过包围域 S 的封闭围线 C 所流出液体的量 Q (液体的消耗);(2)速度向量沿着围线 C 的环流 Γ .若流场无源泉、无漏孔且无旋度,则函数 u 和 v 满足什么样的方程式?

解 (1) 考虑包含着点 $D(x, y)$ 的两边长分别为 Δx 与 Δy 的小矩形元 $ABCD$ (图8.75).

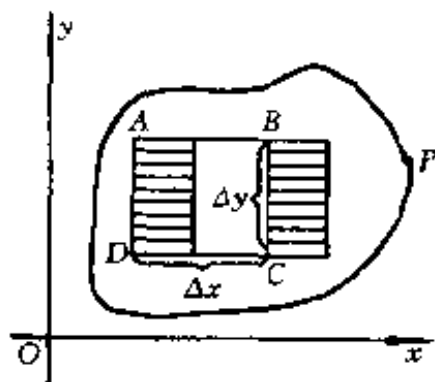


图 8.75

在单位时间内沿 Ox 轴方向从 AD 边流入的量

为 $u(x, y) \cdot \Delta y$ (为简单起见,设密度 $\rho = 1$),而同时从 BC 边流出的量为 $u(x + \Delta x, y) \Delta y$.于是,在单位时间内,沿 Ox 轴方向从单位面积的小正方形内流出的量为

$$\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x \Delta y} \Delta y.$$

当 $\Delta x \rightarrow 0$ 时, 此比值的极限 $\frac{\partial u}{\partial x}$ 就是在点 (x, y) 沿 Ox 轴方向的发散强度. 类似地, $\frac{\partial v}{\partial y}$ 就是在点 (x, y) 沿 Oy 轴方向的发散强度. 于是, 在点 (x, y) 处液体的发散强度为 $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, 而对于面积元 $dxdy$ 的流量即为

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dxdy.$$

因此, 总的流量为

$$Q = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dxdy.$$

另一解法: 令点 P 为围线 C 上的任一点, $\vec{n}$ 为向外法线, 考虑曲线元素 ds . 单位时间内通过 ds 弧段的流量为

$$dQ = \omega_n ds,$$

其中 ω_n 为点 P 处的流速 $\vec{\omega}$ 在法向量 $\vec{n}$ 上的投影: $\omega_n = \vec{\omega} \cdot \vec{n}$. 于是, 所求的通过曲线 C 的流量为

$$Q = \int_C \omega_n ds.$$

但是, $\omega_n = \vec{\omega} \cdot \vec{n} = u \cos(n, x) + v \cos(n, y) = u \frac{dy}{ds} - v \frac{dx}{ds}$, 故得

$$Q = \int_C u dy - v dx.$$

应用格林公式, 即得

$$Q = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx dy.$$

(2) $d\Gamma = \vec{\omega} \cdot d\vec{r} = udx + vdy$, 故

$$\Gamma = \int_C udx + vdy = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy.$$

若流场无源泉无漏孔及无旋度, 则对于流场中任何围线 C 及其所包围的域 S , 均有

$$Q = 0 \text{ 及 } \Gamma = 0.$$

于是, 在流场中的每一点, 均有

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ 及 } \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

或 $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \text{ 及 } \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y},$

这就是 u, v 所应满足的方程.

*) 编者注: 从原书答案来看, 本题叙述有误. 最后的问题中“流体是不可压缩”应改为“流场无源泉、无漏孔”, 而在题目开始, 应假定流体不可压缩.

* *) 参看 4324 题的推导.

4457. 证明: 场

$$\vec{a} = yz(2x + y + z)\vec{i} + xz(x + 2y + z)\vec{j} + xy(x + y + 2z)\vec{k}$$

是有势场, 并求这个场的势.

