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Categories and Modules

With *K*-theory in View

A. J. BERRICK &
M. E. KEATING

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A. J. Berrick & M. E. Keating

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To

Sandra, Genevieve and Anthony

and

Valerie and Christopher

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Preface

This book is an introduction to the theory of categories, together with applications of this theory to some constructions with rings and modules. We start with a discussion of categories in general, and then concentrate on the types of category – additive, abelian – which enjoy to an increasing extent the properties of categories of modules. Our applications are the Morita theory, the localization and completion of rings and modules, and finally some 'local-global' methods, in which the properties of a module are compared to those of its localizations and completions. We also develop the tools that we need for these applications, namely, the tensor product and limits, both direct and inverse.

The selection and presentation of our material is motivated by the needs of algebraic K-theory. Indeed, this book started out as some preliminary remarks within a text on that subject. Thus the categorical foundations are those needed to set up algebraic K-theory, and our applications are chosen since they underly some fundamental results in algebraic K-theory. However, the content of this text will be, we hope, of interest to a wider readership than potential K-theorists.

Here is a more detailed survey of the material that we cover. Our first chapter sets out the basics of category theory. There are three fundamental definitions, those of a category, a functor, and a natural transformation, and we show how to use these notions to define universal objects and universal constructions. An understanding of these is important for two reasons. On the one hand, many definitions in module theory (kernels, cokernels, ...) can be reinterpreted in the language of universal objects and so extended to more general situations; on the other, many of the objects in K-theory itself arise as universal objects in one category or another.

A category of modules has a richer structure than an abstract category, since the additive structure on modules extends to their homomorphisms.

The second chapter shows how this extra structure can be axiomatized and thus imposed on abstract categories. We first analyse the properties of the groups of homomorphisms in a module category to obtain a list of axioms that define additive and abelian categories; these are abstract categories that share most of the important properties of module categories. The final topic in this chapter stems from the fact that modules can be arranged in short exact sequences, which is basic to the definitions in algebraic K-theory. We are thus led to the idea of an exact category, which is a category together with a nicely behaved collection of short exact sequences. Such categories are the domains on which K-theory operates.

The larger part of this text is more directly concerned with categories of modules, rather than categories in the abstract. In the third chapter, we introduce the tensor product and use it to change the ring of scalars of a module. This leads to the Morita theory (Chapter 4), which tells us when two rings have, in some sense, the same module theory.

In Chapters 5 and 6, we look at several related methods for constructing new rings and categories from old, by taking direct and inverse limits and by performing localizations and completions.

Finally, we exploit these techniques to obtain some local-global results, mainly in the theory of orders. These tackle the problem of gaining information about the projective modules over a ring from knowledge of the modules over its localizations and completions. Roughly speaking, the structure of a projective module is transparent locally, that is, over the completions of a ring. The game is to assemble this information to give the structure over the original ring. The full story requires K-theory for its telling.

Although the novelty of a book such as this lies mainly in its selection and presentation of material, we do make some innovations and give some new arguments, including the following.

- In our treatment of modules, we deal both with categories of left modules and with categories of right modules, and it seems expedient to have separate systems of notation for their homomorphisms, according to the rule that homomorphisms should act 'opposite scalars'. This separation leads us to two types of abstract category, namely a left category and a right category, in which the rules for composition of morphisms follow those of the corresponding module category. These two types do not seem to have been distinguished explicitly before. We introduce the term *chirality* for the 'handedness' of a category, and with each category we associate a formal *mirror* category of opposite chirality.
- In K-theory, the term 'exact category' is used for a category together with

some specified exact sequences, which is different to the usage of the term by people who are primarily concerned with category theory. To complicate the issue a bit more, there are two types of exact category in K-theory. The first type has sufficient structure for the purposes of 'lower' K-theory, while the second type has extra structure that is needed to support 'higher' K-theory. We therefore coin the terms 'G-exact category' and 'Q-exact category' for the two types of exact category used in K-theory.

- We give direct proofs of some results that are normally obtained by using the methods of homological algebra, which appears in this text only incidentally. These results mainly concern flat modules (particularly section 3.2), and the local-global properties of modules over orders (section 7.3).
- Recently, Quillen has developed a module theory for nonunital rings. We incorporate some of his results, mostly as exercises.
- The treatments of localization of categories (section 6.3), projective module lifting (section 7.2) and local-global methods (section 7.3) use the minimum of apparatus, so as to make them more accessible to a beginning graduate student.

This text requires the reader to have prior knowledge of some basic results in ring theory, module theory and number theory, since these provide the examples that motivate and illustrate our discussions. These prerequisites are generally met in the first year of graduate studies, although it is unlikely that any one graduate course includes all of them. We have therefore collected them together in a companion volume, [BK: IRM]. Much of the ring theory and module theory can also be found in the expository works [Cohn 1979], [Cohn 1982], [Jacobson 1985] and [Jacobson 1980], with some extra topics that can be met in [Cohn 1985], [McConnell & Robson 1987] or [Rowen 1988]. Sources for the number theory are [Fröhlich & Taylor 1991] and [Marcus 1977]. However, for brevity, we usually give references only to [BK: IRM].

We try to keep to a fairly leisurely pace throughout this text, and, when in doubt, we tend to include an explanation rather than exclude it. Some of the more frequently used prerequisites are recapitulated for convenience.

The symbol $\square$ indicates either the end of a proof, or that a proof is immediate from the preceding discussion. The symbol $\bigcirc$ after a statement indicates that the assertion is quoted from another source, rather than being a consequence of arguments in this text.

There are about two hundred exercises for more ambitious or enthusiastic readers. Some of them (among the exercises!) are fairly routine, others are rather deeper and explore further developments of our results or give brief introductions to topics that are not covered in the body of the text. We also

use the exercises to provide examples and counterexamples that indicate the limitations of our results.

A number of historical remarks are distributed within the text. Readers in search of further details may wish to consult [Bourbaki 1994], [Dieudonné 1989], [Mac Lane 1975] and [Weibel].

We again thank our departments and our families, for their patience and encouragement during the lengthy gestation period of this text and its companion [BK: IRM]. David Tranah, Roger Astley and the TeXline at the Cambridge University Press have provided us with helpful advice.

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1
CATEGORIES

Category theory provides the language for the discussions in this text and it is also an inescapable foundation for any treatment of K-theory. In this chapter, we set out the fundamental ideas of category theory, and we give a first application of the power of categorical methods. These ideas are introduced in three stages, each of which occupies one section. To start, we define categories themselves. Next, we consider functors, which are the tools for moving between different categories, and then natural transformations, which allow us to compare different functors. In the final, fourth, section of the chapter, we apply these basic notions to investigate universal constructions and universal objects. The idea of 'universality' reveals the common properties of apparently diverse objects, such as free modules and free groups, and it provides the framework for many of the definitions and constructions that we make later in this text.

1.1 FUNDAMENTAL PROPERTIES OF CATEGORIES

We commence our discussion of the theory of categories with the axiomatic definition of a category, together with a selection of techniques for the manufacture of new categories from old.

Because our main interest in this text is the application of category theory to module theory, many of our illustrations are obtained by considering various types of module. We shall also see some categories based on other algebraic entities such as sets and groups.

The principal innovation in our discussion stems from our need to use both right modules and left modules. Our view is that it is best to write operators opposite scalars as far as is practicable, so that a homomorphism between right modules is to be a left operator, while a homomorphism of left modules is to be a right operator. We also wish to compose homomorphisms in the

natural way, by 'associativity' – more formal definitions are given below in (1.1.3) and (1.1.4). The consequence is that we obtain two versions of the axiom for the composition of morphisms in an abstract category. Thus we arrive at two kinds of abstract category, a 'right' category and a 'left' category, which are modeled on the corresponding categories of modules. We call this phenomenon *chirality*. It is worth remarking that the distinction between left and right modules was made at a very early stage in the modern development of module theory [Noether & Schmeidler 1920].

The distinction between right and left categories does not seem to have been made explicit before now, and it could be avoided by using a technique given in (1.1.5). However, it seems to us to be more natural to allow both notations for categories, rather than suppressing one notation artificially.

We also discuss some points from set theory which arise from future applications in K-theory, where one needs to be able to work with 'small' categories, that is, categories whose objects can all be taken to be members of some set, which may itself be very big. In general, the objects of a category need not be contained in a set. We are particularly indebted to Wilfrid Hodges for his helpful comments on these questions.

1.1.1 The definition

Informally, a category consists of a collection of mathematical entities, such as the right R-modules over a given ring R, which can be recognised as sharing a common structure, together with a collection of mappings between these entities that respect this structure; for R-modules, we would expect these to be the R-module homomorphisms.

The entities which share the common structure are known as the 'objects' of the category, while the structure-preserving maps are the 'morphisms' of the category. Thus, to define a *category* $\mathcal{C}$ in general, we must specify the following data.

Cat 1. A class $\mathrm{Ob}\,\mathcal{C}$. Members of $\mathrm{Ob}\,\mathcal{C}$ are called the *objects* of $\mathcal{C}$.

Cat 2. For each ordered pair C, D of objects of $\mathcal{C}$, there is a set $\mathrm{Mor}_{\mathcal{C}}(C, D)$; the elements of $\mathrm{Mor}_{\mathcal{C}}(C, D)$ are called the *morphisms* from C to D in $\mathcal{C}$.

It may happen that $\mathrm{Mor}_{\mathcal{C}}(C, D)$ is the empty set.

Given $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, the object C is called the *domain* of α and D the *codomain*. An arrow

$$\alpha : C \longrightarrow D$$

is often used to indicate that α is a morphism from C to D.

These objects and morphisms obey some axioms, but before listing these we note two deeply entrenched conventions and give some examples.

The first convention is to write $C \in \mathcal{C}$ rather than $C \in \mathrm{Ob}\,\mathcal{C}$ when C is an object of $\mathcal{C}$. This reflects the fact that a category is usually considered to be defined in terms of the class $\mathrm{Ob}\,\mathcal{C}$. For this reason, a category is often named after its objects.

Second, the term 'morphism' is used mostly when we are considering categories in the abstract; in concrete situations, the morphisms are often the homomorphisms between some familiar objects, such as modules, groups or rings. In such a case, one writes $\mathrm{Hom}(C, D)$ rather than $\mathrm{Mor}_{\mathcal{C}}(C, D)$.

1.1.2 *Some examples*

$\mathcal{S}_{ET}$ The class $\mathrm{Ob}\,\mathcal{S}_{ET}$ of objects is the class of all sets and the set of morphisms from X to Y is the set of all mappings from X to Y. (Some authors denote this category $\mathcal{E}_{NS}$).

We write $\mathrm{Map}(X, Y)$ rather than $\mathrm{Mor}(X, Y)$.

$\mathcal{G}_{P}$ The objects are the groups, with group homomorphisms as morphisms.

$\mathcal{R}_{ING}$ The objects are the rings (which we require to have an identity element), the morphisms from R to S being the ring homomorphisms, which, we insist, send the identity of R to the identity of S.

$\mathcal{R}_{NG}$ The objects are now the nonunital rings, which are rings except that they need not possess an identity element. A morphism from R to S is a homomorphism of nonunital rings, which is the same as a ring homomorphism except that there can be no requirement that the identity of R is sent to the identity of S, even when R and S are actually rings.

$\mathcal{M}_{OD_R}$ Given a ring R, we form the category of all right R-modules. Thus $\mathrm{Ob}(\mathcal{M}_{OD_R})$ is the class of all right R-modules and the set of morphisms from M to N is the set of all R-module homomorphisms from M to N, which we write $\mathrm{Hom}(M, N)$, $\mathrm{Hom}_R(M, N)$ or $\mathrm{Hom}(M_R, N_R)$ according to context.

${}_R\mathcal{M}_{OD}$ Similarly, we form the category of left R-modules and homomorphisms of left R-modules.

$\mathcal{A}_B$ The category of abelian groups – the objects are abelian groups, the morphisms are group homomorphisms.

Since an abelian group has a uniquely defined structure as a right $\mathbb{Z}$-module ([BK: IRM] (1.2.2)), $\mathcal{A}_B$ is $\mathcal{M}_{OD_{\mathbb{Z}}}$ with another name. Despite

the fact that any right $\mathbb{Z}$-module can be regarded equally as a left $\mathbb{Z}$-module, with $am = ma$ for any integer a and element m of M, it is not quite true that $\mathcal{A}_B$ is the same as the category ${}_{\mathbb{Z}}\mathcal{M}_{OD}$ of left $\mathbb{Z}$-modules. The reason for the distinction will be discussed further in (1.1.5).

$\mathcal{T}_{OP}$ This important non-algebraic example of a category has topological spaces as objects and continuous maps as morphisms.

Λ It is sometimes convenient to regard a partially ordered set as a category. A *partially ordered* set is a set Λ together with an order relation $\leq$ on Λ which satisfies the following requirements.

PO 1. **Reflexivity:**
$\lambda \leq \lambda$ for each $\lambda \in \Lambda$.

PO 2. **Transitivity:**
if $\lambda \leq \mu$ and $\mu \leq \nu$ for λ, μ and ν in Λ, then $\lambda \leq \nu$ also.

A partially ordered set is said to be *proper* if the following axiom also holds.

PO 3. If $\lambda \leq \mu$ and $\mu \leq \lambda$ for λ and μ in Λ, then $\lambda = \mu$.

Here are two examples that will be generalized in (5.1).

(a) Let Λ be the set of nonzero ideals of $\mathbb{Z}$, ordered by reverse inclusion:

$$I \leq J \iff J \subseteq I.$$

Then Λ is a proper partially ordered set.

(b) Take Σ to be the set of nonzero elements of $\mathbb{Z}$, ordered by division:

$$a \leq b \iff ax = b \text{ for some } x \in \mathbb{Z}.$$

Then Σ is partially ordered, but not proper.

Given a partially ordered set Λ, we can view Λ as a category whose objects are the elements $\lambda, \mu, \ldots$ of Λ. If $\lambda \leq \mu$, then $\mathrm{Mor}(\lambda, \mu)$ contains a single element $\iota^{\lambda\mu}$, and $\mathrm{Mor}(\lambda, \mu)$ is empty otherwise.

$\mathcal{O}_{RD}$ The class of all ordered sets can be considered to be a category. We say that a set Λ is *totally ordered*, or simply *ordered*, if it is a proper partially ordered set in which any two members are comparable, that is, the following axiom holds.

TO. If λ and μ are in Λ, then either $\lambda \leq \mu$ or $\mu \leq \lambda$.

Then the objects of $\mathcal{O}_{RD}$ are the ordered sets Λ and a morphism $\alpha : \Lambda \to \Sigma$ of ordered sets is a mapping that preserves the order: if $\lambda \leq \mu$ in Λ, then $\alpha\lambda \leq \alpha\mu$ in Σ.

1.1.3 The axioms

Now that we have some examples in mind, we list the axioms that the morphisms of an abstract category $\mathcal{C}$ are required to satisfy. These correspond to the properties of the homomorphisms in the category $\mathcal{M}_{OD_R}$, provided that we write homomorphisms on the left and that we use the consequent natural convention for composing homomorphisms. Thus, if α is a right R-module homomorphism from M_R to N_R and m is in M, then αm denotes the image of m in N, and if β is a homomorphism from N_R to P_R, then

$$(\beta\alpha)m = \beta(\alpha m).$$

Cat 3. For each object C of $\mathcal{C}$, there is a distinguished morphism

$$id_C \in \mathrm{Mor}_{\mathcal{C}}(C, C),$$

called the *identity* morphism of C.

Cat 4. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$, then there is a morphism

$$\beta\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, E),$$

called the *product* or *composite* of α and β; α and β are said to be *composable*.

Cat 5. For any morphism $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,

$$(id_D)\alpha = \alpha = \alpha(id_C).$$

Cat 6. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$ and $\gamma \in \mathrm{Mor}_{\mathcal{C}}(E, F)$, then

$$\gamma(\beta\alpha) = (\gamma\beta)\alpha.$$

With regard to Cat 4, note that the product of two morphisms α, β is defined precisely when the codomain of α is the same as the domain of β. Observe also that the manner of writing the product (as $\beta\alpha$ rather than $\alpha\beta$) forms a part of this axiom.

Statements such as Cat 6 are sometimes formulated as 'the product is associative when defined'.

1.1.4 Chirality

It seems to us that a category satisfying the axioms above should, strictly speaking, be called a *right* category, since the form of the axioms mimics the natural form of composition of homomorphisms of *right* R-modules when these homomorphisms are written on the left, as is our convention.

When we consider left modules, we prefer to put homomorphisms on the

right. Thus, given a homomorphism of left R-modules $\alpha : {}_RM \to {}_RN$ and an element m in M, α sends m to $m\alpha$ rather than αm. If $\beta : {}_RN \to {}_RP$ is also a homomorphism, then the product $\alpha\beta$ is given by the rule

$$m(\alpha\beta) = (m\alpha)\beta.$$

With this convention, we obtain a modified list of axioms that defines a *left* category, as follows.

Cat 4^ℓ. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$, then there is a morphism

$$\alpha\beta \in \mathrm{Mor}_{\mathcal{C}}(C, E).$$

Cat 5^ℓ. For any morphism $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,

$$(id_C)\alpha = \alpha = \alpha(id_D).$$

Cat 6^ℓ. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$ and $\gamma \in \mathrm{Mor}_{\mathcal{C}}(E, F)$, then

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

The prime example of a left category is ${}_R\mathcal{M}\mathrm{od}$, the category of all left R-modules. Also, most group theorists in effect treat $\mathcal{G}_{\mathcal{P}}$ as a left category.

As we noted in (1.1.2), a partially ordered set Λ can be viewed as a category. It is convenient to regard Λ as a left category, since the left-category composition reads

$$\iota^{\lambda\mu}\iota^{\mu\nu} = \iota^{\lambda\nu},$$

where $\iota^{\lambda\mu}$ is the unique morphism from λ to μ (when $\lambda \leq \mu$).

We introduce the term *chirality* to distinguish the two kinds of category, a left or right category having correspondingly left or right chirality.

1.1.5 The mirror

There is a purely formal method that allows us to switch between categories of opposite chiralities. Given a left category $\mathcal{C}$, we manufacture a right category $\mathcal{C}^\odot$, the *mirror* of $\mathcal{C}$, as follows. The objects of $\mathcal{C}^\odot$ are symbols $C^\odot$, corresponding bijectively to the objects C of $\mathcal{C}$, and the morphisms from $C^\odot$ to $D^\odot$ in $\mathcal{C}^\odot$ are symbols $\alpha^\odot$ in bijective correspondence with the morphisms α from C to D in $\mathcal{C}$. Thus for each morphism

$$\alpha : C \longrightarrow D \text{ in } \mathcal{C},$$

there is exactly one morphism

$$\alpha^\odot : C^\odot \longrightarrow D^\odot \text{ in } \mathcal{C}^\odot.$$

The product in $\mathcal{C}^{\odot}$ is defined by putting

$$\beta^{\odot}\alpha^{\odot} = (\alpha\beta)^{\odot},$$

where $\beta : D \to E$ is a morphism in $\mathcal{C}$ that is composable with α.

Observe that $\mathcal{C}$ and $\mathcal{C}^{\odot}$ are genuinely distinct categories, rather than the same category with two notational conventions, since the order of a pair of morphisms in their product is specified by the axioms.

It is clear that given a right category $\mathcal{D}$, we can construct a left category $\mathcal{D}^{\odot}$ by a similar method, and that there is a formal identification of $\mathcal{C}^{\odot\odot}$ with $\mathcal{C}$ for any category $\mathcal{C}$.

In [BK: IRM] (1.2.6), we introduced a method of turning a left R-module M into a right R°-module M°, where R° is the opposite ring to R. A typical element of R° has the form r° for r in R, and addition and multiplication in the opposite ring are given by

$$r^{\circ} + s^{\circ} = (r+s)^{\circ} \quad \text{and} \quad r^{\circ}s^{\circ} = (sr)^{\circ}.$$

The elements m° of M° are in bijective correspondence with those of M, and the addition and scalar multiplication for M° are defined by

$$m^{\circ} + n^{\circ} = (m+n)^{\circ} \quad \text{and} \quad m^{\circ}r^{\circ} = (rm)^{\circ}.$$

If $\alpha : M \to N$ is a homomorphism of left R-modules, we define a homomorphism α° of right R°-modules by

$$\alpha^{\circ}m^{\circ} = (m\alpha)^{\circ}.$$

Thus we have an identification

$$({}_R\mathcal{M}\mathrm{od})^{\odot} = \mathcal{M}\mathrm{od}_{(R^{\circ})},$$

which shows that the use of the mirror category is a generalization of the use of the opposite ring. This identity was used implicitly in Exercise 1.2.14 of [BK: IRM], where we explored the consequences of this technique for switching between left R-modules and right R°-modules. Some authors use this method to avoid dealing both with left and with right modules.

Suppose now that the ring R is commutative. Then $R = R^{\circ}$, so that $({}_R\mathcal{M}\mathrm{od})^{\odot} = \mathcal{M}\mathrm{od}_R$. Again, this identity is used implicitly to write scalars and operators on the same side in many elementary textbooks that are concerned only with commutative rings.

We can now elucidate the comment in (1.1.2) about the category $\mathcal{A}\mathrm{b}$ of abelian groups. For, we regard $\mathcal{A}\mathrm{b}$ as the right category $\mathcal{M}\mathrm{od}_{\mathbb{Z}}$, which is the mirror of the category ${}_{\mathbb{Z}}\mathcal{M}\mathrm{od}$ of left $\mathbb{Z}$-modules.

We remind the reader that if a category has no obvious chirality, for example, $\mathcal{G}_{\mathcal{P}}$ or $\mathcal{R}_{ING}$, we always take it to be a right category. However, many group theorists prefer to view $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{A}_{B}$ as left categories.

Remark. Some authorities in category theory do not regard a category and its mirror as distinct categories, but simply as alternative notations for composition of morphisms in the one category. From this point of view, a left R-module and the corresponding right R°-module are the same object, in differing notations. However, a ring and its opposite are definitely different rings, as can be seen from the use of opposites in Brauer theory ([Cohn 1979], §10.3). We therefore feel that the distinction between a ring and its opposite should be extended to that between a category and its mirror.

1.1.6 The opposite category

We now give the definition of the *opposite* category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$. This is distinct from the notion of the mirror category that we introduced above.

Given a right category $\mathcal{C}$, the objects C^{op} of the right category $\mathcal{C}^{\mathrm{op}}$ are in bijective correspondence with the objects C of $\mathcal{C}$, and for each pair of objects C^{op}, D^{op} of $\mathcal{C}^{\mathrm{op}}$, there is a bijection between the morphisms α in $\mathrm{Mor}_{\mathcal{C}}(C, D)$ and the morphisms α^{op} in $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(D^{\mathrm{op}}, C^{\mathrm{op}})$; thus $\alpha : C \to D$ corresponds to $\alpha^{\mathrm{op}} : D^{\mathrm{op}} \to C^{\mathrm{op}}$. If $\beta : D \to E$ is a morphism in $\mathcal{C}$, then composition in $\mathcal{C}^{\mathrm{op}}$ is given by the rule

$$(\alpha^{\mathrm{op}})(\beta^{\mathrm{op}}) = (\beta\alpha)^{\mathrm{op}}.$$

(The definition for left categories is left to the reader, who, we are confident, will get it right ...)

Note that $\mathcal{C}^{\mathrm{op}}$ has the same chirality as $\mathcal{C}$.

1.1.7 The principle of duality

If we can make a definition or state a result by using only the objects and morphisms of a category $\mathcal{C}$, then we obtain a *dual* definition or result in the opposite category $\mathcal{C}^{\mathrm{op}}$ in which objects $C, D, \ldots$ of $\mathcal{C}$ are replaced by the corresponding objects $C^{\mathrm{op}}, D^{\mathrm{op}}, \ldots$ of $\mathcal{C}^{\mathrm{op}}$ and similarly morphisms $\alpha, \beta, \ldots$ are replaced by $\alpha^{\mathrm{op}}, \beta^{\mathrm{op}}, \ldots$. An example is provided by the relationship between the definitions of projective and injective modules.

A projective module may be defined as a right R-module P for which the following holds.

Pro. Given any surjective R-module homomorphism $\pi : M \to P$, there is a splitting homomorphism $\sigma : P \to M$, that is, $\pi\sigma = id_P$.

On the other hand, a right R-module I is injective if instead we have the following.

Inj. Given any injective R-module homomorphism $\mu : I \to M$, there is a splitting homomorphism $\rho : M \to I$, that is, $\rho\mu = id_I$.

(More details are given in §2.5 of [BK: IRM].)

It is reasonable to assume that if we can generalize surjective and injective homomorphisms of modules to abstract categories, then a surjective map in $\mathcal{C}$ will correspond to an injective map in the opposite category $\mathcal{C}^{\mathrm{op}}$, and so a projective object in $\mathcal{C}$ will correspond to an injective object in the opposite category. (This assumption is justified in the next chapter – see (2.2.2).) Thus, provided that we confine ourselves to statements that involve only objects and morphisms, we need only prove results about projective objects, since the corresponding results for injective objects are then true by duality.

This technique for pairing together definitions, arguments, etc. in a category and its opposite is the *principle of duality*, and the phrase *by duality* is used to indicate this method of argument.

1.1.8 Subcategories

A *subcategory* $\mathcal{D}$ of a (right) category $\mathcal{C}$ is defined by the following data.

Sub 1. A subclass $\mathrm{Ob}\,\mathcal{D}$ of $\mathrm{Ob}\,\mathcal{C}$, which specifies the objects that belong to $\mathcal{D}$.

Sub 2. For each pair of objects C, D of $\mathcal{D}$, a subset $\mathrm{Mor}_{\mathcal{D}}(C, D)$ of $\mathrm{Mor}_{\mathcal{C}}(C, D)$; these are the morphisms from C to D in $\mathcal{D}$.

Sub 3. If C is an object of $\mathcal{D}$, then the identity morphism id_C in $\mathrm{Mor}_{\mathcal{C}}(C, C)$ shall belong to $\mathrm{Mor}_{\mathcal{D}}(C, C)$ (where it is again the identity morphism).

Sub 4. If $\alpha \in \mathrm{Mor}_{\mathcal{D}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{D}}(D, E)$, then

$$\beta\alpha \in \mathrm{Mor}_{\mathcal{D}}(C, E),$$

(where the product is the product in $\mathcal{C}$).

It is clear that a subcategory of a right category is itself a right category with identities and products inherited from $\mathcal{C}$.

The corresponding definitions for a left category are obvious.

Examples of subcategories are $\mathcal{A}_B$ in $\mathcal{G}_P$ and $\mathcal{R}_{ING}$ in $\mathcal{R}_{NG}$. Note that any group homomorphism from G to H is also a morphism in $\mathcal{A}_B$ if the groups G and H happen to be abelian. In contrast, a morphism in $\mathcal{R}_{NG}$ from a ring R to a ring S need not be allowable as a morphism in $\mathcal{R}_{ING}$. This can be seen

by computing $\mathrm{Mor}(0, S)$ in the two cases: if S is a ring other than the zero ring, then $\mathrm{Mor}(0, S) = \{0\}$ in the first case but $\mathrm{Mor}(0, S) = \emptyset$ in the second.

1.1.9 Full subcategories

A *full* subcategory $\mathcal{D}$ of $\mathcal{C}$ is one in which

$$\mathrm{Mor}_{\mathcal{D}}(C, D) = \mathrm{Mor}_{\mathcal{C}}(C, D)$$

for any two objects of $\mathcal{D}$.

It is obvious that any subclass $\mathcal{D}$ of the class of objects of $\mathcal{C}$ gives rise to a unique full subcategory of $\mathcal{C}$.

This method of constructing subcategories of $\mathcal{M}_{OD_R}$, for various rings R, will be very useful to us in the sequel. For this reason, we define a *module category* to be any category which is a full subcategory of $\mathcal{M}_{OD_R}$ (or ${}_R\mathcal{M}_{OD}$) for some ring R.

Here are some important examples.

$\mathcal{F}_{REE_R}$, the category of all free right R-modules.
$\mathcal{P}_{ROJ_R}$, the category of all projective right R-modules.
$\mathcal{M}_R$, the category of finitely generated right R-modules.
$\mathcal{F}_R$, the category of free right modules R^k of finite rank.
$\mathcal{P}_R$, the category of finitely generated projective right R-modules.

The corresponding subcategories of ${}_R\mathcal{M}_{OD}$ are denoted ${}_R\mathcal{F}_{REE}$, ${}_R\mathcal{P}_{ROJ}$, ${}_R\mathcal{M}$, ${}_R\mathcal{F}$ and ${}_R\mathcal{P}$ respectively.

1.1.10 Some remarks on set theory and small categories

In our definition of a category, we use the expression *'class'*, rather than the word *'set'* which the reader might have expected. The reason for this is that we wish to make a naïve distinction between sets, on which all mathematical constructions are allowed, and 'non-sets', which are too large to permit some operations. The aim is to avoid Russell's Paradox and similar traps: if X is a set of sets, it is sensible to ask whether or not the set X is itself in X. The paradox arises by taking X to be the set of all sets that do not contain themselves. Our avoiding action is to declare that this particular choice of X is a class but not a set; essentially, X is too large to be considered to be a set. The kind of set theory that we have in mind is expanded in detail in [Herrlich & Strecker 1979], [van Dalen, Doets & de Swart 1979], and [Levy 1979].

If the class $\mathrm{Ob}\,\mathcal{C}$ of objects of the category $\mathcal{C}$ is in fact a set, then the category is said to be *small*.

In K-theory there are technical reasons for working with small categories, or at least, categories which are equivalent to small categories in a sense to be made precise in (1.3.15). The purpose of the following remarks is to show that the categories of greatest concern to us can indeed be taken to be small.

To illustrate this point, consider $\mathcal{M}_R$, the category of finitely generated R-modules. From one point of view, the objects in $\mathcal{M}_R$ must form a class, rather than a set, for the trivial reason that any set can be regarded artificially as the single element of a zero module. However, we can impose a very natural restriction on the objects which are permitted in $\mathcal{M}_R$ to ensure that it is small, as follows.

An object M in $\mathcal{M}_R$ is the homomorphic image of some free module R^n of finite rank ([BK: IRM] Lemma 2.5.7), and so $M \cong R^n/\operatorname{Ker}\alpha$ for a suitable surjective homomorphism α, by the Induced Mapping Theorem ([BK: IRM] (1.2.11)). But, by construction ([BK: IRM] (1.2.10)), the quotient module $R^n/\operatorname{Ker}\alpha$ is a set of subsets of R^n. We can therefore regard each object of $\mathcal{M}_R$ as being a set of subsets of the countably infinite cartesian product $\prod_\omega R$. Since a fundamental axiom of set theory assures us that the class of subsets of a set is again a set, we see that $\mathcal{M}_R$ is small.

An alternative way of interpreting this argument is that it shows that $\mathcal{M}_R$ has a set of isomorphism classes of objects, and therefore that $\mathcal{M}_R$ is equivalent to a small category. In applications, it makes no difference whether we take $\mathcal{M}_R$ in the 'large', or replace it by some convenient, equivalent, small category, which we give the same name.

The category $\mathcal{M}_R$ is typical of the type of category that we deal with. The objects (and morphisms) of these categories are constructed from a given set of objects by operations which formally involve taking sets of subsets of cartesian products of the form $\prod_{\lambda\in\Lambda} X_\lambda$, where each X_λ is a set and Λ is some index set. Such a method of construction must generate a set of objects.

More detailed accounts of the bearing of set theory on category theory can be found in [Cameron 1999], [Herrlich & Strecker 1979], [Mac Lane 1971], §1.6, and [Schubert 1972], §3.

We now show how to construct some new categories from given categories.

1.1.11 Product categories

Given a pair of (right) categories $\mathcal{C}$ and $\mathcal{D}$, we form the *product* category $\mathcal{C}\times\mathcal{D}$ as follows. The objects of $\mathcal{C}\times\mathcal{D}$ are pairs (C, D), where C is an object of $\mathcal{C}$ and D is an object of $\mathcal{D}$, and a morphism

$$(\gamma,\delta):(C,D)\longrightarrow(C'',D'')$$

in $\mathcal{C} \times \mathcal{D}$ is a pair of morphisms

$$\gamma : C \longrightarrow C'' \text{ and } \delta : D \longrightarrow D'',$$

belonging to $\mathcal{C}$ and $\mathcal{D}$ respectively.

Composition is given by the rule

$$(\gamma, \delta)(\gamma', \delta') = (\gamma\gamma', \delta\delta'),$$

where $(\gamma', \delta') : (C', D') \to (C, D)$.

It is easily verified that $\mathcal{C} \times \mathcal{D}$ is a right category, the identity map on a pair (C, D) being the morphism (id_C, id_D). It is clear that the product category is small if both $\mathcal{C}$ and $\mathcal{D}$ are small.

If instead $\mathcal{C}$ and $\mathcal{D}$ are both left categories, we make the obvious change in the definition of composition so that $\mathcal{C} \times \mathcal{D}$ is also a left category. Should we need to consider products of categories of mixed chiralities, we fix the notation (by using mirrors) so that the product is always a right category. The reader is invited to write out the various forms for composition. An illustration is encountered in (1.3.6).

We next show that if $\mathcal{C}$ and $\mathcal{D}$ are both module categories, so is their product. Given rings R and S, their *direct product* is

$$R \times S = \{(r, s) \mid r \in R, s \in S\},$$

with addition and multiplication defined componentwise:

$$(r, s) + (r', s') = (r + r', s + s'), \quad (r, s) \cdot (r, s) = (rr', ss').$$

We state the result only for right categories; the reader may supply the variations needed for left categories and for mixed chiralities. (Hint: 'it's all done by mirrors'.)

1.1.12 Lemma

Let $\mathcal{C}$ be a subcategory of $\mathcal{M}\text{od}_R$ and let $\mathcal{D}$ be a subcategory of $\mathcal{M}\text{od}_S$. Then $\mathcal{C} \times \mathcal{D}$ is a subcategory of $\mathcal{M}\text{od}_{R \times S}$.

Proof

An object (C, D) in the product category can be made into an $R \times S$-module by the rule $(c, d) \cdot (r, s) = (cr, ds)$, and a morphism in $\mathcal{C} \times \mathcal{D}$ will then be an $R \times S$-module homomorphism. □

Remark. It is not difficult to show that an $R \times S$-module M must be (isomorphic to) a direct sum $L \oplus N$, in which L is an R-module and N is an S-module. Furthermore, a homomorphism of $R \times S$-modules $L \oplus N \to L' \oplus N'$ must arise from a pair of homomorphisms $L \to L'$ and $N \to N'$. (Details are spelt out

in [BK: IRM] (2.6.6)ff.) Thus the categories $\mathcal{M}_{OD_{R\times S}}$ and $\mathcal{M}_{OD_R} \times \mathcal{M}_{OD_S}$ are in effect the same, but we cannot substantiate this claim formally until we have the language of equivalences at our disposal – see (1.3.16) and (1.3.17).

1.1.13 Infinite products

The generalization of the definition of a direct product to an arbitrary set of categories $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$, where Λ is any ordered index set, is evident. An object of the product category $\prod_\Lambda \mathcal{C}_\lambda$ is a sequence (C_λ), where each C_λ is an object of the corresponding category $\mathcal{C}_\lambda$, and a morphism $(\gamma_\lambda) : (C_\lambda) \to (C'_\lambda)$ is given by a sequence of morphisms $\gamma_\lambda : C_\lambda \to C'_\lambda$, $\lambda \in \Lambda$.

If each $\mathcal{C}_\lambda$ is a right or left category, then the product of morphisms in $\prod_\Lambda \mathcal{C}_\lambda$ is defined so that $\prod_\Lambda \mathcal{C}_\lambda$ is also right or left accordingly; again, if the chiralities are mixed, we make the product a right category.

In the case that the index set is finite, we usually take it to be $\{1, \ldots, n\}$ for some natural number n, and write the product as $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$. When the component categories $\mathcal{C}_\lambda$ are all the same category, $\mathcal{C}$ say, we use the natural notations $\mathcal{C}^n$ and $\mathcal{C}^\Lambda$.

Products of more than two categories arise naturally when we consider direct products of rings. If $R = R_1 \times \cdots \times R_n$ is a finite direct product of rings and C_i is a category of R_i-modules for each i, an obvious generalization of (1.1.12) shows that $C_1 \times \cdots \times C_n$ is a category of right R-modules.

1.1.14 Morphism categories

We can form new categories by regarding some of the morphisms of a given category as themselves objects of a category. Such categories are called *morphism categories*. We give the constructions only for right categories, the left-handed versions being obvious.

The first of this type is the category $\mathcal{M}_{OR}\mathcal{C}$, the *morphism category* of the category $\mathcal{C}$.

The objects of $\mathcal{M}_{OR}\mathcal{C}$ are triples (C, D, χ) where χ is a morphism from C to D in $\mathcal{C}$, and a morphism from (C, D, χ) to (C'', D'', χ'') is a morphism $(\gamma, \delta) : (C, D) \to (C'', D'')$ in the product category $\mathcal{C}\times\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & C'' \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi''} \\ D & \xrightarrow{\delta} & D'' \end{array}$$

commutes; that is, $\chi''\gamma = \delta\chi$.

Clearly, $\mathcal{M}or_R\, \mathcal{C}$ is a right category, which can be regarded as a subcategory of $\mathcal{C} \times \mathcal{C}$ by ignoring the morphism χ. If $\mathcal{C}$ is a module category, say a subcategory of $\mathcal{M}od_R$, then $\mathcal{M}or_R\, \mathcal{C}$ is also a module category, being a subcategory of $\mathcal{M}od_{R\times R}$.

It is also clear that $\mathcal{M}or_R\, \mathcal{C}$ is small if $\mathcal{C}$ is small.

Further morphism categories arise as full subcategories of $\mathcal{M}or_R\, \mathcal{C}$ which are defined by placing restrictions on the morphism appearing in (C, D, χ). Here are a few examples.

$\mathcal{E}nd\, \mathcal{C}$, the *endomorphism* category. The objects are morphisms (C, C, χ), that is, $\chi : C \to C$ must be an *endomorphism* of C.

$\mathcal{I}so\, \mathcal{C}$, the *isomorphism* category, which has objects (C, D, χ) where χ is an *isomorphism* – in an arbitrary category, this means that there is a morphism $\chi^{-1} : D \to C$ with $\chi^{-1}\chi = id_C$ and $\chi\chi^{-1} = id_D$, or, in other words, χ is *invertible*. We may then write $C \cong D$ to indicate that C and D are isomorphic.

$\mathcal{A}ut\, \mathcal{C}$, the *automorphism* category. Here, objects have the form (C, C, χ) with χ invertible.

Exercises

1.1.1 **Discrete categories**

A category $\mathcal{C}$ is *discrete* if

$$\mathrm{Mor}(C, D) = \begin{cases} \{id_C\} & \text{if } C = D, \\ \emptyset & \text{if } C \neq D. \end{cases}$$

Show that

(a) any set X can be regarded as a discrete category,

(b) if Y is any subclass of the objects of a given category $\mathcal{D}$, then Y defines a discrete subcategory of $\mathcal{D}$.

1.1.2 Let λ and μ be elements of a partially ordered set Λ. Show that the following are equivalent.

(a) $\lambda \cong \mu$ when we regard Λ as a category.

(b) Both $\lambda \leq \mu$ and $\mu \leq \lambda$.

1.1.3 **A group as a category**

Given a group G, we can define two categories, called $\mathcal{B}G$ and $\mathcal{E}G$, as follows.

$\mathcal{B}G$: this has one object, G itself, and $\mathrm{Mor}(G,G) = \{g \mid g \in G\}$. Composition is multiplication.

$\mathcal{E}G$: here, $\mathrm{Ob}\,\mathcal{E}G = \{g \mid g \in G\}$ is the set of elements of G, and $\mathrm{Mor}(g,h) = \{hg^{-1}\}$ has one element.

What are the subcategories of $\mathcal{B}G$ and of $\mathcal{E}G$?

Investigate the various morphism categories associated with $\mathcal{B}G$ and $\mathcal{E}G$.

For example, observe that the class $\mathrm{Mor}\,\mathcal{C}$ of all morphisms of a category $\mathcal{C}$ forms a group precisely when $\mathcal{C}$ may be identified with the category $\mathcal{B}(\mathrm{Mor}\ \mathcal{C})$.

1.1.4 **A monoid as a category**

A *monoid* (or *semigroup with identity*) is a set M with an associative multiplication for which there is an identity element 1 (see [BK: IRM] (1.1.1)). Repeat, as far as possible, the previous exercise, substituting a monoid M in place of a group. Since an element of M need not be invertible, we must take the morphisms in $\mathcal{E}M$ to be

$$\mathrm{Mor}(m,n) = \{x \in M \mid xm = n\},$$

so that $id_m = 1$ always.

Show that $m \cong 1$ in $\mathcal{E}M$ if and only if m has a (twosided) inverse in M.

1.1.5 **Opposites of groups**

Groups G and H are said to be *anti-isomorphic* if there is a bijective map $\phi : G \to H$ with $\phi(gh) = \phi(h)\phi(g)$ for all g, h in G.

Show that $\iota : g \to g^{-1}$ is an anti-isomorphism of G, and that ι is an isomorphism of G if and only if G is abelian.

Define the *opposite* of a group to be a formal set

$$G^{\mathrm{op}} = \{g^{\mathrm{op}} \mid g \in G\}$$

in bijective correspondence with G, with multiplication given by

$$g^{\mathrm{op}}h^{\mathrm{op}} = (hg)^{\mathrm{op}}.$$

Verify that G^{op} is a group and that

$$\tau : G \longrightarrow G^{\mathrm{op}}, \quad g \mapsto g^{\mathrm{op}},$$

is an anti-isomorphism.

Show also that any anti-isomorphism $\phi : G \to H$ factors as a product $\phi = \overline{\phi}\tau$ where $\overline{\phi}$ is a group isomorphism.

Deduce that $\mathrm{Mor}((\mathcal{B}G)^{\odot})$ and $\mathrm{Mor}((\mathcal{B}G)^{\mathrm{op}})$ are both isomorphic to G^{op}.

Remark. The existence of an anti-isomorphism between a group and its opposite is sometimes used implicitly to identify the two groups. An example is given by the symmetric group S_n, the group of all permutations of the set $\{1, \ldots, n\}$. If we consider the permutations to act as right operators, we obtain one version of the symmetric group, while if we view the permutations as left operators, we obtain the opposite group. Usually, a group theorist will take the right-handed version to be S_n, so that the left-handed version is then properly S_n^{op}.

A practical situation which requires both the left and the right handed versions of S_n arises in Chapter 12 of [Loday 1992].

1.1.6 **Morphisms on morphisms**

Describe the category $\mathcal{M}or^2\mathcal{C} = \mathcal{M}or\,(\mathcal{M}or\,\mathcal{C})$. Compare the categories $\mathcal{A}ut\,(\mathcal{M}or\,\mathcal{C})$ and $\mathcal{M}or\,(\mathcal{A}ut\mathcal{C})$, etc.

1.1.7 Compare $(\mathcal{C}^{\mathrm{op}})^{\odot}$ and $(\mathcal{C}^{\odot})^{\mathrm{op}}$.

1.1.8 Show that if each category $\mathcal{C}_\lambda$ is a module category, then the product $\prod_\Lambda \mathcal{C}_\lambda$ is also a module category.

1.1.9 **Groupoids**

A *groupoid* is a small category G in which every morphism is invertible.

Let $\{v_\lambda \mid \lambda \in \Lambda\}$ be the set of objects of G, where Λ is some set. Show that $G_\lambda = \mathrm{End}(v_\lambda, v_\lambda)$ is a group for each λ; G_λ is called the *vertex group* at λ.

Show also that if $\mathrm{Mor}(v_\lambda, v_\mu) \neq \emptyset$, then the groups G_λ and G_μ are isomorphic.

(Any set $\{G_\lambda\}$ of groups can be viewed as defining a groupoid in which $\mathrm{Mor}(v_\lambda, v_\mu) = \emptyset$ if $\lambda \neq \mu$.)

1.2 FUNCTORS

A major impact of category theory on mathematics is its emphasis on describing the morphisms between mathematical objects, rather than considering the objects in isolation. This philosophy can be applied to the study of categories themselves, leading to the definition of a functor, that is, a morphism between categories.

The language of functors plays an indispensable role in the remainder of this text, and in K-theory in general.

Up to this point, our distinction between right and left categories has not had much influence on the text, since we have been able to develop the theory

for right categories and dismiss the left-handed version as obvious. However, the fact that categories may have different chiralities does now have an effect on our notation. The reason is that we have to take into account the possibility that a functor may well relate categories of different chiralities. The outcome is that some entrenched conventions must be treated with caution, since a covariant functor may act in a way that is superficially similar to a contravariant functor.

Nevertheless, we continue to work with right categories as far as possible, leaving statements and proofs in the left-handed case to the reader, unless there are particular points which we need to emphasise.

1.2.1 Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ associates with each object C of $\mathcal{C}$ an object FC in $\mathcal{D}$, and with each morphism

$$\alpha : C' \longrightarrow C \text{ in } \mathcal{C}$$

it associates a morphism

$$F\alpha : FC' \longrightarrow FC \text{ in } \mathcal{D}.$$

Thus F is a collection of mappings of two types (all of which are, by custom, denoted by the same letter F): there is one map of the first type

$$F : \mathrm{Ob}\,\mathcal{C} \longrightarrow \mathrm{Ob}\,\mathcal{D},$$

and for each pair C' and C of objects of $\mathcal{C}$ there is a map of the second type

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC).$$

These mappings are subject to the axioms

Fun 1 $F(id_C) = id_{FC}$

and

Fun 2 if $\mathcal{C}$ and $\mathcal{D}$ are both right categories and

$$\alpha : C' \longrightarrow C \text{ and } \beta : C \longrightarrow C''$$

are morphisms in $\mathcal{C}$, then

$$F(\beta\alpha) = F(\beta)F(\alpha) : FC' \longrightarrow FC''.$$

A *contravariant functor* reverses the direction of morphisms. Thus for each morphism

$$\alpha : C' \longrightarrow C \text{ in } \mathcal{C}$$

there is a morphism

$$F\alpha : FC \longrightarrow FC' \text{ in } \mathcal{D}.$$

The second axiom now reads

Fun 2^o If $\mathcal{C}$ and $\mathcal{D}$ are both right categories and

$$\alpha : C' \longrightarrow C \text{ and } \beta : C \longrightarrow C''$$

are morphisms in $\mathcal{C}$, then

$$F(\beta\alpha) = F(\alpha)F(\beta) : FC'' \longrightarrow FC'.$$

The unqualified term *functor* will mean a covariant functor, unless the context demands otherwise.

1.2.2 Some examples

Here are some basic examples.

(i) The *identity* functor $\mathrm{Id}_{\mathcal{C}}$ from a category $\mathcal{C}$ to itself; this 'does nothing' to any object or morphism.

(ii) If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, there is the *inclusion* functor

$$\mathrm{Inc} : \mathcal{C}' \longrightarrow \mathcal{C};$$

again this leaves objects and morphisms unchanged.

(iii) On any category, there is a standard contravariant functor, the *opposite functor*

$$\mathrm{Op} : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}},$$

which has $\mathrm{Op}(C) = C^{\mathrm{op}}$ and $\mathrm{Op}(\alpha) = \alpha^{\mathrm{op}}$ – see (1.1.6).