解 由于对空间任一点 (x, y, z) 均有

$$\begin{aligned} \text{rota} = & \left\{ \frac{\partial}{\partial y} [xy(x + y + 2z)] \right. \\ & \left. - \frac{\partial}{\partial x} [xz(x + 2y + z)] \right\} \vec{i} \end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{\partial}{\partial z} [yz(2x + y + z)] \right. \\
& - \frac{\partial}{\partial x} [xy(x + y + 2z)] \left. \right\} j \\
& + \left\{ \frac{\partial}{\partial x} [xz(x + 2y + z)] \right. \\
& - \frac{\partial}{\partial y} [yz(2x + y + z)] \left. \right\} k \\
& = 0,
\end{aligned}$$

故 $\vec{a}$ 为有势场.

又由于势 u 满足

$$\begin{aligned}
du &= \vec{a} \cdot d\vec{r} \\
&= yz(2x + y + z)dx + xz(x + 2y + z)dy \\
&\quad + xy(x + y + 2z)dz \\
&= xyz(dx + dy + dz) + (x + y + z) \\
&\quad \cdot (yzdx + zxdy + xydz) \\
&= xyzd(x + y + z) + (x + y + z)d(xyz) \\
&= d[xyz(x + y + z)],
\end{aligned}$$

故势 $u = xyz(x + y + z) + C$, 其中 C 为任意常数.

4458. 求由位于坐标原点的质量 m 所产生的引力场

$$\vec{a} = -\frac{m}{r^3}\vec{r}$$

的势.

解 由于

$$\begin{aligned}
du &= \vec{a} \cdot d\vec{r} = -\frac{m}{r^3}(xdx + ydy + zdz) \\
&= -\frac{m}{2r^2}d(r^2)
\end{aligned}$$

$$= -\frac{m}{r^2}dr = d\left(\frac{m}{r}\right),$$

故势 $u = \frac{m}{r} + C$ (C 为任意常数), 通常取 $u = \frac{m}{r}$ ($r \neq 0$).

4459. 求位置在 $M_i (i = 1, 2, \dots, n)$ 各点的质量系 $m_i (i = 1, 2, \dots, n)$ 所产生引力场的势.

解 引力场 $\vec{a} = -\sum_{i=1}^n \frac{m_i}{r_i^3} \vec{r}_i$, 其中 r_i 为动点 M 与 M_i 之间的距离. 由于

$$du = \vec{a} \cdot d\vec{r} = d\left(\sum_{i=1}^n \frac{m_i}{r_i}\right),$$

故势 $u = \sum_{i=1}^n \frac{m_i}{r_i} + C$ (C 为任意常数), 通常取 $u = \sum_{i=1}^n \frac{m_i}{r_i}$.

4460. 证明: 场 $\vec{a} = f(r)\vec{r}$ (其中 $f(r)$ 是单值连续函数) 是有势场. 求这个场的势.

解 利用 4436 题 (σ) 的结果, 即知 $\text{rot}(f(r)\vec{r}) = \vec{0}$. 故 $\vec{a}$ 为有势场. 又由于

$$\begin{aligned} du &= \vec{a} \cdot d\vec{r} = xf(r)dx + yf(r)dy + zf(r)dz \\ &= \frac{1}{2}f(r)d(r^2) = rf(r)dr, \end{aligned}$$

故势 $u = \int_{r_0}^r tf(t)dt$, 其中 $r = \sqrt{x^2 + y^2 + z^2}$.

4461. 证明公式

$$\text{grad}_r \left\{ \iiint_V \rho(Q) \frac{dV}{r} \right\} = - \iint_S \rho(Q) \vec{n} \frac{dS}{r}$$

$$+ \iiint_V \text{grad}_Q \rho(Q) \frac{dV}{r},$$

其中 S 为包含体积 V 的曲面, $\vec{n}$ 为曲面 S 的外法线, r 为点 $P(x, y, z)$ 与点 $Q(\xi, \eta, \zeta)$ 两点间的距离.

证 首先指出, 题中需假定 $\rho(Q)$ 在 V 上具有连续的导函数.

i) 先设点 $P(x, y, z)$ 在 V 之外. 令

$$f(x, y, z) = \iiint_V \rho(Q) \frac{dV}{r}. \quad (1)$$

显然, 右端积分的被积函数对参变量 x, y, z 都具有连续的偏导函数, 故可在积分号下求导数, 得

$$\text{grad}_P f = \iiint_V \rho(Q) \text{grad}_P \frac{1}{r} dV. \quad (2)$$

又由于

$$\text{grad}_P \frac{1}{r} = - \frac{\vec{r}}{r^3} = - \text{grad}_Q \frac{1}{r}, \vec{r} = \vec{QP}.$$

代入(2)式, 得

$$\text{grad}_P f = - \iiint_V \rho(Q) \text{grad}_Q \frac{1}{r} dV. \quad (3)$$

在公式(4408 题(Γ))

$$\text{grad}_Q(\varphi\psi) = \varphi \text{grad}_Q \psi + \psi \text{grad}_Q \varphi$$

中, 令 $\varphi = \rho(Q)$, $\psi = \frac{1}{r}$, 再代入(3)式, 得

$$\begin{aligned} \text{grad}_P f = & - \iiint_V \text{grad}_Q \left(\frac{\rho(Q)}{r} \right) dV \\ & + \iiint_V \text{grad}_Q \rho(Q) \frac{dV}{r}. \end{aligned} \quad (4)$$

根据奥氏公式, 有

$$\iiint_V \operatorname{grad}_Q \left| \frac{\rho(Q)}{r} \right| dV = \iint_S \rho(Q) \vec{n} \frac{dS}{r}. \quad (5)$$

将上式代入(4)式,即得

$$\operatorname{grad}_P f = - \iint_S \rho(Q) \vec{n} \frac{dS}{r} + \iiint_V \operatorname{grad}_Q \rho(Q) \frac{dV}{r}.$$

ii) 现设点 $P(x, y, z)$ 在 V 的内部. 仍按(1)式令 $f(x, y, z)$. 注意, 这时(1)式右端的积分为广义重积分(点 P 为瑕点); 但易知它收敛, 因为在以 P 点为中心, ε 为半径的球域 V_ε 上的积分满足 ($M = \max_{Q \in V} |\rho(Q)|$)

$$\begin{aligned} \left| \iiint_{V_\varepsilon} \frac{\rho(Q)}{r} dV \right| &\leq \iiint_{V_\varepsilon} \frac{|\rho(Q)|}{r} dV \leq M \iiint_{V_\varepsilon} \frac{dV}{r} \\ &= M \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \int_0^\varepsilon \frac{r^2}{r} dr = 2M\pi\varepsilon^2 \rightarrow 0 \quad (\text{当 } \varepsilon \rightarrow +0 \text{ 时}). \end{aligned}$$

时).

我们证明, 这时仍可将(1)式的积分在积分号下求导数而得(2)式. 事实上, 由于

$$\begin{aligned} \left| \iiint_{V_\varepsilon} \rho(Q) \frac{\partial(\frac{1}{r})}{\partial x} dV \right| &\leq \iiint_{V_\varepsilon} \left| \rho(Q) \frac{\partial(\frac{1}{r})}{\partial x} \right| dV \\ &= \iiint_{V_\varepsilon} \left| -\rho(Q) \frac{x-\xi}{r^3} \right| dV \leq M \iiint_{V_\varepsilon} \frac{dV}{r^2} \\ &= M \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \int_0^\varepsilon \frac{r^2}{r^2} dr = 4M\pi\varepsilon, \end{aligned}$$

故积分

$$\iiint_V \rho(Q) \frac{\partial(\frac{1}{r})}{\partial x} dV$$

关于 x 一致收敛. 于是, (1)式右端的积分可在积分号下

关于 x 求偏导函数,得

$$\frac{\partial f}{\partial x} = \iiint_V \rho(Q) \frac{\partial(\frac{1}{r})}{\partial x} dV. \quad (6)$$

同理可得

$$\frac{\partial f}{\partial y} = \iiint_V \rho(Q) \frac{\partial(\frac{1}{r})}{\partial y} dV, \quad (7)$$

$$\frac{\partial f}{\partial z} = \iiint_V \rho(Q) \frac{\partial(\frac{1}{r})}{\partial z} dV. \quad (8)$$