(iv) The *forgetful* or *underlying* functor

$$\Upsilon : \mathcal{C} \longrightarrow \mathcal{D};$$

this arises when each object and morphism of the category $\mathcal{C}$ can be regarded as belonging to another category $\mathcal{D}$ by forgetting some of its structure. Thus there are forgetful functors from $\mathcal{M}_{OD_R}$ to $\mathcal{A}_B$, from $\mathcal{A}_B$ to $\mathcal{G}_P$ and from $\mathcal{G}_P$ to $\mathcal{S}_{ET}$.

1.2.3 Free modules

Another important functor is

$$\mathrm{Fr}_R : \mathcal{S}ET \longrightarrow \mathcal{M}od_R,$$

which associates to a set X the free right R-module $\mathrm{Fr}_R(X)$ on X, where R is a fixed ring. Since this functor will be encountered frequently in the sequel, we give a sketch of its construction. A fuller account is given in [BK: IRM] (2.1.16).

An element m of $\mathrm{Fr}_R(X)$ is a formal sum

$$m = \sum_{x \in X} x r_x(m) \text{ with } r_x(m) \in R,$$

in which only a finite set of coefficients $r_x(m)$ may be nonzero. (In other words, m is a function from X to R whose value at x we prefer to write as $r_x(m)$ rather than $m(x)$, and which takes the value 0 except possibly at a finite number of members of X.) Two such sums m, n are the same precisely when $r_x(m) = r_x(n)$ for all x in X, and addition and scalar multiplication are defined by the rules

$$m + n = \sum_{x \in X} x(r_x(m) + r_x(n)) \quad \text{and} \quad mr = \sum_{x \in X} x(r_x(m)r).$$

It is easy to verify that $\mathrm{Fr}_R(X)$ is a right R-module. We view X as a subset of $\mathrm{Fr}_R(X)$ by identifying an element x of X with the module element that has $r_x = 1$ and $r_y = 0$ for $y \neq x$.

Now, given a mapping $\alpha : X \to Y$ of sets, there is one and only one way to extend α to a homomorphism

$$\mathrm{Fr}_R(\alpha) : \mathrm{Fr}_R(X) \longrightarrow \mathrm{Fr}_R(Y),$$

since we must have

$$\mathrm{Fr}_R(\alpha)\left(\sum_{x \in X} x r_x(m)\right) = \sum_{x \in X} \alpha(x) r_x(m).$$

It is straightforward to check that Fr_R is a functor as claimed.

1.2.4 The product

Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors. Then the *composite* or *product* functor

$$GF : \mathcal{C} \longrightarrow \mathcal{E}$$

is given by

$$GF(C) = G(F(C))$$

and

$$GF(\alpha) = G(F(\alpha)).$$

The product GF is sometimes written $G \circ F$ if we wish to emphasise that we are performing a composition of functors.

It is easy to verify that GF will be covariant if F and G are either both covariant or both contravariant, and contravariant otherwise.

Notice that the opposite functor $\mathrm{Op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ provides a method of interchanging contravariant and covariant functors, since a contravariant functor $G : \mathcal{D} \to \mathcal{C}$ defines a covariant functor $\mathrm{Op} \circ G : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$, and vice versa.

1.2.5 Restriction

Given a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$, the *restriction* of F to $\mathcal{C}'$ is the composite

$$F \circ \mathrm{Inc} : \mathcal{C}' \longrightarrow \mathcal{D}.$$

If $\mathcal{D}'$ is a subcategory of $\mathcal{D}$ such that FC and $F\alpha$ are in $\mathcal{D}'$ for every object and morphism of $\mathcal{C}'$, we can also regard F as a functor from $\mathcal{C}'$ to $\mathcal{D}'$, which we again call the restriction of F.

In applications, $\mathcal{D}'$ will often be a full subcategory of $\mathcal{D}$; in this case, F restricts to a functor from $\mathcal{C}'$ to $\mathcal{D}'$ if and only if $F(\mathrm{Ob}\,\mathcal{C}')$ is contained in $\mathrm{Ob}\,\mathcal{D}'$.

To save notation, a restriction of a functor is usually denoted by the same symbol as the original functor.

1.2.6 Appearances and chirality

So far, we have discussed functors between right categories only. When we have to consider left categories as well, we find that the appearance of axiom Fun 2 depends on the chiralities of the categories involved, as well the variance of the functor itself.

To help the reader through this tangle, we list the various possibilities; here, F is a functor from $\mathcal{C}$ to $\mathcal{D}$, and, as hitherto, $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C', C)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(C, C'')$.

Chirality of $\mathcal{C}$	Chirality of $\mathcal{D}$	F covariant	F contravariant
left	left	$F(\alpha\beta) = F(\alpha)F(\beta)$	$F(\alpha\beta) = F(\beta)F(\alpha)$
left	right	$F(\alpha\beta) = F(\beta)F(\alpha)$	$F(\alpha\beta) = F(\alpha)F(\beta)$
right	left	$F(\beta\alpha) = F(\alpha)F(\beta)$	$F(\beta\alpha) = F(\beta)F(\alpha)$
right	right	$F(\beta\alpha) = F(\beta)F(\alpha)$	$F(\beta\alpha) = F(\alpha)F(\beta)$

We have listed these variations in detail since they are ignored in all texts on category theory that we have come across. Presumably, this is because it is the usual practice in category theory to deal only with what we have called here right categories.

When concentrating on right categories only, it is tempting to characterize covariant and contravariant in terms of the appearance of axiom Fun 2: *in this case*, covariant functors are those which 'preserve' products, while contravariant functors 'reverse' products. However, as we have seen, this characterization does not adequately describe the intrinsic distinction between covariant and contravariant functors. It is an artefact of the notation and so needs to be treated with caution. The essential difference is that the *direction* of a morphism is preserved in one case but reversed in the other.

In an ideal world, there would be a systematic notation devised to handle functors of differing variances between categories of differing chiralities. In reality we are obliged to use well-established notations for functors that take no account of chirality and variance.

The above table can be simplified by introducing the notion of the *chirality* of a functor. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *cochiral* if the categories $\mathcal{C}$ and $\mathcal{D}$ have the same chirality, and *contrachiral* otherwise. Then, for any composable morphisms α, β in $\mathcal{C}$, we have

$$F(\alpha\beta) = F(\alpha)F(\beta)$$

whenever F has the same variance as chirality, that is, both are co- or both are contra-, and

$$F(\alpha\beta) = F(\beta)F(\alpha)$$

whenever the variance of F differs from its chirality.

One example of a product reversing functor is the opposite functor Op (1.2.2)(iii) which is contravariant and cochiral.

Another such functor is the *mirror* functor. For any category $\mathcal{C}$, we define $\mathrm{Mir} : \mathcal{C} \to \mathcal{C}^{\odot}$ from $\mathcal{C}$ to its mirror category $\mathcal{C}^{\odot}$ (1.1.5) by putting $\mathrm{Mir}(C) = C^{\odot}$ and $\mathrm{Mir}(\alpha) = \alpha^{\odot}$. This functor is evidently covariant and contrachiral.

Reinterpreting the definition of the product in $\mathcal{C}^{\odot}$, we find that

$$\mathrm{Mir}(\alpha\beta) = \mathrm{Mir}(\beta)\,\mathrm{Mir}(\alpha).$$

As we noted above, the opposite functor can be used to convert a contravariant functor into a covariant functor. Similarly, $\mathrm{Mir} : \mathcal{D} \to \mathcal{D}^{\odot}$ converts a contrachiral functor $F : \mathcal{C} \to \mathcal{D}$ to a cochiral functor $\mathrm{Mir} \circ F : \mathcal{C} \to \mathcal{D}^{\odot}$, and likewise a cochiral functor can be changed into a contrachiral functor.

Thus, a judicious use of the functors Op and Mir allows one to work only with covariant and cochiral functors.

1.2.7 The morphism functors

Let $\mathcal{C}$ be a (right) category. The morphism sets $\mathrm{Mor}(L, X) = \mathrm{Mor}_{\mathcal{C}}(L, X)$ can be viewed as defining two functors, $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$, from $\mathcal{C}$ to the category $\mathcal{S}_{ET}$ of sets, by holding one term of $\mathrm{Mor}(L, X)$ constant, and allowing the other to vary, as the notation is meant to suggest. The functors arising from Mor in one way or another are called *morphism functors.*

Choose an object L of $\mathcal{C}$. The functor

$$\mathrm{Mor}(L, -) : \mathcal{C} \longrightarrow \mathcal{S}_{ET}$$

sends an object X to the set $\mathrm{Mor}(L, X)$, and for a morphism $\xi : X \to Y$ in $\mathcal{C}$, the map

$$\mathrm{Mor}(L, \xi) : \mathrm{Mor}(L, X) \longrightarrow \mathrm{Mor}(L, Y)$$

is given by

$$\mathrm{Mor}(L, \xi)\alpha = \xi\alpha : L \longrightarrow Y \text{ for } \alpha : L \longrightarrow X \text{ in } \mathrm{Mor}(L, X),$$

so that $\mathrm{Mor}(L, -)$ is a covariant functor from $\mathcal{C}$ to $\mathcal{S}_{ET}$.

(We write $\mathrm{Mor}(L, X)$ and $\mathrm{Mor}(L, \xi)$ in preference to $\mathrm{Mor}(L, -)(X)$ and $\mathrm{Mor}(L, -)(\xi)$.)

If $\vartheta : Y \to Z$ is another morphism, then

$$\mathrm{Mor}(L, \vartheta\xi)\alpha = \vartheta\xi\alpha = \mathrm{Mor}(L, \vartheta)\,\mathrm{Mor}(L, \xi)\alpha.$$

On the other hand, suppose that we fix the object X. Then $\mathrm{Mor}(-, X)(L) = \mathrm{Mor}(L, X)$ for any object L of $\mathcal{C}$, and for a morphism $\lambda : L \to M$, we define

$$\mathrm{Mor}(\lambda, X) : \mathrm{Mor}(M, X) \longrightarrow \mathrm{Mor}(L, X)$$

by

$$\mathrm{Mor}(\lambda, X)\beta = \beta\lambda \text{ for } \beta \in \mathrm{Mor}(M, X),$$

which shows that $\mathrm{Mor}(-, X)$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{S}_{ET}$.

If $\mu : M \to N$ is another morphism, then, for any $\gamma : N \to X$,

$$\mathrm{Mor}(\mu\lambda, X)(\gamma) = \gamma\mu\lambda = \mathrm{Mor}(\lambda, X)\,\mathrm{Mor}(\mu, X)\gamma.$$

When the fixed object L or X need not be mentioned, it is usual to make the abbreviations $\mathrm{Mor}(L, \xi) = \xi_*$ and $\mathrm{Mor}(\lambda, X) = \lambda^*$. The formulas for products then become

$$(\xi\vartheta)_* = \xi_*\vartheta_* \text{ for } \xi : X \longrightarrow Y \text{ and } \vartheta : Y \longrightarrow Z$$

and

$$(\mu\lambda)^* = \lambda^*\mu^* \text{ for } \lambda : L \longrightarrow M \text{ and } \mu : M \longrightarrow N.$$

Note that these formulas do depend on the fact that both the categories $\mathcal{C}$ and $\mathcal{S}_{ET}$ are assumed to be right categories, that is, the functors $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$ are cochiral. If instead we take $\mathcal{C}$ to be a left category, then the product formulas read

$$(\vartheta\xi)_* = \xi_*\vartheta_* \text{ for } \xi : X \longrightarrow Y \text{ and } \vartheta : Y \longrightarrow Z$$

and

$$(\lambda\mu)^* = \lambda^*\mu^* \text{ for } \lambda : L \longrightarrow M \text{ and } \mu : M \longrightarrow N,$$

despite the fact that the variances of the functors $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$ are unchanged. This is an instance of the variability of notation that we discussed in the preceding section.

1.2.8 The category $\mathcal{C}_{AT}$

Given a class of small categories which contains all the small categories that we wish to discuss, we can form the category $\mathcal{C}_{AT}$ of all such categories. The objects of this category are themselves categories $\mathcal{C}, \mathcal{D}, \ldots$, and the set of morphisms from $\mathcal{C}$ to $\mathcal{D}$ is the set $[\mathcal{C}, \mathcal{D}]$ of all covariant functors from $\mathcal{C}$ to $\mathcal{D}$.

Note that an isomorphism in the category $\mathcal{C}_{AT}$ is an invertible functor. Accordingly, we call any invertible functor an *isomorphism.*

1.2.9 Fibre categories

For a functor $F : \mathcal{D} \to \mathcal{C}$, there is a counterpart of the fundamental concept of the kernel of a module homomorphism, namely, that of a right-fibre category. However, as a general category has no distinguished object corresponding to

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the zero element of a module, we must define a right-fibre category $C\backslash F$ (of F at C) for each object C in $\mathcal{C}$.

An object of the *right-fibre category* consists of a pair (D, α), where D is an object of $\mathcal{D}$ and $\alpha : C \to F(D)$ is a morphism of $\mathcal{C}$; such a pair is said to be *under C with respect to F*, or *F-under C*.

A morphism in $C\backslash F$ from (D, α) to (D', α') is given by a morphism $\beta : D \to D'$ in $\mathcal{D}$ such that $\alpha' = F(\beta) \circ \alpha$. Diagrammatically, this may be visualized as:

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & F(D) \\ & \searrow^{\alpha'} & \downarrow{\scriptstyle F(\beta)} \\ & & F(D') \end{array} \qquad \begin{array}{c} D \\ \downarrow{\scriptstyle \beta} \\ D' \end{array}$$

The definition of the dual notion of the *left-fibre category* F/C of pairs (D, α) *F-over C*, should be apparent.

The justification for the name right-fibre category (replacing comma category, surely one of the least descriptive names ever in mathematics) comes from Theorem B of [Quillen 1973] in the homotopy theory of categories, which, however, lies beyond the scope of this book. We use right-fibre categories in section 1.4, where we consider universal constructions.

Here are some simple examples which shed light on the definition, as does Exercise 1.2.1.

1.2.10 Examples

1. Let $\mathcal{C}$ be the category having a unique object C and a unique morphism, id_C. Then, for the unique functor $F : \mathcal{D} \to \mathcal{C}$, the category $C\backslash F$ is isomorphic to the category $\mathcal{D}$ through the functors

$$(D, id_C) \leftrightarrow D, \quad g \leftrightarrow g.$$

2. For the identity functor $\mathrm{Id}_{\mathcal{C}}$ of an arbitrary category $\mathcal{C}$ with C an object of $\mathcal{C}$, the right-fibre category $C\backslash \mathrm{Id}_{\mathcal{C}}$ is just the category of objects of $\mathcal{C}$ *under* C, which is sometimes called the *slice* category $C\backslash\mathcal{C}$. Similarly, there is the category $\mathcal{C}/C = \mathrm{Id}_{\mathcal{C}}/C$ of objects *over* C. These categories are widely used by topologists.
3. Given a group G, form the category $\mathcal{B}G$ with one object G, the morphisms in $\mathcal{B}G$ being simply the elements of the group, as in Exercise 1.1.3. If H

is also a group viewed as a category in this way, a functor F from $\mathcal{B}H$ to $\mathcal{B}G$ is the same as a group homomorphism γ from H to G.

The right-fibre category $G\backslash\gamma$ consists of pairs (H, g) where H is the single object of $\mathcal{B}H$ and g is an arbitrary element of G; we can therefore identify $G\backslash\gamma$ with the set $\{g\}$ of elements of G. A morphism from g to g' is an element h of H with $\gamma(h) \circ g = g'$.

Thus the set of morphisms in $G\backslash\gamma$ from id_G to itself is the kernel of γ, while the set of morphisms from id_G to g' is $\{h \mid \gamma(h) = g'\}$, the inverse image of g'.

In particular, $G\backslash \mathrm{Id}_{\mathrm{Cat}(G)}$ is the category $\mathcal{E}G$ of Exercise 1.1.3.

Exercises

1.2.1 Let X and Y be sets regarded as discrete categories, as in Exercise 1.1.1.

Show that a functor from X to Y is simply a map $f : X \to Y$.

Verify that for any $y \in Y$, the fibre $y\backslash f$ is $f^{-1}(y)$, the inverse image of y.

1.2.2 We may regard a group G as a category, $\mathcal{B}G$ or $\mathcal{E}G$, in two ways as in Exercise 1.1.3. In (1.2.10), we noted that a functor from $\mathrm{Cat}(H)$ to $\mathrm{Cat}(G)$ is the same thing as a group homomorphism from H to G.

Are there any functors from $\mathcal{B}G$ to $\mathcal{E}G$ or vice versa?

Describe the functors from $\mathcal{E}H$ to $\mathcal{E}G$. In particular, show that each element of the group G acts, by multiplication, as a functor on $\mathcal{E}G$ that is transitive on the objects.

Remark. So, if we identify the objects of $\mathcal{E}G$ with one another via this multiplication, we obtain $\mathcal{B}G$. Given a method of passing from groups to topological spaces, this procedure may be used to form the classifying space BG of the group G [Segal 1968].

1.2.3 Recall (or look up in a standard text on group theory) that the commutator subgroup $[G, G]$ of a group G is the normal subgroup of G generated by all commutators $ghg^{-1}h^{-1}$, and that the *commutator quotient group* $G_{ab} = G/[G, G]$ is an abelian group. Sometimes the group G_{ab} is said to be G *made abelian.*

Define $\mathrm{Ab} : \mathcal{G}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{B}}$ by $\mathrm{Ab}(G) = G_{ab}$, where $\mathcal{G}_{\mathcal{P}}$ is the category of groups and $\mathcal{A}_{\mathcal{B}}$ is the category of abelian groups. Verify that, with the obvious action on homomorphisms, Ab is a functor, sometimes called *abelianization.*

1.3 NATURAL TRANSFORMATIONS

Functors themselves can be related to one another by natural transformations. This ability to compare functors enables us to define the concepts of multifunctors and of adjoint functors, vital for our discussion of universal properties in the next section. It also leads to the notion of a functor category, consisting of all the functors from one category to another. When the domain category has an especially simple form, this gives the important special case of a diagram in a category.

We also consider equivalences between categories; these are analogous to isomorphisms between modules or groups in the sense that equivalent categories can be regarded as identical for most purposes. The definition of isomorphism of categories, given in (1.2.8) above, is usually too restrictive for applications.

1.3.1 Natural transformations

Natural transformations are the morphisms between functors themselves.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be covariant functors. A *natural transformation*

$$\eta : F \longrightarrow G$$

consists of a family of morphisms

$$\eta_C : FC \longrightarrow GC,$$

one for each object C of $\mathcal{C}$, such that for each morphism

$$\alpha : C' \longrightarrow C,$$

the diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

commutes. Somewhat informally, a typical member of this family is sometimes said to be a natural transformation.

If, instead, F and G are both contravariant, we require the following dia-

gram to commute:

$$\begin{array}{ccc} FC & \xrightarrow{\eta_C} & GC \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC' & \xrightarrow{\eta_{C'}} & GC' \end{array}$$

(There is little to be gained by attempting to compare functors of different variances, as can be seen from the example of 'unnatural' behaviour that is given in Exercise 1.3.18.)

1.3.2 Examples

(i) Basic and important examples of natural transformations arise from the morphism functors (1.2.7) associated with a (right) category $\mathcal{C}$.

Let $\lambda : L \to M$ and $\xi : X \to Y$ be morphisms in $\mathcal{C}$ and recall that

$$\xi_* = \xi_*^L : \mathrm{Mor}(L, X) \longrightarrow \mathrm{Mor}(L, Y)$$

is given by

$$\xi_*(\alpha) = \xi\alpha : L \longrightarrow Y \text{ for } \alpha : L \longrightarrow X \text{ in } \mathrm{Mor}(L, X),$$

and

$$\lambda^* = \lambda_X^* : \mathrm{Mor}(M, X) \longrightarrow \mathrm{Mor}(L, X)$$

is given by

$$\lambda^*\beta = \beta\lambda \text{ for } \beta \text{ in } \mathrm{Mor}(M, X).$$

It is easy to verify that there is a commutative square

$$\begin{array}{ccc} \mathrm{Mor}(M, X) & \xrightarrow{\xi_*^M} & \mathrm{Mor}(M, Y) \\ \downarrow{\scriptstyle \lambda_X^*} & & \downarrow{\scriptstyle \lambda_Y^*} \\ \mathrm{Mor}(L, X) & \xrightarrow{\xi_*^L} & \mathrm{Mor}(L, Y) \end{array}$$

which can be interpreted in two ways, one as showing that ξ_* is a natural transformation from $\mathrm{Mor}(-, X)$ to $\mathrm{Mor}(-, Y)$ and the other, that λ^* is a natural transformation from $\mathrm{Mor}(M, -)$ to $\mathrm{Mor}(L, -)$.

(ii) Let R be a ring and let $M_n(R)$ be the ring of $n \times n$ matrices over R. The inclusion map

$$\iota_R : R \longrightarrow M_n(R)$$

sends an element r of R to the scalar matrix $\mathrm{diag}(r, \ldots, r)$. (The structure of matrix rings is discussed at length in section 2.2 of [BK: IRM].)

If $f : R \to S$ is a ring homomorphism, there is an obvious commuting diagram

$$\begin{array}{ccc} R & \xrightarrow{\iota_R} & M_n(R) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle M_n(f)} \\ S & \xrightarrow{\iota_S} & M_n(S) \end{array}$$

of ring homomorphisms. Thus ι is a natural transformation from the identity functor on the category $\mathcal{R}_{ING}$ to the functor $M_n(-)$ on $\mathcal{R}_{ING}$. Typically, one says just that the embedding ι is natural.

(iii) A ring R contains a smallest subring, namely its *prime ring*, which is the subring $\langle 1_R \rangle$ comprising the multiples $n.1_R$, $n \in \mathbb{Z}$, of the multiplicative identity 1_R of R. The assignment $P : R \mapsto \langle 1_R \rangle$ is clearly a functor from the category $\mathcal{R}_{ING}$ of rings to itself. Then the embedding of $\langle 1_R \rangle$ in R is a natural transformation from P to the identity functor because a ring homomorphism $f : R \to S$ gives a commuting square

$$\begin{array}{ccc} \langle 1_R \rangle & \hookrightarrow & R \\ {\scriptstyle Pf}\downarrow & & \downarrow{\scriptstyle f} \\ \langle 1_S \rangle & \hookrightarrow & S \end{array}$$

(iv) Given a nonunital ring R, the *enveloping ring* or *unitalization* of R is

$$\overline{R} = \{(r, a) \mid r \in R, a \in \mathbb{Z}\},$$

which is a ring under the addition and multiplication given by

$$(r, a) + (s, b) = (r + s, a + b) \quad \text{and} \quad (r, a) \cdot (s, b) = (rs + br + as, ab).$$

The identity of $\overline{R}$ is $(0, 1)$. Then, in the category $\mathcal{R}_{NG}$ of nonunital rings, the embedding of R in its enveloping ring is natural.

(v) In the category of commutative domains, the embedding of a commutative domain in its field of fractions is also natural – see [BK: IRM] (1.1.12).

(vi) For another example, where neither functor is the identity functor, we use the operation of transposition on matrices. Given a matrix $A = (a_{ij})$ over a ring R, the *transpose* of A is defined to be the matrix $A^t = (a^\circ_{ji})$, which has entries in the opposite ring R°. Applying this definition to matrices over R° rather than R itself, we obtain a map

$$(-)^t : M_n(R^\circ) \longrightarrow (M_n(R))^\circ$$

which is readily seen to be a ring isomorphism. (Further details can be found in Exercises 1.2.13 and 1.2.14 of [BK: IRM].)

Given a ring homomorphism $f : R \to S$ there is a commutative square

$$\begin{array}{ccc} M_n(R^\circ) & \xrightarrow{(-)^t} & (M_n(R))^\circ \\ \downarrow & & \downarrow \\ M_n(S^\circ) & \xrightarrow{(-)^t} & (M_n(S))^\circ \end{array}$$

so that $(-)^t$ is a natural transformation between the functors $M_n((-)^\circ)$ and $(M_n(-))^\circ$.

(vii) For a ring R, the general linear group $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices over R, that is, the group of units of the matrix ring $M_n(R)$.

Evidently, $\mathrm{GL}_n(R)$ can be embedded in $\mathrm{GL}_{n+1}(R)$ by the insertion of an extra (last) row and column with diagonal entry 1 and other entries 0 to each matrix A in $GL_n(R)$:

$$A \longmapsto \left(\begin{array}{c|c} & 0 \\ A & \vdots \\ & 0 \\ \hline 0 \cdots 0 & 1 \end{array}\right).$$

We abbreviate the right-hand matrix to $A \oplus 1$. Since

$$AB \oplus 1 = (A \oplus 1)(B \oplus 1),$$

we obtain a group homomorphism $- \oplus 1 : \mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$. Moreover, $- \oplus 1$ is natural because any ring homomorphism $f : R \to S$ gives

rise to a commuting square

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{-\oplus 1} & \mathrm{GL}_{n+1}(R) \\ \Big\downarrow{\scriptstyle \mathrm{GL}_n(f)} & & \Big\downarrow{\scriptstyle \mathrm{GL}_{n+1}(f)} \\ \mathrm{GL}_n(S) & \xrightarrow{-\oplus 1} & \mathrm{GL}_{n+1}(S) \end{array} .$$

(viii) We continue with the above notation, but now restrict our attention to the category of commutative rings. We can then define the determinant map $\det : M_n(R) \to R$ by the usual formula involving sums and differences of products of entries. (See [Cohn 1982], Chapter 7 for example.) Because the determinant preserves products, that is, $\det(AB) = \det(A)\det(B)$ for $n \times n$ matrices A and B, it restricts to a homomorphism of the groups of units $\det : \mathrm{GL}_n(R) \to U(R)$. Again, this homomorphism is natural, meaning that there is a commuting square

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{\det} & U(R) \\ \Big\downarrow{\scriptstyle \mathrm{GL}_n(f)} & & \Big\downarrow{\scriptstyle U(f)} \\ \mathrm{GL}_n(S) & \xrightarrow{\det} & U(S) \end{array}$$

whenever f is a ring homomorphism.

1.3.3 Natural isomorphisms

If for each object C of $\mathcal{C}$ the morphism η_C is an isomorphism in $\mathcal{D}$, we say that the functors F and G are *naturally isomorphic* through the *natural isomorphism* η. The notation is $F \simeq G$. The terms *naturally equivalent* and *natural equivalence* are also used, especially in older texts.

Among the above examples, that in (vi) is a natural isomorphism because the matrix transpose operation is evidently invertible. In (ii) and (viii), we clearly have natural isomorphisms when $n = 1$.

1.3.4 Matrices and bases of free modules: a summary

Our next example gives a categorical interpretation of some results on the matrix representation of homomorphism between free modules. To get started, we need some facts about free modules and their bases which are given in detail in [BK: IRM], particularly §2.2. For future ease of reference in this

volume, we review these facts in greater generality than we need for our immediate purposes.

Let R be a ring and let M be a right R-module. A *basis* of M is an ordered subset $\{f_\lambda\}_{\lambda\in\Lambda}$ (where Λ is an ordered indexing set), which satisfies the following property.

Bas. Given any element m of M, there are *unique* elements $r_\lambda(m)$, $\lambda \in \Lambda$, of R such that

$$m = \sum_{\lambda\in\Lambda} f_\lambda r_\lambda(m).$$

The set Λ may be infinite, but at most a finite number of coefficients $r_\lambda(m)$ can be nonzero for any particular m.

An arbitrary module M need not have a basis. If M does have a basis, then M is a *free right* R-module.

Note that a basis must be ordered. An 'unordered basis' is called a *free generating set*. Thus, the free module $\mathrm{Fr}_R(X)$ on an unordered set X (1.2.3) has X as a free generating set. If we impose an ordering on X, then we obtain a basis of $\mathrm{Fr}_R(X)$.

When the index set Λ is finite, we usually take it to be $\{1,\dots,m\}$ for some integer m and write the basis as

$$\{f_j\} = \{f_1,\dots,f_m\}.$$

We then say that the free module has *finite rank*.

Now let R^n be the standard free right R-module of finite rank n, which we view as the 'space of column vectors' of length n, with entries in R. The evident choice of 'unit' vectors $\{e_1,\dots,e_n\}$ gives the *standard basis* $\{e_j\}$ of R^n. Next, take any finite basis

$$\{f_j\} = \{f_1,\dots,f_m\}$$

of R^n. If it must be the case that $m = n$, the ring R is said to have *invariant basis number*. However, we wish to work with an arbitrary ring, so we must allow the possibility that $m \neq n$. Some examples and results on invariant basis number and the lack of it are to be found in §2.3 of [BK: IRM].

An element x of R^n has a *coordinate vector* ${}_F(x)$ with respect to the basis $F = \{f_j\}$, which is given by the formula

$${}_F(x) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \quad \text{where } x = f_1y_1 + \cdots + f_my_m,\ y_1,\dots,y_m \in R.$$

If R^ℓ is another standard free right R-module with basis $\{g_j\} = \{g_1, \ldots, g_k\}$ and $\alpha : R^n \to R^\ell$ is a homomorphism of right R-modules, we associate to α a $k \times m$ matrix $A = (a_{ij})$, which is determined by the formula

$$\alpha(f_j) = \sum_{i=1}^{k} g_i a_{ij}, \quad j = 1, \ldots, m.$$

The coordinate vector ${}_G(\alpha x)$ of αx with respect to $G = \{g_j\}$ is then given by the relation

$${}_G(\alpha x) = A_F(x).$$

The association of a matrix to a homomorphism is easily seen to be multiplicative; that is, if $\beta : R^p \to R^n$ is an R-module homomorphism that is composable with α, and if B is the matrix of β with respect to a basis $\{h_j\}$ of R^p and the given basis $\{f_j\}$ of R^n, then the matrix of $\alpha\beta$ with repect to the pair $\{h_j\}$, $\{g_j\}$ is AB.

The association is also additive, for if α and α' are homomorphisms from R^n to R^ℓ with matrices A and A' respectively (relative to the pair of bases $\{f_j\}$, $\{g_j\}$), then the matrix of $\alpha + \alpha'$ will be $A + A'$ (relative to the same pair of bases).

In particular, fix an n-element basis of R^n and use it at 'both ends'. Then the matrix A of an endomorphism α will be $n \times n$, and the map $\alpha \mapsto A$ is a ring isomorphism from $\mathrm{End}(R^n)$ to the matrix ring $M_n(R)$.

1.3.5 Example: matrices and bases of free modules

Now we interpret the above discussion in terms of categories and functors. We define $\mathcal{B}_{ASES_R}$ to be the category whose objects are free right R-modules of the form R^n equipped with a preferred basis $\{f_j\}$, that is, a pair $(R^n, \{f_j\})$. The morphisms in $\mathcal{B}_{ASES_R}$ are just the R-module homomorphisms from R^n to R^ℓ (here the choice of basis is ignored). The full subcategory $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ of $\mathcal{B}_{ASES_R}$ has as objects the pairs $(R^n, \{e_j\})$ where $\{e_j\}$ is the standard basis of R^n. (We use the notation $\{e_j\}$ ambiguously for a standard basis of any size.)

The fact that the operation of taking a matrix of a homomorphism is multiplicative now reveals that we obtain a functor

$$S : \mathcal{B}_{ASES_R} \longrightarrow \mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$$

by setting

$$S(R^n, \{f_j\}) = (R^m, \{e_j\}), \quad m \text{ being the number of members of } F = \{f_j\},$$

and

$$S(\alpha) = A, \text{ the matrix of } \alpha.$$

For each object $(R^n, F = \{f_j\})$ of $\mathcal{B}_{ASES_R}$, there is an R-module isomorphism η_F from R^n to R^m which associates to each member x of R^n its coordinate vector ${}_F(x)$: thus

$$\eta_F(x) = {}_F(x).$$

(Properly speaking, η should have as suffix the pair of objects between which we map, but the notation is then unwieldy.)

To interpret these isomorphisms as defining a natural isomorphism of functors, we let T be the composite functor $T = \text{Inc} \circ S$, where Inc is the inclusion of $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ in $\mathcal{B}_{ASES'_R}$. For any morphism $\alpha : (R^n, F = \{f_j\}) \to (R^l, G = \{g_j\})$ in $\mathcal{B}_{ASES_R}$, there is a commutative diagram:

$$\begin{array}{ccc} (R^n, \{f_j\}) & \xrightarrow{\eta_F} & (R^m, \{e_j\}) \\ \alpha \downarrow & & \downarrow T(\alpha) \\ (R^\ell, \{g_j\}) & \xrightarrow{\eta_G} & (R^k, \{e_j\}) \end{array}$$

which shows $\eta : \text{Id} \to T$ to be a natural transformation, as desired. Note that the homomorphism $T(\alpha)$ is simply left multiplication by the matrix A of α.

1.3.6 *Multifunctors*

It is often necessary to consider functors of several variables. The guiding example is $\text{Mor}_{\mathcal{C}}(-,-)$ for a (right) category $\mathcal{C}$, which attaches a set $\text{Mor}_{\mathcal{C}}(C, D)$ to each pair of objects C, D of $\mathcal{C}$. By the remarks in (1.2.7), $\text{Mor}_{\mathcal{C}}(-, D)$ is a contravariant functor, while $\text{Mor}_{\mathcal{C}}(C, -)$ is covariant. So we wish to recognize $\text{Mor}_{\mathcal{C}}(-,-) : \mathcal{C} \times \mathcal{C} \to \mathcal{S}_{ET}$ as simultaneously functorial in both variables, even though it is not actually a functor on the product category $\mathcal{C} \times \mathcal{C}$. The point is that, because of its mixed variances, the would-be functor $\text{Mor}(-,-)$ does not obey either of the composition laws Fun 2 or Fun 2^o. The same difficulty arises when we consider $\text{Op} \times \text{Id} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}^{\text{op}} \times \mathcal{C}$.

A related problem arises when we wish to take the variables from categories of differing chiralities, since the law of composition in the product of categories of mixed chiralities can be rather convoluted. For instance, let $\mathcal{C}$ be a right category, so that its mirror $\mathcal{C}^\odot$ (1.1.5) is a left category. Our rule then is to view the product category $\mathcal{C} \times \mathcal{C}^\odot$ as a right category (1.1.11).

To see how this works, recall that a morphism $\beta^\odot : B^\odot \to D^\odot$ in $\mathcal{C}^\odot$ is

a formal symbol corresponding to a morphism $\beta : B \to D$ in $\mathcal{C}$, and that the composition of a pair of composable morphisms $\beta^{\odot}$ and $\delta^{\odot}$ in $\mathcal{C}^{\odot}$ is $\beta^{\odot}\delta^{\odot} = (\delta\beta)^{\odot}$.

Thus, if we have a pair of composable morphisms $(\alpha, \beta^{\odot})$, $(\gamma, \delta^{\odot})$ in $\mathcal{C} \times \mathcal{C}^{\odot}$, their product is

$$(\gamma, \delta^{\odot})(\alpha, \beta^{\odot}) = (\gamma\alpha, \beta^{\odot}\delta^{\odot}) = (\gamma\alpha, (\delta\beta)^{\odot}).$$

Methods of handling mixed chiralities become important later (3.1.9), when we consider tensor products. Thus we need a definition which accommodates both mixed variances and mixed chiralities.

Our uniform treatment of the various possible mixtures of variances and chiralities is based on two observations. Firstly, if F is a contravariant functor on $\mathcal{C}$, then $F \circ \mathrm{Op}$ is a covariant functor on $\mathcal{C}^{\mathrm{op}}$; and secondly, if $F : \mathcal{C} \to \mathcal{D}$ is a contrachiral functor, then $F \circ \mathrm{Mir} : \mathcal{C}^{\odot} \to \mathcal{D}$ is a cochiral functor. (Here we rely on the obvious identifications $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ and $(\mathcal{C}^{\odot})^{\odot} = \mathcal{C}$ for any category $\mathcal{C}$.)

Let $\mathcal{D}, \mathcal{C}_1, \ldots, \mathcal{C}_k$ be a finite set of (right) categories. Suppose that

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \longrightarrow \mathcal{D}$$

is given by a function

$$F : \mathrm{Ob}(\mathcal{C}_1) \times \cdots \times \mathrm{Ob}(\mathcal{C}_k) \longrightarrow \mathrm{Ob}(\mathcal{D})$$

and, for each pair C_i', C_i of objects of each $\mathcal{C}_i$, a function

$$F : \mathrm{Mor}_{\mathcal{C}_1} \times \cdots \times \mathrm{Mor}_{\mathcal{C}_k} \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(C_1', \ldots, C_k'), F(C_1, \ldots, C_k)).$$

Then we say that F is a *multifunctor* or a *functor in k variables* on $\mathcal{C}_1, \ldots, \mathcal{C}_k$ provided that for some (necessarily unique) choice of functors δ_i, ϵ_i on $\mathcal{C}_i$, with each δ_i as either Id or Op and each ϵ_i as either Id or Mir,

$$F \circ (\delta_1\epsilon_1 \times \cdots \times \delta_k\epsilon_k) : \epsilon_1\delta_1(\mathcal{C}_1) \times \cdots \times \epsilon_k\delta_k(\mathcal{C}_k) \longrightarrow \mathcal{D}$$

is a covariant and cochiral functor.

If δ_i is Id, then F is said to be *covariant in the i th variable*, while if δ_i is Op, F is said to be *contravariant in the i th variable.*

Likewise, if ϵ_i is Id, then F is said to be *cochiral in the i th variable*, while if ϵ_i is Op, F is *contrachiral in the i th variable.*

The terms *bifunctor*, *trifunctor*,... are also used when $k = 2, 3, \ldots$. Note that a multifunctor in one variable is simply a functor of the given variance and chirality.

Thus $\mathrm{Mor}_{\mathcal{C}}(-,-)$ is a bifunctor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{S}\mathit{ET}$ which is contravariant in

the first variable and covariant in the second. It is cochiral in both variables. A version which exhibits mixed chiralities is given in Exercise 2.1.1.

Natural transformations between multifunctors, of the same variance and chirality, are defined in the obvious way.

It is also possible to describe a multifunctor 'term-by-term'. Let F be a multifunctor, and choose any index $h = 1, \ldots, k$ and any set of objects $\{C_i \in \mathcal{C}_i \mid i \neq h\}$. Then F restricts to a functor

$$F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k) : \mathcal{C}_h \longrightarrow \mathcal{D},$$

which is covariant or contravariant according as δ_h is Id or Op, and cochiral or contrachiral according as ϵ_h is Id or Mir.

Furthermore, any collection of morphisms

$$\{\beta_i : \delta_i(B_i) \longrightarrow \delta_i(C_i) \in \delta_i(\mathcal{C}_i) \mid i \neq h\}$$

gives rise to a natural transformation

$$F(B_1, \ldots, B_{h-1}, -, B_{h+1}, \ldots, B_k) \longrightarrow F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k).$$

Conversely, given a fixed index h, a multifunctor can be defined by a collection of functors of the form $F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k)$, one for each set of objects from the categories $\mathcal{C}_i$, $i \neq h$, provided that the appropriate natural transformations exist.

1.3.7 *Adjoint functors*

We now introduce a concept which is very important in both the theory and the applications of categories, particularly in connection with universal constructions, which we consider in detail in the next section.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be (covariant) functors. We can form two bifunctors

$$\mathrm{Mor}_{\mathcal{C}}(-, G(-)) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{S}et$$

and

$$\mathrm{Mor}_{\mathcal{D}}(F(-), -) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{S}et,$$

both of which are contravariant in the first variable and covariant in the second.

The functors F and G are said to form an *adjoint pair*, or to be *adjoint*, if there is a natural isomorphism

$$\eta : \mathrm{Mor}_{\mathcal{D}}(F(-), -) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{C}}(-, G(-)).$$

If this is the case, F is said to be *left adjoint* to G and G is said to be *right adjoint* to F. (The terms 'left' and 'right' here are convincing only for covariant functors.)

Thus, to show that F and G are adjoint, it is necessary and sufficient to give a family of bijections

$$\eta_{C,D} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D))$$

which are natural in C and D.

The following example typifies the set-up.

1.3.8 Example: free modules

Let X be a set and let $\mathrm{Fr}_R(X)$ be the free right R-module on X. Then $\mathrm{Fr}_R : \mathcal{S}et \to \mathcal{M}od_R$ is a covariant functor (1.2.3). In the other direction, let $\Upsilon : \mathcal{M}od_R \to \mathcal{S}et$ be the forgetful functor, which assigns to a module M its underlying set $\Upsilon(M)$ of elements, as in (1.2.2)(iv). It is easy to see that Fr_R is left adjoint to Υ.

1.3.9 Functor categories

Given a small category $\mathcal{C}$ and an arbitrary category $\mathcal{D}$, the covariant functors from $\mathcal{C}$ to $\mathcal{D}$ can themselves be regarded as the objects of a category $[\mathcal{C}, \mathcal{D}]$. Such categories $[\mathcal{C}, \mathcal{D}]$ and their subcategories are called *functor* categories.

The objects of $[\mathcal{C}, \mathcal{D}]$ are the covariant functors

$$F : \mathcal{C} \longrightarrow \mathcal{D},$$

and a morphism between functors F and G is a natural transformation

$$\eta : F \longrightarrow G.$$

We write $\mathrm{Nat}(F, G)$ for the set of all such natural transformations. This is indeed a set, since $\eta = \{\eta_C \mid C \in \mathcal{C}\}$ is specified by a collection of data which is indexed by the set $\mathrm{Ob}\,\mathcal{C}$, and each η_C must belong to the set $\mathrm{Mor}(FC, GC)$. (The target category $\mathcal{D}$ need not be small.)

To see that we actually have a (right) category, we note that each functor F has an *identity* natural transformation id_F given by

$$(id_F)_C = id_{FC} \text{ for each } C \in \mathcal{C},$$

and that for $\eta \in \mathrm{Nat}(F, G)$ and $\zeta \in \mathrm{Nat}(G, H)$, their product

$$\zeta\eta : F \longrightarrow H$$

is defined by

$$(\zeta\eta)_C = \zeta_C\eta_C \text{ for each } C \in \mathcal{C}$$

when $\mathcal{D}$ is a right category.

The category $[\mathcal{C}^{\mathrm{op}}, \mathcal{D}]$ is the category of all contravariant functors from $\mathcal{C}$ to $\mathcal{D}$.

1.3.10 *Diagrams*

A useful application of the preceding discussion occurs when functors from $\mathcal{C}$ to $\mathcal{D}$ are interpreted as defining *diagrams* in $\mathcal{D}$.

A diagram in a category $\mathcal{D}$ is the same thing as the *graph* of a functor from some category $\mathcal{C}$ to $\mathcal{D}$. That is, it consists of a collection of objects and morphisms of $\mathcal{D}$ which are the images of the objects and morphisms of $\mathcal{C}$ and which satisfy the appropriate commutativity conditions. Although in principle any category $\mathcal{C}$ defines diagrams, the notion is most often used when $\mathcal{C}$ has few objects and morphisms.

For example, let $[n]$ be the totally ordered set $\{0, \ldots, n\}$, which we view as a category with exactly one morphism from i to j whenever $i \leq j$. We can display $[n]$ as a diagram

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \quad \cdots \quad \longrightarrow n-1 \longrightarrow n,$$

in which the 'generating' morphisms are shown; the remaining morphisms in $[n]$ are identities or products of those indicated. (Note that this category is labelled according to the number of generating morphisms, rather than the number of objects. This is due to its topological applications. For example, see Exercise 1.3.17 below.)

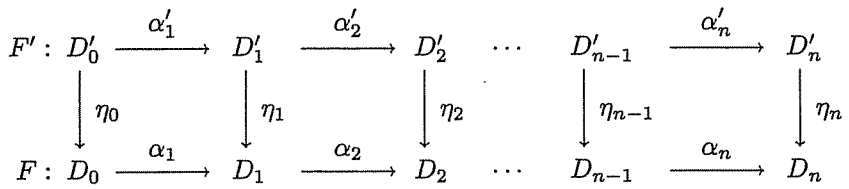
A functor from $[n]$ to $\mathcal{D}$, that is, an object F of $[[n], \mathcal{D}]$, is then a sequence of morphisms in $\mathcal{D}$

$$F : D_0 \xrightarrow{\alpha_1} D_1 \xrightarrow{\alpha_2} D_2 \quad \cdots \quad D_{n-1} \xrightarrow{\alpha_n} D_n$$

and a morphism η from F' (with the obvious labelling) to F is a set of morphisms in $\mathcal{D}$

$$\eta_i : D'_i \longrightarrow D_i, \quad i = 0, \ldots, n,$$

such that the diagram



commutes, that is, each of its squares commutes.

Note that the category $[[0], \mathcal{D}]$ is just $\mathcal{D}$, while $[[1], \mathcal{D}]$ is the morphism category $\mathcal{M}or\,\mathcal{D}$ under another name. Notice also that an object in $[[2], \mathcal{D}]$ can be interpreted as a commutative triangle.

1.3.11 Equivalence of categories

We now consider the circumstances in which two categories can be regarded as the same.

The most obvious of such circumstances is that the categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*, that is, we have a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with $GF = \mathrm{Id}_{\mathcal{C}}$ and $FG = \mathrm{Id}_{\mathcal{D}}$. This means that F and G are mutually inverse isomorphisms (1.2.8). If we have such a pair of functors, they induce mutually inverse bijections between the classes $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{D})$. But this is a relatively uncommon event, since the definition of the 'value' $F(C)$ of a functor F on an object C very often requires the choice of one object from many isomorphic candidates.

More likely, and still acceptable as a means of comparing categories, is for functors F and G to induce bijections of the collections of isomorphism classes of objects of $\mathcal{C}$ and $\mathcal{D}$.

For example, for a field $\mathcal{K}$, each isomorphism class of objects of $\mathcal{S}_{TAN}\mathcal{B}_{ASES_{\mathcal{K}}}$ is a singleton, comprising only the standard basis of $\mathcal{K}^n$, whereas $\mathcal{B}_{ASES_{\mathcal{K}}}$ has many objects isomorphic to the standard basis of $\mathcal{K}^n$. Yet because *every* object of $\mathcal{B}_{ASES_{\mathcal{K}}}$ is isomorphic to the standard basis, the subcategory $\mathcal{S}_{TAN}\mathcal{B}_{ASES_{\mathcal{K}}}$ is a good working approximation to $\mathcal{B}_{ASES_{\mathcal{K}}}$.

Therefore, the appropriate notion is *equivalence* of categories: two categories $\mathcal{C}$ and $\mathcal{D}$ are (naturally) equivalent if there are (covariant) functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ so that GF is naturally isomorphic to $\mathrm{Id}_{\mathcal{C}}$ and FG is naturally isomorphic to $\mathrm{Id}_{\mathcal{D}}$. The functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are then said to be mutually inverse *equivalences* (of $\mathcal{C}$ and $\mathcal{D}$); the notation is $\mathcal{C} \simeq \mathcal{D}$.

If F and G are contravariant, they are called *dualities*, and the categories are said to be *dual*.

1.3.12 Example: standard bases

In (1.3.5), we introduced the category $\mathcal{B}_{ASES_R}$ whose objects are free R-modules R^n together with a chosen basis, and its full subcategory $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ of free modules with standard bases. These categories are equivalent since, in our previous notation, we have shown that $T = \mathrm{Inc} \circ S$ is isomorphic to the identity functor on $\mathcal{B}_{ASES_R}$, while $F \circ \mathrm{Inc}$ is clearly the identity functor on $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$.

1.3.13 Faithful, full and dense

In general, a category $\mathcal{C}$ is equivalent to a full subcategory $\mathcal{C}'$ precisely when there is a natural way of associating to each object of $\mathcal{C}$ a representative in $\mathcal{C}'$ of its $\mathcal{C}$-isomorphism class. This idea may be developed as follows.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if the map

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC)$$

is an injection for every pair of objects in $\mathcal{C}$; F is *full* if the above map is a surjection, and F is *dense* or *representative* if for each object D in $\mathcal{D}$, we have $D \cong FC$ for some C in $\mathcal{C}$.

With these definitions, the following result is straightforward. (More details can be found in [Herrlich & Strecker 1979], (14.11), and [Mitchell 1965], II, particularly Proposition 10.1.)

1.3.14 Proposition

The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there is a full, faithful and dense functor from $\mathcal{C}$ to $\mathcal{D}$. □

1.3.15 Skeletons

A category $\mathcal{S}$ is called *skeletal* if no two of its objects are isomorphic. Trivial examples are provided by the categories $[n]$, $n \geq 0$.

Given any category $\mathcal{C}$, we can form a full skeletal subcategory $\mathrm{Sk}(\mathcal{C})$ of $\mathcal{C}$ by choosing one object from each isomorphism class of the objects of $\mathcal{C}$; naturally, $\mathrm{Sk}(\mathcal{C})$ is called the *skeleton* of $\mathcal{C}$. The skeleton is not, in general, unique. However, the inclusion functor from $\mathrm{Sk}(\mathcal{C})$ to $\mathcal{C}$ is evidently full, faithful and dense, so $\mathrm{Sk}(\mathcal{C})$ is equivalent to $\mathcal{C}$.

Notice that $\mathrm{Sk}(\mathcal{C})$ is small precisely when the class of isomorphism classes of objects in $\mathcal{C}$ is in fact a set. (This condition suffices for applications in K-theory.) Thus the outcome of the discussion in (1.1.10) can be rephrased

as saying that, for a ring R, the category $\mathcal{M}_R$ of finitely generated right R-modules has a small skeleton.

Next we present some equivalences which sharpen the relationships between product and morphism categories on the one hand, and module categories on the other, which we first touched on in (1.1.12) and (1.1.14).

1.3.16 Theorem

Let R and S be rings. Then the direct product category $\mathcal{M}_{OD_R} \times \mathcal{M}_{OD_S}$ is equivalent to $\mathcal{M}_{OD_{R\times S}}$.

Proof

As we noted in (1.1.12), an object (M, N) in the product category can be regarded as an $R \times S$-module by the rule $(m, n) \cdot (r, s) = (mr, ns)$, and a morphism in $\mathcal{M}_{OD_R} \times \mathcal{M}_{OD_S}$ is an $R \times S$-module homomorphism.

On the other hand, any $R \times S$-module is isomorphic to one of the form $M \oplus N$, where M is an R-module and N is an S-module, and the direct sum is the direct sum of abelian groups; furthermore, this decomposition respects homomorphisms (see [BK: IRM] (2.6.6), (2.6.7)).

It is now evident that the equivalence is given by

$$(M, N) \mapsto M \oplus N \text{ and } (\alpha, \beta) \mapsto \alpha \oplus \beta. \qquad \square$$

Recall that $\mathcal{M}_R$ and $\mathcal{P}_R$ are respectively the categories of finitely generated right R-modules and finitely generated projective right R-modules. The components of a finitely generated or projective module will again be finitely generated or projective, as the case may be, since the equivalence given in the previous theorem evidently preserves surjections and direct sums (see Corollary 2.6.9 in [BK: IRM] for more detail). Thus the following result is immediate.

1.3.17 Corollary

There are equivalences of categories

$$\mathcal{M}_R \times \mathcal{M}_S \simeq \mathcal{M}_{R\times S}$$

and

$$\mathcal{P}_R \times \mathcal{P}_S \simeq \mathcal{P}_{R\times S}. \qquad \square$$

Next we describe the morphisms of $\mathcal{M}_{OD_R}$ in terms of modules over a triangular matrix ring.

1.3.18 Theorem

Let $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ *be the ring of upper triangular* 2×2 *matrices over a ring* R. *Then the categories* $\mathcal{M}_{OD_T}$ *and* $\mathcal{M}_{OR}(\mathcal{M}_{OD_R})$ *are equivalent.*

Proof

Let L be a (right) T-module. Since the direct product $R \times R$ is the subring $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ of T, we can write $L = M \oplus N$, where both M and N are R-modules, as in (1.3.16). Using the action of T on $M \oplus N$, define $\alpha : M \to N$ by

$$(m, 0) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0, \alpha m).$$

Then α is easily seen to be a homomorphism of R-modules, and the action of T is given by

$$(*) \qquad (m, n) \begin{pmatrix} r & u \\ 0 & s \end{pmatrix} = (mr, \alpha mu + ns).$$

Let L' be another T-module, let $L' = M' \oplus N'$ be its decomposition and α' the corresponding homomorphism. A T-module homomorphism $\lambda : L' \to L$ induces R-module homomorphisms $\mu : M' \to M$ and $\nu : N' \to N$.

Calculating $\lambda(m', 0) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $m' \in M'$, in two ways, we find that $\nu\alpha' = \alpha\mu$, which shows that we can define a functor $F : \mathcal{M}_{OD_T} \to \mathcal{M}_{OR}(\mathcal{M}_{OD_R})$ by sending L to α.

Conversely, given an R-module homomorphism α from M to N, it is clear how to use the formula $(*)$ to turn $M \oplus N$ into a T-module L, thus defining a functor G in the reverse direction. Evidently, F and G are mutually inverse equivalences. □

Exercises

1.3.1 Let X and Y be sets regarded as discrete categories, and maps between them regarded as functors, as in Exercise 1.2.1.

Show that there is a natural transformation between two maps $f, g : X \to Y$ if and only if $f = g$.

1.3.2 **Yoneda's Lemma**

Let $F : \mathcal{C} \to \mathcal{S}_{ET}$ be a covariant functor, and let L be an object of $\mathcal{C}$ and ℓ an element of $F(L)$. For each object X of $\mathcal{C}$, define

$$\eta_X(\ell) : \mathrm{Mor}_{\mathcal{C}}(L, X) \longrightarrow F(X)$$

by

$$\eta_X(\ell)(\alpha) = F(\alpha)(\ell).$$

Verify that

$$\eta(\ell) = \{\eta_X(\ell) \mid X \text{ an object of } \mathcal{C}\}$$

defines a natural transformation from $\mathrm{Mor}_{\mathcal{C}}(L, -)$ to $F(-)$.

Show that the map $\ell \mapsto \eta(\ell)$ defines a bijection from $F(L)$ to $\mathrm{Nat}(\mathrm{Mor}(L, -), F)$, with inverse given by the map which associates to a natural transformation $\zeta : \mathrm{Mor}_{\mathcal{C}}(L, -) \to F(-)$ the element $\zeta_L(id_L)$ of $F(L)$.

By taking $F(-) = \mathrm{Mor}(M, -)$, show that every natural transformation from $\mathrm{Mor}(L, -)$ to $\mathrm{Mor}(M, -)$ has the form μ^* for some $\mu : M \to L$.

Repeat the exercise for contravariant functors from $\mathcal{C}$ to $\mathcal{S}_{ET}$.

Remark. A covariant functor $F : \mathcal{C} \to \mathcal{S}_{ET}$ is said to be *representable* if it is naturally isomorphic to $\mathrm{Mor}(L, -)$ for some object L in $\mathcal{C}$. The analogous definition is made for contravariant functors.

1.3.3 **Isomorphisms of categories**

Isomorphisms of categories come in four varieties, since the mutually inverse functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ may be either both covariant or contravariant and either both cochiral or contrachiral.

Show that the identity functor, the opposite functor Op, the mirror functor Mir and the composite Mir $\circ$ Op provide examples of all four varieties.

Show also that an arbitrary isomorphism can be converted into a covariant, cochiral isomorphism by composition with one of the isomorphisms listed above.

Now let G and H be groups viewed as one-element categories $\mathcal{B}G$ and $\mathcal{B}H$ (Exercise 1.1.3), and let $F : \mathcal{B}G \to \mathcal{B}H$ be an isomorphism of categories. Show that if F is covariant then F corresponds to an isomorphism of groups in the usual sense, while if F is contravariant, F corresponds to an anti-isomorphism of groups.

1.3.4 We again regard a group G as a category, $\mathcal{B}G$ or $\mathcal{E}G$, in two ways, as in Exercise 1.1.3. Given groups G and H, describe any natural transformations and isomorphisms that there are between a pair of functors between two of the categories $\mathcal{B}G$, $\mathcal{B}H$, $\mathcal{E}G$ and $\mathcal{E}H$.

1.3.5 Let $\mathrm{Ab} : \mathcal{G}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{B}}$ be the abelianization functor between the category $\mathcal{G}_{\mathcal{P}}$ of groups and the category $\mathcal{A}_{\mathcal{B}}$ of abelian groups (Exercise 1.2.3),

and let $\Upsilon : \mathcal{A}_B \to \mathcal{G}_P$ be the inclusion functor. Show that Ab, Υ is an adjoint pair.

1.3.6 (a) Show that if F, G and F', G' are both adjoint pairs, then $F'F$, GG' is also an adjoint pair, whenever the composite functors are defined.

(b) Show that if F, G is an adjoint pair, then each of F and G determines the other to within natural isomorphism.

1.3.7 Show that if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are mutually inverse equivalences, then F, G is an adjoint pair.

1.3.8 Let $A : \mathcal{C}' \to \mathcal{C}$ and $B : \mathcal{D} \to \mathcal{D}''$ be (covariant) functors between small categories. Define functors $[A, -] : [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}', \mathcal{D}]$ and $[-, B] : [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}'']$, and verify that $[-, -]$ is a bifunctor from $\mathcal{C}_{AT} \times \mathcal{C}_{AT}$ to $\mathcal{C}_{AT}$.

1.3.9 (a) In the notation of the preceding exercise (1.3.8) show that if $G : \mathcal{D}'' \to \mathcal{D}$ is right adjoint to $F : \mathcal{D} \to \mathcal{D}''$, then $[\mathcal{C}, F]$, $[\mathcal{C}, G]$ is an adjoint pair.

(b) Each object D in the category $\mathcal{D}$ defines a *constant functor* $\mathrm{Cnst}_D : \mathcal{C} \to \mathcal{D}$ sending each object of $\mathcal{C}$ to D and each morphism of $\mathcal{C}$ to id_D. Show that this assignment defines another constant functor $\mathrm{Cnst}_{\mathcal{D}} : \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$ with the property that, for any functor $G : \mathcal{D}'' \to \mathcal{D}$, the square

$$\begin{array}{ccc} \mathcal{D}'' & \xrightarrow{\mathrm{Cnst}_{\mathcal{D}''}} & [\mathcal{C}, \mathcal{D}''] \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle [\mathcal{C}, G]} \\ \mathcal{D} & \xrightarrow{\mathrm{Cnst}_{\mathcal{D}}} & [\mathcal{C}, \mathcal{D}] \end{array}$$

commutes.

(c) Suppose that $L_{\mathcal{D}} : [\mathcal{C}, \mathcal{D}] \to \mathcal{C}$ is left adjoint to $\mathrm{Cnst}_{\mathcal{D}}$ with $L_{\mathcal{D}''}$ similarly left adjoint to $\mathrm{Cnst}_{\mathcal{D}''}$. By combining Exercise 1.3.6 with the above, show that if $F : \mathcal{D} \to \mathcal{D}''$ has a right adjoint, then $F \circ L_{\mathcal{D}}$ is naturally isomorphic to $L_{\mathcal{D}''} \circ [\mathcal{C}, F]$. In other words, the square obtained from (b) above by replacing the constant functors by their left adjoints also commutes, up to natural isomorphism.

Remark. When $L_{\mathcal{D}} : [\mathcal{C}, \mathcal{D}] \to \mathcal{C}$ is left adjoint to $\mathrm{Cnst}_{\mathcal{D}}$, then, for any functor $G : \mathcal{C} \to \mathcal{D}$, the object $L_{\mathcal{D}}(G)$ is called the *colimit* of G – see (5.1.16) and Exercise 5.1.7.

1.3.10 Let Ω be the *loop* category: Ω has one object $\bullet$, and one generating

morphism $\alpha : \bullet \to \bullet$, so that $\mathrm{Mor}(\bullet, \bullet) = \{id, \alpha, \alpha^2, \ldots\}$. Show that $[\Omega, \mathcal{D}] = \mathcal{E}_{ND}(\mathcal{D})$ for any category $\mathcal{D}$.

Find a functor $[1] \to \Omega$, where $[1]$ is the category $0 \to 1$, that gives rise to the inclusion of $\mathcal{E}_{ND}(\mathcal{D})$ in $\mathcal{M}_{OR}(\mathcal{D})$.

Devise categories $\mathcal{C}$ such that $[\mathcal{C}, \mathcal{D}]$ is (i) $\mathcal{I}_{SO}\,\mathcal{D}$, (ii) $\mathcal{A}_{UT}\,\mathcal{D}$.

1.3.11 Let $[1]^2 = [1] \times [1]$, where $[1]$ is the category $0 \to 1$, and let $\mathcal{D}$ be an arbitrary category. Show that $[1]^2$ is the 'generic' commutative square

$$\begin{array}{ccc} (0,1) & \to & (1,1) \\ \uparrow & \nearrow & \uparrow \\ (0,0) & \to & (1,0) \end{array}$$

and that $[[1]^2, \mathcal{D}]$ is the category of all commutative squares in $\mathcal{D}$.

Show also that $[[1]^2, \mathcal{D}] = \mathcal{M}_{OR}{}^2(\mathcal{D})$, that is, $\mathcal{M}_{OR}(\mathcal{M}_{OR}(\mathcal{D}))$.

Interpret $[1]^n$ and $[[1]^n, \mathcal{D}]$.

Is the category $[2]^2 = [2] \times [2]$ equivalent to $[4]$?

1.3.12 Show that $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ (1.3.5) is isomorphic to the category whose objects are the natural numbers and whose morphisms from n to m are the $m \times n$ matrices over R.

1.3.13 Let $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ and $\mathcal{B}_{ASES_R}$ be as in (1.3.5) and (1.3.12). Show that $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ is a skeletal subcategory of $\mathcal{B}_{ASES_R}$ if and only if the ring R has invariant basis number.

1.3.14 Given an arbitrary ordered set Λ, the *standard free* right R-module R^Λ is the module of 'column vectors' (r_λ) indexed by the members λ of Λ. The *standard basis* of R^Λ is $\{e_\lambda\}$, where the 'unit vector' e_λ has λ-entry 1 and all other entries 0.

Let $\mathcal{S}_{TAN}\mathcal{B}_{ASES}{}^\infty_R$ and $\mathcal{B}_{ASES}{}^\infty_R$ be as $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ and $\mathcal{B}_{ASES_R}$, but with infinite index sets also allowed. Extend the discussion which we gave in (1.3.5) to $\mathcal{S}_{TAN}\mathcal{B}_{ASES}{}^\infty_R$ and $\mathcal{B}_{ASES}{}^\infty_R$. (The definition and properties of matrices of homomorphisms with respect to infinite bases are fairly straightforward – see (2.2.13) of [BK: IRM].)

1.3.15 Let ${}^*\mathcal{F}_{REE_R}$ denote the category of based free (right) R-modules: an object is a pair (F, B) where F is free and B is a basis, and a morphism from (F', B') to (F, B) is an R-homomorphism from F' to F which restricts to an order-preserving map from B' to B. Thus ${}^*\mathcal{F}_{REE_R}$ is the category of functors from the category $\mathcal{O}_{RD}$ of ordered sets to $\mathcal{F}_{REE_R}$. Let $\mathrm{Fr} : \mathcal{O}_{RD} \to {}^*\mathcal{F}_{REE_R}$ associate with Λ the standard free module R^Λ with basis $\{e_\lambda\}$ and let $S : {}^*\mathcal{F}_{REE_R} \to \mathcal{O}_{RD}$ send (F, B) to the ordered set that indexes B.

Show that ${}^*\mathcal{F}_{REER}$ and $\mathcal{O}_{RD}$ are equivalent categories, with $S \circ \mathrm{Fr} = \mathrm{Id}$ but $\mathrm{Fr} \circ S \neq \mathrm{Id}$.

Define a dense and faithful functor $I : \mathcal{O}_{RD} \to \mathcal{S}_{TAN}\mathcal{B}_{ASES}^{\infty}_{R}$.

1.3.16 Prove the following generalization of (1.3.18).

For any ring R, there is a natural isomorphism between the categories $[[n-1], \mathcal{M}_{ODR}]$ and $\mathcal{M}_{OD T_n}$, where T_n is the ring of upper triangular $n \times n$ matrices over R.

(More results in a similar vein are given by [Mitchell 1965], IX §7ff.)

1.3.17 Let Δ (denoted Δ^* in [May 1967]) be the category whose objects are the categories $[n]$, $n = 0, 1, 2, \ldots$ and whose morphisms are the functors among these categories. For want of a name, we may think of Δ as the *proto-simplicial category* (justification in (c) below).

(a) For $0 \leq i \leq n$, define $\delta_i : [n-1] \to [n]$ and $\sigma_i : [n+1] \to [n]$ by

	$j < i$	$j = i$	$j > i$	**Effect**
$\delta_i(j)$:	j	$j+1$	$j+1$	omit i
$\sigma_i(j)$:	j	j	$j-1$	repeat i

Show that any non-identity morphism $\mu : [n] \to [m]$ in Δ can be factorized uniquely in the following way. Let the objects of $[m]$ that are not in the image of μ be $i_1 > i_2 > \cdots > i_s$, and let $\mu(j) = \mu(j+1)$ precisely when j is one of $j_1 < j_2 < \cdots < j_t$. (Thus $n + s = m + t$.) Then

$$\mu = \delta_{i_1} \cdots \delta_{i_s} \sigma_{j_1} \cdots \sigma_{j_t}.$$

(b) Algebraic topologists define a *simplicial object* in a category $\mathcal{C}$ to be a contravariant functor from Δ to $\mathcal{C}$. Show that this amounts to specifying objects $S_0, S_1, S_2, \ldots$ in $\mathcal{C}$, and two kinds of morphisms, *face operators* $d_i : S_n \to S_{n-1}$ and *degeneracy operators* $s_i : S_n \to S_{n+1}$ $(0 \leq i \leq n)$, and commuting diagrams corresponding to relations

(i)
$$d_i d_j = d_{j-1} d_i \quad \text{for } i < j$$

(ii)
$$s_i s_j = s_{j+1} s_i \quad \text{for } i \leq j$$

(iii)
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ id & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$$

This apparatus is informally represented as

$$S_0 \;\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}\; S_1 \;\underset{\longrightarrow}{\overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}}\; S_2 \cdots$$

where the face operators map from right to left, and the degeneracy operators from left to right. When $\mathcal{C} = \mathcal{S}_{\mathcal{ET}}$, we obtain a simplicial set, when $\mathcal{C} = \mathcal{G}_{\mathcal{P}}$ a simplicial group, etc.

(c) The terminology comes from the following situation. As in [Spanier 1966](3.1), a *simplicial complex* is a set V (the set of *vertices*) together with a distinguished set S (the set of *simplices*) of finite subsets of V, such that

(i) every singleton in V is in S, and
(ii) every nonempty subset of a simplex is a simplex.

Show how an ordering of the vertices gives rise to a simplicial set, and conversely, how a simplicial set describes an ordered simplicial complex, where the face and degeneracy operators are given by

$$d_i(v_0, \ldots, v_n) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n),$$
$$s_i(v_0, \ldots, v_n) = (v_0, \ldots, v_i, v_i, v_{i+1}, \ldots, v_n).$$

(See [May 1967] Ch. 1 for further discussion and topological applications. See also [Hovey 1999] and [Jardine 1996] for highly readable accounts of the relation to Quillen's closed model categories and K-theory.)

1.3.18 **A non-natural isomorphism: duality**

After so many examples of natural behaviour, the reader may begin to wonder if anything is unnatural. Here is an example which anticipates our general discussion of duality in (4.1.1).

Given a finite-dimensional vector space V over a field $\mathcal{K}$, the *dual* of V is $V^* = \mathrm{Hom}_{\mathcal{K}}(V, \mathcal{K})$. Verify that this is also a $\mathcal{K}$-space, with the addition and scalar multiplication :

$$(\alpha + \beta)(v) = \alpha(v) + \beta(v), \quad (k \cdot \alpha)(v) = k(\alpha(v)),$$

where $\alpha, \beta \in V^*$, $v \in V$, $k \in \mathcal{K}$.

Let $\{f_1, \ldots, f_n\}$ be a basis of V. The *dual basis* $\{f_1^*, \ldots, f_n^*\}$ of V^* is defined by

$$f_j^*(f_i) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Check that the dual basis is indeed a basis, and so there is an isomorphism $\eta_V : (V, \{f_j\}) \xrightarrow{\cong} (V^*, \{f_j^*\})$.

Show that this isomorphism cannot be natural as defined here, because the assignment $\Delta : \mathcal{B}_{ASES_{\mathcal{K}}} \to \mathcal{B}_{ASES_{\mathcal{K}}}$, $\Delta((V, \{f_j\})) = (V^*, \{f_j^*\})$, corresponds to a *contravariant* functor on $\mathcal{B}_{ASES_{\mathcal{K}}}$; it is a morphism functor $\mathrm{Mor}(-, \mathcal{K})$ as in (1.2.7).

One might try to rescue some naturality by hoping that squares of the form

$$\begin{array}{ccc} (V, \{f_j\}) & \xrightarrow{\eta_V} & (V^*, \{f_j^*\}) \\ \theta \downarrow & & \uparrow \theta^* \\ (W, \{g_j\}) & \xrightarrow{\eta_W} & (W^*, \{g_j^*\}) \end{array}$$

commute.

Take $(V, \{f_j\})$ to be $(\mathcal{K}^1, 1)$ and write $\theta(1) = t_1 f_1 + \cdots + t_n f_n$. Verify that $\theta^* \circ \eta_V \circ \theta(1) = (t_1^2 + \cdots + t_n^2)1^*$, and hence that such squares are not commutative in general.

If we go one step further and define $\Delta^2(V) = (V^*)^*$, the *double dual*, we obtain a covariant functor on the category of all finite-dimensional vector spaces over $\mathcal{K}$, (with morphisms the $\mathcal{K}$-linear transformations of such spaces).

Define $\epsilon_V : V \to V^{**}$ by

$$(\epsilon_V(v))(\alpha) = \alpha(v) \quad \text{for } v \in V,\ \alpha \in V^*.$$

Show that each ϵ_V is an injective linear transformation, and hence an isomorphism by dimension counting, and that the family $\{\epsilon_V\}$ is a natural isomorphism between the identity functor and Δ^2.

1.4 UNIVERSAL OBJECTS

An underlying principle in category theory is that an object should be considered in relation to the other objects in the category, rather than being viewed in isolation. This perspective leads to the idea of a universal object, which, informally, is an object that is defined by the requirement that it should have a specified relationship to the other objects in the category. When such a requirement has been determined, it can often be seen that apparently distinct kinds of object are defined by the same requirement, but in different categories.

An illustration will be provided by free objects. Although the free group on

a set X and the free R-module on X have different appearances, both share the same universal property, but in the categories $\mathcal{G}_{RP}$ of groups and $\mathcal{M}_{OD_R}$ of R-modules respectively. We can therefore define a free object on X in any suitable category $\mathcal{C}$ as an object, in $\mathcal{C}$, with this property, and we can attempt to determine its structure. In such generality, free objects may not exist.

The description of an object in terms of a universal property also provides a means of transferring definitions from one category, in which it is easy to define a particular type of object, to another, where the definition is not immediate. For example, in the module category $\mathcal{M}_{OD_R}$, the kernel of a homomorphism is defined explicity in terms of the elements of a module. However, a kernel can also be described axiomatically as a universal object in a category associated with $\mathcal{M}_{OD_R}$, which enables us to extend the definition of a kernel to more general types of category, and to verify whether or not kernels exist in the new setting. This analysis will be very useful in the next chapter.

In fact, many of the definitions in the remainder of this text, and also in K-theory, are best treated as the specification for some universal object.

In this section, we explain the meanings of the terms 'universal object' (in (1.4.2) below), 'universal construction' (1.4.8) and 'universal property' (1.4.16), and present some important examples.

1.4.1 Initial objects

First, we present some basic definitions that underly the theory. An *initial object* of an (abstract) category $\mathcal{C}$ is an object I in $\mathcal{C}$ with the property that there is exactly one morphism

$$\iota_C : I \longrightarrow C$$

for each object C in $\mathcal{C}$; thus

$$\mathrm{Mor}(I, C) = \{\iota_C\} \text{ for each } C.$$

In the category $\mathcal{G}_{RP}$ of groups, the trivial group is initial, in $\mathcal{M}_{OD_R}$ the zero module is initial, and in the category $\mathcal{R}_{ING}$ of rings (with identity), $\mathbb{Z}$ is initial. On the other hand, the infinite cyclic group Z is not initial in $\mathcal{G}_{RP}$, since there is always more than one homomorphism from Z to a nontrivial group.

An example of a category without an initial object is provided by $\mathcal{F}_{IELD}$, the full subcategory of $\mathcal{R}_{ING}$ whose objects are the fields. This follows from the observation that no field can be mapped homomorphically into both $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

An initial object need not be absolutely unique. For example, there are many manifestations of the zero module since the zero element of any module serves. However, an initial object is essentially unique.

To see this, we note that the morphism ι_I must be the identity morphism on I, and that if $\alpha : C \to C''$ is any morphism in $\mathcal{C}$, then $\alpha\iota_C = \iota_{C''}$. Thus, if I and I' are both initial objects of $\mathcal{C}$, then there are unique morphisms

$$\iota : I \longrightarrow I' \quad \text{and} \quad \iota' : I' \longrightarrow I,$$

which must satisfy the relations

$$\iota'\iota = id_I \quad \text{and} \quad \iota\iota' = id_{I'},$$

that is, ι and ι' are mutually inverse (and unique) isomorphisms.

This type of uniqueness is sometimes summarised in the phrase 'an initial object is *unique up to unique isomorphism*'. Because of it, one often speaks of 'the' initial object rather than 'an' initial object.

In many explicit situations, an initial object will appear to be absolutely unique since there is an obvious choice for one. An example is given by the cokernel $\mathrm{Cok}\alpha$ of a homomorphism $\alpha : M \to N$ between R-modules. This is almost invariably defined to be the set of cosets of N modulo $\mathrm{Im}(\alpha)$, given the appropriate module structure. However, it can be thought of as the initial object in the category of homomorphic images N'' of N such that the surjection $\pi : N \twoheadrightarrow N''$ composes with α to yield the zero map $0 = \pi\alpha$.

1.4.2 Universal objects

The philosophy underlying the term 'universal object' is as follows.

We wish to specify a type of object in a category $\mathcal{C}$ by its relationships to the totalility of objects of $\mathcal{C}$. Such relationships manifest themselves as a set of conditions which may or may not be satisfied by an object of $\mathcal{C}$. In turn, it is often possible to characterize those objects C of $\mathcal{C}$ which do satisfy the given conditions in terms of associated objects C' of a related or ancillary category $\mathcal{C}'$. If it happens that C' is an initial object in $\mathcal{C}'$, then the corresponding object C of $\mathcal{C}$ is called a universal object of $\mathcal{C}$, for the given conditions.

Since an initial object is unique up to unique isomorphism, the same will be true of a universal object. Thus one often refers to 'the' universal object with some specified properties. An example will illustrate these ideas better than formalism alone. We consider a characterization of free modules.

1.4.3 Free modules revisited

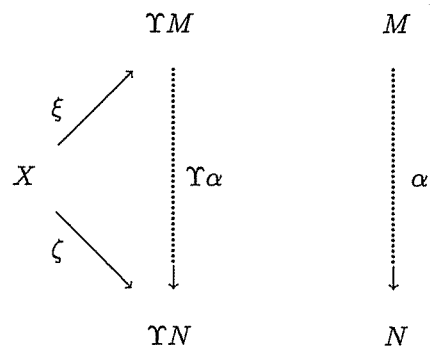
Let X be a set and let R be a ring, and consider the following condition on a pair consisting of a right R-module M together with a specified subset $\{m_x\}_{x\in X}$ of M.

Given any set $\{n_x \mid x \in X\}$ of elements n_x of any right R-module N, there exists a unique R-module homomorphism $\alpha : M \to N$ with $\alpha m_x = n_x$ for all x in X.

We show that the free right R-module $\mathrm{Fr}_R(X)$ on X (1.2.3) is the universal object in $\mathcal{M}_{OD_R}$ which satisfies this condition.

To construct the appropriate ancillary category, observe that specifying a set $\{n_x\}$ in N amounts to the same thing as defining a mapping of sets $\zeta : X \to N$. We therefore consider X as an object of the category $\mathcal{S}_{ET}$ of sets and their mappings, and use the forgetful functor $\Upsilon : \mathcal{M}_{OD_R} \to \mathcal{S}_{ET}$. If M satisfies our condition, then there must be a map $\xi : X \to M$ and a homomorphism $\alpha : M \to N$ with $\alpha\xi = \zeta$.

We therefore arrive at the category $X\backslash\Upsilon$ of (right) R-modules under X via Υ (1.2.9). The objects of this category are pairs (M, ξ) in which M is an R-module and $\xi : X \to \Upsilon M$ is a mapping (of sets), and a morphism $\alpha : (M, \xi) \to (N, \zeta)$ is given by a commutative triangle



where α is an R-module homomorphism.

The free module $\mathrm{Fr}_R(X)$, together with the map $\xi_0 : X \to \mathrm{Fr}_R(X)$ that simply identifies an element x of X as a free generator of $\mathrm{Fr}_R(X)$, as in (1.2.3), gives an initial element of $X\backslash\Upsilon$. Thus $\mathrm{Fr}_R(X)$, or more properly, the pair $(\mathrm{Fr}_R(X), \xi_0)$, is universal among R-modules which are Υ-under X.

Moreover, any free module F with a set of free generators $\{b_x\}$ labelled by X also gives an initial object (F, ξ) of $X\backslash\Upsilon$, with ξ sending x to b_x. (See [BK: IRM] (2.1.19) ff. for further details.)

Although the map ξ_0 is an essential part of the information, it is often omitted in practice, so that the free module $\mathrm{Fr}_R(X)$ is itself called the universal object. The utility of the notion of a universal object should be apparent. Replacing $\mathcal{M}_{OD_R}$ by $\mathcal{G}_P$, we are led to recover the free group on X, and if

instead we subsitute $\mathcal{A}_B$, we obtain the free abelian group on X (see [Cohn 1982], §9). Indeed, for an arbitrary category $\mathcal{C}$, one has a specification for the 'free object in $\mathcal{C}$ on X', whenever this exists.

To describe a universal object in a category $\mathcal{C}$, how should we choose the ancillary category $\mathcal{C}'$ whose initial object defines the desired universal object in $\mathcal{C}$? The above example gives a clue. It suggests that $\mathcal{C}'$ is likely to be chosen as a suitable right-fibre category.

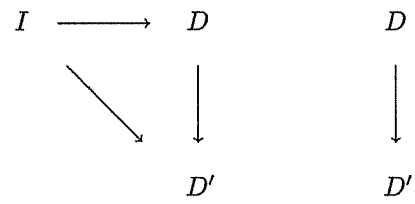
Our first result shows this idea to be formally correct in that an initial object must arise in a right-fibre category. Thus we define a *universal object* to be an initial object in a right-fibre category.

1.4.4 Lemma

If a category $\mathcal{D}$ has an initial object, then it is (isomorphic to) a right-fibre category.

Proof

Let $I \in \mathcal{D}$ be initial. Then $\mathcal{D}$ is isomorphic to the right-fibre category $I\backslash Id_{\mathcal{D}}$, via the correspondence which associates an object D of $\mathcal{D}$ with the object $I \to D$ of $I\backslash Id_{\mathcal{D}}$, and a morphism $D \to D'$ of $\mathcal{D}$ with the morphism



of $I\backslash Id_{\mathcal{D}}$. □

The converse to this lemma is false, as may be seen from Exercise 1.4.1.

In general, the right-fibre categories that we seek will arise from adjoint pairs of functors, as we now proceed to demonstrate.

We start by reminding ourselves of the definition of an adjoint pair (1.3.7). A pair $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ of (covariant) functors between the categories $\mathcal{C}$ and $\mathcal{D}$ is an adjoint pair if there is a natural isomorphism φ between the bifunctors $\mathrm{Mor}_{\mathcal{D}}(F(-), -)$ and $\mathrm{Mor}_{\mathcal{C}}(-, G(-))$.

To define such an isomorphism φ, we must exhibit a family of bijections $\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \to \mathrm{Mor}_{\mathcal{C}}(C, G(D))$ which are natural in C and D. We argue in two stages. In the first, we fix C and allow D to vary. After a preliminary lemma to decide when $\varphi_{(C,D)}$ is natural in D, we show that $\varphi_{(C,D)}$ being a bijection for all D corresponds to the existence of an initial object in a suitable right-fibre category.

1.4.5 Lemma

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, let C be an object of $\mathcal{C}$ and let $F(C)$ be an object of $\mathcal{D}$. Suppose that for each object D in $\mathcal{D}$ there is a mapping of sets

$$\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{C}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D)).$$

Then the following are equivalent:

(a) *for all $g \in \mathrm{Mor}_{\mathcal{D}}(F(C), D)$,*

$$\varphi_{(C,D)}(g) = G(g) \circ \varphi_{(C,F(C))}(id_{F(C)});$$

(b) *φ is natural in D.*

Proof

(a) $\Rightarrow$ (b). To show that φ_D is natural in D, we have to verify the criterion of (1.3.2)(i), that for any morphism $\delta : D \to D'$ in $\mathcal{D}$, the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(F(C), D) & \xrightarrow{\phi_{(C,D)}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D)) \\ \Big\downarrow \delta_* & & \Big\downarrow G(\delta)_* \\ \mathrm{Mor}_{\mathcal{D}}(F(C), D') & \xrightarrow{\phi_{(C,D')}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D')) \end{array}$$

is commutative. Take any $g : F(C) \to D$. Then, combining (a) with the definitions in (1.2.7), we have

$$\begin{aligned} G(\delta)_*(\varphi_{(C,D)}(g)) &= G(\delta) \circ G(g) \circ \varphi_{(C,F(C))}(id_{F(C)}) \\ &= G(\delta g) \circ \varphi_{(C,F(C))}(id_{F(C)}) \\ &= \varphi_{(C,D')}(\delta g) \\ &= \varphi_{(C,D')}(\delta_*(g)). \end{aligned}$$

(b) $\Rightarrow$ (a). In the above diagram, we set $D = F(C)$ and $\delta = g : F(C) \to D$. Then commutativity yields

$$\begin{aligned} \varphi_{(C,D)}(g) &= \varphi_{(C,D)}(g \circ id_{F(C)}) \\ &= \varphi_{(C,D)}(g_*(id_{F(C)})) \\ &= G(g)_*(\varphi_{(C,F(C))}(id_{F(C)})). \end{aligned}$$

□

1.4.6 Lemma

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, let C be an object of $\mathcal{C}$ and let $F(C)$ be an object of $\mathcal{D}$, with $u : C \to G(F(C))$ a morphism in $\mathcal{C}$. Then the following assertions are equivalent.

(i) *$(F(C), u)$ is an initial object in the right-fibre category $C\backslash G$.*

(ii) *There is a natural isomorphism of functors*

$$\varphi_{(C,-)} : \mathrm{Mor}_{\mathcal{D}}(F(C), -) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(-))$$

such that $u = \varphi_{(C,F(C))}(id_{F(C)})$.

Proof

We first use the preceding lemma to make explicit the effect of the natural transformation φ occurring in (ii) on any $g \in \mathrm{Mor}_{\mathcal{D}}(F(C), D)$. On the one hand, if (ii) holds, then by the lemma,

$$\varphi_{(C,D)}(g) = G(g) \circ \varphi_{(C,F(C))}(id_{F(C)}) = G(g) \circ u.$$

On the other hand, if we start with u, then for each D in $\mathcal{D}$ we may define a mapping

$$\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D))$$

by the equation

$$\varphi_{(C,D)}(g) = G(g) \circ u.$$

Since then

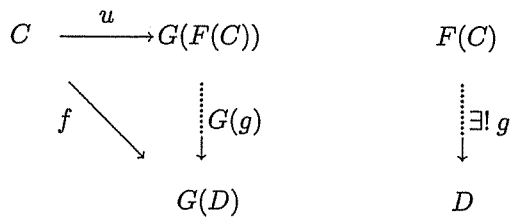
$$\varphi_{(C,F(C))}(id_{F(C)}) = G(id_{F(C)}) \circ u = u,$$

it follows from the lemma that this mapping is a natural transformation with respect to D. So in either event, it suffices to consider the natural transformation $\varphi_{(C,-)}$ defined by

$$\varphi_{(C,D)}(g) = G(g) \circ u.$$

Then it is easy to see that this natural transformation is an isomorphism if and only if (i) holds, since both properties correspond to the existence of a unique morphism g from $(F(C), u)$ to (D, f) for each object (D, f) of $C\backslash G$,

as described by the diagram



(Here the triangle commutes, and the exclamation mark expresses uniqueness rather than astonishment.) □

We now extend the preceding lemma by showing that the existence of an initial object in $C\backslash G$ for every object C can be neatly reformulated in the language of adjoint functors.

1.4.7 Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then the following assertions are equivalent.

(i) *For any object C of $\mathcal{C}$ there exists an object $F(C)$ of $\mathcal{D}$ and a morphism $u_C : C \to G(F(C))$ in $\mathcal{C}$ such that $(F(C), u_C)$ is an initial object in the right-fibre category $C\backslash G$.*

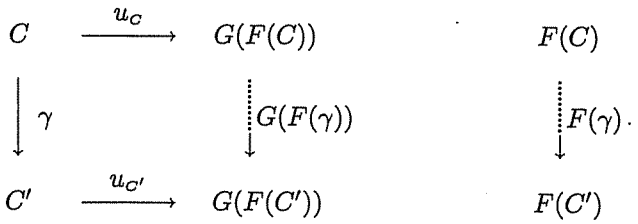
(ii) *G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$.*

Proof

(ii) $\Rightarrow$ (i) follows immediately from the above lemma.

(i) $\Rightarrow$ (ii): As the notation suggests, the effect of the functor F on an object C of $\mathcal{C}$ is to send it to $F(C)$. We must specify the effect of F on morphisms and verify functoriality.

Let $\gamma : C \to C'$ be a morphism in $\mathcal{C}$. The composite $u_{C'} \circ \gamma$ gives an object $(F(C'), u_{C'} \circ \gamma)$ of the right-fibre category $C\backslash G$, so we may define $F(\gamma) : F(C) \to F(C')$ as the unique morphism from $(F(C), u_C)$ to $(F(C'), u_{C'} \circ \gamma)$ in $C\backslash G$; this definition can be encapsulated by the diagram



in which the square is commutative.

Appealing to uniqueness again, we see that $F(id_C) = id_{F(C)}$ and that $F(\gamma'\gamma) = F(\gamma')F(\gamma)$ for a morphism $\gamma' : C' \to C''$. Thus F is a functor.

By the preceding lemma, the map $\varphi_{(C,D)}$ is a bijection for each pair C, D, and φ is natural in D. The remaining point to check is that it is also natural in C.

By (1.3.2)(i), this amounts to checking that, for any morphism $\gamma : C \to C'$ in $\mathcal{C}$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(F(C), D) & \xrightarrow{\phi_{(C,D)}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D)) \\ \uparrow{\scriptstyle F(\gamma)^*} & & \uparrow{\scriptstyle \gamma^*} \\ \mathrm{Mor}_{\mathcal{D}}(F(C'), D) & \xrightarrow{\phi_{(C',D)}} & \mathrm{Mor}_{\mathcal{C}}(C', G(D)) \end{array}$$

(Note that the vertical arrows point upwards.)

Take any $h \in \mathrm{Mor}_{\mathcal{D}}(F(C'), D)$. Then

$$\begin{aligned} \varphi_{(C,D)}(F(\gamma)^*(h)) &= \varphi_{(C,D)}(h \circ F(\gamma)) \\ &= G(h \circ F(\gamma)) \circ u_C \\ &= G(h) \circ GF(\gamma) \circ u_C, \end{aligned}$$

while

$$\begin{aligned} \gamma^*(\varphi_{(C',D)}(h)) &= \gamma^*(G(h) \circ u_{C'}) \\ &= G(h) \circ u_{C'} \circ \gamma. \end{aligned}$$

However, $u_{C'} \circ \gamma = GF(\gamma) \circ u_C$ by construction; so the result follows. □

1.4.8 *Universal constructions*

In the light of this theorem we may now make the formal definition that a *universal construction* is a functor which has a right adjoint.

We have already noted in (1.3.8) above that the forgetful functor $\Upsilon : \mathcal{M}od_R \to \mathcal{S}et$ has as its left adjoint the functor $\mathrm{Fr}_R : \mathcal{S}et \to \mathcal{M}od_R$ which associates to each set X the free right R-module $\mathrm{Fr}_R(X)$ on X. Similarly, the free group and free abelian group constructions defined on $\mathcal{S}et$ are examples of universal constructions whose right adjoints are the forgetful functors from $\mathcal{G}p$ and $\mathcal{A}b$ to $\mathcal{S}et$.

An extensive catalogue of examples may be found in [Mac Lane 1971] and [Herrlich & Strecker 1979].

1.4.9 Terminal objects

Before we give more examples of universal objects, we mention another manner in which they can arise.

A *terminal object* in a category $\mathcal{C}$ is an object T such that $\mathrm{Mor}(C,T)$ contains exactly one morphism $\tau_C : C \to T$ for each C in $\mathcal{C}$. As with initial objects, a terminal object is unique to within unique isomorphism, if one exists at all.

It sometimes happens that the description of a universal object which satisfies some set of conditions is more naturally given in terms of a terminal object of a convenient ancillary category, rather than an initial object as in (1.4.2) above. We therefore extend the definition of a universal object to include those arising in this way. (Some authors prefer to use the term *couniversal object* for such objects.) Notice that the object T in $\mathcal{C}$ is terminal if and only if the corresponding object T^{op} is initial in the opposite category $\mathcal{C}^{\mathrm{op}}$ (1.1.6), so it is possible to avoid the use of terminal objects by changing the ancillary category. However, it would be rather artificial to do this.

1.4.10 The direct sum revisited

Our next examples of universal objects arise by generalizing the definitions of direct sums and products of modules to arbitrary categories. We draw on some well-known basic definitions and results that are discussed in detail in section 2.1 of [BK: IRM].

The most elementary definition of the direct sum $M = M' \oplus M''$ of right R-modules M' and M'' is in terms of ordered pairs:

$$M = \{(m', m'') \mid m' \in M', \ \ m'' \in M''\}$$

with the expected componentwise addition and scalar multiplication. However, this definition does not lend itself to generalization, so we must recast it in terms of homomorphisms. When $M = M' \oplus M''$, there are homomorphisms

$$\sigma' : M' \longrightarrow M \quad \text{and} \quad \sigma'' : M'' \longrightarrow M$$

given by inclusion on the first and second summands respectively, and homomorphisms

$$\pi' : M \longrightarrow M' \quad \text{and} \quad \pi'' : M \longrightarrow M''$$

given by the corresponding projections.

These homomorphisms satisfy the relations

$$\pi'\sigma' = id_{M'}, \ \ \pi''\sigma'' = id_{M''} \ \text{ and } \ \sigma'\pi' + \sigma''\pi'' = id_M.$$

Conversely, the existence of such a set of homomorphisms for a module M shows that it is isomorphic to the direct sum of M' and M'' (Proposition 2.1.7 of [BK: IRM]).

A further characterization is in terms of the existence of split short exact sequences [BK: IRM] (2.4.5).

In a category where we can speak about sums of maps or about exact sequences, we can investigate whether there are pairs of morphisms analogous to the projections π', π'' and the inclusions σ', σ''. We pursue this line of attack in the next chapter.

In the most general setting, however, sums of maps and exact sequences are not available to us. It is then more productive to focus on the pair of projections, or alternatively the pair of inclusion maps, from the standpoint of universal objects. The former approach gives rise to the general definition of the product of two objects, while the latter leads to the coproduct. These two generalizations of 'direct sum' need not coincide.

1.4.11 The product

Let $\mathcal{C}$ be any category and let C' and C'' be any two objects of $\mathcal{C}$. Define a category $\mathcal{P}_{\mathcal{RD}}(C', C'')$ as follows. The objects of $\mathcal{P}_{\mathcal{RD}}(C', C'')$ are all the triples (A, α', α'') where $\alpha' : A \to C'$ and $\alpha'' : A \to C''$ are morphisms in $\mathcal{C}$, and a morphism $\xi : (A, \alpha', \alpha'') \to (B, \beta', \beta'')$ is given by a morphism $\xi : A \to B$ in $\mathcal{C}$ so that $\beta'\xi = \alpha'$ and $\beta''\xi = \alpha''$. Thus a morphism is given by a commutative diagram

$$\begin{array}{ccccc} C' & \xleftarrow{\alpha'} & A & \xrightarrow{\alpha''} & C'' \\ \| & & \downarrow \xi & & \| \\ C' & \xleftarrow{\beta'} & B & \xrightarrow{\beta''} & C'' \end{array}$$

A *product* of C' and C'' is a terminal object $(C' \prod C'', \pi', \pi'')$ in the category $\mathcal{P}_{\mathcal{RD}}(C', C'')$, if such exists. It is usual in everyday language to refer to the corresponding universal object $C' \prod C''$ of $\mathcal{C}$ as the product, suppressing the morphisms π' and π''.

In $\mathcal{M}_{\mathcal{OD}_R}$, the direct sum $M' \oplus M''$ is the product in this sense, while in the category $\mathcal{G}_{\mathcal{P}}$ of groups, the product of G' and G'' is the usual cartesian product $G' \times G''$ with componentwise multiplication.

1.4.12 The coproduct

The analogous definition in terms of inclusions leads to the coproduct. Again, let C' and C'' be two fixed objects of a category $\mathcal{C}$. The category $\mathcal{C}_{OPRD}(C', C'')$ has as objects triples (A, α', α'') where now $\alpha' : C' \to A$ and $\alpha'' : C'' \to A$, and a morphism ξ from (A, α', α'') to (B, β', β'') is given by a morphism $\xi : A \to B$ with $\xi\alpha' = \beta'$ and $\xi\alpha'' = \beta''$. A *coproduct* of C' and C'' is an initial object $(C' \coprod C'', \sigma', \sigma'')$ in this category, or alternatively, the corresponding universal object $C' \coprod C''$ in $\mathcal{C}$.