由(6), (7), (8) 三式, 即得(2) 式, 仿 i) 段办法, 可得(3) 式与(4) 式(注意, 仿前, 可知(4) 式右端两个积分都收敛). 但不能直接对 V 应用奥氏公式而得(5) 式, 因为有瑕点 P , 但显然可对 $V - V_\epsilon$ 应用奥氏公式, 得

$$\iiint_{V-V_\epsilon} \text{grad}_Q \left(\frac{\rho(Q)}{r} \right) dV = \iint_{S+S_\epsilon} \rho(Q) \vec{n} \frac{dS}{r}, \quad (9)$$

其中 S_ϵ 为球域 V_ϵ 的边界(球面), 在 S_ϵ 上的 $\vec{n}$ 是指向点 P 的. 由于

$$\begin{aligned} \left| \iint_{S_\epsilon} \rho(\theta) \vec{n} \frac{dS}{r} \right| &\leq \sqrt{3} \iint_{S_\epsilon} |\rho(\theta)| \frac{dS}{r} \leq \sqrt{3} M \iint_{S_\epsilon} \frac{dS}{r} \\ &= \frac{\sqrt{3} M}{\epsilon} \iint_{S_\epsilon} dS = \frac{\sqrt{3} M}{\epsilon} \cdot 4\pi\epsilon^2 = 4\sqrt{3} \pi M \epsilon, \end{aligned}$$

故

$$\lim_{\epsilon \rightarrow +0} \iint_{S_\epsilon} \rho(\theta) \vec{n} \frac{dS}{r} = 0.$$

于是, 在(9) 式两端令 $\epsilon \rightarrow +0$ 取极限, 即得(5) 式. 以

(5) 式代入(4)式,最后得所要证的公式

$$\begin{aligned} \operatorname{grad}_P \left\{ \iiint_V \rho(Q) \frac{dV}{r} \right\} = & - \iint_S \rho(Q) \vec{n} \frac{dS}{r} \\ & + \iiint_V \operatorname{grad}_Q \rho(Q) \frac{dV}{r}. \end{aligned}$$

证毕.

4462. 证明:若 $\vec{a} = \operatorname{grad} u$, 其中

$$u(x, y, z) = - \frac{1}{4\pi} \iiint_{(\infty)}^{+\infty} \frac{\rho(\xi, \eta, \zeta)}{r} d\xi d\eta d\zeta$$

及 $r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}$,

则 $\operatorname{div} \vec{a} = \rho(x, y, z)$

(假定对应的积分有意义).

证 首先指出,为保证题述的广义重积分(既是无穷积分,又是瑕积分)的存在性以及下面要用到的积分号下求导数的合理性,一般我们需假定: $\rho(\xi, \eta, \zeta)$ 在全空间具有连续的偏导函数,并且当 $R = \sqrt{\xi^2 + \eta^2 + \zeta^2}$ 充分大时($R \geq R_0$),有

$$|\rho(\xi, \eta, \zeta)| \leq \frac{M}{R^{2+\alpha}}, \quad (1)$$

其中 $M > 0, \alpha > 0$ 是两个常数.

考虑空间任一点 $P_0(x_0, y_0, z_0)$ 用 V_0 表示以 P_0 为中心,1 为半径的单位球域.我们先限制点 $P(x, y, z)$ 只在 V_0 中变动.又用 V_1 表示以 P_0 为中心,2 为半径的球域, V_2 表示整个空间去掉 V_1 所剩下的部分(无界域).令

$$u_1(x, y, z) = \iiint_{V_1} \frac{\rho(\xi, \eta, \zeta)}{r} d\xi d\eta d\zeta, \quad (2)$$

$$u_2(x, y, z) = \iiint_{V_2} \frac{\rho(\xi, \eta, \zeta)}{r} d\xi d\eta d\zeta. \quad (3)$$

于是,

$$u(x, y, z) = -\frac{1}{4\pi} [u_1(x, y, z) + u_2(x, y, z)]. \quad (4)$$

(2) 式右端为瑕积分, 在 4461 题证明的第 ii) 段中已证它是收敛的; (3) 式右端为无穷积分, 下面证明它收敛.