In $\mathcal{M}_{OD_R}$, the direct sum $M' \oplus M''$ is the coproduct of the modules M' and M'' as well as being their product. On the other hand, the coproduct in the category of groups is the *free product* $G' * G''$ of G' and G'', which assertion is easily seen from the explicit description of the free product. The elements of $G' * G''$ are formal products $h_1 \cdots h_k$ with each h_i in either G' or G'', of arbitrary length $k \geq 1$; the product is given by

$$(h_1 \cdots h_k)(h'_1 \cdots h'_{k'}) = h_1 \cdots h_k h'_1 \cdots h'_{k'}$$

and the inverse by

$$(h_1 \cdots h_k)^{-1} = h_k^{-1} \cdots h_1^{-1}.$$

A detailed account of the construction of the free product can be found in, for example, [Scott 1964], Chapter 8.

Underlying the coincidence of the product and coproduct in $\mathcal{M}_{OD_R}$ is the fact that $\mathrm{Hom}(M, N)$ is an abelian group for any two modules M, N. We look at this property in more detail in the next chapter.

1.4.13 Arbitrary products and coproducts

We next outline briefly the definition of products and coproducts for an arbitrary set of objects.

Let Λ be any ordered set and let $C(\Lambda) = \{C'_\lambda \mid \lambda \in \lambda\}$ be a set of objects of some category $\mathcal{C}$ which are indexed by Λ. To define the product, we introduce the category $\mathcal{P}_{RD}(C(\Lambda))$ whose objects are sets $\{\alpha_\lambda : A \to C_\lambda\}$ of morphisms in $\mathcal{C}$, a morphism

$$\xi : \{\alpha_\lambda : A \longrightarrow C_\lambda\} \longrightarrow \{\beta_\lambda : B \longrightarrow C_\lambda\}$$

being given by a morphism $\xi : A \to B$ with $\alpha_\lambda = \beta_\lambda \xi$ for all λ.

Then the *product* $\prod_\Lambda C_\lambda$ is the universal object in $\mathcal{C}$ corresponding to a terminal object in $\mathcal{P}_{RD}(C(\Lambda))$.

Dually, we define the category $\mathcal{C}_{OPRD}(C(\Lambda))$ to have objects $\{\alpha_\lambda : C_\lambda \to A\}$

and evident morphisms, and the *coproduct* $\coprod_\Lambda C_\lambda$ to be the universal object of $\mathcal{C}$ given by an initial object in $\mathcal{C}_{OPRD}(C(\Lambda))$.

When the index set Λ is infinite, the product and coproduct differ even in the category $\mathcal{M}_{OD_R}$. Using the definitions ([BK: IRM] (2.1.11)), the reader should have no difficulty verifying that the product of a set of modules $\{M_\lambda\}$ is the direct product $\prod_\Lambda M_\lambda$, while the coproduct is the direct sum $\bigoplus_\Lambda M_\lambda$. (However, the direct sum and product coincide for any finite index set.)

1.4.14 *Zero objects*

An object 0 of a category $\mathcal{C}$ which is both initial and terminal is said to be a *zero object* of $\mathcal{C}$. In this case, the unique morphisms $0 \to C$ and $C \to 0$ are both denoted 0, so that both $\mathrm{Mor}_\mathcal{C}(0, C) = \{0\}$ and $\mathrm{Mor}_\mathcal{C}(C, 0) = \{0\}$. Clearly, a zero object is unique up to unique isomorphism if it exists.

The zero module is the zero object in the category $\mathcal{M}_{OD_R}$ and the trivial group is the zero object in $\mathcal{G}_\mathcal{P}$. On the other hand, $\mathcal{R}_{ING}$ has no zero object, since its initial and terminal objects $\mathbb{Z}$ and 0 are different.

When $\mathcal{C}$ has a zero object 0, we can define the zero morphism between any two objects C and D to be the composite

$$0 : C \longrightarrow 0 \longrightarrow D.$$

This morphism is easily seen to be independent of the choice of zero object, and has the property that

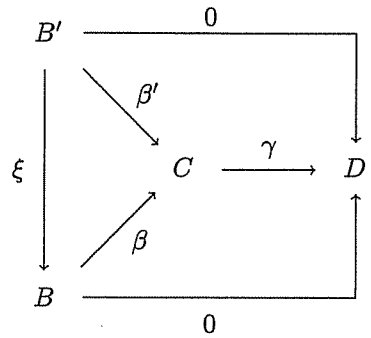
$$0\beta = 0 : B \longrightarrow D \quad \text{and} \quad \delta 0 = 0 : C \longrightarrow E$$

for any morphisms $\beta : B \to C$ and $\delta : D \to E$.

1.4.15 *Kernels and cokernels*

The definition of a kernel and cokernel of a morphism can be extended to any category with a zero object, although at this level of abstraction, a given morphism need have neither a kernel nor cokernel.

Let $\gamma : C \to D$ be a morphism in such a category $\mathcal{C}$, and define a new category $\mathcal{K}_{ER}(\gamma)$ as follows. The objects of $\mathcal{K}_{ER}(\gamma)$ are all the pairs (B, β) where B is an object of $\mathcal{C}$ and $\beta : B \to C$ is a morphism with $\gamma\beta = 0$. A morphism $\xi : (B', \beta') \to (B, \beta)$ is a morphism $\xi : B' \to B$ in $\mathcal{C}$ such that $\beta\xi = \beta'$. The morphisms can be described by commutative diagrams



Evidently, $\mathcal{K}_{ER}(\gamma)$ is a subcategory of the left-fibre category Id_C/C (see Exercise 1.4.4 below).

A *kernel* of γ is a pair $(\mathrm{Ker}\,\gamma, \kappa)$ which is terminal in $\mathcal{K}_{ER}(\gamma)$, if such an object exists. There is some redundancy in this notation, since $\mathrm{Ker}\,\gamma$ is the domain of κ. Thus one may validly refer to the morphism κ as the kernel of γ. Alternatively, given the object $\mathrm{Ker}\,\gamma$, there is often a canonical choice of κ (most commonly, κ is taken to be an inclusion map, which is usually possible when $\mathcal{C}$ is a subcategory of $\mathcal{S}_{ET}$). In this case, it is the object $\mathrm{Ker}\,\gamma$ that is called the kernel of γ.

To define the cokernel of γ, we introduce a category $\mathcal{C}_{OK}(\gamma)$ with objects (E, ϵ) where $\epsilon\gamma = 0$, and morphisms $\nu : (E, \epsilon) \to (E'', \epsilon'')$ given by morphisms $\nu : E \to E''$ with $\nu\epsilon = \epsilon''$. A *cokernel* of γ is then an initial object $(\mathrm{Cok}\,\gamma, \chi)$ in this category, if there is one.

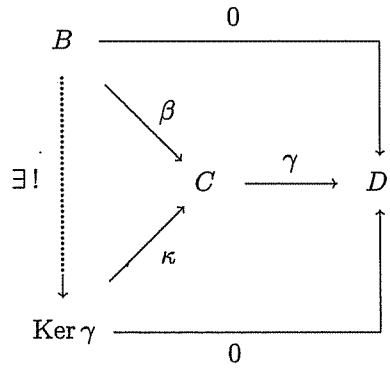
It is easy to check that in $\mathcal{M}_{OD_R}$, the usual kernel and cokernel remain such in this abstract sense. Because the maps κ and χ are then always taken to be inclusion and the standard surjection respectively, it is usual to omit them.

Further discussion of the circumstances in which categories may fail to contain kernels and cokernels is given in the next chapter. An immediate (but contrived) example is provided by $\mathcal{I}_{NF}\mathcal{A}_B$, the full subcategory of $\mathcal{A}_B$ whose objects are the infinite abelian groups. Since $\mathcal{A}_B$ is $\mathcal{M}_{OD_\mathbb{Z}}$ under another name, it has kernels and cokernels as expected. Take A to be the multiplicative group of nonzero real numbers, and let σ be the endomorphism $\sigma(x) = x^2$ of A. Then σ has neither kernel nor cokernel in $\mathcal{I}_{NF}\mathcal{A}_B$.

1.4.16 Universal properties

At this point we introduce a third term commonly found in the literature, along with 'universal object' and 'universal construction'. A *universal property* is the property that characterizes a universal object; it is frequently

described by means of a diagram. Such a diagram typically highlights the morphism whose existence and uniqueness determines the universal object. For example, one may say that the kernel of a morphism $\gamma : C \to D$ enjoys the universal property



that makes the above diagram commutative. Here, the existence and uniqueness of the vertical arrow from B to $\operatorname{Ker}\gamma$ which makes the above diagram commutative expresses the fact that the pair $(\operatorname{Ker}\gamma, \kappa)$ is terminal in the category $\mathcal{K}\varepsilon_R(\gamma)$.

1.4.17 Pleasure versus guilt: the universal dilemma

Being natural, functors have their emotional side too. Typically, as we have just seen, a universal property is pleasantly depicted by a commuting diagram (leading to the existence of a unique morphism). It is said that the universal object 'enjoys' the universal property. The existence of an adjoint functor, on the other hand, often involves a tortuous check; a functor is said to 'admit' an adjoint. See Exercise 1.4.8 for an example of this phenomenon.

1.4.18 Kernels of natural transformations

We can also define kernels, cokernels, and so on, in the functor category $[\mathcal{C}, \mathcal{D}]$ provided they exist in $\mathcal{D}$. Suppose, for example, that any morphism α in $\mathcal{D}$ has a kernel $(\operatorname{Ker}\alpha, \kappa(\alpha))$. We construct for each $\eta \in \operatorname{Nat}(F, G)$ a kernel $(\operatorname{Ker}\eta, \kappa)$ as follows.

The object $\operatorname{Ker}\eta$ in $[\mathcal{C}, \mathcal{D}]$ is to be a functor from $\mathcal{C}$ to $\mathcal{D}$. For each object C of $\mathcal{C}$ we take the evident definition

$$(\operatorname{Ker}\eta)(C) = \operatorname{Ker}\eta_C.$$

(Here, $\operatorname{Ker}\eta_C$ is any fixed choice from among the mutually isomorphic objects in $\mathcal{D}$ which represent the kernel of η_C.)

To define $(\operatorname{Ker}\eta)(\alpha)$ for a morphism $\alpha : C' \to C$, we first recall that the diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

is commutative. Thus the definition of a kernel as a terminal object (1.4.15) shows that there is an induced morphism

$$(\operatorname{Ker}\eta)(\alpha) : (\operatorname{Ker}\eta)(C') \longrightarrow (\operatorname{Ker}\eta)(C)$$

which fits into the commutative diagram

$$\begin{array}{ccccc} (\operatorname{Ker}\eta)(C') & \xrightarrow{\kappa(\eta_{C'})} & FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle (\operatorname{Ker}\eta)(\alpha)}\vdots\downarrow & & {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ (\operatorname{Ker}\eta)(C) & \xrightarrow{\kappa(\eta_C)} & FC & \xrightarrow{\eta_C} & GC \end{array}$$

It is now easy to confirm that, with the above choice of $(\operatorname{Ker}\eta)(\alpha)$, $\operatorname{Ker}\eta$ is indeed a functor.

The natural transformation

$$\kappa : \operatorname{Ker}\eta \longrightarrow F$$

is given by $\kappa_C = \kappa(\eta_C) : (\operatorname{Ker}\eta)(C) \to F(C)$. A routine verification shows that we have a kernel for η, as claimed.

1.4.19 Some history

A nice description of the origin of category theory is given in Chapter 1, §5E of [Dieudonné 1989], which we summarize. In the early 1940s, Eilenberg and Mac Lane were seeking to make precise the idea of a 'natural' transformation between various constructions in algebraic topology. Typically, such constructions start with a topological space and yield a group (fundamental group, homology group, etc.). The authors sought a framework for saying that a map of topological spaces gives rise to a homomorphism of groups in a well-regulated manner. In [Eilenberg & Mac Lane 1942], they achieved this

aim by introducing the notion of a 'functor', as they called a function F that not only operates on a set C (with extra structure) to give a new set $F(C)$, but also on a (structure-preserving) function $\gamma : C \to D$ to give a function $F(\gamma) : F(C) \to F(D)$. Then the idea of a natural transformation can be made precise, as in (1.3).

Subsequently, in [Eilenberg & Mac Lane 1945], the concept of a category was introduced to formalize the viewpoint that objects that share a common structure should not be considered in isolation, but together with the mappings that respect this structure. Once this level of abstraction had been attained, it became possible to introduce categories, such as opposite categories, morphism categories, etc., whose objects are no longer 'sets-with-structure'. In turn, the freedom to construct categories led to the characterization of universal constructions as initial or terminal objects in a suitably manufactured category. The notion of 'universality' first appears in [Samuel 1948], while the description of universal constructions in terms of adjoint functors (1.4.8) is due to [Kan 1958].

Exercises

1.4.1 Let $\mathcal{D}$ be the category with one object X and one morphism f other than the identity id_X, subject to $f \circ f = f$, say. Let $\mathcal{C}$ be the subcategory obtained by removing the morphism f, and let $G : \mathcal{D} \to \mathcal{C}$ be the constant functor (in fact, the only possible functor). Show that the right-fibre category $X\backslash G$ is isomorphic to $\mathcal{D}$. Evidently $\mathcal{D}$ lacks an initial object, so the same is true of $X\backslash G$.

The reader is invited to construct other right-fibre categories which lack an initial object. However, initial objects are surprisingly common in any choices of right-fibre category that one is likely to make.

1.4.2 Show that if (K, κ) is the kernel of a morphism γ in a category $\mathcal{C}$, then $(K^{\mathrm{op}}, \kappa^{\mathrm{op}})$ is the cokernel of γ^{op} in $\mathcal{C}^{\mathrm{op}}$ (and conversely).

Show also that if C, D are objects of $\mathcal{C}$ with product $C \prod D$ in $\mathcal{C}$, then $(C \prod D)^{\mathrm{op}} = C^{\mathrm{op}} \coprod D^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$ (and conversely).

1.4.3 Let R and S be rings. Show that the direct product $R \times S$ is a product in $\mathcal{R}_{ING}$ but not a coproduct.

1.4.4 Sometimes one has to be quite ingenious to present a universal construction as an initial or terminal object in a right- or left-fibre category. To discuss kernels and cokernels in this light, let $\gamma : C \to D$ be a morphism in a category $\mathcal{C}$ which has zero object 0. As noted, the category $\mathcal{K}_{ER}\gamma$ is a subcategory of the left-fibre category $Id_C/C = \mathcal{C}/C$.

Define a functor $\gamma_* : \mathcal{C}/C \to \mathcal{C}/D$ by $\gamma_*(B, \beta) = (B, \gamma\beta)$ and, for any morphism $\xi : (B', \beta') \to (B, \beta)$ in $\mathcal{C}/C$ (that is, $\beta\xi = \beta'$),

$$\gamma_*\xi = \xi : (B', \gamma\beta') \longrightarrow (B, \gamma\beta).$$

Show that the left-fibre category $\gamma_*/(0 \to D)$, where $(0 \to D)$ is the initial object of the category $\mathcal{C}/D$, has as objects commuting squares of the form

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \gamma \\ 0 & \longrightarrow & D \end{array}$$

Deduce that $\mathcal{K}\mathit{er}\,\gamma$ is isomorphic to $\gamma_*/(0 \to D)$.

Provide the dual description of $\mathcal{C}\mathit{ok}\,\gamma$.

1.4.5 What is the universal property enjoyed by the abelianization G_{ab} of a group G?

1.4.6 Let $\gamma : G \to F$ be a homomorphism of groups. Show that the usual definition of $\operatorname{Ker}\gamma$ does give a kernel in the category $\mathcal{G}_{\mathcal{P}}$.

Suppose that H is a normal subgroup of G, and let η be the inclusion homomorphism. Show that $\operatorname{Cok}\eta = G/H$.

Let H be an arbitrary subgroup of G. Then the *normal closure* H^G of H is the smallest normal subgroup of G which contains H. Verify that the inclusion homomorphism $\eta : H \to G$ has $\operatorname{Cok}\eta = G/H^G$.

Deduce that for an arbitrary homomorphism $\eta : H \to G$ in $\mathcal{G}_{\mathcal{P}}$, $\operatorname{Cok}\eta = G/(\operatorname{Im}\eta)^G$.

1.4.7 In the category $\mathcal{R}_{\mathcal{NG}}$ of nonunital rings (that is, 'rings' which need not have an identity element and homomorphisms which need not preserve it), show that the zero ring 0 is a zero object.

Let $\theta : R \to S$ be a homomorphism of nonunital rings. Show that the usual kernel $\operatorname{Ker}\theta$ is a kernel in $\mathcal{R}_{\mathcal{NG}}$, and that $\operatorname{Ker}\theta$ is an ideal of R (where we extend the definition of ideals to nonunital rings in the obvious way).

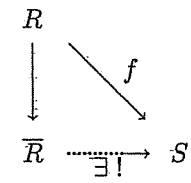
Show that if I is an ideal of S, then there is a residue nonunital ring R/I which is a cokernel for the inclusion homomorphism of I in R.

Given a sub-nonunital ring S' of S, let $I(S')$ be the smallest ideal of S which contains S'. Show that $\operatorname{Cok}\theta = S/I(\operatorname{Im}\theta)$.

1.4.8 Consider the functor from $\mathcal{R}_{\mathcal{NG}}$ to $\mathcal{R}_{\mathcal{ING}}$ which takes a nonunital ring R to its enveloping ring $\overline{R}$ (1.3.2)(iv). Show that its right adjoint

is the forgetful functor from $\mathcal{R}_{ING}$ to $\mathcal{R}_{NG}$ which simply ignores the multiplicative identity of a ring.

To describe the associated universal property, let $f : R \to S$ be a nonunital ring homomorphism. Show that if S is a ring, then f factors uniquely through the standard embedding of R in $\overline{R}$. Diagrammatically, we have:



1.4.9 **Sets**

Show that the empty set $\emptyset$ is an initial object in $\mathcal{S}_{ET}$ (note that $\mathrm{Map}(\emptyset, X)$ contains $|X|^0$ members since $0 = |\emptyset|$). Show also that $\mathcal{S}_{ET}$ has no terminal object.

Prove that the product (in $\mathcal{S}_{ET}$) of a set $\{X_\lambda \mid \lambda \in \Lambda\}$ of sets is the cartesian product $\prod_\Lambda X_\lambda$.

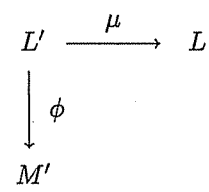
Define the *disjoint union* $\bigsqcup_\Lambda X_\lambda$ to be the union $\bigcup_\Lambda (X_\lambda, \lambda)$, where (X_λ, λ) is the set of pairs (x_λ, λ) with $x_\lambda \in X_\lambda$. (The purpose of this construction is to remove any 'accidental' overlap between the sets X_λ; note that $(X_\lambda, \lambda) \cap (X_\mu, \mu) = \emptyset$ for $\lambda \neq \mu$, even if X_λ and X_μ have common members.)

Show that the coproduct of the set of sets $\{X_\lambda\}$ is $\bigsqcup_\Lambda X_\lambda$.

1.4.10 **Push-outs**

This and the following exercises indicate how the definitions of push-outs and pull-backs can be generalized from modules to arbitrary categories. The constructions for modules are given in detail in [BK: IRM] (2.4.8)ff.

Given a diagram



of right R-modules, the push-out (of M' and L over L') is

$$M' \oplus_{L'} L = (M' \oplus L)/\{(\phi\ell', -m\ell') \mid \ell' \in L'\}.$$

Furthermore, there are homomorphisms $\overline{\mu} : M' \to M' \oplus_{L'} L$ and $\overline{\phi} : L \to M' \oplus_{L'} L$ which give a commutative diagram

$$\begin{array}{ccc} L' & \xrightarrow{\mu} & L \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} \\ M' & \xrightarrow{\overline{\mu}} & M' \oplus_{L'} L \end{array}$$

and if

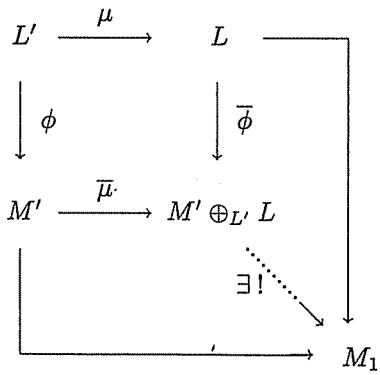
$$\begin{array}{ccc} L' & \xrightarrow{\mu} & L \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_1} \\ M' & \xrightarrow{\mu_1} & M_1 \end{array}$$

is a commutative diagram, then there is a unique homomorphism

$$\xi : M' \oplus_{L'} L \longrightarrow M_1$$

with $\xi\overline{\mu} = \mu_1$ and $\xi\overline{\phi} = \phi_1$.

Thus the universal property may be described by the diagram



Show that the push-out arises through an initial object in an appropriate category, and hence extend the definition of a push-out to an arbitrary category $\mathcal{C}$.

Prove further that a push-out in $\mathcal{C}$ may be regarded as a coproduct in a suitable right-fibre category, and conversely, that if $\mathcal{C}$ has an initial object, then a coproduct is a special case of a push-out.

1.4.11 **Pull-backs**

Suppose that, in a category $\mathcal{C}$, we are given a diagram of the form

$$\begin{array}{ccc} & & L'' \\ & & \downarrow \phi \\ M & \xrightarrow{\beta} & M'' \end{array}$$

where β and θ are morphisms.

Arguing by duality, define the *pull-back* $M \times_{M''} L''$ of M and L'' over M'' in terms of a universal property.

Confirm that in the category of right R-modules, the pull-back is given explicitly (as in (2.4.8) of [BK: IRM]) by the formula

$$M \times_{M''} L'' = \{(m, \ell'') \in M \oplus L'' \mid \beta m = \theta \ell''\}.$$

Show also that if $\mathcal{C}$ has a terminal object, then a product is a special case of a pull-back.

1.4.12 Determine the push-outs and pull-backs in the category $\mathcal{S}_{ET}$ (see Exercise 1.4.9 above).

1.4.13 Let $\beta : M \to M''$ and $\delta : M'' \to N''$ be morphisms in a category with a zero object, and let $\alpha : N' \to M''$ be the kernel of δ. By comparing universal properties, show that, whenever the pull-back exists, the canonical morphism $\overline{\alpha} : M \times_{M''} N' \to M$ is the kernel of $\delta\beta : M \to N''$. (See Exercise 2.4.7 of [BK: IRM] for a discussion of this situation in $\mathcal{M}_{OD_R}$.)

1.4.14 For this, one needs some routine facts about products of groups. These can be found in [Scott 1964] Chapter 8, for example.

Show that the pull-back in $\mathcal{G}_P$ is the restricted direct product and that the push-out is the amalgamated free product.

Remark. There are corresponding but more complicated results for rings, where it is usually preferable to work in a category of *A-algebras*: for an arbitrary commutative ring A, an A-algebra is a ring R that has A contained in its centre. Some details can be found in [Rowen 1988], §§1.4, 1.9, and [Eisenbud 1995] A6.3.

1.4.15 Let $\mathcal{C}_{AT}$ be a category of small categories. Show that for categories $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{C}_{AT}$, their direct product $\mathcal{C} \times \mathcal{D}$ is a product in $\mathcal{C}_{AT}$ in the sense of (1.4.11).

2

CATEGORIES AND EXACT SEQENCES

The definition of an abstract category imposes few conditions on the objects and morphisms of a category, with the consequence that a great diversity of mathematical entities can be regarded as categories. In contrast, a category of modules has a richer structure than the minimum required for the definition of a category. The set of homomorphisms $\mathrm{Hom}(M, N)$ between two modules is always an abelian group, and we can recognise whether or not a homomorphism is an injection or surjection. Furthermore, the category may or may not contain such things as direct sums, kernels, projective modules, etc., depending on the nature of the modules that belong to it.

Our aim in this chapter is to analyse the extra structure that is available to module categories, and thereby to formulate sets of conditions which ensure that an abstract category behaves as a module category. We start by investigating the functorial properties of the abelian groups $\mathrm{Hom}(M, N)$ and we see how to reformulate the basic properties of modules and homomorphisms in terms of these functorial properties. We then use these interpretations to define a series of types of abstract category – preadditive, additive, abelian – which increasingly resemble module categories. Ultimately, we arrive at the notion of an exact category, which is a category that has enough structure for the purposes of algebraic K-theory.

2.1 THE HOMOMORPHISM FUNCTORS

In this section, we take a break from the theory of categories in general and return to the consideration of module categories. Our aim is to investigate the functorial properties of the group of homomorphisms $\mathrm{Hom}(M_R, N_R)$ as M and N vary through the category $\mathcal{M}_{OD_R}$ of right R-modules, where R is a ring.

We find that certain properties, such as the injectivity or surjectivity of

an R-module homomorphism, or the injectivity or projectivity of a module, can be reinterpreted in terms of the behaviour of the functors $\mathrm{Hom}(M,-)$ and $\mathrm{Hom}(-,N)$. Such results are useful not only for their applications in module theory, but also because they suggest a method whereby concepts such as injectivity and projectivity can be extended to abstract categories, provided the morphism sets in these categories share enough of the properties of $\mathrm{Hom}(-,-)$. This theme will be developed in the next section. The Hom functors are basic to the study of homology, and their introduction in [Eilenberg & Mac Lane 1942] predates category theory.

As usual, we work with categories of right modules, except in a few places where we are obliged to consider left modules. The transcription of the main part of our discussion to left modules is not quite immediate, although straightforward. The complication arises since we usually view $\mathrm{Hom}(M,N)$ as a member of the right category $\mathcal{A}_B$, regardless of the chirality of the modules M and N. There are also some awkward but essentially trivial variations in notation which arise when we need to regard $\bar{\mathrm{H}}\mathrm{om}(M,N)$ as a left module over some ring.

2.1.1 *Basic properties*

We start with a discussion of the functorial properties of the sets of homomorphisms between modules.

Let R be a ring and let L_R and X_R be right R-modules. Since $\mathrm{Hom}(L_R,X_R)$ is simply the set $\mathrm{Mor}_{\mathcal{C}}(L,X)$ where the category $\mathcal{C}$ is the category $\mathcal{M}_{OD_R}$ of right R-modules, the discussion in (1.2.7) shows that Hom defines a pair of functors $\mathrm{Hom}(-,X)$ and $\mathrm{Hom}(L,-)$, which are respectively contravariant and covariant.

Explicitly, given a right R-module homomorphism $\lambda : L \to M$, the map

$$\lambda^* : \mathrm{Hom}(M_R,X_R) \longrightarrow \mathrm{Hom}(L_R,X_R)$$

is given by $\lambda^*(\xi) = \xi\lambda$. If $\mu : M \to N$ is another right R-module homomorphism, then

$$(\mu\lambda)^* = (\lambda)^*(\mu)^* : \mathrm{Hom}(N_R,X_R) \longrightarrow \mathrm{Hom}(L_R,X_R).$$

On the other hand, for any R-module homomorphism $\xi : X \to Y$, the map

$$\xi_* : \mathrm{Hom}(L_R,X_R) \longrightarrow \mathrm{Hom}(L_R,Y_R)$$

is given by $\xi_*(\alpha) = \xi\alpha$, and $(\eta\xi)_* = \eta_*\xi_*$ for an R-module homomorphism $\eta : Y \to Z$.

Each set $\mathrm{Hom}(L_R, X_R)$ is an abelian group, with addition given by the rule

$$(\alpha + \alpha')\ell = \alpha\ell + \alpha'\ell \text{ for } \ell \in L \text{ and } \alpha, \alpha' \in \mathrm{Hom}(L_R, X_R),$$

and it is easy to verify that the maps λ^* and ξ_* are homomorphisms of abelian groups, that is,

$$\lambda^*(\alpha + \alpha') = \lambda^*\alpha + \lambda^*\alpha' \text{ and } \xi_*(\alpha + \alpha') = \xi_*\alpha + \xi_*\alpha'$$

for $\alpha, \alpha' \in \mathrm{Hom}(L_R, X_R)$.

Thus $\mathrm{Hom}(-, X)$ and $\mathrm{Hom}(L, -)$ are functors with values in the category $\mathcal{A}\mathcal{B}$ of abelian groups rather than the category $\mathcal{S}\mathcal{E}\mathcal{T}$ of sets. Furthermore, the equalities

$$(\lambda + \lambda')^* = \lambda^* + \lambda'^* \text{ and } (\xi + \xi')_* = \xi_* + \xi'_*$$

hold for pairs of homomorphisms $\lambda, \lambda' : L \to M$ and $\xi, \xi' : X \to Y$ and, as in (1.3.2)(i), each pair of homomorphisms $\lambda : L \to M$ and $\xi : X \to Y$ gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(M_R, X_R) & \xrightarrow{\xi_*} & \mathrm{Hom}(M_R, Y_R) \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda^*} \\ \mathrm{Hom}(L_R, X_R) & \xrightarrow{\xi_*} & \mathrm{Hom}(L_R, Y_R) \end{array}$$

which can be interpreted as showing that the homomorphisms λ^* define a natural transformation from the functor $\mathrm{Hom}(M_R, -)$ to the functor $\mathrm{Hom}(L_R, -)$, or, equally, that the homomorphisms ξ_* give a natural transformation from $\mathrm{Hom}(-, X_R)$ to $\mathrm{Hom}(-, Y_R)$.

2.1.2 Exact sequences

We now interpret the exactness of sequences of modules in terms of the Hom functors. By definition, a sequence

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N$$

of right R-modules and right R-module homomorphisms is *exact* if

$$\mathrm{Ker}(\mu) = \mathrm{Im}(\lambda).$$

More generally, a sequence of right R-modules

$$\cdots \longrightarrow M_{i-1} \xrightarrow{\alpha_{i-1}} M_i \xrightarrow{\alpha_i} M_{i+1} \longrightarrow \cdots$$

is called an exact sequence if it is exact at each of its *terms* M_i. Such a sequence may have finite length, or it may be infinite. (A more detailed treatment of exact sequences can be found in [BK: IRM], particularly section 2.4.) Thus the sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M$$

is exact precisely when the homomorphism λ is injective, while

$$M \xrightarrow{\mu} N \longrightarrow 0$$

is exact precisely when μ is surjective.

2.1.3 Proposition

(i) *The sequence*

$$0 \longrightarrow X \xrightarrow{\xi} Y$$

is exact in $\mathcal{M}od_R$ if and only if

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y)$$

is an exact sequence of abelian groups for any right R-module L.

(i)$^{\mathrm{op}}$ *The sequence*

$$M \xrightarrow{\mu} N \longrightarrow 0$$

is exact in $\mathcal{M}od_R$ if and only if

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X)$$

is an exact sequence of abelian groups for any right R-module X.

Proof

We prove (i) only, (i)$^{\mathrm{op}}$ being similar. Suppose that ξ is injective and that $\xi_*\alpha = 0, \alpha : L \to X$. By definition, this means that $\xi\alpha = 0$, and so $\alpha = 0$.

Conversely, suppose that ξ_* is injective. Choose any element $x \in \mathrm{Ker}\,\xi$, put $L = xR$ and let $\alpha : L \to X$ be the inclusion map. Then $\xi_*\alpha = 0$, hence $\alpha = 0$, and so $xR = 0$, which means that $x = 0$. □

Next we examine what happens when we extend the exact sequences by one term. The assertion that

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$$

is an exact sequence of R-modules tells us that X is isomorphic via ξ to the kernel of η, while the exactness of

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

is equivalent to the fact that N is isomorphic via μ to the cokernel of λ.

2.1.4 Proposition

(i) *A sequence*

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$$

of right R-modules is exact if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z)$$

is exact for any right R-module L.

(i)$^{\mathrm{op}}$ *A sequence*

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

of right R-modules is exact if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X) \xrightarrow{\lambda^*} \mathrm{Hom}(L, X)$$

is exact for any right R-module X.

Proof

We prove (i) only. Suppose that $0 \to X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$ is exact. By the previous result, ξ_* is injective, so it is enough to check exactness at $\mathrm{Hom}(L, Y)$. For any $\alpha \in \mathrm{Hom}(L, X)$, we have $\eta_*\xi_*\alpha = \eta\xi\alpha = 0$ since $\eta\xi = 0$. Suppose that $\beta \in \mathrm{Hom}(L, Y)$ and $\eta_*\beta = 0$. Then $\eta\beta = 0$, and so there is a homomorphism $\alpha' : L \to \mathrm{Ker}\,\eta$ such that $\kappa\alpha' = \beta$, where κ is the canonical injection of $\mathrm{Ker}\,\eta$ into Y. But, as we noted above, $\mathrm{Ker}\,\eta$ is isomorphic via ξ to X, so we can construct a homomorphism $\alpha : L \to X$ with $\beta = \xi\alpha = \xi_*\alpha$.

Conversely, suppose that the Hom sequence is exact. Again, the previous result tells us that ξ is injective. Substituting X for L, we see that $0 = \eta_*\xi_*(id_X) = \eta\xi$, that is, $\xi X \subseteq \mathrm{Ker}\,\eta$.

To obtain equality, take an element $y \in \mathrm{Ker}\,\eta$ and put $L = yR$. Let $\beta : L \to Y$ be the inclusion map. Since $\eta_*\beta = 0$, $\beta = \xi_*\alpha = \xi\alpha$ for some $\alpha : L \to X$, which shows that $y \in \xi X$. □

2.1.5 Short exact sequences

A *short exact sequence* is a sequence

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

which is exact at each of the terms M', M and M''. There is an evident extension of the statement of the preceding result in which all the sequences are taken to be short exact sequences, but the result then fails, as can be seen from the following example.

Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

in which ι is left multiplication by 2: $z \mapsto 2z$, and π is the canonical surjection. This induces a Hom sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\iota^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0.$$

Now $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ itself under the map which assigns to an endomorphism α of $\mathbb{Z}$ its value $\alpha(1)$ at 1, and it is not hard to see that the endomorphism of $\mathbb{Z}$ induced by ι^* is ι again and hence non-surjective. Thus the Hom sequence is not exact on the right.

For the dual example, we look at

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\iota_*} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi_*} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

The terms $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ are evidently 0, while $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. So again the sequence fails to be exact on the right.

There are several interesting circumstances in which the Hom sequences are guaranteed to be short exact sequences. The first is the most immediate extension of the last proposition. To state the result, we need the following definition. A short exact sequence of R-modules (or of abelian groups)

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is said to be a *split exact sequence* if we can find homomorphisms

$$\gamma : M \longrightarrow M' \text{ and } \delta : M'' \longrightarrow M$$

such that

$$\beta\delta = id_{M''}, \quad \gamma\alpha = id_{M'} \text{ and } \alpha\gamma + \delta\beta = id_M.$$

It can be shown ([BK: IRM] Theorem 2.4.5) that it is enough to know the

existence of only one of the maps γ or δ, the other then being constructible, and that when the sequence is split, we have $M \cong M' \oplus M''$.

Conversely, if we are given a direct sum decomposition of M, then we can reconstruct a split exact sequence as above.

2.1.6 Proposition

(a) *The following statements are equivalent.*

(i) *The sequence*

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

of right R-modules is split exact.

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z) \longrightarrow 0$$

is split exact for any right R-module L.

(iii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z) \longrightarrow 0$$

is exact for any right R-module L.

$(a)^{\mathrm{op}}$ *The following statements are equivalent.*

(i) *The sequence*

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

of right R-modules is split exact.

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X) \xrightarrow{\lambda^*} \mathrm{Hom}(L, X) \longrightarrow 0$$

is split exact for any right R-module X.

(iii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X) \xrightarrow{\lambda^*} \mathrm{Hom}(L, X) \longrightarrow 0$$

is exact for any right R-module X.

Proof

We prove (a) only. Suppose that (i) holds, that is,

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

is split exact. Then there are R-module homomorphisms $\vartheta : Y \to X$ and $\zeta : Z \to Y$ such that

$$\vartheta\xi = id_X,\ \eta\zeta = id_Z \text{ and } \xi\vartheta + \zeta\eta = id_Y.$$

Since the identity map id_A of a module induces the identity map $(id_A)_*$ on a homomorphism group $\mathrm{Hom}(L, A)$, we have

$$\vartheta_*\xi_* = id,\ \eta_*\zeta_* = id \text{ and } \xi_*\vartheta_* + \zeta_*\eta_* = id,$$

with the identity maps appropriately interpreted. This shows that (ii) holds.

Trivially (ii) gives (iii). Assume that (iii) holds. Taking the special case $L = Z$, we see that there is some homomorphism $\zeta : Z \to Y$ with $\xi_*(\zeta) = id_Y$, that is, $\xi\zeta = id_Y$, which establishes that the module sequence is split exact. □

2.1.7 Projective and injective modules

An alternative procedure for ensuring that the Hom sequences are short exact is to impose conditions on the 'fixed' term rather than the exact sequence of 'variables'. This leads to a characterization of projective modules and their duals, the injective modules.

2.1.8 Proposition

(a) *Let L be a right R-module. Then the following are equivalent.*

(i) *L is projective.*

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z) \longrightarrow 0$$

is exact for every exact sequence

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

of right R-modules.

(a)$^{\mathrm{op}}$ *Let X be a right R-module. Then the following are equivalent.*

(i) *X is injective.*

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X) \xrightarrow{\lambda^*} \mathrm{Hom}(L, X) \longrightarrow 0$$

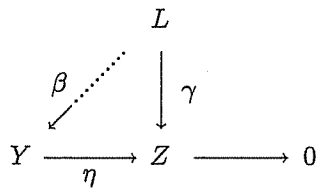
is exact for every exact sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

of right R-modules.

Proof

For (a), we note that the surjectivity of η_* is the same as the assertion that, for any homomorphism $\gamma : L \to Z$, there is a homomorphism $\beta : L \to Y$ which makes the following diagram commute:



But this is a standard criterion for L to be projective (see [BK: IRM] Theorem 2.5.4).

For (a)$^{\mathrm{op}}$, use the dual characterization of injective modules (Exercise 2.5.7 of [BK: IRM]). □

2.1.9 Homomorphism functors arising from bimodules

So far, we have regarded $\mathrm{Hom}(M_R, X_R)$ simply as an abelian group, so that both the functors $\mathrm{Hom}(-, X_R)$ and $\mathrm{Hom}(M_R, -)$ take values in the right category $\mathcal{A}_B$. However, it sometimes happens that we wish to take account of a further module structure on $\mathrm{Hom}(M_R, X_R)$ under which it may be most naturally viewed as a left module, and then our functors will take values in a left category. We therefore introduce a separate notation to alert the reader to our changed point of view.

Suppose that ${}_SX_R$ is an S-R-bimodule for some rings R and S. Then $\mathrm{Hom}(M_R, X_R)$ has a natural left S-module structure given by

$$(s\alpha)(m) = s(\alpha m), \quad s \in S,\ m \in M,\ \alpha : M \to X.$$

Thus $\mathrm{Hom}(-, X_R)$ is a contravariant functor from $\mathcal{M}_{OD_R}$ to ${}_S\mathcal{M}_{OD}$, which we write as $H_X(-)$ to distinguish it from its more common manifestation as a functor with values in $\mathcal{A}_B$.

If N is another right R-module and $\mu : M \to N$ is an R-module homomorphism, we define

$$H_X(\mu) : H_X(N) \longrightarrow H_X(M) \text{ by } \alpha H_X(\mu) = \alpha\mu.$$

In the language of (1.2.6), H_X is a contravariant and contrachiral functor from $\mathcal{M}\mathit{od}_R$ to ${}_S\mathcal{M}\mathit{od}$, with $H_X(\mu\nu) = H_X(\mu)H_X(\nu)$ for composable μ and ν.

Now assume that M is a T-R-bimodule for some ring T. Then $\mathrm{Hom}(M_R, X_R)$ becomes a right T-module for any right R-module X, under the rule $(\alpha t)m = \alpha(tm)$ for $\alpha \in \mathrm{Hom}(M_R, X_R), t \in T, m \in M$.

We write $H^M(-) = \mathrm{Hom}(M_R, -)$. The action of this functor on morphisms is given by $H^M(\xi)\alpha = \xi\alpha$, where $\alpha \in \mathrm{Hom}(M_R, X_R)$ and $\xi \in \mathrm{Hom}(X_R, Y_R)$, and Y is a right R-module. We see that $H^M(\eta)H^M(\xi) = H^M(\eta\xi)$ whenever the composition is defined, and that $H^M(-)$ is a covariant, cochiral functor from $\mathcal{M}\mathit{od}_R$ to $\mathcal{M}\mathit{od}_T$.

Note that these constructions include the case that X or M is simply an R-module, since any right R-module can be viewed as a $\mathbb{Z}$-R-bimodule. The functor H^M is then the same as the functor $\mathrm{Hom}(M_R, -)$, both having values in the right category $\mathcal{A}\mathit{b}$, which is $\mathcal{M}\mathit{od}_{\mathbb{Z}}$ under another name. However, the functor H_X takes values in the mirror category ${}_{\mathbb{Z}}\mathcal{M}\mathit{od}$ of $\mathcal{A}\mathit{b}$, so that $\mathrm{Mor}(-, X_R) = \mathrm{Mir} \circ H_X(-)$: compare with (1.1.5) and (1.2.6).

We can also regard both M and X simultaneously as being variables for Hom, which is then a bifunctor from $\mathcal{M}\mathit{od}_R \times \mathcal{M}\mathit{od}_R$ to $\mathcal{A}\mathit{b}$. More generally, Hom gives rise to bifunctors from $\mathcal{M}\mathit{od}_R \times {}_S\mathcal{M}\mathit{od}_R$ to ${}_S\mathcal{M}\mathit{od}$ and from ${}_T\mathcal{M}\mathit{od}_R \times \mathcal{M}\mathit{od}_R$ to $\mathcal{M}\mathit{od}_T$. These will have mixed variance and chirality.

2.1.10 *The extension functors*

As we saw in (2.1.4), an exact sequence

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

of right R-modules gives rise to an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z)$$

for any R-module L. By the example in (2.1.5), the Hom sequence cannot always be extended on the right to form a short exact sequence.

This fault can be repaired by introducing the *extension functors* $\mathrm{Ext}^n_R(-,-)$, $n \geq 1$, which, like $\mathrm{Hom}_R(-,-)$, are bifunctors from $\mathcal{M}\mathit{od}_R \times \mathcal{M}\mathit{od}_R$ to $\mathcal{A}\mathit{b}$.

There are then long exact sequences of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(L,X) & \longrightarrow & \mathrm{Hom}(L,Y) & \longrightarrow & \mathrm{Hom}(L,Z) & \longrightarrow \\ & \longrightarrow & \mathrm{Ext}^1(L,X) & \longrightarrow & \mathrm{Ext}^1(L,Y) & \longrightarrow & \mathrm{Ext}^1(L,Z) & \longrightarrow \\ & \longrightarrow & \mathrm{Ext}^2(L,X) & \longrightarrow & \cdots & & & \end{array}$$

which may continue for ever to the right, depending on the nature of the ring R and the particular modules involved. There are analogous long exact sequences in which the roles of the variables are exchanged.

The term 'extension functor' is used since the elements of $\mathrm{Ext}^1_R(L,X)$ describe all the possible extensions of L by X, that is, all short exact sequences

$$0 \longrightarrow X \longrightarrow M \longrightarrow L \longrightarrow 0$$

with M a right R-module, under a suitable equivalence relation. The higher extension groups $\mathrm{Ext}^n_R(L,X)$ have a similar interpretation.

The construction of these functors is outlined in Exercise 2.1.7 below.

Exercises

2.1.1 Let ${}_RM$ and ${}_RX$ be left R-modules. Following (2.1.1), define functors $\mathrm{Hom}({}_RM,-) : {}_R\mathcal{M}_{OD} \to \mathcal{A}_B$ and $\mathrm{Hom}(-,{}_RX) : {}_R\mathcal{M}_{OD} \to \mathcal{A}_B$, and determine their chiralities.

Following (2.1.9), show that a bimodule ${}_RM_S$ defines a functor $H^M(-) : {}_R\mathcal{M}_{OD} \to {}_S\mathcal{M}_{OD}$ and that a bimodule ${}_RX_T$ defines a functor $H_X(-) : {}_R\mathcal{M}_{OD} \to \mathcal{M}_{ODT}$. Give the chiralities of these functors.

Thus $\mathrm{Hom}(-,-)$ affords an example of a bifunctor of mixed variance and chirality.

2.1.2 Let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a set of right R-modules, where the index set Λ may be infinite. Show that, for any right R-modules L and X, there are isomorphisms of abelian groups

$$\mathrm{Hom}(\textstyle\bigoplus_\Lambda M_\lambda, X) \cong \prod_\Lambda \mathrm{Hom}(M_\lambda, X)$$

and

$$\mathrm{Hom}(L, \textstyle\prod_\Lambda M_\lambda) \cong \prod_\Lambda \mathrm{Hom}(L, M_\lambda)$$

Show that the inverse of the second of the above isomorphisms restricts to an injective homomorphism

$$\textstyle\bigoplus_\Lambda \mathrm{Hom}(L, M_\lambda) \longrightarrow \mathrm{Hom}(L, \bigoplus_\Lambda M_\lambda),$$

which is in fact an isomorphism when L is finitely generated.

Warning. The description of the fourth term $\mathrm{Hom}(\prod_\Lambda M_\lambda, X)$ is

far from straightforward, since it requires some notion of convergence. A related question is treated in Exercise 2.3.1 of [BK: IRM].

2.1.3 Regard the ring R as an R-R-bimodule, and let M be a right R-module. Define a map $\epsilon : M_R \to \mathrm{Hom}(R_R, M_R)$ by $\epsilon(m)(r) = mr$. Show that ϵ is an isomorphism of right R-modules, and deduce that the functor $\mathrm{Hom}(R_R, -) = H^R(-)$ is naturally isomorphic to the identity functor on $\mathcal{M}_{OD_R}$.

Verify the left-handed version of this result.

(The map ϵ is sometimes called *left multiplication by* m; its inverse is *evaluation at* 1.)

2.1.4 Show that an arbitrary direct product of injective R-modules is again injective.

2.1.5 **Injectives exist**

This longish exercise outlines the proof that, for any ring R and right R-module M, there is an injective right R-module N and an injective R-module homomorphism from M to N.

(i) Let Q be any divisible $\mathbb{Z}$-module. Then it is known that Q is an injective $\mathbb{Z}$-module (see Exercise 2.5.11 of [BK: IRM]). Put $\widehat{R} = \mathrm{Hom}(R_{\mathbb{Z}}, Q_{\mathbb{Z}})$. Verify that $\widehat{R}$ is a right R-module by $(\rho r)(x) = \rho(rx)$, for ρ in $\widehat{R}$, r, x in R.

(ii) Let $\mathfrak{a}$ be a right ideal of R, let $\iota : \mathfrak{a} \to R$ be the inclusion map, and let $\varphi : \mathfrak{a} \to \widehat{R}$ be an R-module homomorphism. It is a fact that $\widehat{R}$ is injective if for any such φ there is a homomorphism $\psi : R \to \widehat{R}$ with $\psi\iota = \varphi$ – see Exercise 2.5.8 in [BK: IRM].
Let $a \in \mathfrak{a}$. Verify that $\gamma : \mathfrak{a} \to Q$, $\gamma(a) = (\varphi a)(1_R)$, is a homomorphism of $\mathbb{Z}$-modules. Choose $\delta : R_{\mathbb{Z}} \to Q_{\mathbb{Z}}$ such that $\delta\iota = \gamma$ and put $(\psi r)(x) = \delta(rx)$. Verify that $\psi r \in \widehat{R}$ and that ψ is as required.