令

$r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$, $R_1 = \max\{R_0, 2(r_0 + 1)\}$, 则当 $R \geq R_1$ 时, 有 $R \geq R_0$ (从而 (1) 式满足), 且 $R \geq 2(r_0 + 1)$. 以 Q 表示点 (ξ, η, ζ) , O 表示原点 $(0, 0, 0)$. 由于三角形两边之和大于第三边, 故 (注意 $P \in V_0$).

$$R = \overline{OQ} \leq \overline{OP} + \overline{PQ} \leq r_0 + 1 + r \leq \frac{R}{2} + r,$$

从而

$$\begin{aligned} & \iiint_{\xi^2 + \eta^2 + \zeta^2 \geq R_1^2} \left| \frac{\rho(\xi, \eta, \zeta)}{r} \right| d\xi d\eta d\zeta \\ & \leq M \iiint_{\xi^2 + \eta^2 + \zeta^2 \geq R_1^2} \frac{d\xi d\eta d\zeta}{r R^{2+\alpha}} \\ & \leq 2M \iiint_{\xi^2 + \eta^2 + \zeta^2 \geq R_1^2} \frac{d\xi d\eta d\zeta}{R^{3+\alpha}} \\ & = 2M \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \int_{R_1}^{+\infty} \frac{R^2}{R^{3+\alpha}} dR \\ & = 2M \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \int_{R_1}^{+\infty} \frac{dR}{R^{1+\alpha}} \end{aligned}$$

$$= \frac{8M\pi}{\alpha R_1^\alpha} < +\infty, \quad (5)$$

故(3)式右端的无穷积分收敛.

由(4)式知 $u(x, y, z)$ 有定义. 由于 $\operatorname{div}(\operatorname{grad} u) = \Delta u$, 故我们只要证明

$$\Delta u = \rho(x, y, z). \quad (6)$$

我们证明(3)式右端的无穷积分可在积分号下求导数两次:

$$\frac{\partial u_2}{\partial x} = \iiint_{V_2} \rho(\xi, \eta, \zeta) \frac{\partial}{\partial x} \left(\frac{1}{r} \right) d\xi d\eta d\zeta, \quad (7)$$

$$\frac{\partial^2 u_2}{\partial x^2} = \iiint_{V_2} \rho(\xi, \eta, \zeta) \frac{\partial^2}{\partial x^2} \left(\frac{1}{r} \right) d\xi d\eta d\zeta. \quad (8)$$

为此, 只要证明(7)式右端的积分和(8)式右端的积分都关于 $(x, y, z) \in V_0$ 一致收敛. 由于

$$\frac{\partial}{\partial x} \left(\frac{1}{r} \right) = \frac{\xi - x}{r^3}, \quad \frac{\partial^2}{\partial x^2} \left(\frac{1}{r} \right) = -\frac{1}{r^3} + \frac{3(\xi - x)^2}{r^5},$$

故仿(5)式之推导, 可得: 当 $R_2 > R_1 = \max\{R_0, 2(r_0 + 1)\}$ 时, 对一切 $(x, y, z) \in V_0$, 有

$$\begin{aligned} & \iiint_{\xi^2 + \eta^2 + \zeta^2 \leq R_2^2} \left| \rho(\xi, \eta, \zeta) \frac{\partial}{\partial x} \left(\frac{1}{r} \right) \right| d\xi d\eta d\zeta \\ & \leq M \iiint_{\xi^2 + \eta^2 + \zeta^2 \geq R_2^2} \frac{d\xi d\eta d\zeta}{r^2 R_2^{1+\alpha}} \leq 4M \iiint_{\xi^2 + \eta^2 + \zeta^2 \geq R_2^2} \frac{d\xi d\eta d\zeta}{R_2^{4+\alpha}} \\ & = \frac{16M\pi}{(1+\alpha)R_2^{1+\alpha}}, \\ & \iiint_{\xi^2 + \eta^2 + \zeta^2 \geq R_2^2} \left| \rho(\xi, \eta, \zeta) \frac{\partial^2}{\partial x^2} \left(\frac{1}{r} \right) \right| d\xi d\eta d\zeta \end{aligned}$$