(iii) Let M be any right R-module. Using the fact that there is an isomorphism of $\mathbb{Z}$-modules $M \cong \mathbb{Z}^\Lambda/A$ for some free abelian group $\mathbb{Z}^\Lambda$ and subgroup A of $\mathbb{Z}^\Lambda$, and the divisibility of the group $\mathbb{Q}$ of additive rationals, show that M is a subgroup of a divisible abelian group Q.

(iv) Confirm that there is a chain of injective R-module homomorphisms

$$M \rightarrowtail \mathrm{Hom}(R, M) \rightarrowtail \mathrm{Hom}(R, Q) = \widehat{R}.$$

Remark. It can be shown that an R-module M can be embedded in a 'minimal' injective R-module $E(M)$; this means that there is an injective R-module homomorphism $\epsilon : M \to E(M)$ such that for

any other injective R-module homomorphism $\eta : M \to I$ with I an injective module, there is an injective homomorphism $\rho : E(M) \to I$ with $\eta = \rho\epsilon$. As a universal object, the module $E(M)$ is unique up to isomorphism and is known as the *injective hull* of M. The construction can be found in most texts on ring theory, for instance [Rotman 1979], §3.

2.1.6 Let Q be a divisible abelian group. For a left R-module N, let $\Delta(N) = \mathrm{Hom}(N_{\mathbb{Z}}, Q_{\mathbb{Z}})$. Verify that $\Delta(N)$ is a right R-module, with $\Delta(R) = \widehat{R}$ as in the previous exercise.

Confirm that Δ is a contravariant functor from ${}_R\mathcal{M}\mathit{od}$ to $\mathcal{M}\mathit{od}_R$ and that Δ restricts to a functor from ${}_R\mathcal{P}\mathit{roj}$ to $\mathcal{I}\mathit{nj}_R$, the category of injective right R-modules.

2.1.7 **The group Ext^1_R**

This substantial exercise gives one construction, using the Baer sum, of the groups $\mathrm{Ext}^1_R(M'', M')$. Our treatment is based on Chapter III, §2, of [Mac Lane 1975]. Alternative definitions, and further details, can be found in most texts on homological algebra (see, for example, [Weibel 1995]).

Our arguments depend on some basic methods of manipulating short exact sequences that are described in more detail in section 2.4 of [BK: IRM]. Let

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence and let $\theta : L'' \to M''$ be a homomorphism. Then the pull-back of M and L'' over M'' gives an exact sequence

$$\theta^*\mathbf{E} \quad 0 \longrightarrow M' \xrightarrow{\mu} M \times_{M''} L'' \xrightarrow{\overline{\beta}} L'' \longrightarrow 0,$$

the *pull-back* of $\mathbf{E}$ along θ. (An alternative expression is that $\theta^*\mathbf{E}$ is obtained from $\mathbf{E}$ by *base change*.)

Similarly, given a homomorphism $\phi : M' \to N'$, the push-out of N' and M over M' gives a short exact sequence

$$\phi_*\mathbf{E} \quad 0 \longrightarrow N' \xrightarrow{\overline{\alpha}} N' \oplus_{M'} M \xrightarrow{\nu} M'' \longrightarrow 0,$$

the *push-out* of $\mathbf{F}$ along ϕ. (Alternatively, $\phi_*\mathbf{E}$ is said to be obtained from $\mathbf{E}$ by *cobase change*.)

Two short exact sequences

$$\mathbf{E}_1 \qquad 0 \longrightarrow M' \xrightarrow{\alpha_1} M_1 \xrightarrow{\beta_1} M'' \longrightarrow 0$$

and

$$\mathbf{E}_2 \qquad 0 \longrightarrow M' \xrightarrow{\alpha_2} M_2 \xrightarrow{\beta_2} M'' \longrightarrow 0$$

are *equivalent* if there is an isomorphism $\chi : M_1 \to M_2$ with $\chi\alpha_1 = \alpha_2$ and $\beta_2\chi = \beta_1$. The notation is $\mathbf{E}_1 \equiv \mathbf{E}_2$.

For fixed M' and M'', equivalence of short exact sequences is an eqivalence relation on the set of all short exact sequences $\mathbf{E}$, and it is compatible with pull-backs and push-outs.

We write $[\mathbf{E}]$ for the class of $\mathbf{E}$ and we define $\mathrm{Ext}^1_R(M'', M')$ to be the set of all such equivalence classes.

The direct sum $\mathbf{E}_1 \oplus \mathbf{E}_2$ is defined to be the short exact sequence

$$0 \longrightarrow M' \oplus M' \xrightarrow{\alpha_1 \oplus \alpha_2} M_1 \oplus M_2 \xrightarrow{\beta_1 \oplus \beta_2} M'' \oplus M'' \longrightarrow 0.$$

Define an operation, the *Baer sum*, on $\mathrm{Ext}^1_R(M'', M')$, by

$$[\mathbf{E}_1] + [\mathbf{E}_2] = [(\Sigma')_*(\Delta'')^*(\mathbf{E}_1 \oplus \mathbf{E}_2)],$$

where the diagonal homomorphism

$$\Delta'' : M'' \longrightarrow M'' \oplus M'' \text{ has } \Delta'' m'' = (m'', m'')$$

and the codiagonal homomorphism

$$\Sigma' : M' \oplus M' \longrightarrow M' \text{ has } \Sigma'(m', n') = m' + n'.$$

(i) Verify that the Baer sum depends only on the equivalence classes of $\mathbf{E}_1$ and $\mathbf{E}_2$, so that it is a well-defined law of composition on $\mathrm{Ext}^1_R(M'', M')$.

(ii) Confirm that the middle term of $\mathbf{E}_1 + \mathbf{E}_2$ is

$$(M_1 \times_{M''} M_2)/\{(\alpha_1 m', -\alpha_2 m') \mid m' \in M'\}.$$

(iii) Verify that the composition is associative and commutative.

(iv) Deduce that $\mathrm{Ext}^1_R(M'', M')$ is an abelian group, with zero element corresponding to the standard split exact sequence, and

$$-[\mathbf{E}] = [(-id)_*\mathbf{E}]$$

where $-id$ is the negative of the identity map on M'.

(v) Let $\phi, \psi : L'' \to M''$. Verify the following chain of equivalences:

$$\begin{aligned}\phi^*\mathbf{E} + \psi^*\mathbf{E} &\equiv (\Sigma')_*(\Delta'')^*(\phi^*\mathbf{E} \oplus \psi^*\mathbf{E}) \\ &\equiv (\Sigma')_*(\Delta'')^*(\phi \oplus \psi)^*(\mathbf{E} \oplus \mathbf{E}) \\ &\equiv (\Sigma''(\phi \oplus \psi)\Delta'')^*\mathbf{E} \\ &\equiv (\phi + \psi)^*\mathbf{E}.\end{aligned}$$

Deduce that $\mathrm{Ext}^1_R(-, M')$ is a contravariant functor. Show also that $\mathrm{Ext}^1_R(M'', -)$ is a covariant functor and that $\mathrm{Ext}^1_R(-,-)$ is a bifunctor.

(vi) Show that $\mathrm{Ext}^1_R(P, X) = 0$ for all R-modules X if and only if P is a projective R-module.

(vii) Let

$$\mathbf{Y} \qquad 0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

be an exact sequence of right R-modules. For $\lambda \in \mathrm{Hom}(L, Z)$, define

$$\partial(\lambda) = [\lambda^*\mathbf{Y}] \in \mathrm{Ext}^1_R(Z, X),$$

and, anticipating the Snake Lemma (Exercise 2.3.13), verify that

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}_R(L, X) &\longrightarrow \mathrm{Hom}_R(L, Y) \longrightarrow \mathrm{Hom}_R(L, Z) \\ \xrightarrow{\partial} \mathrm{Ext}^1_R(L, X) &\longrightarrow \mathrm{Ext}^1_R(L, Y) \longrightarrow \mathrm{Ext}^1_R(L, Z)\end{aligned}$$

is an exact sequence of abelian groups.

(viii) Derive the corresponding exact sequence with fixed second variable, and show that $\mathrm{Ext}^1_R(L, J) = 0$ for all R-modules L if and only if J is an injective R-module.

(ix) Here is an inductive definition of $\mathrm{Ext}^n_R(L, X)$ for $n \geq 2$. Choose a short exact sequence

$$0 \longrightarrow S \longrightarrow P \longrightarrow L \longrightarrow 0$$

with P projective (such a sequence exists; we may take P to be free by Lemma 2.5.7 of [BK: IRM]), and put

$$\mathrm{Ext}^n_R(L, X) = \mathrm{Ext}^{n-1}_R(S, L) \text{ for } n \geq 2.$$

It can be shown that the right-hand side is independent of the choice of the short exact sequence above.

It is also possible to make a definition in terms of the second variable by taking a short exact sequence

$$0 \longrightarrow X \longrightarrow J \longrightarrow C \longrightarrow 0$$

with J injective (see Exercise 2.1.5 above). Much hard work reveals that the two definitions coincide, or, more properly, that the functors obtained are naturally isomorphic. Full details are given in [Mac Lane 1975] and [Rotman 1979].

2.2 ADDITIVE CATEGORIES

When [Eilenberg & Mac Lane, 1945] had given the abstract definition of a category, a natural next step was to seek the conditions on the morphism sets of an abstract category which enable the definitions and constructions of module theory to be developed in that category. The exploration led to the definitions of additive, exact (in the non-K-theory sense) [Buchsbaum 1955] and abelian categories [Grothendieck 1957]. These abstractions were motivated by the desire to apply homological methods in categories that arise in algebraic geometry, such as categories of sheaves and of vector bundles (which we do not encounter in this text), as well as intellectual curiosity. The widest class of categories that share some properties of modules comprises the preadditive categories. In such a category $\mathcal{C}$, each morphism set $\mathrm{Mor}_{\mathcal{C}}(L, X)$ is an abelian group and the functors arising from morphisms must respect addition. These requirements suffice to permit the recognition of direct sums, exact sequences, kernels and cokernels, and projective and injective objects in $\mathcal{C}$. However, there need not be any nontrivial examples of such things within the category.

The next step is the definition of an additive category, which is required to contain a direct sum of any two of its objects. Many interesting module categories are additive categories, for example, the categories $\mathcal{M}_R$ of finitely generated modules and $\mathcal{P}_R$ of finitely generated projectives. Also, many of the categories of interest in K-theory are additive categories.

There is some variation in the literature as to the precise definition of a preadditive category, as there is with the definitions of (pre)additive subcategories of (pre)additive categories. Indeed, some authorities do not make formal definitions. Even where terms like additive and abelian categories have standard meanings, there is a complex web of interacting axioms woven through the literature. Our own choice of axioms has been guided by intuitive appeal rather than minimality.

Throughout this section, we work with right categories, it being clear that

there is a corresponding discussion for left categories. Modules are right modules over an arbitrary ring R, unless otherwise specified.

To make the comparison and transcription of results easier, we often use the same symbols – $L, M, \ldots, X, Y, \ldots$ – for objects of an abstract category $\mathcal{C}$ as we used for modules in the preceding section.

2.2.1 Preadditive categories

A category $\mathcal{C}$ is called *preadditive* if the following conditions are satisfied.

Ad 1. For each pair of objects M and N of $\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(M,N)$ is an abelian group.

Ad 2. If $\lambda, \lambda' : L \to M$ and $\mu, \mu' : M \to N$ are morphisms in $\mathcal{C}$, then

$$\mu(\lambda + \lambda') = \mu\lambda + \mu\lambda' \text{ and } (\mu + \mu')\lambda = \mu\lambda + \mu'\lambda.$$

Ad 3. $\mathcal{C}$ contains a zero object 0 such that $\mathrm{Mor}_{\mathcal{C}}(0,N) = 0$ and $\mathrm{Mor}_{\mathcal{C}}(M,0) = 0$, the trivial group, for all objects M and N.

Here, a zero object is as defined in (1.4.14). There is (usually) more than one zero object, but any two zero objects are isomorphic by a unique isomorphism, so we often refer to 'the' zero object of $\mathcal{C}$.

Note that the set $\mathrm{End}_{\mathcal{C}}(M)$ of endomorphisms of any object M in $\mathcal{C}$ must be a ring.

Of course, $\mathcal{A}_{B}$ is a preadditive category, and so is the category $\mathcal{R}_{NG}$ of nonunital rings. However, the category $\mathcal{R}_{ING}$ of rings is *not* preadditive because the zero mapping is not in general a ring homomorphism. It is clear that $\mathcal{M}_{OD_R}$ is preadditive for any ring R, and it is easy to see that the basic functorial properties of $\mathrm{Hom}_R(-,-)$, given in (2.1.1), transcribe to $\mathrm{Mor}_{\mathcal{C}}(-,-)$. We list them for convenience.

A morphism $\lambda : L \to M$ in $\mathcal{C}$ defines a map

$$\lambda^* : \mathrm{Mor}_{\mathcal{C}}(M,X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L,X) \text{ by } \lambda^*(\xi) = \xi\lambda.$$

From Ad 2, λ^* is a homomorphism of abelian groups. For another morphism $\mu : M \to N$,

$$(\mu\lambda)^* = \lambda^*\mu^* : \mathrm{Mor}_{\mathcal{C}}(N,X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L,X).$$

In the other variable, a morphism $\xi : X \to Y$ gives a homomorphism of abelian groups

$$\xi_* : \mathrm{Mor}_{\mathcal{C}}(L,X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L,Y) \text{ by } \xi_*(\alpha) = \xi\alpha,$$

and $(\eta\xi)_* = \eta_*\xi_*$ for a morphism $\eta : Y \to Z$.

Thus $\mathrm{Mor}_{\mathcal{C}}(-,X)$ and $\mathrm{Mor}_{\mathcal{C}}(L,-)$ are functors on $\mathcal{C}$, contravariant and covariant respectively, with values in the category $\mathcal{A}_{\mathcal{B}}$ of abelian groups. As with the Hom functor, in a preadditive category we have equalities

$$(\lambda+\lambda')^* = \lambda^* + \lambda'^* \text{ and } (\xi+\xi')_* = \xi_* + \xi'_*$$

for pairs of morphisms $\lambda, \lambda' : L \to M$ and $\xi, \xi' : X \to Y$. Each pair of morphisms $\lambda : L \to M$ and $\xi : X \to Y$ gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}(M,X) & \xrightarrow{\xi_*} & \mathrm{Mor}(M,Y) \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda^*} \\ \mathrm{Mor}(L,X) & \xrightarrow{\xi_*} & \mathrm{Mor}(L,Y) \end{array}$$

showing that the homomorphisms λ^* define a natural transformation from the functor $\mathrm{Mor}\,(M,-)$ to the functor $\mathrm{Mor}\,(L,-)$ and that the homomorphisms ξ_* give a natural transformation from $\mathrm{Mor}\,(-,X)$ to $\mathrm{Mor}\,(-,Y)$.

2.2.2 *Preadditive subcategories*

Next we define a preadditive subcategory of a preadditive category. There is no general agreement as to the proper definition, so we have chosen one suited to our immediate purposes.

A subcategory $\mathcal{D}$ of a preadditive category $\mathcal{C}$ is said to be a *preadditive subcategory* if

PAS 1. $\mathrm{Mor}_{\mathcal{D}}(L,X)$ is a subgroup of $\mathrm{Mor}_{\mathcal{C}}(L,X)$ for any pair of objects L and X of $\mathcal{D}$, and

PAS 2. $\mathcal{D}$ contains a zero object of $\mathcal{C}$.

Evidently, such a subcategory $\mathcal{D}$ is itself a preadditive category.

The zero object of a preadditive category can be viewed as defining a preadditive subcategory with one object. Less trivially, a nonzero object X of $\mathcal{C}$ gives a minimal full preadditive subcategory $\mathcal{P}_{\mathcal{READD}}(X)$ of $\mathcal{C}$; the set of objects of $\mathcal{P}_{\mathcal{READD}}(X)$ is $\{0,X\}$ and the set of morphisms is $\mathrm{End}_{\mathcal{C}}(X)$ together with the requisite zero morphisms.

2.2.3 *Monomorphisms and epimorphisms*

We now use the Mor functors to transcribe definitions and results from module categories $\mathcal{M}_{\mathcal{OD}_R}$ to arbitrary preadditive categories. The fact that the Mor

functors on a preadditive category take values in $\mathcal{A}_B$ (rather than $\mathcal{S}_{ET}$) is crucial in what follows.

We start with monomorphisms and epimorphisms, which are the analogues of injective and surjective homomorphisms – the reason for the change in terminology will be explained shortly.

By analogy with (2.1.3), we make the following definitions in a preadditive category $\mathcal{C}$.

A morphism $\mu : X \to Y$ in $\mathcal{C}$ is said to be a *monomorphism* if, for any object L of $\mathcal{C}$,

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, X) \xrightarrow{\mu_*} \mathrm{Mor}_{\mathcal{C}}(L, Y)$$

is an exact sequence of abelian groups.

A morphism $\epsilon : M \to N$ in $\mathcal{C}$ is said to be an *epimorphism* if, for any object X of $\mathcal{C}$,

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(N, X) \xrightarrow{\epsilon^*} \mathrm{Mor}_{\mathcal{C}}(M, X)$$

is an exact sequence of abelian groups.

Important examples of monomorphisms and epimorphisms are provided by the following result.

2.2.4 Lemma

If $\mu : X \to Y$ has a left inverse $\rho : Y \to X$ (that is, $\rho\mu = id_X$), then μ is a monomorphism.

If $\epsilon : M \to N$ has a right inverse $\sigma : N \to M$ (that is, $\epsilon\sigma = id_N$), then ϵ is an epimorphism.

Proof

By functoriality of $\mathrm{Mor}_{\mathcal{C}}(L, -)$, $\rho_*\mu_*$ is just the identity homomorphism on the group $\mathrm{Mor}_{\mathcal{C}}(L, X)$. So μ_* is injective. Similarly, ϵ_* is surjective. □

A morphism with a left inverse is often called a split monomorphism, while a morphism with a right inverse is known as a split epimorphism.

Warning. These descriptions are valid only for right categories! The reader may wish to formulate their left category counterparts. (The definitions of epimorphism and monomorphism can be extended to arbitrary categories; see Exercise 2.2.4.)

In a module category $\mathcal{M}_{OD_R}$, the basic result (2.1.3) shows that a monomorphism is the same thing as an injective morphism while an epimorphism is just a surjective morphism. However, these coincidences need not happen in an abstract category, which explains the need for new terminology.

The most blatant examples in which a monomorphism cannot be an injection are provided by categories whose objects are not sets. More seriously, it may happen that the objects of a preadditive category are sets, but that a monomorphism need not be an injection or an epimorphism need not be a surjection. Such pathologies are somewhat tricky to find.

We give examples of both phenomena.

2.2.5 *Example: the opposite category*

Recall from (1.1.6) that the opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ has objects C^{op} in bijective correspondence with the objects C of $\mathcal{C}$, and that for each pair of objects C^{op}, D^{op} of $\mathcal{C}^{\mathrm{op}}$, there is a bijection between the morphisms α in $\mathrm{Mor}_{\mathcal{C}}(C, D)$ and the morphisms α^{op} in $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(D^{\mathrm{op}}, C^{\mathrm{op}})$.

The composition in $\mathcal{C}^{\mathrm{op}}$ is given by the rule

$$(\alpha^{\mathrm{op}})(\beta^{\mathrm{op}}) = (\beta\alpha)^{\mathrm{op}}.$$

If $\mathcal{C}$ is preadditive, then so also is $\mathcal{C}^{\mathrm{op}}$, under the natural definition

$$\alpha^{\mathrm{op}} + \beta^{\mathrm{op}} = (\alpha + \beta)^{\mathrm{op}}.$$

A morphism $\xi : X \to Y$ in $\mathcal{C}$ induces

$$\xi_* : \mathrm{Mor}_{\mathcal{C}}(L, X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, Y),$$

while its opposite counterpart $\xi^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ gives

$$(\xi^{\mathrm{op}})^* : \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, L^{\mathrm{op}}) \longrightarrow \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(Y^{\mathrm{op}}, L^{\mathrm{op}}).$$

For any $\lambda : L \to X$, we have $(\xi^{\mathrm{op}})^*(\lambda^{\mathrm{op}}) = (\xi_*(\lambda))^{\mathrm{op}}$, so that $(\xi^{\mathrm{op}})^*$ and ξ_* are either both injective or both not injective.

Thus ξ^{op} is an epimorphism exactly when ξ is a monomorphism. Similarly, ξ^{op} is a monomorphism precisely when ξ is an epimorphism.

Thus, if $\mathcal{C}$ contains both monomorphisms and epimorphisms, so does $\mathcal{C}^{\mathrm{op}}$. However, the objects of $\mathcal{C}^{\mathrm{op}}$ are not sets, so its monomorphisms cannot be injective mappings, nor can its epimorphisms be surjections.

2.2.6 *Example: topological abelian groups*

This example relies on some elementary topology which we do not develop in this text. (The definitions of terms below can be found in [Willard 1970] §13.) As well as providing an epimorphism which is not a surjection, it shows that an epimorphism in a subcategory may cease to be an epimorphism in a larger category.

Let $\mathcal{HA}_B$ be the category of Hausdorff topological abelian groups. Thus an object A of $\mathcal{HA}_B$ is a Hausdorff topological space which is also an abelian group in such a way that the group operations are continuous maps, and a morphism from A to B in $\mathcal{HA}_B$ is a continuous map which is also a group homomorphism. We exploit the following result *ibid,* (13.14): if $f, g : X \to Y$ are continuous maps of topological spaces with Y Hausdorff, and f and g agree on a dense subset of X, then $f = g$.

Now the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are both objects of $\mathcal{HA}_B$ with their customary additions and topologies, and moreover $\mathbb{Q}$ is dense in $\mathbb{R}$. The result quoted above shows that the inclusion map $\iota : \mathbb{Q} \to \mathbb{R}$ is an epimorphism in $\mathcal{HA}_B$, although not a surjection. Note that ι is also a monomorphism and, of course, an injection.

In particular, we have an example of a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.

The quotient group $\mathbb{R}/\mathbb{Q}$ can be given a topology (indiscrete) so that it becomes the cokernel of ι in the category $\mathcal{TOPA}_B$ of all topological abelian groups, but it is not Hausdorff (see [Higgins 1974] Chapter II, §3 and Proposition 5).

We also note that $\mathcal{HA}_B$ is a preadditive subcategory of $\mathcal{TOPA}_B$ but that ι is no longer an epimorphism in $\mathcal{TOPA}_B$.

2.2.7 Kernel and cokernel

Before we define short exact sequences in a preadditive category, we record the next, easily verified lemma, which shows that the concepts of kernel and cokernel in a preadditive category correspond to their counterparts in $\mathcal{A}_B$ via the morphism functors.

2.2.8 Lemma

Let $\alpha : A \to B$ be a morphism in a preadditive category $\mathcal{C}$. Then the following assertions hold.

(i) *$\kappa : K \to A$ is a kernel for α if and only if, for any L in $\mathcal{C}$,*

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, K) \xrightarrow{\kappa_*} \mathrm{Mor}_{\mathcal{C}}(L, A) \xrightarrow{\alpha_*} \mathrm{Mor}_{\mathcal{C}}(L, B)$$

is an exact sequence of abelian groups. Thus, for any choice of $\mathrm{Ker}\,\alpha$,

$$\mathrm{Mor}_{\mathcal{C}}(L, \mathrm{Ker}\,\alpha) = \mathrm{Ker}\,\mathrm{Mor}_{\mathcal{C}}(L, \alpha).$$

(*Note that α_* is an alternative notation for* $\mathrm{Mor}_{\mathcal{C}}(L, \alpha)$.)

(ii) $\gamma : B \to C$ *is a cokernel for* α *if and only if, for any* X *in* $\mathcal{C}$,

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, X) \xrightarrow{\gamma^*} \mathrm{Mor}_{\mathcal{C}}(B, X) \xrightarrow{\alpha^*} \mathrm{Mor}_{\mathcal{C}}(A, X)$$

is an exact sequence of abelian groups. Thus, for any choice of $\mathrm{Cok}\,\alpha$,

$$\mathrm{Mor}_{\mathcal{C}}(\mathrm{Cok}\,\alpha, X) = \mathrm{Ker}\,\mathrm{Mor}_{\mathcal{C}}(\alpha, X),$$

(*where* α^* *is an alternative notation for* $\mathrm{Mor}_{\mathcal{C}}(\alpha, X)$). □

An immediate consequence is that a morphism is a monomorphism precisely when its kernel is the zero object, and an epimorphism precisely when its cokernel is the zero object.

2.2.9 Short exact sequences

A *short exact sequence* in a preadditive category is a sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

in which α is a kernel for β and β is a cokernel for α. Thus, in particular, α is a monomorphism and β is an epimorphism.

An easy exercise is that β is an isomorphism if and only if A is the zero object, while α is an isomorphism just when $C = 0$.

The distinction between an epimorphism and a surjection means that some caution must be exercised when working with short exact sequences in a general preadditive category. The unexpected phenomena that can arise are illustrated by the inclusion map $\iota : \mathbb{Q} \to \mathbb{R}$.

As we noted in (2.2.6), ι is both an epimorphism and monomorphism in the category $\mathcal{HA}_{\mathcal{B}}$ of Hausdorff abelian topological groups, but it is not an isomorphism. Thus the sequence

$$0 \longrightarrow \mathbb{Q} \xrightarrow{\iota} \mathbb{R} \xrightarrow{0} 0 \longrightarrow 0$$

is not short exact. In fact, $\mathbb{Q}$ cannot be the kernel of $\mathbb{R} \to 0$ because the identity map on $\mathbb{R}$ cannot be factored through $\mathbb{Q}$.

2.2.10 Projective and injective objects

There are three equivalent definitions for a projective object in an arbitrary preadditive category $\mathcal{C}$, which generalize well-known characterizations of projective modules.

Proj 1. The generalization of our basic definition for modules (1.1.7): an object P of $\mathcal{C}$ is *projective* (in $\mathcal{C}$) if every short exact sequence

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} P \longrightarrow 0$$

in $\mathcal{C}$ *splits* at P, meaning that there exists $\sigma : P \to M$ in $\mathcal{C}$ with $\pi\sigma = id_P$.

Proj 2. P is projective if and only if, whenever the row is exact, the following diagram can be completed (so as to provide a commuting triangle):

$$\begin{array}{ccccc} & & P & & \\ & \overset{\beta}{\swarrow} & \downarrow{\scriptstyle \gamma} & & \\ Y & \xrightarrow[\eta]{} & Z & \longrightarrow & 0 \end{array}$$

(The equivalence of Proj 1 and Proj 2 for modules is shown in Theorem 2.5.4 of [BK: IRM].)

Proj 3. P is projective if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Mor}\,(P, X) \xrightarrow{\xi_*} \mathrm{Mor}\,(P, Y) \xrightarrow{\eta_*} \mathrm{Mor}\,(P, Z) \longrightarrow 0$$

is exact for every short exact sequence

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

in $\mathcal{C}$ (see (2.1.8) for the module version).

The proof of the fact that these definitions are mutually equivalent is much the same as that for modules, and is left to the reader. Likewise, the reader should have no problem in formulating and verifying the equivalence of three definitions for an injective object.

However, there is no reason why a preadditive category should contain any nontrivial projective or injective object. For example, the category $\mathcal{FA}_{\mathcal{B}}$ of finite abelian groups (which can also be described as the category of finitely generated torsion $\mathbb{Z}$-modules or the category of Artinian $\mathbb{Z}$-modules) has no nonzero projectives – this can be seen as in Exercise 2.2.10 below.

More seriously, it may happen that a projective object in a subcategory is not projective in a larger category. For an illustration, consider the category $\mathcal{ASS}_{\mathbb{Z}}$ of Artinian semisimple $\mathbb{Z}$-modules. Such a module is a finite sum of irreducible $\mathbb{Z}$-modules, that is, abelian groups $\mathbb{Z}/p\mathbb{Z}$ of prime order for various primes p (see (2.3.20 - F) below). A fundamental result, the Artinian Splitting

Theorem† ([BK: IRM] (4.1.17)) tells us that, for any module M in $\mathcal{AS}_{\mathbb{Z}}$, any submodule or quotient module of M must be a direct summand, and so every module in $\mathcal{AS}_{\mathbb{Z}}$ is both projective and injective as an object of that category. In contrast, no such (nonzero) module can be projective or injective in the category $\mathcal{FA}_{\mathcal{B}}$.

Although the concept of exactness was introduced in [Hurewicz 1941], in the context of homology groups of topological spaces, the term 'exact sequence' did not appear until [Kelley & Pitcher 1947]. The crucial role of exact sequences in contemporary module theory and homology theory is evident from almost any modern text on these subjects.

2.2.11 *Additive categories*

As we remarked in (1.4.10), the direct sum $M' \oplus M''$ of two modules M' and M'' is characterized up to isomorphism by the existence of homomorphisms

$$\sigma' : M' \longrightarrow M,\ \sigma'' : M'' \longrightarrow M,\ \pi' : M \longrightarrow M' \text{ and } \pi'' : M \longrightarrow M''$$

satisfying the relations

$$\pi'\sigma' = id_{M'},\ \pi''\sigma'' = id_{M''} \text{ and } \sigma'\pi' + \sigma''\pi'' = id_M.$$

These relations make sense in any preadditive category $\mathcal{C}$, and so we may define the *direct sum* of objects M' and M'' of $\mathcal{C}$ to be an object M of $\mathcal{C}$ which satisfies these relations. However, a preadditive category need not contain the direct sum of a pair of objects; an easy example is given by $\mathcal{P}_{\mathcal{READD}}(\mathbb{Z})$ in $\mathcal{A}_{\mathcal{B}}$. These observations lead to the following definition.

A category $\mathcal{C}$ is said to be *additive* if it is a preadditive category satisfying the next condition.

Ad 4. Any two objects of $\mathcal{C}$ have a direct sum in $\mathcal{C}$.

As in the case of modules (the same arguments apply), such a direct sum is both a product and a coproduct and so it is unique to within unique isomorphism. The notation $M' \oplus M''$ is used to indicate any convenient choice of the direct sum – in most contexts, there is an obvious candidate for this choice.

It can be shown (Exercise 2.2.6) that a product in a preadditive category must be a direct sum, as must be a coproduct. Conversely, when products and coproducts exist and they are the same as direct sums in a category $\mathcal{C}$ satisfying Ad 3, then Ad 1 and Ad 2 hold, so that $\mathcal{C}$ is additive. (See [Gabriel 1962] for details.)

† The Artinian Splitting Theorem holds over any ring R.

We also note in passing that when the direct sum $M' \oplus M''$ exists, so also does $M'' \oplus M'$, which is isomorphic to $M' \oplus M''$ via a unique isomorphism. Thus one may speak of the direct sum of an unordered pair of objects.

Note also that it is possible to characterize direct sums in terms of *split exact sequences*, again mimicking the result for modules.

A short exact sequence

$$0 \longrightarrow M' \xrightarrow{\sigma'} M \xrightarrow{\pi'} M'' \longrightarrow 0$$

in a preadditive category is *split* if there are morphisms σ'' and π'' as above. As with modules, it is enough to know that one of the splitting morphisms σ'' or π'' exists, since the other can then be constructed, and we then find that $M \cong M' \oplus M''$.

2.2.12 Additive subcategories

A subcategory $\mathcal{D}$ of an additive category $\mathcal{C}$ is an *additive subcategory* if, as well as being a preadditive subcategory, $\mathcal{D}$ is additive.

Let M', M'' be a pair of objects in $\mathcal{D}$. Any direct sum of M' and M'' in $\mathcal{D}$ is necessarily their direct sum in $\mathcal{C}$ also, but the favoured choice $M' \oplus M''$ of the direct sum in $\mathcal{C}$ need not itself be an object of $\mathcal{D}$. However, there is to be some choice, necessarily isomorphic to this, which, together with its associated inclusions and projections $\sigma', \sigma'', \pi', \pi''$, does lie in $\mathcal{D}$.

As with preadditive categories, we can define a minimal additive subcategory $\mathcal{A}_{\mathcal{DD}}(X)$ of an additive category $\mathcal{C}$ in terms of a nonzero object X of $\mathcal{C}$ by taking $\mathcal{A}_{\mathcal{DD}}(X)$ to be the full subcategory of $\mathcal{C}$ with objects the finite direct sums X^n, $n \geq 1$, together with the zero object.

2.2.13 Examples

Before developing the theory of additive categories, we give some more examples.

All the module categories introduced in (1.1.9), namely, $\mathcal{P}_{\mathcal{ROJ}R}$, $\mathcal{F}_{\mathcal{REE}R}$, $\mathcal{M}_R$, $\mathcal{F}_R$ and $\mathcal{P}_R$, are additive subcategories of $\mathcal{M}_{\mathcal{OD}R}$; the verifications rely on simple facts about direct sums which are mostly immediate consequences of the definitions. For projective modules, we use the fact that $M' \oplus M''$ is projective if and only if both summands are projective ([BK: IRM] Corollary 2.5.6); and for free modules, we note that the zero module is free on the empty set, and that if M and N are free on sets X and Y respectively, then $M \oplus N$ is isomorphic to the free module on the disjoint union $X \sqcup Y$ ([BK: IRM] Exercise

2.1.8). Also, by Lemma 2.5.7 of [BK: IRM], $M' \oplus M''$ is finitely generated if and only if its summands are finitely generated, which equivalence is easy to check directly.

We record the results as follows.

2.2.14 Lemma

The categories $\mathcal{P}_{ROJ_R}$, $\mathcal{F}_{REE_R}$, $\mathcal{M}_R$, $\mathcal{F}_R$ and $\mathcal{P}_R$ are additive subcategories of $\mathcal{M}_{OD_R}$. If S is a subring of R, then $\mathcal{M}_{OD_R}$ is an additive subcategory of $\mathcal{M}_{OD_S}$. □

Next, we consider product categories.

2.2.15 Lemma

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. Then $\mathcal{C} \times \mathcal{D}$ is an additive category.

Proof

The sum of morphisms (α, β) and (α', β') from (C, D) to (C', D') in $\mathcal{C} \times \mathcal{D}$ is given componentwise:

$$(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta').$$

In effect, there is an identity

$$\mathrm{Mor}_{\mathcal{C}}(C, C') \times \mathrm{Mor}_{\mathcal{D}}(D, D') = \mathrm{Mor}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D'))$$

which gives us the abelian group structure that we require in $\mathcal{C} \times \mathcal{D}$. Note that the zero is the pair $(0, 0)$ and the direct sum is

$$(C, D) \oplus (C', D') = (C \oplus C', D \oplus D').$$ □

2.2.16 Morphism categories

Equally straightforward is the extension of the above result to morphism categories. Let $\mathcal{C}$ be an additive category. If (α, β) and (α', β') are both morphisms from (C, D, γ) to (C', D', γ') in $\mathcal{M}_{OR_{\mathcal{C}}}$, then so is their sum. Composition of morphisms is distributive over addition because this is already true in the product category $\mathcal{C} \times \mathcal{C}$, and the zero is $(0, 0, id_0)$.

The direct sum in $\mathcal{M}_{OR_{\mathcal{C}}}$ is given by

$$(C, D, \gamma) \oplus (C', D', \gamma') = (C \oplus C', D \oplus D', \gamma \oplus \gamma'),$$

where

$$\gamma \oplus \gamma' = \sigma_D \gamma \pi_C + \sigma_{D'} \gamma' \pi_{C'} : C \oplus C' \to D \oplus D',$$

π_C and $\pi_{C'}$ being the split epimorphisms associated with $C \oplus C'$, and σ_D and $\sigma_{D'}$ being the split monomorphisms associated with $D \oplus D'$. In a module category, this becomes the familiar formula

$$(\gamma \oplus \gamma')(c, c') = (\gamma c, \gamma' c') \text{ for all } c \in C, c' \in C'.$$

We record the consequences of our discussion.

2.2.17 Lemma

Suppose that $\mathcal{C}$ is an additive category. Then the categories $\mathcal{M}or_{\mathcal{C}}$, $\mathcal{E}nd_{\mathcal{C}}$, $\mathcal{I}so_{\mathcal{C}}$ and $\mathcal{A}ut_{\mathcal{C}}$ are additive categories. □

2.2.18 Additive functors

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. For each pair of objects C', C in $\mathcal{C}$, the set $\mathrm{Mor}_{\mathcal{C}}(C', C)$ is an additive group, as is $\mathrm{Mor}_{\mathcal{D}}(D', D)$ for each pair D', D in $\mathcal{D}$. In this circumstance, it is natural to consider functors between $\mathcal{C}$ and $\mathcal{D}$ that respect the additive structures on the sets of morphisms.

A covariant functor F from $\mathcal{C}$ to $\mathcal{D}$ is said to be *additive* if the map

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC)$$

is a group homomorphism for all objects C' and C in $\mathcal{C}$:

$$F(\alpha + \beta) = F\alpha + F\beta \quad \text{always.}$$

Additive contravariant functors are defined similarly, as are additive multifunctors on products of additive categories.

First examples are the morphism functors themselves. For a fixed object C of an additive category, $\mathrm{Mor}_{\mathcal{C}}(C, -)$ is a covariant additive functor from $\mathcal{C}$ to $\mathcal{A}_{\mathcal{B}}$ and $\mathrm{Mor}_{\mathcal{C}}(-, C)$ is a contravariant additive functor from $\mathcal{C}$ to $\mathcal{A}_{\mathcal{B}}$.

It is clear that if $\mathcal{C}'$ is an additive subcategory of the additive category $\mathcal{C}$, then the inclusion functor is additive. If F is an additive functor from $\mathcal{C}$ to $\mathcal{D}$, then the restriction of F to $\mathcal{C}'$ will also be additive.

An important property of an additive functor $F : \mathcal{C} \to \mathcal{D}$ is that it preserves zero morphisms, zero objects and direct sums.

2.2.19 Lemma

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories, and let $F : \mathcal{C} \to \mathcal{D}$ be an additive

functor. If 0 *is a zero morphism or object in* $\mathcal{C}$*, then* $F0$ *is a zero morphism or object in* $\mathcal{D}$*.*

Proof

Temporarily, we indicate a zero object in $\mathcal{C}$ by Z to distinguish it from the various zero maps that we use.

If $0 : C' \to C$ is any zero map in $\mathcal{C}$, then $F0$ must be the zero map from FC' to FC in $\mathcal{D}$, since F is additive. As $id_Z = 0$ on Z itself, $id_{FZ} = F(id_Z) = 0$ on FZ. Hence for a zero object Z' in $\mathcal{D}$, the unique maps $FZ \to Z'$ and $Z' \to FZ$ are mutually inverse isomorphisms. □

2.2.20 Theorem

Let $\mathcal{C}$ *and* $\mathcal{D}$ *be additive categories and let* $F : \mathcal{C} \to \mathcal{D}$ *be an additive functor. Then for any two objects* C' *and* C'' *of* $\mathcal{C}$*,*

$$F(C' \oplus C'') \cong F(C') \oplus F(C'') \text{ in } \mathcal{D}.$$

Proof

Recall from (2.2.11) that the direct sum of the pair of objects C' and C'' in $\mathcal{C}$ can be characterized as an object C of $\mathcal{C}$ for which there are morphisms

$$\sigma' : C' \longrightarrow C,\ \sigma'' : C'' \longrightarrow C,\ \pi' : C \longrightarrow C' \text{ and } \pi'' : C \longrightarrow C''$$

satisfying the relations

$$\pi'\sigma' = id_{C'},\ \pi''\sigma'' = id_{C''} \text{ and } \sigma'\pi' + \sigma''\pi'' = id_C.$$

Since F preserves sums and products, and converts identities to identities and zeroes to zeroes, the maps $F\sigma'$, $F\sigma''$, $F\pi'$ and $F\pi''$ satisfy the equations that show FC to be a direct sum also. □

2.2.21 Functor categories

Recall from (1.3.9) that, given a small category $\mathcal{C}$ and an arbitrary category $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ has as objects the covariant functors from $\mathcal{C}$ to $\mathcal{D}$, the morphisms from F to G in $[\mathcal{C}, \mathcal{D}]$ comprising the set $\mathrm{Nat}(F, G)$ of all natural transformations from F to G.

Our aim now is to show that properties of $\mathcal{D}$ can be carried over to $[\mathcal{C}, \mathcal{D}]$ by making *pointwise* definitions; the category $\mathcal{C}$ can be completely arbitrary.

The essential observation is that a set $\mathrm{Nat}(F, G)$ of morphisms in $[\mathcal{C}, \mathcal{D}]$ is in fact a disjoint union of sets $\mathrm{Mor}_{\mathcal{D}}(F(C), G(C))$, one for each object C of $\mathcal{C}$,

so that any property or condition on $\mathrm{Nat}(F,G)$ can be verified, or imposed, separately in each $\mathrm{Mor}_{\mathcal{D}}(F(C),G(C))$.

Suppose that $\mathcal{D}$ is additive. If η and ζ are in $\mathrm{Nat}(F,G)$, the pointwise definition of their sum $\eta+\zeta$ in $\mathrm{Nat}(F,G)$ is

$$(\eta+\zeta)_C = \eta_C + \zeta_C \text{ for all } C \in \mathcal{C},$$

and the zero natural transformation 0 in $\mathrm{Nat}(F,G)$ is given by

$$0_C = 0 : FC \longrightarrow GC.$$

Thus $\mathrm{Nat}(F,G)$ is an abelian group for any pair of functors.

Since $\mathcal{D}$ has a zero object, we can define the *zero functor*

$$0 : \mathcal{C} \longrightarrow \mathcal{D}$$

by

$$0(C) = 0 \text{ for all } C \text{ in } \mathcal{C},$$

which is a zero object in $[\mathcal{C},\mathcal{D}]$.

We can also define the *direct sum* of two functors F and G in $[\mathcal{C},\mathcal{D}]$ in the obvious way:

$$(F \oplus G)(C) = F(C) \oplus G(C) \text{ for all } C \text{ in } \mathcal{C}.$$

To verify that this actually gives a direct sum, we again use the characterization of direct sums in a preadditive category which we gave in (2.2.11). For each object C, there are morphisms

$$\alpha_C : FC \longrightarrow FC \oplus GC, \quad \beta_C : FC \oplus GC \longrightarrow GC,$$

$$\gamma_C : FC \oplus GC \longrightarrow FC \text{ and } \delta_C : GC \longrightarrow FC \oplus GC$$

with

$$\beta_C\alpha_C = 0,\ \gamma_C\delta_C = 0,\ \beta_C\delta_C = id_{GC},\ \gamma_C\alpha_C = id_{FC}$$

and

$$\alpha_C\gamma_C + \delta_C\beta_C = id_{FC\oplus GC},$$

which gives a set $\alpha,\beta,\gamma,\delta$ of natural transformations that exhibit $F \oplus G$ as a direct sum in $[\mathcal{C},\mathcal{D}]$.

Thus we have the following result.

2.2.22 Proposition

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be an additive category. Then the functor category $[\mathcal{C},\mathcal{D}]$ is an additive category. □

Exercises

2.2.1 Let $\mathcal{D}$ be a preadditive subcategory of the preadditive subcategory $\mathcal{C}$. Show that $\mathrm{End}_{\mathcal{D}}(X)$ is a subring of $\mathrm{End}_{\mathcal{C}}(X)$ for every object X in $\mathcal{D}$.

Given an object X of $\mathcal{C}$ and a subring D of $\mathrm{End}_{\mathcal{C}}(X)$, define a minimal preadditive subcategory $\mathcal{D}$ of $\mathcal{C}$ with $\mathrm{End}_{\mathcal{D}}(X) = D$.

2.2.2 Given a nonzero object X of an additive category $\mathcal{C}$ and a subring D of $\mathrm{Mor}_{\mathcal{C}}(X, X)$, define a minimal additive subcategory $\mathcal{D}$ of $\mathcal{C}$ with $\mathrm{End}_{\mathcal{D}}(X) = D$.

Show that the morphisms in $\mathcal{D}$ can be represented as sets $M_{m,n}(D)$ of $m \times n$ matrices over D.

Hint. Exercise 2.1.6′ of [BK: IRM] helps.

2.2.3 There seems to be a curious asymmetry in the exact sequences of abelian groups obtained in (2.2.3) above, in that they begin, but need not end, with the zero group. Show that this is inevitable for the following reason.

In a preadditive (right) category, the following statements are equivalent.

(i) $\varphi : X \to Y$ has a right inverse;
(ii) for all $L \in \mathcal{C}$, $\mathrm{Mor}(L, X) \xrightarrow{\varphi_*} \mathrm{Mor}(L, Y) \to 0$ is exact;
(iii) for all $L \in \mathcal{C}$, $\varphi_* : \mathrm{Mor}(L, X) \to \mathrm{Mor}(L, Y)$ has a right inverse.

There is an obvious dualization of the above.

Deduce that the following are equivalent.

(i) φ is an isomorphism;
(ii) $\mathrm{Mor}(-, \varphi)$ is always an isomorphism in $\mathcal{A}_{\mathcal{B}}$;
(iii) $\mathrm{Mor}(\varphi, -)$ is always an isomorphism in $\mathcal{A}_{\mathcal{B}}$.

2.2.4 **Epimorphisms and monomorphisms in arbitrary categories**

Let $\mathcal{C}$ be a (right) category. A morphism $\gamma : C \to D$ in $\mathcal{C}$ is defined to be a *monomorphism* if the following holds.

If β, $\beta' : B \to C$ are two morphisms with $\gamma\beta = \gamma\beta'$, then $\beta = \beta'$.

Verify that a monomorphism in a preadditive category is again a monomorphism in this sense. Observe too that if γ has a left inverse, then γ is a monomorphism.

Give a definition of an epimorphism by duality, that is, γ is an epimorphism in $\mathcal{C}$ if and only if γ^{op} is a monomorphism in $\mathcal{C}^{\mathrm{op}}$.

Verify directly that the two definitions of epimorphism coincide in a preadditive category, and that any morphism with a right inverse is an epimorphism.

Show that the inclusion map $\iota : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the (nonadditive) category $\mathcal{R}_{ING}$ of rings with identity.

2.2.5 Show that $C \oplus 0 \cong C$ for any object C of a preadditive category $\mathcal{C}$.

2.2.6 Let C, C' and C'' be objects of a preadditive category $\mathcal{C}$. Verify that the following statements are equivalent, as asserted in (2.2.11).

(i) C is the product of C' and C''.
(ii) C is the coproduct of C' and C''.
(iii) C is the direct sum of C' and C''.

Generalize this result to an arbitrary finite set of objects in $\mathcal{C}$.

2.2.7 Let $\mathcal{C}$ be a preadditive category, with $\alpha : A \to B$, $\beta : B \to C$ and $\gamma : C \to D$ composable morphisms in $\mathcal{C}$.

Show that if β and γ are epimorphisms, then $\gamma\beta$ is an epimorphism, and if $\gamma\beta$ is an epimorphism, then γ is an epimorphism.

Prove also that if α and β are monomorphisms, so is $\beta\alpha$, and if $\beta\alpha$ is a monomorphism, so also is α.

Give counterexamples to the claims that if $\gamma\beta$ is an epimorphism, then β is an epimorphism, and if $\beta\alpha$ is a monomorphism, so also is β.