$$\begin{aligned}
&\leq 4M \iiint_{\xi^2 + \eta^2 + \zeta^2 \geq R_2^2} \frac{d\xi d\eta d\zeta}{r^3 R^{2+\alpha}} \\
&\leq 32M \iiint_{\xi^2 + \eta^2 + \zeta^2 \geq R_2^2} \frac{d\xi d\eta d\zeta}{R^{5+\alpha}} \\
&= \frac{128M\pi}{(2+\alpha)R_2^{2+\alpha}}.
\end{aligned}$$

由此可知, (7) 式右端的积分和 (8) 式右端的积分都关于 $(x, y, z) \in V_0$ 一致收敛. 因此, (7) 式与 (8) 式当 $(x, y, z) \in V_0$ 时成立. 同理可证, 当 $(x, y, z) \in V_0$ 时, 有

$$\frac{\partial u_2}{\partial y^2} = \iiint_{V_2} \rho(\xi, \eta, \zeta) \frac{\partial^2}{\partial y^2} \left(\frac{1}{r} \right) d\xi d\eta d\zeta, \quad (9)$$

$$\frac{\partial u_2}{\partial z^2} = \iiint_{V_2} \rho(\xi, \eta, \zeta) \frac{\partial^2}{\partial z^2} \left(\frac{1}{r} \right) d\xi d\eta d\zeta. \quad (10)$$

将 (8), (9), (10) 三式相加, 即得 (注意到 $\Delta \left(\frac{1}{r} \right) = 0$)

$$\Delta u_2 = \iiint_{V_2} \rho(\xi, \eta, \zeta) \Delta \left(\frac{1}{r} \right) d\xi d\eta d\zeta = 0. \quad (11)$$

下面再求 $\Delta u_1 = \operatorname{div}(\operatorname{grad} u_1)$. 由 4461 题的结果知

$$\operatorname{grad} u_1 = - \iint_{S_1} \rho(Q) \vec{n} \frac{dS}{r} + \iiint_{V_1} \operatorname{grad}_Q \rho(Q) \frac{dV}{r}, \quad (12)$$

其中 S_1 表示 V_1 的边界 (球面). 显然, 当 $P(x, y, z) \in V_0$ 时, (12) 式右端的第一个积分 (面积分) 的被积函数具有对于 x, y 及 z 的连续偏导函数, 故可在积分号下求对于 x, y 及 z 的偏导函数. 另外, 仿照 4461 题 ii) 段之证可知 (12) 式右端的第二个积分 (三重积分) 也可在积分号下求对于 x, y 及 z 的偏导函数. 于是, 得

$$\begin{aligned}\operatorname{div}(\operatorname{grad} u_1) = & - \iint_{\vec{S}_1} \operatorname{div}_P \left[\frac{\rho(Q) \vec{n}}{r} \right] dS \\ & + \iiint_{V_1} \operatorname{div}_P \left[\frac{1}{r} \operatorname{grad}_Q \rho(Q) \right] dV. \quad (13)\end{aligned}$$

利用公式 $\operatorname{div}(va) = v \operatorname{div} a + \vec{a} \cdot \operatorname{grad} v$ (4424 题(B)), 可知(注意到 $\rho(Q) \vec{n}$ 及 $\operatorname{grad}_Q \rho(Q)$ 均与 P 无关)

$$\begin{aligned}\operatorname{div}_P \left[\frac{\rho(Q) \vec{n}}{r} \right] &= \rho(Q) \vec{n} \cdot \operatorname{grad}_P \left(\frac{1}{r} \right) \\ &= - \rho(Q) \vec{n} \cdot \operatorname{grad}_Q \left(\frac{1}{r} \right) = - \rho(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right), \\ \operatorname{div}_P \left[\frac{1}{r} \operatorname{grad}_Q \rho(Q) \right] &= \operatorname{grad}_Q \rho(Q) \cdot \operatorname{grad}_P \left(\frac{1}{r} \right) \\ &= - \operatorname{grad}_Q \rho(Q) \cdot \operatorname{grad}_Q \left(\frac{1}{r} \right).\end{aligned}$$