2.2.8 Let $A_1, \ldots, A_k$ be objects of an additive category $\mathcal{A}$. Show that $A_1 \oplus \cdots \oplus A_k$ is a projective object in $\mathcal{A}$ if and only $A_1, \ldots, A_k$ are each projective objects in $\mathcal{A}$. (This generalizes a well-known result for modules (Theorem 2.5.5 of [BK: IRM]).)

2.2.9 **Projectives depend on the category**

Generalize (2.2.10) to show that, for any ring R, every module in $\mathcal{ASS}_R$ is projective as an object in that category.

Suppose that R is right Artinian but not semisimple. Find a module in $\mathcal{ASS}_R$ which is not projective in $\mathcal{M}_{OD_R}$.

Hint. [BK: IRM], section 4.3 is releyant, particularly Exercise 4.3.7.

2.2.10 **A category without projectives**

Let $\mathcal{FA}_B$ be the category of finite abelian groups. It is well known that any finite abelian group is the direct sum of finite cyclic groups (see Theorem 6.3.24 of [BK: IRM], for example). So if $\mathcal{FA}_B$ contains a nonzero projective, there must be a nontrivial finite cyclic projective, by Exercise 2.2.8 above. However, this is easily seen to be impossible – consider the natural surjection of $\mathbb{Z}/a^2\mathbb{Z}$ to $\mathbb{Z}/a\mathbb{Z}$ for any positive integer a.

2.2.11 **The opposite category**

Show that if $\mathcal{C}$ is an additive category, so likewise is its opposite

category $\mathcal{C}^{\mathrm{op}}$, where addition of morphisms is given by $\alpha^{\mathrm{op}} + \beta^{\mathrm{op}} = (\alpha + \beta)^{\mathrm{op}}$ as in (2.2.5).

Show that C^{op} is a projective object in $\mathcal{C}^{\mathrm{op}}$ if and only if C is an injective object in $\mathcal{C}$, and vice-versa.

Give an example of an additive category which contains nontrivial injective objects but only the zero projective object.

2.2.12 Just as a monoid can be regarded as the same thing as a category with one object (Exercise 1.1.4), a ring may be taken to be the same as a preadditive category with one nonzero object. Viewing a ring R in this way, show that right R-modules correspond to contravariant, cochiral additive functors from R to $\mathcal{M}\mathit{od}_{\mathbb{Z}}$, while left R-modules correspond to covariant, contrachiral additive functors from R to ${}_{\mathbb{Z}}\mathcal{M}\mathit{od}$. Show also that R-module homomorphisms correspond to natural transformations.

2.3 ABELIAN CATEGORIES

For the next stage in our examination of categories whose objects in some way resemble modules, we consider abelian categories. Such categories must contain the kernel and cokernel of each of their morphisms, and most constructions that can be carried out in the full module categories $\mathcal{M}\mathit{od}_R$ are reproducible in an abelian category. Indeed, there are Embedding Theorems, which show that an arbitrary abelian category can be realized as a subcategory of $\mathcal{M}\mathit{od}_R$ for some ring R. However, an abelian category may lack nontrivial examples of projective or injective objects.

We show that the various morphism categories associated to abelian categories are again abelian, as are the functor and product categories. We also introduce the idea of a direct sum of categories, which leads to an interesting categorical interpretation of the module theory of a Dedekind domain.

2.3.1 The definition

Although we have defined kernels and cokernels in an arbitrary category with zero object, and thus in an additive category, a morphism in an additive category need not have a kernel or cokernel in that category. Indeed, the category $\mathcal{P}_{A[\epsilon]}$ of finitely generated projective modules over a ring of dual numbers $A[\epsilon]$ (where $\epsilon^2 = 0$) fails to contain either the kernel or the cokernel of the multiplication homomorphism $\epsilon : A[\epsilon] \to A[\epsilon]$, since each is isomorphic to A, which is not projective over $A[\epsilon]$.

The requirement that a category should contain kernels and cokernels leads to the next definition.

Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is an *abelian category* if the following conditions are satisfied.

Ab 1. If $\lambda : L \to M$ is a morphism in $\mathcal{C}$, then $\mathcal{C}$ contains a kernel $(\operatorname{Ker}\lambda, \kappa)$ and a cokernel $(\operatorname{Cok}\lambda, \chi)$ of λ. (This means not only that the objects $\operatorname{Ker}\lambda$ and $\operatorname{Cok}\lambda$ are in $\mathcal{C}$, but also that the morphisms

$$\kappa : \operatorname{Ker}\lambda \longrightarrow L \quad \text{and} \quad \chi : M \longrightarrow \operatorname{Cok}\lambda$$

belong to $\mathcal{C}$.)

Ab 2. If $\kappa : K \to L$ is a monomorphism in $\mathcal{C}$, then (K, κ) is a kernel of some morphism $\lambda : L \to M$ in $\mathcal{C}$.

Ab 2^{op}. If $\chi : M \to C$ is an epimorphism in $\mathcal{C}$, then (C, χ) is a cokernel of some morphism $\lambda : L \to M$ in $\mathcal{C}$.

Statement Ab 2 is often phrased as 'every monomorphism is normal'. Alternative axiom sets abound in the literature. For example, see [Freyd 1964] or [Herrlich & Strecker 1979] for the following.

2.3.2 Theorem

Suppose that $\mathcal{C}$ is an additive category satisfying Ab 1. Then the following statements are equivalent.

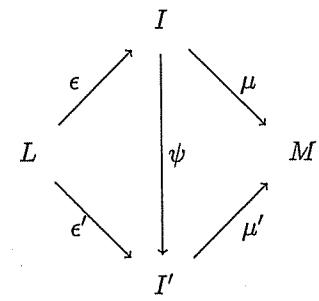
(i) *Ab 2 and Ab 2^{op} hold in $\mathcal{C}$.*

(ii) *For any morphism $\lambda : L \to M$ with kernel $(\operatorname{Ker}\lambda, \kappa)$ and cokernel $(\operatorname{Cok}\lambda, \chi)$, the morphism*

$$\psi : \operatorname{Cok}\kappa \longrightarrow \operatorname{Ker}\chi$$

(which exists by the universal properties of kernels and cokernels) is an isomorphism.

(iii) *Every morphism $\lambda : L \to M$ has a unique (epi, mono) factorization; that is, there exists an epimorphism $\epsilon : L \to I$ and monomorphism $\mu : I \to M$ with $\mu\epsilon = \lambda$ such that, if $\epsilon' : L \to I'$, $\mu' : I' \to M$ is another (epi, mono) factorization of λ, then there is a unique morphism $\psi : I \to I'$ making*



commute; moreover, ψ is an isomorphism. ○

The isomorphic objects $\operatorname{Ker}\chi$ and I of this theorem are commonly referred to as the image $\operatorname{Im}\lambda$ of λ.

2.3.3 Module-like behaviour

In essence, the above result tells us that the Induced Mapping Theorem ([BK: IRM] (1.2.11)) must hold in an abelian category. It is then possible to translate many general results about modules into the context of an abstract abelian category. For example, the Isomorphism Theorems ([BK: IRM] Exercise 1.2.2) can be reproduced. However, the proofs are more delicate since it is not possible to work with elements. Instead, the universal properties of kernels, cokernels, etc. must be used. Arguments of this type are developed in [Mac Lane 1975], Chapter IX and [Mac Lane 1971], Chapter VIII.

On the other hand, there are some features of module categories which need not be reproducible in an abelian category. These usually relate to the existence of an object with some specific property. For instance, an abelian category need not contain any projective object, as is the case with the category $\mathcal{FA}_{\mathcal{B}}$ of finite abelian groups (see Exercise 2.2.10).

2.3.4 Example

For an example of an additive category that satisfies axiom Ab 1 but not Ab 2, consider the full additive subcategory of $\mathcal{A}_{\mathcal{B}}$ whose objects comprise all finite direct sums of the form $\mathbb{Z}^m \oplus \mathbb{Q}^n$ with $m, n \geq 0$. Here the inclusion monomorphism $\mathbb{Z} \to \mathbb{Q}$ is not a kernel.

We now look for abelian categories of modules. The first result simplifies the checking in many cases of interest. (Its converse, while also true, is less elementary – see Exercise 2.3.8 for the details.)

2.3.5 Theorem

Let R be a ring and let $\mathcal{C}$ be a full additive subcategory of $\mathcal{M}od_R$ such that, for any homomorphism $\lambda : L \to M$ of R-modules in $\mathcal{C}$, the R-modules $\operatorname{Ker}\lambda$ *and* $\operatorname{Cok}\lambda$ *are both in $\mathcal{C}$.*

Then $\mathcal{C}$ is abelian.

Proof

The R-modules $\operatorname{Ker}\lambda$ and $\operatorname{Cok}\lambda$ evidently satisfy the universal properties needed to make them the kernel and cokernel of λ in $\mathcal{C}$. So Ab 1 holds. To see that Ab 2 holds, consider a monomorphism $\kappa : K \to L$ in $\mathcal{C}$. As we remarked in (2.2.3), κ is an injective mapping. Now $\mathcal{C}$ must also contain $\operatorname{Cok}\kappa = L/\operatorname{Im}\kappa$, and so K is the kernel of the canonical module epimorphism $\lambda : L \to L/\operatorname{Im}\kappa$, which is in $\mathcal{C}$. The proof of Ab 2^{op} is similar. □

For the next result, recall that, given a ring R, a right R-module M is *Noetherian* if every R-submodule of M is finitely generated. The ring R itself is *right Noetherian* if every right ideal of R is finitely generated. An account of Noetherian rings and modules is given in [BK: IRM], §3.1.

2.3.6 Corollary

Let R be a ring. Then the category $\mathcal{M}_R$ of finitely generated R-modules is an additive subcategory of $\mathcal{M}od_R$, and $\mathcal{M}_R$ is abelian precisely when the ring R is right Noetherian.

Proof

The sole obstruction to $\mathcal{M}_R$ being abelian is that it might not contain the kernel of some homomorphism, that is, there may be a finitely generated module M having a submodule which is not finitely generated. By Corollary 3.1.7 of [BK: IRM], this cannot happen if R is right Noetherian. In the reverse direction, if $\mathcal{M}_R$ is abelian, then every ideal of R is in $\mathcal{M}_R$. □

2.3.7 More examples

The category $\mathcal{HA}_B$ of Hausdorff abelian groups is an example of a full additive subcategory of $\mathcal{M}od_{\mathbb{Z}}$ which is not abelian. Observe that the cokernel (in $\mathcal{T}op\mathcal{A}_B$) of the inclusion monomorphism $\mathbb{Q} \to \mathbb{R}$ does not lie in $\mathcal{HA}_B$.

There are some subcategories of $\mathcal{M}od_R$ that are abelian for an arbitrary ring R. Let $\mathcal{N}oeth_R$ denote the full subcategory of $\mathcal{M}od_R$ whose objects are the Noetherian right R-modules, let $\mathcal{A}rt_R$ denote the full subcategory whose objects are the Artinian right R-modules and let $\mathcal{ASS}_R$ be the full subcategory given by the Artinian semisimple modules. Each of these categories is abelian

since it contains any submodule or quotient module of any of its members – see [BK: IRM], Proposition 3.1.2, Proposition 4.1.5 and the Artinian Splitting Theorem (4.1.17) respectively. We record our assertion formally.

2.3.8 Theorem

The categories $\mathcal{N}_{OETH_R}$, $\mathcal{A}_{RT_R}$ and $\mathcal{ASS}_R$ are abelian. □

By [BK: IRM] Corollary 4.1.18, there is an inclusion of categories

$$\mathcal{ASS}_R \subseteq \mathcal{A}_{RT_R}$$

and, by the same result together with the definition of Noetherian, there are inclusions

$$\mathcal{ASS}_R' \subseteq \mathcal{N}_{OETH_R} \subseteq \mathcal{M}_R.$$

In general, $\mathcal{A}_{RT_R}$ is not a subcategory of $\mathcal{N}_{OETH_R}$. An example (due to P. M. Cohn) of a cyclic Artinian, non-Noetherian module is exhibited in Exercise 4.1.7 of [BK: IRM].

Next, we give the conditions for equality between these various categories.

2.3.9 Proposition

(i) *$\mathcal{N}_{OETH_R} = \mathcal{M}_R$ if and only if R is right Noetherian.*
(ii) *$\mathcal{A}_{RT_R} = \mathcal{M}_R$ if and only if R is right Artinian.*
(iii) *$\mathcal{ASS}_R = \mathcal{M}_R$ if and only if R is an Artinian semisimple ring (in which case $\mathcal{ASS}_R = \mathcal{P}_R$, the category of finitely generated projective right R-modules).*
(iv) *Suppose that R is right Artinian. Then the ring $R/\operatorname{rad}(R)$ is Artinian semisimple, where $\operatorname{rad}(R)$ is the Jacobson radical, and*

$$\mathcal{ASS}_R = \mathcal{M}_{R/\operatorname{rad}(R)}.$$

Proof

(i) Since R itself is in $\mathcal{M}_R$, the equality of categories makes R right Noetherian. Conversely, if R is right Noetherian then R^k is Noetherian for any natural number k and hence any finitely generated module M is also Noetherian – see [BK: IRM] (3.1.4).

(ii) The argument is similar to the proof of (i); we use the fact that if R is right Artinian, then so is any finitely generated right R-module ([BK: IRM] (4.1.7)).

(iii) Suppose that R is an Artinian semisimple ring. By the Wedderburn-Artin Theorem ([BK: IRM] (4.2.3)), R is a direct product of matrix rings over division rings, and so the categories are equal since every finitely generated

R-module is both semisimple and projective ([BK: IRM] (4.2.6)). Conversely, if the equalities hold, then R is Artinian semisimple by definition.

(iv) By [BK: IRM] (4.3.15), $R/\operatorname{rad}(R)$ is Artinian semisimple, and, by [BK: IRM] (4.3.16), a right R-module M is semisimple if and only if $M \cdot \operatorname{rad}(R) = 0$. Thus a semisimple R-module can be regarded as an $R/\operatorname{rad}(R)$-module by the scalar multiplication

$$m \cdot \overline{r} = m \cdot r \quad \text{for } r \in R,$$

and conversely, this rule can be used to make any $R/\operatorname{rad}(R)$-module into a semisimple R-module. □

2.3.10 Product and morphism categories

Next, we show that product and morphism categories of abelian categories are also abelian. As we already know that such categories are additive, it suffices to verify the axioms Ab 1, Ab 2 and Ab 2^{op}, which will follow at once from explicit descriptions of kernels, cokernels and so forth.

Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A morphism in $\mathcal{C} \times \mathcal{D}$ is a pair

$$(\alpha, \beta) : (C, D) \longrightarrow (C', D'),$$

with α a morphism in $\mathcal{C}$ and β a morphism in $\mathcal{D}$. Thus

$$\operatorname{Mor}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) = \operatorname{Mor}_{\mathcal{C}}(C, C') \times \operatorname{Mor}_{\mathcal{D}}(D, D'),$$

and any assertion about a morphism (α, β) in $\mathcal{C} \times \mathcal{D}$ can be verified on its components α, β separately. Thus (α, β) is a monomorphism precisely when α and β are both monomorphisms, and likewise for epimorphisms.

The kernel of (α, β) is $((\operatorname{Ker}\alpha, \operatorname{Ker}\beta), (\kappa, \psi))$, where $(\operatorname{Ker}\alpha, \kappa)$ and $(\operatorname{Ker}\beta, \psi)$ are the respective kernels of α and β, and the cokernel is $((\operatorname{Cok}\alpha, \operatorname{Cok}\beta), (\chi, \zeta))$. In practice, we omit the maps κ and ψ, and write

$$\operatorname{Ker}(\alpha, \alpha') = (\operatorname{Ker}\alpha, \operatorname{Ker}\alpha'),$$

and similarly for the cokernel.

The image is the pair $\operatorname{Im}(\alpha, \beta) = (\operatorname{Im}\alpha, \operatorname{Im}\beta)$, with the obvious implicit morphisms. We summarize as follows.

2.3.11 Proposition

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are abelian categories. Then the category $\mathcal{C} \times \mathcal{D}$ is also abelian.

In particular, a sequence

$$0 \longrightarrow (C', D') \longrightarrow (C, D) \longrightarrow (C'', D'') \longrightarrow 0$$

in $\mathcal{C} \times \mathcal{D}$ is a short exact sequence if and only both component sequences

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

are short exact sequences. □

Similar arguments are used for the morphism categories attached to the category $\mathcal{C}$. In the morphism category $\mathcal{M}or_R\mathcal{C}$, the kernel of

$$(\alpha, \beta) : (C, D, \gamma) \longrightarrow (C', D', \delta)$$

is $(\operatorname{Ker}\alpha, \operatorname{Ker}\beta, \overline{\gamma})$, where $\overline{\gamma}$ is the restriction of γ to $\operatorname{Ker}\alpha$, and the cokernel is $(\operatorname{Cok}\alpha, \operatorname{Cok}\beta, \overline{\delta})$, where $\overline{\delta}$ is induced by δ. Images in $\mathcal{M}or_R\mathcal{C}$ are defined in the expected way. The result is as follows.

2.3.12 Proposition

Let $\mathcal{C}$ be an abelian category. Then the categories $\mathcal{M}or_R\mathcal{C}$, $\mathcal{E}nd\mathcal{C}$, $\mathcal{I}so\mathcal{C}$ and $\mathcal{A}ut\,\mathcal{C}$ are also abelian.

In particular, a sequence

$$0 \longrightarrow (C', D', \gamma') \longrightarrow (C, D, \gamma) \longrightarrow (C'', D'', \gamma'') \longrightarrow 0$$

in $\mathcal{M}or_R\mathcal{C}$ is a short exact sequence if and only both component sequences

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

are short exact sequences in $\mathcal{C}$.

Note that in $\mathcal{E}nd\,\mathcal{C}$ and $\mathcal{A}ut\,\mathcal{C}$, the two component sequences are identical.

□

2.3.13 Module categories

Let R and S be rings. As we saw in (1.3.16), the product category is equivalent to the category $\mathcal{M}od_{R\times S}$, since every object (M, N) in the product category can be regarded as a module over $R \times S$ and every morphism is an $R \times S$-module homomorphism (1.1.12). Thus there are two ways in which, for instance, a kernel can be defined in $\mathcal{M}od_R \times \mathcal{M}od_S$, either by using the abstract machinery or by taking an equivalent object in $\mathcal{M}od_{R\times S}$.

The reader should not have much difficulty in verifying that these approaches give essentially the same objects, to within isomorphisms.

The kernels and cokernels in the (homo)morphism categories $\mathcal{M}or(\mathcal{M}od_R)$, $\mathcal{E}nd(\mathcal{M}od_R)$, $\mathcal{I}so(\mathcal{M}od_R)$ and $\mathcal{A}ut(\mathcal{M}od_R)$ are those in $\mathcal{M}od_{R\times R}$.

2.3.14 Functor categories

We can also give pointwise descriptions of kernels, cokernels, and so on, in a functor category $[\mathcal{C},\mathcal{D}]$ provided the appropriate objects exist in $\mathcal{D}$. For simplicity we suppose that $\mathcal{D}$ is abelian.

For example, the kernel $(\operatorname{Ker}\eta,\kappa)$ of a morphism η from F to G, that is, of a natural transformation $\eta \in \operatorname{Nat}(F,G)$, is obtained as follows.

For each object C of $\mathcal{C}$, write $(\operatorname{Ker}\eta_C,\kappa_C)$ for the kernel of $\eta_C : FC \to GC$. Then the object $\operatorname{Ker}\eta$ in $[\mathcal{C},\mathcal{D}]$ is the functor from $\mathcal{C}$ to $\mathcal{D}$ given by

$$(\operatorname{Ker}\eta)(C) = \operatorname{Ker}(\eta_C : FC \longrightarrow GC)$$

for each object C of $\mathcal{C}$.

To see that $\operatorname{Ker}\eta$ is a functor, note that a morphism $\alpha : C' \to C$ gives a commutative diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

which, together with the definition of a kernel as a universal object (1.4.15), shows that there is an induced morphism

$$(\operatorname{Ker}\eta)(\alpha) : (\operatorname{Ker}\eta)(C') \longrightarrow (\operatorname{Ker}\eta)(C).$$

There is a natural transformation

$$\kappa : \operatorname{Ker}\eta \longrightarrow F$$

given by

$$\kappa_C : (\operatorname{Ker}\eta)(C) \longrightarrow F(C),$$

κ_C as above, and it is straightforward to verify that the pair $(\operatorname{Ker}\eta,\kappa)$ is a kernel for η.

Similarly, cokernels, monomorphisms, epimorphisms, projective and injective objects in $[\mathcal{C},\mathcal{D}]$ are given pointwise by the corresponding objects or morphisms in $\mathcal{D}$.

We record the result for future reference.

2.3.15 Theorem

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be an abelian category. Then the functor category $[\mathcal{C}, \mathcal{D}]$ is an abelian category. □

2.3.16 Direct sums of categories

Since we are able to define the direct sum of two, or more, functors between additive categories, we can interpret the direct product of two additive categories as a *direct sum*, because we can mimic the description of a direct sum of modules in terms of injections and projections (2.2.11).

Throughout the sequel we demand that an *equivalence* between additive categories must be given by an additive functor.

2.3.17 Theorem

Let $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be additive categories. Then the following statements are equivalent.

(i) *There is an equivalence of additive categories*

$$\mathcal{C} \simeq \mathcal{C}_1 \times \mathcal{C}_2.$$

(ii) *There are additive functors*

$$I_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}, \quad I_2 : \mathcal{C}_2 \longrightarrow \mathcal{C}, \quad P_1 : \mathcal{C} \longrightarrow \mathcal{C}_1, \quad P_2 : \mathcal{C} \longrightarrow \mathcal{C}_2$$

such that there are natural isomorphisms

$$I_1 P_1 \oplus I_2 P_2 \simeq \mathrm{Id}_{\mathcal{C}}$$

and

$$P_1 I_1 \simeq \mathrm{Id}_{\mathcal{C}_1}, \quad P_2 I_2 \simeq \mathrm{Id}_{\mathcal{C}_2},$$

and also

$$P_2 I_1 = 0, \quad P_1 I_2 = 0.$$

(I and P stand for inclusion and projection.)

Proof

Suppose that (i) holds, and let $F : \mathcal{C} \to \mathcal{C}_1 \times \mathcal{C}_2$ and $G : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ be the (additive) functors that give the equivalence. If C is an object of $\mathcal{C}$, we have $FC = (C_1, C_2)$ for objects C_1 of $\mathcal{C}_1$ and C_2 of $\mathcal{C}_2$. Similarly, for a morphism α we have $F\alpha = (\alpha_1, \alpha_2)$. Define $P_1 C = C_1$ and $P_1 \alpha = \alpha_1$, and likewise for P_2.

For an object C_1 and morphism α_1 in $\mathcal{C}_1$, let $I_1 C_1 = G(C_1, 0)$ and $I_1 \alpha_1 =$

$G(\alpha_1, 0)$, and similarly for the other variable. The assertion follows, on noting that

$$(C_1, C_2) \cong (C_1, 0) \oplus (0, C_2) \text{ in } C_1 \times C_2.$$

Conversely, if (ii) holds, define $FC = (P_1C, P_2C)$ and $G(C_1, C_2) = I_1C_1 \oplus I_2C_2$, and similarly for morphisms. □

For completeness, we record the extension of the above result to a direct product of any finite set of additive categories. The result is analogous to that for modules (see [BK: IRM], (2.1.7)).

2.3.18 Corollary

Let $\mathcal{C}$ and $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be additive categories. Then the following are equivalent.

(i) *There is an equivalence of additive categories*

$$\mathcal{C} \simeq \mathcal{C}_1 \times \cdots \times \mathcal{C}_n.$$

(ii) *There are additive functors $I_i : \mathcal{C}_i \to \mathcal{C}$ and $P_i : \mathcal{C} \to \mathcal{C}_i$, for $i = 1, \ldots, n$, with natural isomorphisms*

$$I_1P_1 \oplus \cdots \oplus I_nP_n \simeq \mathrm{Id}_{\mathcal{C}}$$

and

$$P_iI_i \simeq \mathrm{Id}_{\mathcal{C}_i} \text{ for all } i,$$

and further

$$P_jI_i = 0 \text{ for } i \neq j.$$ □

2.3.19 Infinite direct sums of categories

Since an additive category has a zero object, it is sensible to speak of the *direct sum* $\bigoplus_\Lambda \mathcal{C}_\lambda$ of an infinite set $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$ of categories, where Λ is an ordered set. The objects of the direct sum are all sequences of the form

$$C_\Lambda = (C_\lambda \mid C_\lambda \in \mathcal{C}_\lambda, \text{ almost all } C_\lambda = 0),$$

and a morphism $\alpha_\Lambda : C'_\lambda \to C_\Lambda$ is a sequence $(\alpha_\lambda : C' \to C_\lambda)$ with each α_λ a morphism in $\mathcal{C}_\lambda$.

It is straightforward to verify that $\bigoplus_\Lambda \mathcal{C}_\lambda$ is again an additive category under the usual componentwise definitions. Thus the zero object is $0_\Lambda = (0)$, and the direct sum in $\bigoplus_\Lambda \mathcal{C}_\lambda$ is given by

$$C'_\Lambda \oplus C''_\Lambda = (C'_\lambda \oplus C''_\lambda).$$

The reader is invited to characterize infinite direct sums in terms of inclusion and projection functors.

2.3.20 *Dedekind domains: a review*

Some interesting direct sum decompositions of categories arise from the structure theory of modules over a Dedekind domain. As we use this structure theory in several places in this text, we turn aside to review its basic definitions and results, which are developed in detail in Chapters 5 and 6 of [BK: IRM].

We start with some ideal theory. Let $\mathcal{O}$ be an arbitrary commutative domain with field of fractions $\mathcal{K}$. A *fractional ideal* of $\mathcal{O}$ is a finitely generated nonzero $\mathcal{O}$-submodule $\mathfrak{a}$ of $\mathcal{K}$. If $\mathfrak{b}$ is also a fractional ideal of $\mathcal{O}$, so is their *product*

$$\mathfrak{ab} = \{a_1b_1 + \cdots + a_kb_k \mid a_1, \ldots, a_k \in \mathfrak{a},\ b_1, \ldots, b_k \in \mathfrak{b},\ \ k \geq 1\}.$$

A fractional ideal $\mathfrak{a}$ is *invertible* if

$$\mathfrak{ab} = \mathcal{O}$$

for some $\mathfrak{b}$.

Our definition of a Dedekind domain ([BK: IRM] (5.1.10)) is that it is a commutative domain $\mathcal{O}$ all of whose fractional ideals are invertible. It can then be shown that $\mathcal{O}$ is a Noetherian ring ([BK: IRM] Exercise 5.1.2). Clearly, the fractional ideals of a Dedekind domain form a group $\mathrm{Frac}(\mathcal{O})$, and the set of nonzero principal ideals $\mathrm{Pr}(\mathcal{O})$ is a subgroup of $\mathrm{Frac}(\mathcal{O})$. The *ideal class group* of $\mathcal{O}$ is the quotient group

$$\mathrm{Cl}(\mathcal{O}) = \mathrm{Frac}(\mathcal{O})/\mathrm{Pr}(\mathcal{O}).$$

The image of $\mathfrak{a}$ in $\mathrm{Cl}(\mathcal{O})$ is written $\{\mathfrak{a}\}$.

It is obvious that a commutative principal ideal domain is the same thing as a Dedekind domain with trivial ideal class group.

We record [BK: IRM] (5.1.19).

A The Unique Factorization Theorem for Ideals

Let $\mathfrak{a}$ be a fractional ideal of a Dedekind domain $\mathcal{O}$. Then there is a set of distinct nonzero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and a set of nonzero integers $n_1, \ldots, n_k$ such that

$$\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}.$$

The prime ideals are uniquely determined apart from the order in which they

are written, and the exponents are uniquely determined once an order has been fixed. ○

Let $\mathcal{O}$ again be an arbitrary commutative domain, and let M be an $\mathcal{O}$-module. The *torsion submodule* $T(M)$ is

$$T(M) = \{m \in M \mid mx = 0 \text{ for some } x \in \mathcal{O},\ x \neq 0\}.$$

We say that M is a *torsion module* if $M = T(M)$ and that M is *torsion-free* if $T(M) = 0$. (These definitions anticipate the more general definitions that will be made in section 6.2.)

Here are the fundamental results for projective modules over a Dedekind domain.

B Theorem

Let $\mathcal{O}$ be a Dedekind domain and let M be a finitely generated $\mathcal{O}$-module. Then M is projective if and only if M is torsion-free (see ([BK: IRM] (6.3.4)).○

C Theorem

Let $\mathcal{O}$ be a Dedekind domain and let M be a finitely generated projective $\mathcal{O}$-module. Then

$$M \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$$

for some fractional ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ of $\mathcal{O}$ ([BK: IRM] (6.1.2)). ○

The next result gives the criterion for two projective modules to be isomorphic ([BK: IRM] (6.1.6)).

D Steinitz' Theorem

Let $P \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$ be a projective module over a Dedekind domain $\mathcal{O}$. Then

$$P \cong \mathcal{O}^{r-1} \oplus \mathfrak{a}_1 \cdots ; \mathfrak{a}_r.$$

If $Q \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s$ is also a projective $\mathcal{O}$-module, then the following statements are equivalent.

(i) $P \cong Q$;

(ii) $r = s$ *and* $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\} = \{\mathfrak{b}_1 \cdots \mathfrak{b}_s\}$ *in* $\mathrm{Cl}(\mathcal{O})$. ○

The integer r is called the *rank* of P and the ideal class $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\}$ is called the *ideal class* of P. A projective module is said to be in *standard form* if $P = \mathcal{O}^{r-1} \oplus \mathfrak{a}$.

The next result is immediate from the observation that $M/T(M)$ is torsion-free.

E Corollary

Let M be a finitely generated module over a Dedekind domain $\mathcal{O}$. Then

$$M \cong T(M) \oplus M/T(M)$$

with $M/T(M)$ projective. ○

To complete the picture, we describe the structure of torsion modules over a Dedekind domain $\mathcal{O}$. A finitely generated torsion $\mathcal{O}$-module M has *annihilator*

$$\mathrm{Ann}(M) = \{x \in \mathcal{O} \mid mx = 0 \text{ for all } m \in M\},$$

which is a nonzero ideal of $\mathcal{O}$. If $\mathrm{Ann}(M) = \mathfrak{p}^n$ for some nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ and natural number n, then M is said to be $\mathfrak{p}$-*primary*.

Since $\mathcal{O}$ is Noetherian, any set of submodules of a finitely generated module M has maximal members, each of which is finitely generated ([BK: IRM], (3.1.6)). However, the sum of any two $\mathfrak{p}$-primary submodules of M is again $\mathfrak{p}$-primary, so that M has a unique maximal $\mathfrak{p}$-primary submodule, $T_{\mathfrak{p}}(M)$. The submodule $T_{\mathfrak{p}}(M)$ is the $\mathfrak{p}$-*primary component* of M; it is again finitely generated.

Then we have the following result ([BK: IRM] (6.3.15), (6.3.20)).

F The Primary Decomposition Theorem

Let M be a finitely generated torsion module over a Dedekind domain $\mathcal{O}$. Then:

(i) *there is a direct sum decomposition*

$$M = \bigoplus_{\mathfrak{p}} T_{\mathfrak{p}}(M),$$

where almost all (that is, all except a finite number) of the $\mathfrak{p}$-primary components of M are zero;

(ii) *for each prime ideal $\mathfrak{p}$ of $\mathcal{O}$,*

$$T_{\mathfrak{p}}(M) = \mathcal{O}/\mathfrak{p}^{\delta(\mathfrak{p},1)} \oplus \cdots \oplus \mathcal{O}/\mathfrak{p}^{\delta(\mathfrak{p},\ell_{\mathfrak{p}})}$$

with $\delta(\mathfrak{p},1) \leq \cdots \leq \delta(\mathfrak{p},\ell_{\mathfrak{p}})$ and $\ell_{\mathfrak{p}} \geq 0$;

(iii) *the collection of integers $\delta(\mathfrak{p},1), \ldots, \delta(\mathfrak{p},\ell_{\mathfrak{p}})$, where $\mathfrak{p}$ ranges over all the nonzero prime ideals of $\mathcal{O}$, is uniquely determined by M and in turn determines M to within isomorphism.* ○

2.3.21 Module categories over Dedekind domains

We can now give the promised direct sum decompositions of categories that arise from the structure theory of modules over a Dedekind domain $\mathcal{O}$.

Since $\mathcal{O}$ is a Noetherian ring, any finitely generated (right) $\mathcal{O}$-module M is also Noetherian, and in particular, any ascending chain of submodules of M has a maximal member, which is again finitely generated ([BK: IRM] (3.1.6)). Thus the category $\mathcal{M}_{\mathcal{O}}$ of finitely generated (right) $\mathcal{O}$-modules is abelian (2.3.6).

Let $\mathcal{T}_{\mathcal{O}}$ be the full subcategory of $\mathcal{M}_{\mathcal{O}}$ whose objects are all the finitely generated torsion $\mathcal{O}$-modules. Since the direct sum of two modules belonging to $\mathcal{T}_{\mathcal{O}}$ is again a module in $\mathcal{T}_{\mathcal{O}}$, we see that $\mathcal{T}_{\mathcal{O}}$ is an additive subcategory of $\mathcal{M}_{\mathcal{O}}$. As a submodule or quotient module of a module in $\mathcal{T}_{\mathcal{O}}$ again belongs to $\mathcal{T}_{\mathcal{O}}$, the kernel and cokernel of a homomorphism in $\mathcal{T}_{\mathcal{O}}$ is also in $\mathcal{T}_{\mathcal{O}}$, so that $\mathcal{T}_{\mathcal{O}}$ is an abelian category (2.3.5).

Write $\mathcal{TF}_{\mathcal{O}}$ for the category of finitely generated torsion-free (right) $\mathcal{O}$-modules. By (2.3.20 - B), $\mathcal{TF}_{\mathcal{O}}$ is the same as $\mathcal{P}_{\mathcal{O}}$, the category of finitely generated projective $\mathcal{O}$-modules.

Let M and N be finitely generated $\mathcal{O}$-modules. Clearly, an $\mathcal{O}$-module homomorphism $\lambda : M \to N$ induces a homomorphism $T(\lambda) : T(M) \to T(N)$. Thus we have a functor

$$T : \mathcal{M}_{\mathcal{O}} \longrightarrow \mathcal{T}_{\mathcal{O}}.$$

There is also a functor $F : \mathcal{M}_{\mathcal{O}} \to \mathcal{P}_{\mathcal{O}}$, which sends M to $M/T(M)$. Now, any module M has a direct sum decomposition $M = T(M) \oplus M'$ with $M' \cong M/T(M)$ projective, but there is no direct sum decomposition of the category $\mathcal{M}_{\mathcal{O}}$ in terms of $\mathcal{P}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$, essentially because there is no canonical choice of M' – see Exercise 2.3.4.

For each nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ let $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$ denote the full subcategory of $\mathcal{T}_{\mathcal{O}}$ given by the $\mathfrak{p}$-primary modules. Again, $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$ is an abelian category, and we have functors

$$T_{\mathfrak{p}} : \mathcal{T}_{\mathcal{O}} \longrightarrow \mathcal{T}_{\mathfrak{p},\mathcal{O}}$$

for each $\mathfrak{p}$.

By (2.3.20 - F), there is a direct sum decomposition of

$$\mathcal{T}_{\mathcal{O}} = \bigoplus_{\mathfrak{p}\in\mathbf{P}} \mathcal{T}_{\mathfrak{p},\mathcal{O}}$$

of the category $\mathcal{T}_{\mathcal{O}}$ into its subcategories $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$.

2.3.22 The Embedding Theorems

There are several results to the effect that an abstract additive or abelian category can be viewed as embedded as a subcategory in a concrete abelian category, such as $\mathcal{A}_B$, or $\mathcal{M}od_R$ for some ring R.

If $\mathcal{C}$ is a small additive category, then $\mathcal{C}$ can be embedded contravariantly as an additive subcategory in the functor category $[\mathcal{C}, \mathcal{A}_B]$ by associating with an object C the morphism functor $\mathrm{Mor}_{\mathcal{C}}(C, -)$ ([Mitchell 1965], IV (2.3)). Since the opposite of an additive category is also additive, this result gives a covariant embedding of $\mathcal{C}$ in the additive category $[\mathcal{C}^{\mathrm{op}}, \mathcal{A}_B]$.

This result applies also to abelian categories, and extends to show that a small additive or abelian category can be embedded in $\mathcal{A}_B$ (*ibid.* IV (2.6)), which result is sometimes known as the Lubkin-Heron-Freyd Representation Theorem, although it is also attributed to Mitchell. (The discoverers of this circle of results are generous with their attributions.)

In [Pareigis 1970] (4.14) Theorem 3, we find the more refined result that, given a small abelian category $\mathcal{C}$, there is a ring R and a covariant full faithful functor (1.3.13) from $\mathcal{C}$ to the category $\mathcal{M}od_R$. Anticipating the next section, we remark that this functor is also 'exact' in that it preserves exact sequences.

The philosophical consequence of these results is that it suffices to verify any sufficiently general statement about abelian categories in the category $\mathcal{A}_B$; this point of view is discussed in [Mitchell 1965], IV. However, it is usually more natural to work directly in a given additive or abelian category.

2.3.23 Example: The Famous Five Lemma

This lemma† is one of a number of useful results which hold in an abstract abelian category, but which are proved more readily by a 'diagram chase' in a module category.

Suppose that we have a commuting diagram, with both rows exact, in an additive category:

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & M_4 & \xrightarrow{\alpha_4} & M_5 \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_5} \\
N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 & \xrightarrow{\beta_3} & N_4 & \xrightarrow{\beta_4} & N_5
\end{array}$$

† Possibly so-named because it leads to the Smuggler's Top Theorem, a truly marvellous result which this footnote is unfortunately too small to contain ([Blyton 1950]).

Then, by embedding the category in $\mathcal{A}_B$, we may work with elements in the various objects. For example:

> *Suppose that γ_2 and γ_4 are epimorphisms, and that γ_5 is a monomorphism. Then γ_3 is also an epimorphism.*

To prove this, let $y_3 \in N_3$. Taking $x_4 \in M_4$ with $\gamma_4 x_4 = \beta_3 y_3 \in N_4$, we have

$$\gamma_5 \alpha_4 x_4 = \beta_4 \beta_3 y_3 = 0.$$

Thus $x_4 \in \operatorname{Ker} \alpha_4 = \operatorname{Im} \alpha_3$, and so we can find $x_3 \in M_3$ with $\alpha_3 x_3 = x_4$; then

$$y_3 - \gamma_3 x_3 \in \operatorname{Ker} \beta_3 = \operatorname{Im} \beta_2 = \operatorname{Im} \beta_2 \gamma_2 = \operatorname{Im} \gamma_3 \alpha_2.$$

So $y_3 \in \operatorname{Im} \gamma_3$ as required.

With slightly less rigmarole, one can prove the dual statement for γ_3 to be a monomorphism. In combination, these diagram chases yield the following highly useful result.

> *If γ_1, γ_2, γ_4 and γ_5 are isomorphisms, then so also is γ_3.*

Other, less widely used, diagram chase theorems to be found in the literature have such evocative names as the Snake Lemma, the Horseshoe Lemma, the Windmill Lemma, the 3×3 Lemma, the Nine Lemma, etc.

Exercises

2.3.1 Show that the opposite category of an abelian category is also abelian.

2.3.2 Let $\alpha : L \to M$ and $\beta : M \to N$ be morphisms in an abelian category. Show (for example, by (2.3.2)), that the following statements are equivalent.

(i) $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is a short exact sequence;

(ii) α is a monomorphism and $\beta = \operatorname{Cok} \alpha$;

(iii) β is an epimorphism and $\alpha = \operatorname{Ker} \beta$.

Recall that our example $0 \to \mathbb{Q} \to \mathbb{R} \to 0 \to 0$ in $\mathcal{HA}_B$ shows that (ii) need not imply (i) when the category is additive but not abelian.

2.3.3 Show that in an abelian category $\mathcal{C}$ any morphism $\alpha : L \to M$ gives

rise to a commuting diagram of the following kind.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\kappa} & L & \xrightarrow{\tau} & I & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longleftarrow & N & \xleftarrow{\chi} & M & \xleftarrow{\nu} & I' & \longleftarrow & 0 \end{array}$$

Here the horizontal sequences are both short exact, θ is an isomorphism, and $\kappa = \operatorname{Ker}\alpha$ and $\chi = \operatorname{Cok}\alpha$. Thus $I = \operatorname{Cok}\operatorname{Ker}\alpha$ is the image $\operatorname{Im}\alpha$ of α.

Generalizing (2.1.2) from module categories to arbitrary abelian categories, we say that a sequence

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \longrightarrow \cdots$$

in $\mathcal{C}$ is *exact at* A_i if $\operatorname{Ker}\alpha_i = \operatorname{Im}\alpha_{i-1}$, and *exact* if it is exact at each term A_i in the sequence. Show that a short exact sequence (as defined in (2.2.7)) is an exact five-term sequence which begins and ends in the zero object 0 of $\mathcal{C}$.

2.3.4 Suppose that an additive category $\mathcal{C}$ is the direct sum of its subcategories $\mathcal{C}_1$ and $\mathcal{C}_2$, and that P_1, P_2 are the projection functors as in (2.3.17). Show that for any pair of objects M, N of $\mathcal{C}$, there is an isomorphism of abelian groups

$$\operatorname{Mor}_{\mathcal{C}}(M,N) \cong \operatorname{Mor}_{\mathcal{C}_1}(P_1M, P_1N) \oplus \operatorname{Mor}_{\mathcal{C}_2}(P_2M, P_2N)$$

which is natural in each term M, N.

Show that, for any nonzero integer a,

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/a\mathbb{Z}) \cong \mathbb{Z}/a\mathbb{Z}$$

and deduce that $\mathcal{M}_{\mathbb{Z}}$ is *not* the direct sum of $\mathcal{T}_{\mathbb{Z}}$ and $\mathcal{TF}_{\mathbb{Z}}$ – see (2.3.21).

2.3.5 Let $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$ be a set of categories, where the index set Λ may be infinite. Show that if each $\mathcal{C}_\lambda$ is additive (or abelian), then the direct product $\prod_\Lambda \mathcal{C}_\lambda$ (1.1.13) is again additive (or abelian).

Describe the projective objects in the direct product $\prod_\Lambda \mathcal{C}_\lambda$ and direct sum $\bigoplus_\Lambda \mathcal{C}_\lambda$ – see (2.3.19).

2.3.6 Let $\mathcal{C}$ be an abelian category. Show that a pull-back in $\mathcal{C}$ can be obtained as the kernel of a suitably defined morphism, and that a push-out can be obtained as a cokernel.

Deduce that $\mathcal{C}$ contains the pull-back of any pull-back diagram of objects and morphisms in $\mathcal{C}$ and likewise for push-outs.

2.3.7 Let $\mathcal{C}$ be a preadditive category. Show that if $\mathcal{C}$ contains all possible pull-backs and push-outs of its objects and morphisms, then $\mathcal{C}$ is abelian.

2.3.8 Here, for a ring T and T-module homomorphism λ, we denote by $\mathrm{Ker}_T \lambda$ and $\mathrm{Cok}_T \lambda$ the kernel and cokernel of λ in the category $\mathcal{M}_{OD_T}$.

By the Embedding Theorems (2.3.22), given an abelian category $\mathcal{C}$, there is a ring S and embedding of $\mathcal{C}$ in $\mathcal{M}_{OD_S}$ such that for any morphism λ in $\mathcal{C}$ both $\mathrm{Ker}_S \lambda$ and $\mathrm{Cok}_S \lambda$ lie in $\mathcal{C}$. Deduce that if $\mathcal{C}$ is also embedded in $\mathcal{M}_{OD_R}$ for another ring R, then $\mathrm{Ker}_R \lambda \cong \mathrm{Ker}_S \lambda$ and $\mathrm{Cok}_R \lambda \cong \mathrm{Cok}_S \lambda$ as abelian groups, and hence $\mathrm{Ker}_R \lambda$ and $\mathrm{Cok}_R \lambda$ lie in $\mathcal{C}$ (at least to within a canonical isomorphism). This proves the converse to (2.3.5).

2.3.9 Let $\mathcal{C}$ be a full subcategory of $\mathcal{M}_{OD_R}$ containing the zero module. Show that the following are equivalent.

(a) $\mathcal{C}$ is abelian.

(b) For any short exact sequence of right R-modules

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

both the following hold:

(i) if $M \in \mathcal{C}$, then both M' and M'' are in $\mathcal{C}$, and

(ii) when the sequence is split, if M' and M'' are in $\mathcal{C}$, then $M \in \mathcal{C}$.

2.3.10 **Extensions depend on the category**

Generalizing (2.3.5), we may define an *abelian subcategory* $\mathcal{C}$ of an abelian category $\mathcal{A}$ to be an additive subcategory with the further property that for any morphism $\gamma : C \to D$ in $\mathcal{C}$, then the kernel and cokernel of γ in $\mathcal{A}$ are (isomorphic to) objects in $\mathcal{C}$. More precisely, if (K, κ) is a kernel for γ in $\mathcal{A}$, then there is an object K' in $\mathcal{C}$, a morphism $\kappa' : K' \to C$ in $\mathcal{C}$ and an isomorphism $\theta : K \to K'$ (in $\mathcal{A}$) such that $\kappa = \kappa'\theta$, and similarly for cokernels.

Verify that an abelian subcategory is indeed abelian.

By the discussion in (2.3.7), for any ring R, the category of Artinian semisimple modules $\mathcal{ASS}_R$ is abelian. Verify that $\mathcal{ASS}_R$ is an abelian subcategory of $\mathcal{M}_{OD_R}$.

Now suppose that L, N lie in $\mathcal{ASS}_R$. By the Artinian Splitting

Theorem ([BK: IRM] (4.1.17)), any short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with term M in $\mathcal{ASS}_R$ must be split. However, this need not be so when L and N are viewed as modules in $\mathcal{M}_{\mathcal{O}D_R}$ and M can lie outside $\mathcal{ASS}_R$.

For an explicit example, take $R = \mathcal{K}[\epsilon]$, the ring of dual numbers over a field $\mathcal{K}$. (This is the polynomial ring whose indeterminate ϵ has $\epsilon^2 = 0$.)

Show that the Jacobson radical of R is $\mathrm{rad}(R) = \epsilon\mathcal{K}[\epsilon]$ and that $\mathcal{ASS}_R = \mathcal{M}_{\mathcal{K}}$, with ϵ acting as 0 on $\mathcal{K}$-spaces (see (2.3.9)(iv)). Now consider the exact sequence of R-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}[\epsilon] \longrightarrow \mathcal{K} \longrightarrow 0.$$

2.3.11 Let A be an object of an additive category $\mathcal{C}$. Write $\sigma_j : A \to A \oplus A$, $\pi_j : A \oplus A \to A$ for the usual monomorphisms and epimorphisms, $j = 1, 2$. Then the *diagonal map* $\Delta_A : A \to A \oplus A$ is defined by universality to be the unique morphism with $\pi_j \Delta_A = id_A$, $j = 1, 2$. Dually, the *codiagonal map* $\nabla_A : A \oplus A \to A$ is defined to be the unique map such that $\nabla_A \sigma_j = id_A$, $j = 1, 2$.

Show that, when $\mathcal{C}$ is abelian, $\Delta_A : A \to A \oplus A$ is the kernel of $\pi_1 - \pi_2 : A \oplus A \to A$, while $\nabla_A : A \oplus A \to A$ is the cokernel of $\sigma_1 - \sigma_2 : A \to A \oplus A$.

Confirm also that these definitions extend those given for modules in Exercise 2.1.7.

Show too that, for morphisms $\alpha : A \to C$ and $\beta : B \to D$ in $\mathcal{C}$,

$$\alpha \oplus \beta : A \oplus B \longrightarrow C \oplus D,$$

as defined in (2.2.16), is the unique morphism such that

$$(\alpha \oplus \beta)\sigma_A = \sigma_C \alpha \text{ and } (\alpha \oplus \beta)\sigma_B = \sigma_D \beta.$$

Further, when $A = B$ and $C = D$, $\alpha + \beta : A \to C$ is uniquely defined by

$$\alpha + \beta = \nabla_C(\alpha \oplus \beta)\Delta_A.$$

[Pareigis 1970] 4.1 develops this line of argument to give an alternative treatment of additive categories.

Finally, prove the converse of (2.2.20): if $F : \mathcal{C} \to \mathcal{D}$ is a functor between additive categories, which preserves finite direct sums of

objects, together with the corresponding monomorphisms and epimorphisms, then F is additive.

2.3.12 **The Three Lemma**

Extend the Three Lemma ([BK: IRM] Exercise 2.4.1) from modules to an arbitrary abelian category $\mathcal{A}$: if $\alpha : L \to M$ and $\beta : M \to N$ are any morphisms in $\mathcal{A}$, there is an induced exact sequence

$$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta\alpha \to \operatorname{Ker}\beta \to \operatorname{Cok}\alpha \to \operatorname{Cok}\beta\alpha \to \operatorname{Cok}\beta \to 0.$$

2.3.13 **The Snake Lemma**

Extend the Snake Lemma ([BK: IRM] Exercise 2.4.2) from modules to an arbitrary abelian category $\mathcal{A}$: given a commuting diagram of objects and morphisms in $\mathcal{A}$ with both rows exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{\mu'} & M & \xrightarrow{\mu} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\
0 & \longrightarrow & N' & \xrightarrow{\nu'} & N & \xrightarrow{\nu} & N'' & \longrightarrow & 0
\end{array}$$

construct a *connecting morphism* $\delta : \operatorname{Ker}\alpha'' \to \operatorname{Cok}\alpha'$ which fits into an exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}\alpha' & \longrightarrow & \operatorname{Ker}\alpha & \longrightarrow & \operatorname{Ker}\alpha'' & & \\
 & & & & & \overset{\delta}{\searrow} & & & \\
 & & \operatorname{Cok}\alpha' & \longrightarrow & \operatorname{Cok}\alpha & \longrightarrow & \operatorname{Cok}\alpha'' & \longrightarrow & 0
\end{array}$$

2.4 EXACT CATEGORIES

We now introduce a structure that plays a fundamental role in K-theory, namely, that of an exact category. This is an additive category $\mathcal{C}$ together with a specified class of 'admissible' exact sequences chosen from the class of all short exact sequences in $\mathcal{C}$. The definitions, and properties, of the K-groups $K_0(\mathcal{C}), K_1(\mathcal{C}), \ldots$ depend not only on the category $\mathcal{C}$ but also on the class $\operatorname{Ex}(\mathcal{C})$. To illustrate this phenomenon, we discuss a few basic properties of the Grothendieck group $K_0(\mathcal{C})$. We state the properties required of $\operatorname{Ex}(\mathcal{C})$ in two parts, one 'elementary', the other 'advanced'. The elementary properties

suffice for many purposes, in particular, the construction of the Grothendieck group. The more refined advanced properties are designed to ensure that Quillen's Q-construction can be carried out, as needed for a definition of higher K-theory. We do not discuss Quillen's construction in this text.

Several authors have formulated definitions of a 'category with admissible exact sequences', sometimes from the point of view of homological algebra, sometimes from that of K-theory. [Mac Lane 1975], Ch. XII §4, introduces *proper* classes of exact sequences in an abelian category. Such a class satisfies conditions GE1–3 of (2.4.1) together with QE1, QE1$^{\mathrm{op}}$, QE2 and QE2$^{\mathrm{op}}$ of (2.4.10). Other variations can be found in [Buchsbaum 1959], [Buchsbaum 1960] and [Heller 1965]. The definition of a 'K-theoretic' exact category that we adopt here appears in [Quillen 1973].

To aid the exposition, we use the term 'G-exact' for a category which satisfies the elementary conditions that allow the construction of the Grothendieck group, and 'Q-exact' for a category satisfying Quillen's conditions. The reason for these neologisms is that the term 'exact category' is already used in category theory with a different meaning ([Mitchell 1965], p. 18), and we feel it useful to preserve a distinction in an expository text. On the other hand, some authors of papers on K-theory use 'exact' to mean 'Q-exact' as in this text.

As usual, we discuss right categories only; it is clear that everything works also for left categories.

2.4.1 G-exact categories

A *G-exact category* is a pair $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ consisting of an additive category $\mathcal{C}$ together with a specified class $\mathrm{Ex}(\mathcal{C})$ of short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

all of whose terms M', M and M'' are in $\mathcal{C}$. The class $\mathrm{Ex}(\mathcal{C})$ must satisfy the following requirements.

GE1. Every split exact sequence in $\mathcal{C}$ belongs to $\mathrm{Ex}(\mathcal{C})$.

GE2. $\mathrm{Ex}(\mathcal{C})$ is *closed under isomorphism*: if we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\mu'} & M & \xrightarrow{\mu} & M'' & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & N' & \xrightarrow{\nu'} & N & \xrightarrow{\nu} & N'' & \longrightarrow & 0 \end{array}$$

in which all the vertical arrows are isomorphisms in $\mathcal{C}$ and the top sequence is in $\mathrm{Ex}(\mathcal{C})$, then so also is the bottom sequence.

GE3. $\mathrm{Ex}(\mathcal{C})$ is *closed under direct sums*: if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so also is

$$0 \longrightarrow M' \oplus N' \longrightarrow M \oplus N \longrightarrow M'' \oplus N'' \longrightarrow 0.$$

The *admissible exact sequences* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ are the short exact sequences belonging to $\mathrm{Ex}(\mathcal{C})$. An *admissible monomorphism* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a monomorphism $\alpha : M' \to M$ in $\mathcal{C}$ that occurs in some short exact sequence belonging to $\mathrm{Ex}(\mathcal{C})$, and an *admissible epimorphism* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is similarly one occurring in an admissible short exact sequence. Thus any admissible monomorphism has a cokernel, which must be an admissible epimorphism, and dually.

In many contexts where there is a standard choice for the class $\mathrm{Ex}(\mathcal{C})$, a G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is referred to simply as $\mathcal{C}$.

2.4.2 Split and repletely G-exact categories

Given an additive category $\mathcal{C}$, there are two extreme choices of $\mathrm{Ex}(\mathcal{C})$ that make $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ into a G-exact category.

On the one hand, we can take $\mathrm{Ex}(\mathcal{C})$ to be the class of all short exact sequences in $\mathcal{C}$. In this case, we say that $\mathcal{C}$ is a *repletely G-exact category*. As we frequently wish to consider an additive category $\mathcal{C}$ to be a repletely G-exact category, we use the same symbol for both and omit $\mathrm{Ex}(\mathcal{C})$ from the notation.

At the other extreme, we can take $\mathrm{Ex}(\mathcal{C})$ to be the class of all split short exact sequences with terms in $\mathcal{C}$. By condition GE1, this is the smallest possible choice for $\mathrm{Ex}(\mathcal{C})$, and a series of routine verifications establishes that axioms GE2 and GE3 hold. In this case, we say that the category is *split G-exact* and we use the notation $\mathcal{C}^{\oplus}$ to indicate that we view $\mathcal{C}$ as a G-exact category in this way.

In some categories, notably the categories $\mathcal{P}_R$ and $\mathcal{F}_R$, these extremal choices coincide, since any short exact sequence in them must be split.

We note next that many familiar module categories are G-exact.

2.4.3 Theorem

For any ring R, the following are repletely G-exact categories of right R-modules.

(i) $\mathcal{M}_{OD_R}$ *itself.*
(ii) $\mathcal{P}_R$, *the category of finitely generated projective modules.*
(iii) $\mathcal{F}_R$, *the category of free modules of finite rank.*
(iv) $\mathcal{A}_{RT_R}$, *the category of Artinian modules.*
(v) $\mathcal{N}_{OETH_R}$, *the category of Noetherian modules.*
(vi) $\mathcal{M}_R$, *the category of finitely generated modules.*
(vii) $\mathcal{ASS}_R$, *the category of Artinian semisimple modules.*

The categories $\mathcal{P}_R$, $\mathcal{F}_R$ and $\mathcal{ASS}_R$ are also split G-exact.

Proof

By (2.2.14), $\mathcal{P}_R$, $\mathcal{F}_R$ and $\mathcal{M}_R$ are additive categories, and by (2.3.8), $\mathcal{N}_{OETH_R}$, $\mathcal{A}_{RT_R}$ and $\mathcal{ASS}_R$ are also additive (in fact, abelian). Assertions (i) to (vii) follow by trivial verifications.

The definition of a projective module shows that any short exact sequence in $\mathcal{P}_R$, and hence in $\mathcal{F}_R$, must be split, while any short exact sequence in $\mathcal{ASS}_R$ is split by the Artinian Splitting Theorem ([BK: IRM] (4.1.17)). □

2.4.4 Relative exact categories

We next give some illustrations of more subtle choices for $\mathcal{C}$ or $\mathrm{Ex}(\mathcal{C})$.

Suppose that S is a subring of R. One can then define a G-exact category $\mathcal{M}_{R,S}$, the *S-relative category*, to be the additive category $\mathcal{M}_R$, but with $\mathrm{Ex}(\mathcal{M}_{R,S})$ consisting of all those short exact sequences in $\mathcal{M}_R$ that are split when regarded as short exact sequences of S-modules. For examples, see Exercise 2.4.5.

Similarly, it is sometimes useful to consider the full subcategory $\mathcal{D}$ of $\mathcal{M}_R$ whose objects are projective as S-modules; then all short exact sequences in $\mathcal{D}$ are split over S.

The common generalization of both these examples is given by an additive subcategory $\mathcal{C}$ of an additive category $\mathcal{A}$ ($\mathcal{M}_S$ in the preceding examples), with $\mathrm{Ex}(\mathcal{C})$ comprising all those short exact sequences in $\mathcal{C}$ that are split in $\mathcal{A}$.

2.4.5 On terminology

The term 'semisimple category' is sometimes used by K-theorists instead of 'split exact category'. However, we feel that 'split' is more descriptive, since

the use of 'semisimple' suggests that results about semisimple modules ought to hold in the category. However, there are many results, such as the Artinian Splitting Theorem ([BK: IRM] (4.1.17)), which hold in $\mathcal{ASS}_R$ but do not necessarily hold in a split exact category such as $\mathcal{P}_R$.

2.4.6 Exact functors

Let $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ and $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ be G-exact categories. An *exact functor* from $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ to $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which satisfies the following conditions.

EFun1. F is an additive functor.
EFun2. If

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is a short exact sequence belonging to $\mathrm{Ex}(\mathcal{C})$, then

$$F\mathbf{E} \qquad 0 \longrightarrow FM' \xrightarrow{F\alpha} FM \xrightarrow{F\beta} FM'' \longrightarrow 0$$

must belong to $\mathrm{Ex}(\mathcal{D})$.

Notice that requirement EFun2 is, in effect, two conditions – firstly, $F\mathbf{E}$ must be a short exact sequence in $\mathcal{D}$, and then it must also belong to the distinguished class $\mathrm{Ex}(\mathcal{D})$. In the literature, $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ and $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ are commonly both repletely G-exact. One then speaks of F simply as being an exact functor from $\mathcal{C}$ to $\mathcal{D}$.

2.4.7 Examples

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are G-exact categories and that both have the same underlying additive category $\mathcal{B}$. Then the identity functor on $\mathcal{B}$ will induce an exact functor from $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ to $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ if and only if $\mathrm{Ex}(\mathcal{C})$ is a subclass of $\mathrm{Ex}(\mathcal{D})$. In particular, the identity functor fails to be an exact functor from $\mathcal{M}_R$ to $\mathcal{M}_R^{\oplus}$ (unless R is Artinian semisimple – see (2.3.9) and (2.4.3)).

Conversely, an exact functor between G-exact categories need not preserve all short exact sequences; examples are given by the homomorphism functors $\mathrm{Hom}(L, -)$ on $\mathcal{M}_{OD_R^{\oplus}}$ with L not projective (2.1.8). However, any additive functor preserves split exact sequences (2.2.20) and so will act as an exact functor on a split G-exact category. We record this fact for reference.

2.4.8 Proposition

Suppose that $\mathcal{C}$ is a split G-exact category and that $\mathcal{D}$ is a G-exact category. Then any additive functor from $\mathcal{C}$ to $\mathcal{D}$ is exact. □

The above result shows that, when $\mathcal{C}$ is split G-exact, condition EFun1 implies EFun2. More surprisingly, when $\mathcal{C}$ and $\mathcal{D}$ are abelian, EFun2 always implies EFun1, as can be seen from Exercise 2.3.11.

2.4.9 The Grothendieck group

We now trespass into K-theory to give the construction of the Grothendieck group of a G-exact category. This will at least serve to motivate our definition of a G-exact category.

Let $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ be a G-exact category, and assume that $\mathcal{C}$ is small, or that $\mathcal{C}$ has a small skeleton (1.3.15). For an object C of $\mathcal{C}$, let $\langle C\rangle$ denote the isomorphism class of C, so that $\langle C\rangle = \langle C'\rangle$ precisely when $C \cong C'$. Then the isomorphism classes of objects of $\mathcal{C}$ constitute a set $\mathrm{Is}(\mathcal{C})$, and we can form the free abelian group $\mathrm{Fr}(\mathcal{C})$ on $\mathrm{Is}(\mathcal{C})$. Let $\mathrm{Rel}(\mathcal{C})$ be the subgroup of $\mathrm{Fr}(\mathcal{C})$ generated by all expressions of the form

$$\langle C\rangle - \langle C'\rangle - \langle C''\rangle\,,$$

one for each short exact sequence

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

belonging to $\mathrm{Ex}(\mathcal{C})$.

The *Grothendieck group* $K_0(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ of $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is the quotient group $\mathrm{Fr}(\mathcal{C})/\mathrm{Rel}(\mathcal{C})$. It is usual to omit to mention the class of exact sequences and write simply $K_0(\mathcal{C})$.

Clearly, $K_0(\mathcal{C})$ is an abelian group because $\mathrm{Fr}(\mathcal{C})$ is. The image of $\langle C\rangle$ in $K_0(\mathcal{C})$ is denoted $[C]$. By construction, a short exact sequence as above gives the equality

$$[C] = [C'] + [C''] \text{ in } K_0(\mathcal{C}).$$

It is evident that an exact functor $F : \mathcal{C} \to \mathcal{D}$ between G-exact categories gives a homomorphism

$$K_0(F) : K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D})$$

of abelian groups.

The Grothendieck group K_0 was introduced by A. Grothendieck in 1957 in letters to J.-P. Serre that formed the basis of [Borel & Serre 1958]. It was

initially applied (in the notation $K(X)$) to the category of coherent sheaves on an algebraic variety X and to the category of vector bundles over a topological space X. Grothendieck chose the symbol K (for the German 'Klasse') to avoid $C(X)$, already in use as the ring of functions on X. For further history, see [Bak 1987].

Although we do not give any computations of Grothendieck groups in this text, we can give a plausibility argument to illustrate how the choice of the class $\mathrm{Ex}(\mathcal{C})$ influences $K_0(\mathcal{C})$.

Consider the category $\mathcal{FA}_B$ of finite abelian groups, which is $\mathcal{A}_{RT\mathbb{Z}}$ in disguise. The Primary Decomposition Theorem (2.3.20 – F) shows that the irreducible modules in $\mathcal{FA}_B$ are the cyclic groups $\mathbb{Z}/p\mathbb{Z}$ of prime order p. Since any finite abelian group A has a composition series with factors of the form $\mathbb{Z}/p\mathbb{Z}$ for various primes p, the symbol $[A]$ in $K_0(\mathcal{FA}_B)$ can be written as a sum of symbols $[\mathbb{Z}/p\mathbb{Z}]$, and it can be shown that $K_0(\mathcal{FA}_B)$ is the free abelian group on the set

$$\{[\mathbb{Z}/p\mathbb{Z}] \mid p \text{ prime}\}.$$

However, if we work in the Grothendieck group $K_0(\mathcal{FA}_B{}^{\oplus})$ of the G-exact category $\mathcal{FA}_B{}^{\oplus}$, in which only split exact sequences are permitted, then the symbol $[A]$ can be reduced to symbols corresponding to indecomposable finite abelian groups, which have the form $\mathbb{Z}/p^k\mathbb{Z}$ for p prime, $k \geq 1$, but no further reduction is possible. As might be anticipated, $K_0(\mathcal{FA}_B{}^{\oplus})$ is the free abelian group on the set

$$\{[\mathbb{Z}/p^k\mathbb{Z}] \mid p \text{ prime}, k \geq 1\}.$$

Granted these computations, we can see that the identity functor on the additive category $\mathcal{FA}_B$ induces an exact functor I from the split G-exact category $\mathcal{FA}_B{}^{\oplus}$ to the repletely G-exact category $\mathcal{FA}_B$, but that

$$K_0(I) : K_0(\mathcal{FA}_B{}^{\oplus}) \longrightarrow K_0(\mathcal{FA}_B)$$

is not injective, since

$$K_0(I)([\mathbb{Z}/p^k\mathbb{Z}]) = k[\mathbb{Z}/p\mathbb{Z}] = K_0(I)([(\mathbb{Z}/p\mathbb{Z})^k]).$$

2.4.10 Q-exact categories

In a fundamental paper, [Quillen 1973], Quillen has shown how to construct higher dimensional analogues, $K_1(\mathcal{C})$, $K_2(\mathcal{C}), \ldots$, of the Grothendieck group of a suitably defined exact category. Although we cannot give any indication of Quillen's construction in this text (see [Rosenberg 1996] pp. 289–297 for a

quick introduction), this is a convenient place to list the additional conditions on the class $\mathrm{Ex}(\mathcal{C})$ that are required by Quillen's theory. We take our definition directly from [Quillen 1973], §2.

A *Q-exact category* is a G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ in which the class $\mathrm{Ex}(\mathcal{C})$ of admissible exact sequences satisfies the following requirements in addition to GE1–3. Recall that a monomorphism or epimorphism in $\mathcal{C}$ is admissible if it occurs in a short exact sequence that belongs to $\mathrm{Ex}(\mathcal{C})$.

QE1. $\mathrm{Ex}(\mathcal{C})$ is *closed under composition of admissible epimorphisms* : if

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

and

$$0 \longrightarrow N' \xrightarrow{\gamma} M'' \xrightarrow{\delta} N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so is the short exact sequence

$$0 \longrightarrow M \times_{M''} N' \longrightarrow M \xrightarrow{\delta\beta} N'' \longrightarrow 0.$$

QE1$^{\mathrm{op}}$. $\mathrm{Ex}(\mathcal{C})$ is *closed under composition of admissible monomorphisms* : if

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\gamma} N \xrightarrow{\delta} N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so is the short exact sequence

$$0 \longrightarrow M' \xrightarrow{\gamma\alpha} N \longrightarrow M'' \oplus_M N \longrightarrow 0.$$

QE2. Suppose that the short exact sequence

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is in $\mathrm{Ex}(\mathcal{C})$ and that $\theta : L'' \to M''$ is a morphism in $\mathcal{C}$. Then the pull-back exact sequence (cf. Exercise 1.4.11 and [BK: IRM] (2.4.10))

$$\theta^*\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\mu} M \times_{M''} L'' \xrightarrow{\lambda} L'' \longrightarrow 0$$

also belongs to $\mathrm{Ex}(\mathcal{C})$. (That is, $\mathrm{Ex}(\mathcal{C})$ is closed under base change.)

QE2$^{\mathrm{op}}$. Suppose that the short exact sequence

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\mu} M \xrightarrow{\lambda} M'' \longrightarrow 0$$

is in $\mathrm{Ex}(\mathcal{C})$ and that $\phi : M' \to N'$ is a morphism in $\mathcal{C}$. Then the push-out exact sequence

$$\phi_*\mathbf{E} \qquad 0 \longrightarrow N' \xrightarrow{\alpha} N' \oplus_{L'} M \xrightarrow{\beta} M'' \longrightarrow 0$$

also belongs to $\mathrm{Ex}(\mathcal{C})$. (That is, $\mathrm{Ex}(\mathcal{C})$ is closed under cobase change.)

QE3. Suppose that the morphism $\beta : M \to M''$ in $\mathcal{C}$ has a kernel in $\mathcal{C}$, and that there is a morphism $\varphi : L \to M$ such that $\beta\varphi : L \to M''$ is an admissible epimorphism. Then β is already an admissible epimorphism.

QE3$^{\mathrm{op}}$. Suppose that the morphism $\alpha : M' \to M$ in $\mathcal{C}$ has a cokernel in $\mathcal{C}$, and that there is a morphism $\psi : M \to N$ such that $\psi\alpha : M' \to N$ is an admissible monomorphism. Then α is already an admissible monomorphism.

2.4.11 Comments on the axioms

(i) Pull-backs and push-outs in an abstract category are defined by their universal properties, which are given for modules in ([BK: IRM] Proposition 2.4.9), and in Exercises 1.4.10 and 1.4.11 for categories in general. The definitions of the pull-back sequence $\theta^*\mathbf{E}$ and the push-out sequence $\phi_*\mathbf{E}$ follow those given for modules in Exercise 2.1.7.

(ii) In the statement of the axioms, we take for granted the fact that various constructions on short exact sequences yield short exact sequences. This is easily verified in a module category. In an abstract category, the verification can be made by using universal properties, or, more conveniently, by invoking the Embedding Theorems (2.3.22).

(iii) Since the underlying category $\mathcal{C}$ is assumed to be additive, rather than abelian, part of the requirement imposed by the axioms is that the various pull-backs and push-outs are objects in $\mathcal{C}$ (note Exercises 2.3.6 and 2.3.7).

(iv) It is not difficult to show that these conditions make the 'elementary' condition GE3 redundant – see Exercise 2.4.4. The fact that the Q-conditions are stronger than the G-conditions is illustrated in Exercise 2.4.11.

We call a Q-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ *repletely Q-exact* when $\mathrm{Ex}(\mathcal{C})$ comprises all short exact sequences in $\mathcal{C}$, and *split Q-exact* when $\mathrm{Ex}(\mathcal{C})$ consists only of the split exact sequences in $\mathcal{C}$.

Although the conditions for a Q-exact category appear formidable, they are usually fairly easy to check in any given instance. For example, the checking of the claims in the following lemma is straightforward.

2.4.12 Lemma

(a) *Any split G-exact category is Q-exact.*

(b) *Let $\mathcal{C}$ be an abelian category and let $\mathrm{Ex}(\mathcal{C})$ be the class of all short exact sequences of $\mathcal{C}$. Then the repletely G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is Q-exact.*

□

Because of (b) above, we use the term *repletely exact abelian* to describe a repletely G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ with $\mathcal{C}$ abelian.

2.4.13 Exact subcategories

To see when a subcategory of a Q-exact category is also Q-exact, we need a preliminary definition. Let $(\mathcal{A}, \mathrm{Ex}(\mathcal{A}))$ be a Q-exact category. Then $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a *sub-Q-exact category* of $(\mathcal{A}, \mathrm{Ex}(\mathcal{A}))$ if the following conditions are satisfied.

SubQ1. $\mathcal{C}$ is a full additive subcategory of $\mathcal{A}$ with the property that whenever

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

is a short exact sequence in $\mathcal{A}$ such that A' and A'' are isomorphic to objects of $\mathcal{C}$, then A is also isomorphic to an object of $\mathcal{C}$.

SubQ2. $\mathrm{Ex}(\mathcal{C})$ is the class of all short exact sequences in $\mathrm{Ex}(\mathcal{A})$ whose terms belong to $\mathcal{C}$.

Condition SubQ1 is often rephrased as: $\mathcal{C}$ is *closed under extensions*, or *extension closed.* Condition SubQ2 implies that the inclusion functor from $\mathcal{C}$ to $\mathcal{A}$ is exact.

In practice, it is usually obvious from the nature of $\mathcal{C}$ that the first condition is satisfied, while the canonical choice of $\mathrm{Ex}(\mathcal{C})$ is that suggested by the second condition. Thus the following theorem gives a widely applicable condition for Q-exactness. The proof is by routine verification of the axioms.

2.4.14 Theorem

Any sub-Q-exact category of a Q-exact category is again Q-exact. □

2.4.15 Corollary

Any sub-Q-exact category of a repletely exact abelian category is Q-exact.

□

[Quillen 1973], §2, observes that the above corollary has a converse: if a Q-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is (skeletally) small, then there exists a repletely exact abelian category in which $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is sub-Q-exact. This is shown by mimicking the proofs of the earlier embedding theorems for additive categories, which were outlined in (2.3.22). The corollary and its converse enable us to define a Q-exact category as a sub-Q-exact category of a repletely exact abelian category. The new definition has the advantage that its axioms are relatively simple.

Observe that a subcategory $\mathcal{C}$ of a Q-exact category $\mathcal{A}$ may be Q-exact without being a sub-Q-exact category of $\mathcal{A}$. For example, the category $\mathcal{ASS}_R$ of Artinian semisimple modules over a ring R is an abelian category, and Q-exact; it is in fact both split and repletely exact. The inclusion functor from $\mathcal{ASS}_R$ to $\mathcal{M}_R$ is exact, but $\mathcal{ASS}_R$ is not a sub-Q-exact category of $\mathcal{M}_R$, unless R is Artinian semisimple (which makes Condition SubQ1 hold.)

The reader may wish to verify the accuracy of the following listing of Q-exact categories. Here, Q-exactness follows immediately from the corollary above.

2.4.16 Theorem

For any ring R, $\mathcal{M}\mathit{od}_R$ is a repletely exact abelian category, and the following are (repletely) sub-Q-exact categories and therefore Q-exact.

(i) *$\mathcal{P}_R$, the category of finitely generated projective modules.*
(ii) *$\mathcal{F}_R$, the category of free modules of finite rank.*
(iii) *$\mathcal{A}\mathit{rt}_R$, the category of Artinian modules.*
(iv) *$\mathcal{N}\mathit{oeth}_R$, the category of Noetherian modules.*
(v) *$\mathcal{M}_R$, the category of finitely generated modules.*
(vi) *$\mathcal{M}\mathit{od}_T$, for any extension ring T of R.* □

Exercises

Note. These exercises refer (mostly) to G-exact categories. The enthusiastic reader may wish to extend them to Q-exact categories where appropriate.

2.4.1 Let $\mathcal{C}$ be a G-exact category. Show that $\mathcal{C}^{\mathrm{op}}$ is also a G-exact category.

2.4.2 Let $\mathcal{C}$ and $\mathcal{D}$ be G-exact categories. Show that there are canonical ways to make $\mathcal{C} \times \mathcal{D}$ and $\mathcal{M}\mathit{od}_R\mathcal{C}$ into G-exact categories.

Prove that $\mathcal{C} \times \mathcal{D}$ is split or repletely G-exact if and only if both $\mathcal{C}$ and $\mathcal{D}$ are split or repletely G-exact respectively.

Generalize these statements for arbitrary finite direct products of G-exact categories, and for infinite direct sums of categories as in (2.3.19).

Let $\mathcal{K}$ be a field. Show that $\mathcal{M}_{OR}(\mathcal{M}_{\mathcal{K}})$ is not split G-exact, even though $\mathcal{M}_{\mathcal{K}}$ is.

2.4.3 Let $\mathcal{C}$ be a G-exact category, and let $N_1 \to N_2$ be an isomorphism in $\mathcal{C}$. Show that if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\mathrm{Ex}(\mathcal{C})$, then the short exact sequences

$$0 \longrightarrow M' \oplus N_1 \longrightarrow M \oplus N_2 \longrightarrow M'' \longrightarrow 0$$

and

$$0 \longrightarrow M' \longrightarrow M \oplus N_1 \longrightarrow M'' \oplus N_2 \longrightarrow 0$$

are also in $\mathrm{Ex}(\mathcal{C})$.

2.4.4 Show that axioms QE1 and QE2 together imply axiom GE3, as follows. Take short exact sequences

(1) $$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and

(2) $$0 \longrightarrow N' \longrightarrow N \xrightarrow{\nu} N'' \longrightarrow 0$$

in $\mathrm{Ex}(\mathcal{C})$, construct firstly the pull-back of (1) by the admissible epimorphism

$$(id,0) : M'' \oplus N'' \longrightarrow M'',$$

and then the pull-back of the resulting sequence by the admissible epimorphism

$$id \oplus \nu : M'' \oplus N \longrightarrow M'' \oplus N''$$

to obtain a short exact sequence in $\mathrm{Ex}(\mathcal{C})$

(3) $$0 \longrightarrow M' \longrightarrow W \xrightarrow{\omega} M'' \oplus N \longrightarrow 0.$$

Show that $\theta = (id \oplus \nu)\omega : W \to M'' \oplus N''$ is admissible, and that the corresponding short exact sequence

$$0 \longrightarrow L \longrightarrow W \xrightarrow{\theta} M'' \oplus N'' \longrightarrow 0$$

is isomorphic to the direct sum of (1) and (2).

2.4.5 Let $\mathcal{O}$ be a commutative domain. A ring R is said to be an *$\mathcal{O}$-order* if there is an injective ring homomorphism from $\mathcal{O}$ into the centre $Z(R)$ of R, and if, as both a left and right $\mathcal{O}$-module, R is finitely generated over $\mathcal{O}$ and torsion-free. The ring $\mathcal{O}$ is called the coefficient ring. If the coefficient ring can be taken as granted, we often refer to R simply as an *order*.

Given an $\mathcal{O}$-order R, let $\mathcal{T}_{or\mathcal{O},R}$, $\mathcal{T}_{\mathcal{O},R}$ and $\mathcal{TF}_{\mathcal{O},R}$ denote the full subcategories of $\mathcal{M}_{od_R}$ whose objects are respectively the $\mathcal{O}$-torsion, finitely generated $\mathcal{O}$-torsion and finitely generated $\mathcal{O}$-torsion-free modules.

Show that all these categories are repletely G-exact categories. In each case, show how to form an $\mathcal{O}$-relative exact category by taking as short exact sequences those which are $\mathcal{O}$-split.

2.4.6 Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between G-exact categories. Show that F and G are both exact if and only if $F \oplus G : \mathcal{C} \to \mathcal{D}$ is exact, and that

$$K_0(F \oplus G) = K_0(F) + K_0(G) : K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D}).$$

2.4.7 Let $\mathcal{C}$ and $\mathcal{D}$ be G-exact categories. Show that

$$K_0(\mathcal{C} \times \mathcal{D}) \cong K_0(\mathcal{C}) \oplus K_0(\mathcal{D}).$$

Generalize this result to an arbitrary finite direct product of categories, and to infinite direct sums of categories (2.3.19).

2.4.8 **Categories with cofibrations: axioms**

In [Waldhausen 1985], Quillen's conditions QE1-QE3 (2.4.10) on the admissible exact sequences of a Q-exact category are replaced by weaker requirements which still suffice for the construction of (higher) K-theory, with the advantage that a wider range of categories can be used. This and the subsequent exercises outline the definitions and some initial calculations.

Let $\mathcal{C}$ be a category with zero object 0. The notion of an admissible monomorphism is replaced by that of a *cofibration* (which need not in fact be a monomorphism), and the counterpart of an admissible exact sequence is a *cofibration sequence*

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

in which α is a cofibration and (M'', β) is a cokernel of α (as defined in (1.4.15)).

The cofibrations are, by definition, the morphisms of a subcategory $\mathrm{co}\mathcal{C}$ of $\mathcal{C}$ which has the same objects as $\mathcal{C}$ and is subject to the following three axioms.

Cof1. Every isomorphism of $\mathcal{C}$ is in $\mathrm{co}\mathcal{C}$.

Cof2. Every morphism $0 \to C$ in $\mathcal{C}$ belongs to $\mathrm{co}\mathcal{C}$.

Cof3. Cofibrations are closed under cobase change, meaning that, for any cofibration $M' \xrightarrow{\alpha} M$ and any morphism $\phi : M' \to N$ in $\mathcal{C}$, the bottom row of the push-out diagram

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \downarrow & & \downarrow \\ N & \xrightarrow{\overline{\alpha}} & N \oplus_{M'} M \end{array}$$

is also in $\mathrm{co}\mathcal{C}$. (This implies that $N \oplus_{M'} M$ is an object of $\mathcal{C}$.)

With these axioms, the pair $(\mathcal{C}, \mathrm{co}\mathcal{C})$ is then called a *category with cofibrations.*

(a) Show that the push-out $0 \oplus_{M'} M$ of a cofibration $\alpha : M' \to M$ and the zero morphism $0 : M' \to 0$ is (isomorphic to) the cokernel $\mathrm{Cok}\,\alpha$ of α. Thus, by Cof3, every cofibration is part of a cofibration sequence

$$M' \xrightarrow{\alpha} M \longrightarrow \mathrm{Cok}\,\alpha.$$

The canonical morphism $M \to \mathrm{Cok}\,\alpha$ is known as a *quotient map.*

(b) Show that cofibration sequences are closed under cobase change, meaning that, for any cofibration sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

and any morphism $\phi : M' \to N$ in $\mathcal{C}$, there is a push-out diagram

$$\begin{array}{ccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\ \downarrow & & \downarrow & & \downarrow \\ N & \xrightarrow{\overline{\alpha}} & N \oplus_{M'} M & \xrightarrow{\overline{\beta}} & \mathrm{Cok}\,\overline{\alpha} \end{array}$$

in which the lower row is also a cofibration sequence and the morphism from M'' to $\mathrm{Cok}\,\overline{\alpha}$ is an isomorphism. (Compare with Axiom QE2$^{\mathrm{op}}$, (2.4.10).)

(c) Show that, by taking the admissible monomorphisms to be the cofibrations, any Q-exact category may be considered to be a category with cofibrations.

2.4.9 **Categories with cofibrations: constructions.**

Let $\mathcal{C}$ be a category with cofibrations. For each natural number n, define $\mathcal{F}_n\mathcal{C}$ to be the category for which an object is a sequence of cofibrations

$$M_0 \longrightarrow M_1 \longrightarrow \quad \cdots \quad \longrightarrow M_n$$

in $\mathcal{C}$, and for which a morphism is a natural transformation of such diagrams. One can also form a category $\mathcal{F}_n^+\mathcal{C}$ equivalent to $\mathcal{F}_n\mathcal{C}$ by taking for an object, in addition to the above data, a quotient map for each cofibration $M_i \to M_j$, $0 \leq i < j \leq m$. The following assertions generalize readily to $n > 1$.

(a) Show that $\mathcal{F}_1\mathcal{C}$ is a category with cofibrations, where a cofibration from $\alpha : M' \to M$ to $\beta : N' \to N$ in $\mathrm{co}\,\mathcal{F}_1\mathcal{C}$ is, by definition, any pair γ, δ of morphisms in $\mathcal{C}$ for which

(i) there is a commutative square

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ N' & \xrightarrow{\beta} & N \end{array}$$

(ii) both γ and the induced morphism $N' \oplus_{M'} M \to N$ are cofibrations in $\mathcal{C}$.

Deduce that δ is a cofibration.

(b) Show that an object of $\mathcal{F}_1^+\mathcal{C}$ is a cofibration sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

in which M'' is a specified cokernel of α.

Use (a) and the equivalence $\mathcal{F}_1^+\mathcal{C} \to \mathcal{F}_1\mathcal{C}$ to regard $\mathcal{F}_1^+\mathcal{C}$ as a category with cofibrations. Show that the three functors

$$s, t, q : \mathcal{F}_1^+\mathcal{C} \longrightarrow \mathcal{C}$$

which map a cofibration sequence to its respective terms

$$M', \qquad M, \qquad M''$$

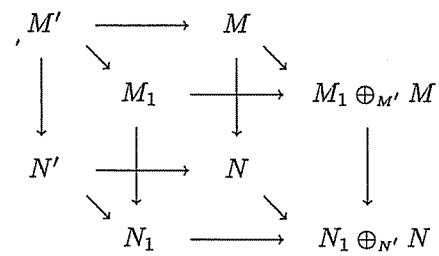
are *exact* in the sense that they preserve zero objects, cofibrations and push-out diagrams.

2.4.10 **Waldhausen categories**

(a) A *Waldhausen category* or *category of weak equivalences* is a category with cofibrations $\mathcal{C}$, together with a specified subcategory $w\mathcal{C}$. The morphisms in $w\mathcal{C}$ are the *weak equivalences*. The following axioms hold.

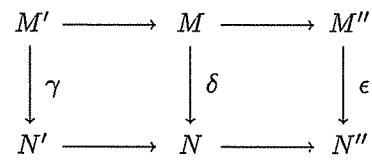
Weq1. Every isomorphism of $\mathcal{C}$ is a morphism of $w\mathcal{C}$.

Weq2. Glueing Axiom. For any commutative diagram in $\mathcal{C}$



in which each horizontal morphism is a cofibration and the three left-most vertical morphisms are weak equivalences, the fourth vertical morphism is also a weak equivalence.

Deduce that if



is a map of cofibration sequences in which γ and δ are weak equivalences, then ϵ is also a weak equivalence.

(b) Define the *Grothendieck group* $K_0(\mathcal{C})$ (which, strictly speaking, should be written $K_0(\mathcal{C}, \mathrm{co}\mathcal{C}, w\mathcal{C})$) of the Waldhausen category $\mathcal{C}$ to be the abelian group generated by an element $[M]$ for each object M of $\mathcal{C}$ and subject to a relation

$$[M] = [M'] + [M'']$$

for each cofibration sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'',$$

and a further relation

$$[M] = [N]$$

for each weak equivalence $M \to N$ in $\mathcal{C}$.

Show that when $\mathcal{C}$ is a Q-exact category considered as a category with cofibrations, and with weak equivalences chosen to be the isomorphisms of $\mathcal{C}$, then the group just defined coincides with the Grothendieck group of $\mathcal{C}$ as defined previously (2.4.9).

2.4.11 **Q-exact is strictly stronger than G-exact**

Let $\mathcal{C}$ be the full subcategory of $\mathcal{A}_B$ whose objects are the finite abelian 2-groups. To choose $\mathrm{Ex}(\mathcal{C})$ such that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a G-exact category, let $\mathrm{Ex}(\mathcal{C})$ consist of all direct sums (as in GE3) of short exact sequences that are either split or have cyclic middle term. Thus, for example, the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sigma} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0,$$

$$\sigma : (a, b) \mapsto (a, 4b) \qquad \pi : (c, d) \mapsto d$$

(using representative integers to indicate residues) is admissible, while

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0,$$

$$\alpha : x \mapsto (x, 2x) \qquad \beta : (y, z) \mapsto 2y - z$$

is not. Show that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is not a Q-exact category, because it falls foul, for example, of QE3.

Let $\phi : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$, $a \mapsto 2a$. Show that $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is isomorphic to the push-out of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ over $\mathbb{Z}/2\mathbb{Z}$ with respect to ϕ, σ.

Deduce that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ also violates $\mathrm{QE2}^{\mathrm{op}}$. Thus, the admissible monomorphisms of $\mathrm{Ex}(\mathcal{C})$ fail even to make $\mathcal{C}$ a category with cofibrations.

3
CHANGE OF RINGS

In this chapter, we introduce a family of constructions that show off the functorial approach to great advantage. The common theme is that they allow us to pass from a category of modules over one ring to a category of modules over another ring. These constructions are based on the tensor product, which we introduce in the first section. Given a ring R, a right R-module M and a left R-module N, their tensor product $M \otimes_R N$ is an abelian group that has a certain universal property. If also M or N is a bimodule, then $M \otimes_R N$ inherits a module structure. Thus, by holding one of M and N fixed and varying the other, we obtain functors between module categories.

An important matter is whether or not these functors are exact; this leads to the concept of a flat module, which always gives an exact functor. Some fundamental examples and properties of flat modules are given in section 2.

One way of changing rings is by means of a ring homomorphism from one ring to another. Classically, such homomorphisms are inclusions and the functors are extension and restriction of scalars. In the third section, we look at these functors for a ring homomorphism in general.

3.1 THE TENSOR PRODUCT

The tensor product is a fundamental tool for constructing functors between various categories of modules. In this section, we give the definition and basic properties of the tensor product $M \otimes_R N$ of a right R-module M and a left R-module N, where R is an arbitrary ring. We then establish the functorial properties of the tensor product, which can be viewed variously as a functor of the second term N, with M being kept fixed, or of the first term M, with N kept fixed, or as a bifunctor. We discuss how the tensor product behaves on various module categories, and we investigate the adjointness between the tensor product and the Hom functor.

Finally, we show that the category of modules over a generalized triangular matrix ring can be realized as a morphism category.

3.1.1 The definition

Let R be an arbitrary ring, let M_R be a right R-module and let ${}_RN$ be a left R-module.

A *biadditive R-balanced map* on the pair M, N is a function β from the cartesian product $M \times N$ to some abelian group A, with the following properties:

$$\beta(m + m', n) = \beta(m, n) + \beta(m', n),$$

$$\beta(m, n + n') = \beta(m, n) + \beta(m, n')$$

and

$$\beta(mr, n) = \beta(m, rn)$$

for all m, m' in M, n, n' in N and r in R.

The *tensor product* $M \otimes_R N$ is defined to be the universal object with respect to such maps (see (1.4.2)). This means that $M \otimes_R N$ is an abelian group and that there is a biadditive R-balanced map

$$\tau : M \times N \longrightarrow M \otimes_R N$$

with the property that, given any β as above, there is a unique homomorphism

$$\alpha : M \otimes_R N \longrightarrow A$$

with $\alpha\tau = \beta$.

For a category-theoretic description, consider the category whose objects are the biadditive R-balanced maps $\beta : M \times N \to A$ on the pair M, N, a morphism from $\beta : M \times N \to A$ to $\beta' : M \times N \to A'$ being a homomorphism of abelian groups $\alpha : A \to A'$ with $\alpha\beta = \beta'$. Then $\tau : M \times N \to M \otimes_R N$ is the initial object in this category. The associated diagram is

$$\begin{array}{ccc} M \times N & \xrightarrow{\ \tau\ } & M \otimes_R N \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

Such a definition does not guarantee the existence of the required universal object, which we now establish.

3.1.2 The construction

To see that the tensor product exists, start with the free abelian group $M\#N$ generated by the cartesian product $M \times N$; elements of $M\#N$ can be uniquely expressed as sums of the form

$$z_1(m_1, n_1) + \cdots + z_k(m_k, n_k),$$

where $z_i \in \mathbb{Z}$, $m_i \in M$, $n_i \in N$ for $i = 1, \ldots, k$, and $k \geq 0$ (the vacuous sum with $k = 0$ is to be read as 0).

Next, let $B(M, N)$ be the subgroup of $M\#N$ which is generated by all expressions of the form

$$(m + m', n) - (m, n) - (m', n),$$

$$(m, n + n') - (m, n) - (m, n'),$$

and

$$(mr, n) - (m, rn),$$

with m, m' in M, n, n' in N and r in R.

We write

$$M \otimes_R N = (M\#N)/B(M, N),$$

and put $\tau(m, n) = m \otimes n$, where $\tau : M \times N \to M \otimes_R N$ is the canonical homomorphism. (Where it is felt safe to do so, we occasionally write $M \otimes N$, omitting the ring R.)

A typical element of the tensor product is then a sum of the form

$$z_1(m_1 \otimes n_1) + \cdots + z_k(m_k \otimes n_k),$$

and the identities

$$(m + m') \otimes n = m \otimes n + m' \otimes n,$$

$$m \otimes (n + n') = m \otimes n + m \otimes n',$$

and

$$mr \otimes n = m \otimes rn$$

hold among the generators of $M \otimes_R N$.

Thus τ is biadditive R-balanced as desired. It is straightforward to check that τ is universal among such maps in the sense described previously and thus that $M \otimes_R N$ is the tensor product.

Although applied mathematicians and physicists have long computed with tensors, the formal definition of the tensor product is comparatively recent.

It was given for abelian groups in [Whitney 1938], and subsequently for arbitrary rings and modules in [Bourbaki 1948]. It should be remarked that the tensor product is not in fact a product or coproduct as defined in (1.4.11) and (1.4.12), but a 'coend'. For a definition of this term and a category-theoretic analysis of the tensor product, see [Mac Lane 1971] p. 222.

3.1.3 Bimodule structures

Commonly, $M \otimes_R N$ is rather more than just an abelian group. For example, when N is an R-S-bimodule, then $M \otimes_R N$ becomes an S-module via

$$(m \otimes n)s = m \otimes ns \text{ for } m \in M,\ n \in N \text{ and } s \in S.$$

To verify this assertion, note that $M\#N$ can be made into a right S-module by setting $(m,n)s = (m,ns)$. Then $B(M,N)$ is an S-submodule of $M\#N$ and so the quotient abelian group $M \otimes_R N$ inherits a right S-module structure as indicated.

This method of constructing actions or mappings on $M \otimes_R N$ can be described informally as specifying what happens to elements of the form $m \otimes n$ and then extending the effect to linear combinations of such elements 'by linearity'. Properly speaking, the action or mapping must first be defined on the generators of $M\#N$, and then it must be verified that $B(M,N)$ is invariant, so that there is a corresponding induced action or mapping on $M \otimes_R N$. However, such verifications are usually trivial.

If M is a T-R-bimodule for some ring T, then $M \otimes_R N$ is a left T-module, and if M is a T-R-bimodule and N an R-S-bimodule, then $M \otimes_R N$ is a T-S-bimodule.

Our first computation is the following result, wherein R is viewed as an R-R-bimodule.

3.1.4 Proposition

Let M_R be a right R-module and let ${}_RN$ be a left R-module. Then

(i) *$M \otimes_R R \cong M$ as right R-modules*
and
(ii) *$R \otimes_R N \cong N$ as left R-modules.*

Proof

(i) Define homomorphisms $\eta : M \otimes_R R \to M$ and $\lambda : M \to M \otimes_R R$, by $m \otimes r \mapsto mr$ and $m \mapsto m \otimes 1$ respectively. Since $m \otimes r = mr \otimes 1$ (and $m1 = m$), η and λ are mutually inverse R-module homomorphisms.

(ii) Similar. □

The next fact can be interpreted as saying that the tensor product is associative.

3.1.5 Proposition

Let R and S be rings and let L_R, ${}_RM_S$ and ${}_SN$ be modules as indicated. Then there is an isomorphism

$$(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N).$$

Proof

The expected isomorphism is that under which $(\ell \otimes m) \otimes n$ corresponds to $\ell \otimes (m \otimes n)$. We must check that this correspondence does in fact give a well-defined homomorphism.