代入(13)式,得

$$\begin{aligned}\Delta u_1 = & \iint_{\vec{S}_1} \rho(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS - \iiint_{V_1} \operatorname{grad}_Q \rho(Q) \\ & \cdot \operatorname{grad}_Q \left(\frac{1}{r} \right) dV. \quad (14)\end{aligned}$$

由于

$$\begin{aligned}\operatorname{div}_Q \left[\rho(Q) \operatorname{grad}_Q \left(\frac{1}{r} \right) \right] &= \rho(Q) \Delta_Q \left(\frac{1}{r} \right) + \operatorname{grad}_Q \rho(Q) \\ &\cdot \operatorname{grad}_Q \left(\frac{1}{r} \right), \text{ 而 } \Delta_Q \left(\frac{1}{r} \right) = 0 (Q \neq P), \text{ 故 (14) 式可写} \\ &\text{为}\end{aligned}$$

$$\Delta u_1 = \iint_{\vec{S}_1} \rho(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS$$

$$= \iiint_{V_1} \operatorname{div}_Q \left[\rho(Q) \operatorname{grad}_Q \left(\frac{1}{r} \right) \right] dV. \quad (15)$$

下面计算(15)式中的三重积分,用 Ω_ϵ 表示以点 $P(x, y, z)$ 为中心, ϵ 为半径的球域,其边界(球面)记为 S_ϵ .对 $V_1 - \Omega_\epsilon$ 应用奥氏公式,得

$$\begin{aligned} & \iiint_{V_1 - \Omega_\epsilon} \operatorname{div}_Q \left[\rho(Q) \operatorname{grad}_Q \left(\frac{1}{r} \right) \right] dV \\ &= \iint_{S_1 + S_\epsilon} \rho(Q) \operatorname{grad}_Q \left(\frac{1}{r} \right) \cdot \vec{n} dS \\ &= \iint_{S_1} \rho(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS + \iint_{S_\epsilon} \rho(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS, \end{aligned} \quad (16)$$

其中 $\vec{n}$ 是向外法线,从而在 S_ϵ 上是指向点 $P(x, y, z)$ 的.由中值定理知

$$\begin{aligned} & \iint_{S_\epsilon} \rho(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS = - \iint_{S_\epsilon} \rho(Q) \frac{\partial}{\partial r} \left(\frac{1}{r} \right) dS \\ &= \iint_{S_\epsilon} \rho(Q) \frac{dS}{r^2} = \frac{1}{\epsilon^2} \iint_{S_\epsilon} \rho(Q) dS \\ &= \frac{1}{\epsilon^2} \cdot \rho(Q_\epsilon) \cdot 4\pi\epsilon^2 \\ &= 4\pi\rho(Q_\epsilon), \end{aligned}$$

其中 Q_ϵ 是球面 S_ϵ 上的某一点.代入(16)式,得

$$\begin{aligned} & \iiint_{V_1 - \Omega_\epsilon} \operatorname{div}_Q \left[\rho(Q) \operatorname{grad}_Q \left(\frac{1}{r} \right) \right] dV \\ &= \iint_{S_1} \rho(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS + 4\pi\rho(Q_\epsilon), \end{aligned}$$

两端令 $\epsilon \rightarrow +0$ 取极限,得

$$\begin{aligned} & \iiint_{V_1} \operatorname{div}_Q \left[\rho(Q) \operatorname{grad}_Q \left(\frac{1}{r} \right) \right] dV \\ &= \iint_{S_1} \rho(Q) \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS + 4\pi \rho(P), \end{aligned}$$

再以此式代入(15)式,得

$$\Delta u_1 = -4\pi \rho(x, y, z). \quad (17)$$

由(17)式,(11)式以及(4)式,即得(6)式.于是,(6)式对一切点 $P(x, y, z) \in V_0$ 成立.由于 V_0 的中心 $P_0(x_0, y_0, z_0)$ 是任意的(可为空间任一点),故知(6)式对空间任一点都成立.证毕.