Take a fixed element n of N. The map

$$\beta(n) : L \times M \to L \otimes_R (M \otimes_S N),$$

$$(\ell, m) \longmapsto \ell \otimes (m \otimes n),$$

is biadditive and R-balanced (since $r(m \otimes n) = rm \otimes n$), and so it induces a homomorphism

$$\beta'(n) : L \otimes_R M \longrightarrow L \otimes_R (M \otimes_S N).$$

Allowing n to vary, we see that the maps $\beta'(n)$ define a biadditive S-balanced map $\beta' : (L \otimes_R M) \times N \to L \otimes_R (M \otimes_S N)$, which induces a homomorphism $\theta : (L \otimes_R M) \otimes_S N \to L \otimes_R (M \otimes_S N)$ as desired.

Reversing this construction, and appealing to the universal property of the tensor product, we confirm that θ has the expected inverse. □

3.1.6 Functorial properties of tensor products

We wish to see how the tensor product behaves as a functor when one of the modules M_R or ${}_RN$ is kept fixed and the other varies through ${}_R\mathcal{M}_D$ or $\mathcal{M}_{D_R}$. As we are mainly concerned with functors on categories of right modules, we give the arguments in the case where the second module in the tensor product is regarded as being constant. The parallel results where the first module is viewed as constant are noted from time to time, the details usually being left to the reader, who will easily get them right.

Let S be an arbitrary ring and suppose that N is an R-S-bimodule. As

noted above, for any right R-module M the tensor product $M \otimes_R N$ is a right S-module under the rule

$$(m \otimes n)s = m \otimes ns, \ s \in S.$$

Thus, for any homomorphism $\alpha : M' \to M$ of right R-modules, the induced homomorphism

$$\alpha \otimes id_N : M' \otimes_R N \longrightarrow M \otimes_R N,$$

defined by

$$(\alpha \otimes id_N)(m' \otimes n) = \alpha m' \otimes n,$$

is a homomorphism of right S-modules.

Write $\alpha \otimes_R N = \alpha \otimes id_N$, so that the symbol $- \otimes_R N$ represents both a mapping from the objects of $\mathcal{M}_{OD_R}$ to those of $\mathcal{M}_{OD_S}$ and a collection of mappings

$$- \otimes_R N : \mathrm{Hom}_R(M', M) \longrightarrow \mathrm{Hom}_S(M' \otimes_R N, M \otimes_R N),$$

one for each pair of R-modules M', M. This information, together with the obvious properties

$$id_M \otimes id_N = id_{M \otimes N}$$

and

$$(\beta\alpha) \otimes id_N = (\beta \otimes id_N)(\alpha \otimes id_N),$$

defines a functor from $\mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_S}$, which we again denote $- \otimes_R N$. Moreover, $- \otimes_R N$ is additive because if $\alpha : M' \to M$ and $\alpha' : M' \to M$ are right R-module homomorphisms, then

$$(\alpha + \alpha') \otimes id_N = \alpha \otimes id_N + \alpha' \otimes id_N.$$

In summary, we have the following result.

3.1.7 Lemma

$- \otimes_R N$ is an additive covariant functor from $\mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_S}$. □

3.1.8 Proposition

Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ be a direct sum of right R-modules, where Λ is any ordered set. Then

$$M \otimes_R N \cong \bigoplus_{\lambda \in \Lambda} (M_\lambda \otimes_R N).$$

Proof

When Λ is finite, the assertion follows immediately from the preceding result, since additive functors preserve finite direct sums (2.2.20).

Now suppose that Λ is infinite. An element of either $M \otimes N$ or of the direct sum $\bigoplus_{\lambda\in\Lambda}(M_\lambda \otimes_R N)$ involves nonzero terms $m_\lambda \in M_\lambda$ for only a finite subset Σ of Λ, which makes it clear that there is a homomorphism

$$\phi : M \otimes_R N \longrightarrow \bigoplus_{\lambda\in\Lambda}(M_\lambda \otimes_R N)$$

and that ϕ is surjective.

Suppose that $\phi(x) = 0$. Then the preimage of $\phi(x)$ in $\bigoplus_\Sigma(M_\sigma \# N)$ belongs to the direct sum of relation subgroups $\bigoplus_\Sigma B(M_\sigma, N)$. Thus the preimage of x is already 0 in $(\bigoplus_\Sigma M_\sigma) \otimes N$, and hence $x = 0$. □

Any left R-module N is automatically an R-$\mathbb{Z}$-bimodule, so our discussion includes the case when N has not been attributed any particular structure as a right module. The functor $- \otimes_R N$ is then a functor from $\mathcal{M}_{OD_R}$ to $\mathcal{A}_B$, the category of abelian groups.

3.1.9 Fixing the first argument

We now review the results and notation that arise when the first module of a tensor product is considered to be an operator on the second.

Let M be a T-R-bimodule for some ring T. Then there is an additive functor

$$M \otimes_R - : {}_R\mathcal{M}_{OD} \longrightarrow {}_T\mathcal{M}_{OD}.$$

On modules,

$$(M \otimes_R -)(N) = M \otimes_R N,$$

with the T-module structure given by

$$t(m \otimes n) = tm \otimes n,$$

and for a homomorphism $\zeta : N' \to N$ of left R-modules,

$$M \otimes \zeta = (id_M \otimes \zeta) : m \otimes n' \longmapsto m \otimes n'\zeta.$$

When $id_M \otimes \zeta$ is to be considered as a left T-module homomorphism, as happens in (5.1.19) for example, we write $id_M \otimes \zeta$ as a right operator, in accord with our convention (1.1.4).

However, sometimes we are given only that M is a right R-module, and we wish to consider M to be a $\mathbb{Z}$-R-bimodule in the natural way. In this case, the functor $M \otimes_R -$ takes values in ${}_\mathbb{Z}\mathcal{M}_{OD}$, but, for applications, it is often

more convenient to compose it with the mirror functor Mir : ${}_{\mathbb{Z}}\mathcal{M}_{OD} \to \mathcal{M}_{OD\mathbb{Z}}$ and view $M \otimes_R -$ as having values in $\mathcal{A}_B = \mathcal{M}_{OD\mathbb{Z}}$.

Thus, for a homomorphism $\zeta : N' \to N$ of left R-modules, $id_M \otimes \zeta$ is to be viewed as a left operator on $M \otimes_R N'$, which is the convention most commonly met in the literature. With this notation, $M \otimes_R -$ is a covariant functor from ${}_R\mathcal{M}_{OD}$ to $\mathcal{M}_{OD\mathbb{Z}}$ that appears to 'reverse products'; in the terminology of (1.2.6), it is a contrachiral functor.

This approach is especially handy when we have both a right R-module homomorphism $\alpha : M' \to M$ and a left R-module homomorphism $\zeta : N' \to N$, in which case the composite

$$\alpha \otimes \zeta = (\alpha \otimes id_N)(id_{M'} \otimes \zeta) : M' \otimes_R N' \longrightarrow M \otimes_R N$$

cannot in general be regarded as a module homomorphism.

We use the convention that $id_M \otimes \zeta$ is to be considered as a left operator for the remainder of the present chapter.

Next, we note some naturality properties of the functors $M \otimes_R -$ and $- \otimes_R N$.

3.1.10 Proposition

Let T, R and S be arbitrary rings.

(i) *Let $\zeta : N' \to N$ be a homomorphism of left R-modules or of R-S-bimodules. Then ζ induces a natural transformation*

$$- \otimes_R \zeta : - \otimes_R N' \longrightarrow - \otimes_R N.$$

(ii) *Let $\alpha : M' \longrightarrow M$ be a homomorphism of right R-modules or of T-R-bimodules. Then α induces a natural transformation*

$$\alpha \otimes_R - : M' \otimes_R - \longrightarrow M \otimes_R -.$$

Proof

The argument is a matter of interpreting the requirements for a natural transformation (1.3.1) in terms of the notation of the tensor product. Fix ζ, and define η_M for each M in $\mathcal{M}_{OD_R}$ by

$$\eta_M = id_M \otimes \zeta : M \otimes_R N' \longrightarrow M \otimes_R N.$$

Given a homomorphism $\alpha : M' \to M$ of right R-modules, we need to check that $(\alpha \otimes N)\eta_{M'} = \eta_M(\alpha \otimes N')$. But both sides are simply $\alpha \otimes \zeta$. □

If we allow both M and N to vary simultaneously, the preceding result can be interpreted as saying that $- \otimes_R -$ is a bifunctor from $\mathcal{M}_{OD_R} \times {}_R\mathcal{M}_{OD}$ to the category $\mathcal{A}_B$ of abelian groups.

We now state a refined version of (3.1.4).

3.1.11 Proposition

(i) *The functor* $- \otimes_R R$ *is naturally isomorphic to the identity functor on* $\mathcal{M}_{OD_R}$.

(ii) *The functor* $R \otimes_R -$ *is naturally isomorphic to the identity functor on* ${}_R\mathcal{M}_{OD}$.

Proof

For each right R-module M, let

$$\eta_M : M \otimes_R R \xrightarrow{\cong} M, \qquad \eta_M(m \otimes r) = mr,$$

be the isomorphism of right R-modules defined in (3.1.4). If $\alpha : M' \to M$ is a homomorphism of right R-modules, then the formula $\alpha(m'r) = \alpha(m')r$ gives

$$\alpha\eta_{M'} = \eta_M(\alpha \otimes id_R);$$

thus η is a natural transformation of functors. □

As $M \otimes_R -$ and $- \otimes_R N$ are additive functors, the next result is immediate from (2.2.20).

3.1.12 Proposition

(a) *Suppose that* $M \cong M' \oplus M''$ *as a right* R*-module and that* N *is an* R-S*-bimodule. Then*

$$M \otimes_R N \cong (M' \otimes_R N) \oplus (M'' \otimes_R N)$$

as a right S*-module.*

(b) *Suppose that* $N \cong N' \oplus N''$ *as a left* R*-module and that* M *is a* T-R*-bimodule. Then*

$$M \otimes_R N \cong (M \otimes_R N') \oplus (M \otimes_R N'')$$

as a left T*-module.* □

Since the isomorphisms in parts (a) and (b) of the above proposition are evidently natural in N and M respectively, the definitions of a direct sum of functors (2.2.21) and of a natural isomorphism (1.3.3) lead to the following functorial interpretation of the proposition.

3.1.13 Proposition

(a) *Suppose that* $M \cong M' \oplus M''$ *as a right* R*-module. Then there is a natural isomorphism*

$$M \otimes_R - \simeq M' \otimes_R - \oplus M'' \otimes_R -$$

of functors from ${}_R\mathcal{M}\mathrm{od}$ *to* $\mathcal{A}\mathrm{b}$.

(b) *Suppose that* $N \cong N' \oplus N''$ *as a left* R*-module. Then there is a natural isomorphism*

$$- \otimes_R N \simeq - \otimes_R N' \oplus - \otimes_R N''$$

of functors from $\mathcal{M}\mathrm{od}_R$ *to* $\mathcal{A}\mathrm{b}$. □

The above proposition easily leads to the next, in the case of finite direct sums. However, we also need to consider direct sums based on an infinite ordered set Λ, and to introduce some notation. Recall that we write the standard free right R-module on Λ as R^Λ. The *standard free left* R-module is written as ${}^\Lambda R$, and we view ${}^\Lambda R$ as the 'space of row vectors' over ${}^\Lambda R$ indexed by Λ. When the index set is the finite set $\{1, \ldots, n\}$, we use the more familiar notations R^n and ${}^n R$ for the free right and left modules respectively. The direct sum of copies of a general R-module M is written M^Λ or M^n, regardless of chirality.

3.1.14 Proposition

(a) *The two functors*

$$\mathcal{M}\mathrm{od}_R \times \mathcal{O}_{\mathcal{R}D} \longrightarrow \mathcal{M}\mathrm{od}_R$$

given respectively by

$$(M, \Lambda) \longmapsto M \otimes_R {}^\Lambda R$$

and

$$(M, \Lambda) \longmapsto M^\Lambda$$

are naturally isomorphic.

(b) *The two functors*

$${}_R\mathcal{M}\mathrm{od} \times \mathcal{O}_{\mathcal{R}D} \longrightarrow {}_R\mathcal{M}\mathrm{od}$$

given respectively by

$$(N, \Lambda) \longmapsto R^\Lambda \otimes_R N$$

and

$$(N, \Lambda) \longmapsto N^\Lambda$$

are naturally isomorphic.

Proof

It suffices to discuss (a). The key point is that the sequences (m_λ), with each $m_\lambda \in M$, $\lambda \in \Lambda$, that make up M^Λ, have only finitely many terms m_λ nonzero. Writing the standard generators of ${}^\Lambda R$ as e_λ, we map $M \otimes_R {}^\Lambda R$ to M^Λ by sending $\sum_i (\sum_\lambda m_i \otimes r_{i\lambda} e_\lambda)$ to $(\sum_i m_i r_{i\lambda})_\lambda$. Clearly this defines a surjective natural transformation, whose injectivity follows from the interchangeability of the finite summations:

$$\sum_i (\sum_\lambda m_i \otimes r_{i\lambda} e_\lambda) = \sum_\lambda (\sum_i m_i r_{i\lambda}) \otimes e_\lambda. \qquad \square$$

We also note an important special case of the above proposition. Let $M_{m,n}(R)$ denote the additive group of $m \times n$ matrices with entries in the ring R. Clearly, $M_{m,n}(R)$ is an $M_m(R)$-$M_n(R)$-bimodule, and we know that R^m is an $M_m(R)$-R-bimodule and that the standard free left R-module ${}^n R$ is an R-$M_n(R)$-bimodule. A direct verification gives the following result.

3.1.15 Corollary

$R^m \otimes_R {}^n R \cong M_{m,n}(R)$ as an $M_m(R)$-$M_n(R)$-bimodule. $\square$

3.1.16 Return of the dyads

In classical tensor analysis, an $m \times n$ matrix that can be expressed as the matrix product of a column vector by a row vector (that is, an $m \times 1$ matrix by a $1 \times n$ matrix) is called a *dyad*, especially when $m = n$. Thus the above result tells us that any matrix is a sum of dyads.

We next investigate the effect of a tensor product when it acts on subcategories of $\mathcal{M}_{OD_R}$. Recall that $\mathcal{M}_R$ is the category of finitely generated right R-modules and that $\mathcal{P}_R$ is the category of finitely generated projective right R-modules.

3.1.17 Lemma

Let N be an R-S-bimodule.

(i) *The functor $- \otimes_R N$ from $\mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_S}$ induces a functor*

$$- \otimes_R N : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

if and only if N is finitely generated as a right S-module.

(ii) *The functor $- \otimes_R N$ from $\mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_S}$ induces a functor*

$$- \otimes_R N : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

if and only if N is finitely generated and projective as a right S-module.

Proof

Necessity is clear in both cases, since $N \cong R \otimes_R N$ must be in the target category. On the other hand, if N has a finite set $\{n_1, \ldots, n_k\}$ of generators as an S-module, and M has generators $\{m_1, \ldots, m_\ell\}$ as an R-module, then $\{m_1 \otimes n_1, \ldots, m_\ell \otimes n_k\}$ generates $M \otimes_R N$, which gives (i).

For (ii), note that any module P in $\mathcal{P}_R$ is a direct summand of R^n for some integer n. But then $P \otimes_R N$ is a direct summand of $R^n \otimes_R N \cong N^n$, which is projective precisely when N is, by a standard result ([BK: IRM] Theorem 2.5.5). □

3.1.18 The adjointness of the functors Hom *and* $\otimes$

The homomorphism functors and the tensor product are connected by an adjointness relation, the existence of which serves to explain and simplify some calculations, especially in the Morita theory.

It will be convenient to depart from our customary labels for rings in order to avoid some confusing substitutions in applications. Let A, B and C be arbitrary rings, and take bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated.

Then $L \otimes_B M$ is an A-C-bimodule, and we can form the abelian group $\mathrm{Hom}_{A-C}(L \otimes_B M, N)$ of A-C-bimodule homomorphisms from $L \otimes_B M$ to N. It is easy to verify that this group is natural with respect to bimodule homomorphisms of each variable, being contravariant in the first two variables and covariant in the third. Thus we have a trifunctor

$$\mathrm{Hom}_{A-C}(- \otimes_B -, -) : {}_A\mathcal{B}_{\mathrm{IMOD}_B} \times {}_B\mathcal{B}_{\mathrm{IMOD}_C} \times {}_A\mathcal{B}_{\mathrm{IMOD}_C} \longrightarrow \mathcal{A}_B.$$

We can also give $\mathrm{Hom}(M_C, N_C)$ the structure of an A-B-bimodule by the usual rule:

$$(a\mu b)(m) = a(\mu(bm)) \text{ for } m \in M,\ \mu \in \mathrm{Hom}(M_C, N_C).$$

We can then form the group $\mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C))$, so obtaining a trifunctor

$$\mathrm{Hom}_{A-B}(-, \mathrm{Hom}(-_C, -_C)) : {}_A\mathcal{B}_{\mathrm{IMOD}_B} \times {}_B\mathcal{B}_{\mathrm{IMOD}_C} \times {}_A\mathcal{B}_{\mathrm{IMOD}_C} \longrightarrow \mathcal{A}_B.$$

A third variation on this theme is to view $\mathrm{Hom}({}_AL, {}_AN)$ as a B-C-bimodule and form the group $\mathrm{Hom}_{B-C}(M, \mathrm{Hom}({}_AL, {}_AN))$. Notice that the order of the bimodules L, M and N is changed. Despite this alteration, we again have a trifunctor

$$\mathrm{Hom}_{B-C}(-, \mathrm{Hom}({}_A-, {}_A-)) : {}_A\mathcal{B}_{\mathrm{IMOD}_B} \times {}_B\mathcal{B}_{\mathrm{IMOD}_C} \times {}_A\mathcal{B}_{\mathrm{IMOD}_C} \longrightarrow \mathcal{A}_B.$$

The Adjointness Theorem tells us that all three general constructions yield essentially the same trifunctor. More precisely, the following result holds.

3.1.19 The Adjointness Theorem for Hom and $\otimes$

Let A, B and C be arbitrary rings. Then there are natural isomorphisms

$$\eta : \mathrm{Hom}_{A-C}(L \otimes_B M, N) \longrightarrow \mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C)),$$

and

$$\psi : \mathrm{Hom}_{A-C}(L \otimes_B M, N) \longrightarrow \mathrm{Hom}_{B-C}(M, \mathrm{Hom}({}_AL, {}_AN))$$

of trifunctors from ${}_A\mathcal{B}_{IMOD_B} \times {}_B\mathcal{B}_{IMOD_C} \times {}_A\mathcal{B}_{IMOD_C}$ to $\mathcal{A}_B$.

In particular, for M in ${}_B\mathcal{B}_{IMOD_C}$, the functor

$$- \otimes_B M : {}_A\mathcal{B}_{IMOD_B} \longrightarrow {}_A\mathcal{B}_{IMOD_C}$$

is left adjoint to

$$\mathrm{Hom}(M_C, -) : {}_A\mathcal{B}_{IMOD_C} \longrightarrow {}_A\mathcal{B}_{IMOD_B},$$

and for L in ${}_A\mathcal{B}_{IMOD_B}$, the functor

$$L \otimes_B - : {}_B\mathcal{B}_{IMOD_C} \longrightarrow {}_A\mathcal{B}_{IMOD_C}$$

is left adjoint to

$$\mathrm{Hom}({}_AL, -) : {}_A\mathcal{B}_{IMOD_C} \longrightarrow {}_B\mathcal{B}_{IMOD_C}.$$

Proof

We establish the first assertion only, the argument for the second being similar.

Take bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ and let α be in $\mathrm{Hom}_{A-C}(L \otimes_B M, N)$. Then a map $\eta\alpha : L \to \mathrm{Hom}(M_C, N_C)$ is defined by setting

$$((\eta\alpha)(\ell))(m) = \alpha(\ell \otimes m) \text{ for each } \ell \in L \text{ and } m \in M.$$

Since C acts on $L \otimes M$ by $(\ell \otimes m)c = \ell \otimes mc$ and α is a right C-module homomorphism (and the tensor product is linear in each variable), every $(\eta\alpha)(\ell)$ is a right C-module homomorphism. It is also clear that $\eta\alpha$ is an additive map on L.

The verification that $\eta\alpha$ is an A-B-bimodule homomorphism is a matter of careful book-keeping. We need to check that for all $\ell \in L$, $a \in A$ and $b \in B$ we have

$$(\eta\alpha)(a\ell b) = a((\eta\alpha)(\ell))b.$$

For any $m \in M$,

$$\begin{aligned}((\eta\alpha)(a\ell b))(m) &= \alpha(a\ell b \otimes m) = \alpha(a(\ell \otimes bm)) \\ &= a(\alpha(\ell \otimes bm)) = a(\eta\alpha(\ell))(bm) = (a((\eta\alpha)(\ell))b)(m),\end{aligned}$$

which is the desired result.

We define the inverse τ of η as follows. Let $\gamma \in \mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C))$, and put $(\tau\gamma)(\ell \otimes m) = \gamma(\ell)(m)$. Routine checking confirms that $\tau\gamma$ arises from a B-balanced biadditive map on $L \times M$ in the expected way and that $\tau\gamma$ is an A-C-bimodule homomorphism.

It is clear that η and τ are mutually inverse isomorphisms for each triple L, M and N, and further routine verification shows them to be natural in each variable. □

3.1.20 An equivalence of categories

Finally, we use the tensor product to generalize the characterization of triangular matrix rings that was given in (1.3.18).

Let R and S be arbitrary rings and let W be an R-S-bimodule. We define the *morphism category relative to* W (a variant of the fibre category of (1.2.9)) to be the category $\mathcal{M}_{OR}(W; R, S)$ whose objects are triples (α, M, N) in which M is a right R-module, N is a right S-module and $\alpha : M \otimes_R W \to N$ is a homomorphism of right S-modules.

A morphism from (α', M', N') to (α, M, N) is a pair (μ, ν) of homomorphisms, $\mu : M' \to M$ and $\nu : N' \to N$, such that $\nu\alpha' = \alpha(\mu \otimes id)$, which means that the following diagram must commute:

$$\begin{array}{ccc} M' \otimes_R W & \xrightarrow{\alpha'} & N' \\ {\scriptstyle \mu \otimes id}\downarrow & & \downarrow{\scriptstyle \nu \otimes id} \\ M \otimes_R W & \xrightarrow{\alpha} & N \end{array}$$

Thus with $W = R = S$, we regain the morphism category $\mathcal{M}_{OR}(\mathcal{M}_{OD_R})$.

The rings R and S and the bimodule W also define the generalized triangular matrix ring $T = \begin{pmatrix} R & W \\ 0 & S \end{pmatrix}$.

3.1.21 Theorem

The categories $\mathcal{M}_{OD_T}$ and $\mathcal{M}_{OR}(W; R, S)$ are equivalent.

Proof

As the details of the argument are very similar to those of the proof of (1.3.18), we merely indicate the correspondence between right T-modules L and homomorphisms $\alpha : M \otimes_R \hat{W} \to N$.

Suppose that L is a right T-module. Since the direct product $R \times S$ is a subring of T, L can be decomposed into a direct sum $M \oplus N$ with M a right R-module and N a right S-module. For m in M and w in W, we can write

$$(m, 0)\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = (0, \alpha(m, w))$$

with $\alpha(m, w) \in N$. It is easy to check that α is biadditive and R-balanced, and so gives rise to a homomorphism $\beta : M \otimes_R W \to N$.

Conversely, given such a homomorphism β, we can define a T-module structure on $M \oplus N$. The remaining details are for the reader. □

Exercises

3.1.1 Let $\beta : M \to M''$ and $\xi : N \to N''$ be surjective homomorphisms of right and left R-modules respectively. Show that $\operatorname{Ker}\beta \otimes \xi$ is the subgroup of $M \otimes N$ generated by the set

$$\{k \otimes n, m \otimes \ell \mid k \in \operatorname{Ker}\beta,\ \ell \in \operatorname{Ker}\xi,\ m \in M,\ n \in N\}.$$

3.1.2 Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of the ring R. Show that $R/\mathfrak{a} \otimes_R R/\mathfrak{b} \cong R/(\mathfrak{a}+\mathfrak{b})$. (The preceding exercise is relevant.)

Hence compute $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z}$ for various integers a and b.

3.1.3 **Dedekind domains**

Let $\mathcal{O}$ be a Dedekind domain. This exercise requires some facts about $\mathcal{O}$-modules which are summarized in (2.3.20).

(a) Let P be a finitely generated projective $\mathcal{O}$-module. Then, by Steinitz' Theorem, P is isomorphic to a projective module in the standard form $P \cong \mathcal{O}^{r-1} \oplus \mathfrak{a}$ where the rank r and the ideal class $\{\mathfrak{a}\}$ are uniquely determined by P. Compute $P \otimes_{\mathcal{O}} Q$ for any finitely generated projective modules P, Q.

(b) Using the Primary Decomposition Theorem and Exercise 3.1.2 above, describe $M \otimes_{\mathcal{O}} N$ for any pair M and N of finitely generated torsion $\mathcal{O}$-modules.

(c) Using the fact that any finitely generated $\mathcal{O}$-module has the form $T \oplus P$ with T torsion and P projective, describe $M \otimes_{\mathcal{O}} N$ for any pair M and N of finitely generated $\mathcal{O}$-modules.

3.1.4 **Associativity of the tensor product**

Let R and S be rings and let L_R, ${}_RM_S$ and ${}_SN$ be modules as indicated.

Show that the isomorphism

$$(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N)$$

of (3.1.5) is natural in each variable.

Deduce that the functors $(- \otimes_R -) \otimes_S -$ and $- \otimes_R (- \otimes_S -)$ from $\mathcal{M}\text{od}_R \times {}_R\mathcal{B}\text{imod}_S \times {}_S\mathcal{M}\text{od}$ to $\mathcal{A}_B$ are naturally isomorphic.

3.1.5 Let A, B, and C be rings with bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated.

Using the Adjointness Theorem (3.1.19) together with the characterizations of projective and injective modules in terms of the Hom functor given in (2.1.8), show that

(i) if L_B and M_C are both projective, then $L \otimes_B M$ is projective as a right C-module (here, take $A = \mathbb{Z}$);

(ii) if M_C is projective and ${}_AN$ is injective, then $\text{Hom}(M_C, N_C)$ is left A-injective (here, take $B = \mathbb{Z}$.)

3.1.6 (Tricky!)

Let T be the ring of upper triangular 2×2-matrices over R. By (3.1.21), there is an equivalence between the categories $\mathcal{M}\text{od}_T$ and $\mathcal{M}\text{od}(\mathcal{M}\text{od}_R)$. Show that the projective T-modules correspond to split surjections $\alpha : P \to Q$ with both P and Q R-projective.

Dualize this statement to one about injectives.

3.2 EXACTNESS OF THE TENSOR PRODUCT

We next investigate whether or not a tensor product functor $- \otimes_R N$ is an exact functor. The answer depends both on the left R-module N that is kept fixed and on the category of right R-modules on which it operates. The modules that always give exact functors form an important class of modules, namely, the flat modules. We give some criteria for a module to be flat, and we prove Villamayor's Lemma, a fundamental result which shows that a flat module enjoys a weak form of the splitting property that defines projective modules. We also give a summary of the properties of the functors Tor_n^R that are used to repair the non-exactness of the tensor product, and we show that the tensor product induces a pairing on $\mathcal{M}\text{od}_R$ for a commutative ring R.

The novelty of our proofs is that they are 'elementary' in that they avoid the use of the machinery of homological algebra.

We also show that, for R commutative, the tensor product defines a binary operation on the category of R-modules.

Unless otherwise stated, the ring R is arbitrary. Our main object of attention is a left module, since we wish to investigate operators on categories of right modules.

3.2.1 *Flat modules*

Let $\mathcal{C}$ be a G-exact category of right R-modules, that is, $\mathcal{C}$ is a subcategory of $\mathcal{M}_{OD_R}$ together with a specified set $\mathrm{Ex}(\mathcal{C})$ of short exact sequences as in (2.4.1), and let N be an R-S-bimodule. The additive functor $-\otimes_R N$ may or may not be an exact functor on $\mathcal{C}$ as defined in (2.4.6). It will be so if $\mathcal{C}$ is split exact, but it is not in general, as the following example for $\mathcal{M}_{OD_{\mathbb{Z}}}$ shows.

For any nonzero integer n, there is an exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

in which the first map is multiplication by n. Tensoring with $\mathbb{Z}/n\mathbb{Z}$ (as a $\mathbb{Z}$-module) gives the sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{0} \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

which evidently fails to be exact at the first term $\mathbb{Z}/n\mathbb{Z}$.

A left R-module N for which $-\otimes_R N : \mathcal{M}_{OD_R} \to \mathcal{M}_{OD_{\mathbb{Z}}}$ is an exact functor is, by definition, a *flat* module. Thus, the left R-module N is flat if for any short exact sequence

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

of right R-modules the sequence

$$0 \longrightarrow M' \otimes_R N \xrightarrow{\alpha \otimes id} M \otimes_R N \xrightarrow{\beta \otimes id} M'' \otimes_R N \longrightarrow 0$$

is also a short exact sequence. The definition of a flat right R-module is analogous.

We have an almost tautologous lemma.

3.2.2 Lemma

Suppose that the R-S-bimodule N is flat as a left R-module and finitely generated as a right S-module (where R and S are arbitrary rings). Then the functor

$$-\otimes_R N : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

is exact. □

Before we make a detailed analysis of the properties of flat modules, we discuss the extent to which exactness is preserved by modules in general.

3.2.3 Proposition

(a) *Let N be a left R-module and let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of right R-modules. Then the sequence

$$M' \otimes_R N \xrightarrow{\alpha \otimes id} M \otimes_R N \xrightarrow{\beta \otimes id} M'' \otimes_R N \longrightarrow 0$$

is exact.

(b) *Let M be a right R-module and let*

$$0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$$

be a short exact sequence of left R-modules. Then the sequence

$$M \otimes_R N' \xrightarrow{id \otimes \alpha} M \otimes_R N \xrightarrow{id \otimes \beta} M \otimes_R N'' \longrightarrow 0$$

is exact.

Proof

(a) The fact that $M'' \otimes N$ is generated by symbols $m'' \otimes n$ shows immediately that $\beta \otimes id$ is surjective. Since

$$0 = \beta\alpha \otimes id = (\beta \otimes id)(\alpha \otimes id),$$

we have

$$\mathrm{Im}(\alpha \otimes id) \subseteq \mathrm{Ker}(\beta \otimes id).$$

We must verify that this inclusion is equality. Write $I = \mathrm{Im}(\alpha \otimes id)$ and let

$$\lambda : (M \otimes_R N)/I \longrightarrow M'' \otimes_R N$$

be the homomorphism induced by β. It suffices to show that λ is an isomorphism, which we do by constructing an inverse.

Recall that $M'' \otimes_R N = M''\#N/B(M'', N)$ (3.1.2). Define

$$\mu : M''\#N \longrightarrow (M \otimes_R N)/I$$

by

$$\mu(m'', n) = m \otimes n + I,$$

where m is any element of M such that $\beta m = m''$.

If also $\beta m_1 = m''$, then exactness at M implies that $m - m_1$ is in $\operatorname{Im}\alpha$ and hence

$$m \otimes n + I = m_1 \otimes n + I.$$

Thus μ is well-defined. Since μ is trivial on $B(M'', N)$, it induces a homomorphism $M'' \otimes_R N \longrightarrow (M \otimes_R N)/I$ which is evidently inverse to λ.

The proof of (b) is analogous. □

3.2.4 Corollary

A left R-module N is flat if and only if for each injective right R-module homomorphism $\alpha : M' \to M$, the group homomorphism

$$\alpha \otimes id_N : M' \otimes_R N \longrightarrow M \otimes_R N,$$

is also an injection. □

The existence of flat modules is guaranteed by the next result.

3.2.5 Corollary

Suppose that N is a projective left R-module. Then N is flat.

Proof

That any free module R^Λ is flat follows from (3.1.14), since any injective right R-homomorphism $M' \to M$ induces an injection $M'^\Lambda \to M^\Lambda$. Now, for the module N, we have $N \oplus Q \cong {}^\Lambda R$ for some left R-module Q and free module ${}^\Lambda R$. Therefore, by (3.1.13),

$$- \otimes_R N \oplus - \otimes_R Q \simeq - \otimes_R {}^\Lambda R.$$

So $- \otimes_R N$ is exact by Exercise 2.4.6. □

3.2.6 The functors Tor_n^R

The failure of the tensor product to be an exact functor can be measured by a sequence of derived functors $\operatorname{Tor}_n^R(-,-)$ for $n \geq 1$. These may be constructed as follows.

Given a right R-module M and a left R-module N, where R is an arbitrary ring, we take a short exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow N \longrightarrow 0$$

of left R-modules with F flat (for example, projective), and define

$$\mathrm{Tor}_1^R(M,N) = \mathrm{Ker}(M \otimes_R L \to M \otimes_R F)$$

and

$$\mathrm{Tor}_n^R(M,N) = \mathrm{Tor}_{n-1}^R(M,L) \text{ for } n > 1.$$

It can be shown that the abelian groups $\mathrm{Tor}_n^R(M,N)$ are independent of the choice of the short exact sequence for N, and that they could equally be defined by taking a similar short exact sequence for M and considering the kernel after tensoring with N. Further, $\mathrm{Tor}_n^R(M,N)$ is a covariant additive functor in each of its arguments, and, given a short exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

of left R-modules, there is a long exact sequence

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & \mathrm{Tor}_{n+1}^R(M,N'') \longrightarrow \\
\longrightarrow & \mathrm{Tor}_n^R(M,N') & \longrightarrow & \mathrm{Tor}_n^R(M,N) & \longrightarrow & \mathrm{Tor}_n^R(M,N'') & \longrightarrow \\
\longrightarrow & \cdots & & & & & \\
 & & & & \cdots & \longrightarrow & \mathrm{Tor}_1^R(M,N'') \longrightarrow \\
\longrightarrow & M \otimes_R N' & \longrightarrow & M \otimes_R N & \longrightarrow & M \otimes_R N'' & \longrightarrow 0,
\end{array}$$

and likewise for the other variable.

Much analysis is required to verify these claims; a good account is given in [Rotman 1979].

Note that the flat left R-modules can be characterized as those modules F for which the functors $\mathrm{Tor}_n^R(-,F)$ $(n \geq 1)$ are identically zero (and correspondingly for right flat modules). Some very elementary calculations with the Tor functors are indicated in Exercise 3.2.7 below, and in Exercise 6.2.1, which justifies the notation 'Tor'. Flat modules made a reticent entrance into the mathematical landscape in Exercise 3 of Ch. VI of [Cartan & Eilenberg 1956], where they are defined in terms of the vanishing of the functors Tor_n. The pivotal role that flat modules now play will become clear in subsequent chapters of this text.

3.2.7 Criteria for flatness and Villamayor's Lemma

We now obtain some useful criteria for flatness and thence Villamayor's Lemma. For the next few results and proofs, it is convenient to say that a functor F *respects the injectivity* of a module monomorphism μ if $F\mu$ is also a monomorphism.

3.2.8 Proposition

(a) *For a left R-module N, the following assertions are equivalent.*

(i) ${}_RN$ *is flat.*

(ii) *The functor $-\otimes_R N$ respects the injectivity of any R-module monomorphism $\alpha : M'_R \to M_R$ with M'_R finitely generated. That is,*

$$\alpha \otimes id : M' \otimes_R N \longrightarrow M \otimes_R N$$

is also injective.

(iii) *The functor $-\otimes_R N$ respects the injectivity of any R-module monomorphism $M'_R \to M_R$.*

(iv) *The functor $- \otimes_R N : \mathcal{M}\text{od}_R \to \mathcal{A}\text{b}$ is exact.*

(b) *For a right R-module M, the following assertions are equivalent.*

(i) M_R *is flat.*

(ii) *The functor $M\otimes_R -$ respects the injectivity of any R-module monomorphism ${}_RN' \to {}_RN$ with ${}_RN'$ finitely generated.*

(iii) *The functor $M\otimes_R -$ respects the injectivity of any R-module monomorphism ${}_RN' \to {}_RN$.*

(iv) *The functor $M \otimes_R - : {}_R\mathcal{M}\text{od} \to \mathcal{A}\text{b}$ is exact.*

Proof

The only outstanding points to check are the implications (ii) $\Rightarrow$ (iii); we deal with (a) only. Consider an injective homomorphism $\alpha : M'_R \to M_R$ with M'_R not necessarily finitely generated, and let $x = \sum_{i=1}^{k}(m'_i \otimes n_i)$ be in $\text{Ker}(\alpha \otimes id)$. Then x also lies in $\text{Ker}(\alpha' \otimes id)$, where α' is the restriction of α to the submodule of M generated by $m'_1, \ldots, m'_k$. This forces $x = 0$, so that $\text{Ker}(\alpha \otimes id) = 0$ as required. □

The following criterion is designed to simplify one's workload in establishing that a given module is flat.

3.2.9 The Flat Test

A left R-module N is flat if and only if $-\otimes_R N$ respects the injectivity of all inclusions $\mu : \mathfrak{a} \to R$ of finitely generated right ideals in R. The corresponding statement holds with left and right interchanged.

Proof

Necessity of the condition is immediate from the preceding proposition. Sufficiency is proved in four stages. First, we note that the argument just given to prove (ii) $\Rightarrow$ (iii) above also shows that $-\otimes_R N$ respects the injectivity of inclusions $\mu : \mathfrak{a} \to R$ of arbitrary right ideals in R. Second, we argue by

induction on n to show that $-\otimes_R N$ respects injectivity for $\mu : M \to R^n$, the case $n = 1$ having just been settled.

For the inductive step, we use the standard embedding of R^{n-1} in R^n. Put $M'' = M/(M \cap R^{n-1})$. Since $M'' \cong (M + R^{n-1})/R^{n-1}$ and $R^n/R^{n-1} \cong R$, there is an injective homomorphism from M'' to R. So there is a commuting diagram with exact rows, the lower of which is split:

$$\begin{array}{ccccccccc} & & (M \cap R^{n-1}) \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & M'' \otimes_R N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R^{n-1} \otimes_R N & \longrightarrow & R^n \otimes_R N & \longrightarrow & R \otimes_R N & \longrightarrow & 0 \end{array}$$

Then the injectivity of the two outside vertical arrows implies that of the middle one.

The next step is to show that $-\otimes_R N$ respects the injectivity of a monomorphism $\mu : M \to F$ where F is an arbitrary free module. Again, it is enough to check the case where the domain M is finitely generated. But then, if $(\mu \otimes id)x = 0$ in $F \otimes_R M$, there is a finitely generated free submodule G of the codomain F such that $(\mu \otimes id)x = 0$ in $G \otimes_R M$ already, since any member of the relation group $B(F, M)$ (3.1.2) can involve only finitely many generators of F. By the previous step, $x = 0$.

For the final step, let $\mu : M' \to M$ be an arbitrary injective homomorphism of right R-modules, and let F be a free right R-module with $\beta : F \to M$ an R-homomorphism onto M, having kernel K, say. Then, in the pull-back

$$\begin{array}{ccc} \overline{F} & \xrightarrow{\overline{\beta}} & M' \\ \overline{\mu} \downarrow & & \downarrow \mu \\ F & \xrightarrow{\beta} & M \end{array}$$

$\overline{\beta}$ also has kernel K, and so there is a commuting diagram with exact rows

$$\begin{array}{ccccccccc} K \otimes_R N & \longrightarrow & \overline{F} \otimes_R N & \longrightarrow & M' \otimes_R N & \longrightarrow & 0 \\ id \downarrow & & \overline{\mu} \otimes id \downarrow & & \mu \otimes id \downarrow & & \\ K \otimes_R N & \longrightarrow & F \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & 0 \end{array}$$

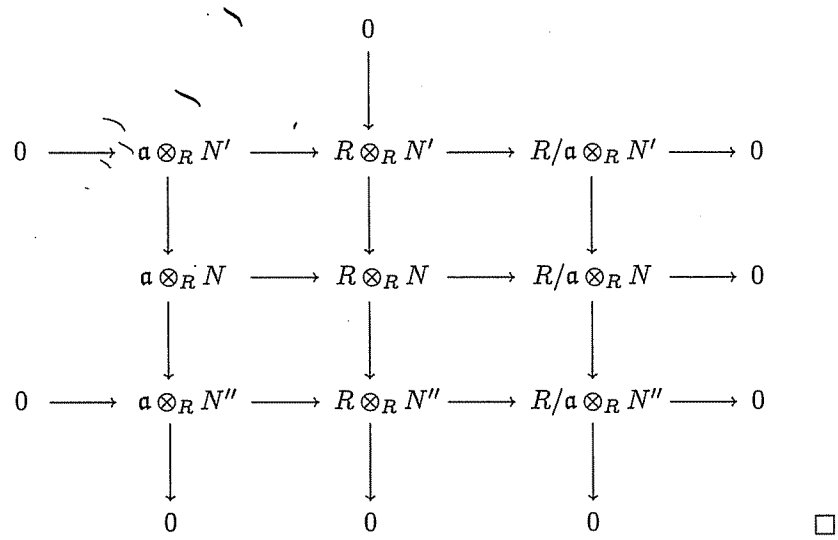
Since $\overline{\mu} \otimes id$ is now known to be injective, so too is $\mu \otimes id$, and the proof is complete. □

3.2.10 Corollary

Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of left R-modules. If N' and N'' are flat, then so too is N.

Proof

Considering any inclusion $\mathfrak{a} \to R$ of a right ideal in R, simply chase the following commutative diagram of exact rows and columns to show that the middle row commences with an injection.



□

3.2.11 Villamayor's Lemma

Let M be a right R-module. Then the following assertions are equivalent.

(i) *M is flat.*

(ii) *Given any pair (ϵ, L), where $\epsilon : F \to M$ is a surjective homomorphism of right R-modules with F free, and L is a finitely generated submodule of $\operatorname{Ker}\epsilon$, then there is an R-homomorphism $\rho : F \to \operatorname{Ker}\epsilon$ which restricts to the identity map on L and which has finitely generated image ρF.*

Proof

First assume that M is flat. We argue by induction on the number of generators $u_1, \ldots, u_n$ of the submodule L.

$n = 1$. Write $u_1 = \sum_{i=1}^{k} x_i r_i$ with $x_1, \ldots, x_n$ elements of a free generating set for F. Let $\mathfrak{a}$ be the left ideal of R generated by $r_1, \ldots, r_k$. Since M is flat,

the obvious map $M \otimes_R \mathfrak{a} \to M \otimes_R R \cong M$ is an injection. So in the following commutative diagram with exact rows, the right vertical arrow is injective.

$$\begin{array}{ccccccccc} & & (\operatorname{Ker}\epsilon)\otimes_R \mathfrak{a} & \longrightarrow & F\otimes_R \mathfrak{a} & \longrightarrow & M\otimes_R \mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\epsilon & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since u_1 is the image of the element $\sum_{i=1}^{k} x_i \otimes r_i$ of $F \otimes_R \mathfrak{a}$, a diagram chase shows that

$$\sum_{i=1}^{k} x_i \otimes r_i = \operatorname{Im}(\sum_{j=1}^{h} y_j \otimes s_j)$$

for elements $y_j \in \operatorname{Ker}\epsilon$ and $s_j \in \mathfrak{a}$. Write $s_j = \sum_{i=1}^{k} a_{ij} r_i$ for $j = 1, \ldots, h$, with each a_{ij} in R, and put $v_i = \sum_{j=1}^{h} y_j a_{ij}$ for each i. Then

$$u_1 = \operatorname{Im}(\sum_{i=1}^{k} v_i \otimes r_i) = \sum_{i=1}^{k} v_i r_i \text{ with each } v_i \in \operatorname{Ker}\epsilon.$$

The R-homomorphism $\rho_1 : F \to \operatorname{Ker}\epsilon$, sending x_i to v_i for $i = 1, \ldots, k$ and each other generator of F to zero, has finitely generated image and $\rho_1 u_1 = u_1$.

$n > 1$. Let $\rho_1 : F \to \operatorname{Ker}\epsilon$ be as above, and let $\rho' : F \to \operatorname{Ker}\epsilon$ be the homomorphism obtained by applying the induction hypothesis to the submodule of $\operatorname{Ker}\epsilon$ generated by $u_2 - \rho_1 u_2, \ldots, u_n - \rho_1 u_n$. Then the R-homomorphism

$$\rho = \rho_1 + \rho'(id - \rho_1) : F \longrightarrow \operatorname{Ker}\epsilon$$

has the desired property.

For the converse, we wish to apply the Flat Test. Therefore let $\mathfrak{a}$ be a left ideal of R (with inclusion $\iota : \mathfrak{a} \to R$) and let $\epsilon : F \to M$ be some surjection from a free module F. We obtain the same diagram as above, but this time seek to establish the injectivity of the right-hand vertical arrow.

Since the free module F is flat by the left-handed version of (3.2.5), $id \otimes \iota : F \otimes_R \mathfrak{a} \to F \otimes_R R \cong F$ is injective. Thus, by an easy diagram chase, the injectivity of the right-hand arrow will follow once we can show that if $\sum_{i=1}^{k} x_i \otimes a_i$ is in $F \otimes_R \mathfrak{a}$ and $\epsilon(\sum_{i=1}^{k} x_i a_i) = 0$, then $\sum_{i=1}^{k} x_i a_i$ lies in $(id \otimes \iota)((\operatorname{Ker}\epsilon) \otimes_R \mathfrak{a})$.

Now by hypothesis there exists an R-homomorphism $\rho : F \to \operatorname{Ker}\epsilon$ which restricts to the identity map on $\sum_{i=1}^{k} x_i a_i$. Then $\sum_{i=1}^{k} \rho x_i \otimes a_i \in (\operatorname{Ker}\epsilon) \otimes_R \mathfrak{a}$

and

$$\sum_{i=1}^{k} \rho(x_i)a_i = \rho(\sum_{i=1}^{k} x_i a_i) = \sum_{i=1}^{k} x_i a_i,$$

as required. □

Note that here L is not quite a direct summand of F. We now show that in favourable circumstances we do obtain a direct summand. The first result is obvious, the next rather deeper.

3.2.12 Corollary

Suppose that $\epsilon : F \twoheadrightarrow M$ *is surjective, with* F *free and* M *flat. If* $\operatorname{Ker}\epsilon$ *is finitely generated, then* $F \cong \operatorname{Ker}\epsilon \oplus M$. □

Although the finitely generated submodule L in Villamayor's Lemma need not be a summand of F, the next result shows that it is contained in a finitely generated summand provided that F is 'big enough'. This result is used in (5.2.7) to help in the derivation of another important characterization of flatness.

3.2.13 Lemma

Suppose that (ϵ, L) *is a pair as in Villamayor's Lemma, and also that* M *is flat and that* $F = \operatorname{Fr}_R(X)$, *where for each* $x \in X$ *there exist infinitely many* $y \in X$ *such that* $\epsilon y = \epsilon x$. *Then there is a finitely generated direct summand* L' *of* F *with*

$$L \subseteq L' \subseteq \operatorname{Ker}\epsilon.$$

Proof

With $\rho : F \to \operatorname{Ker}\epsilon$ as above, let X_1 be the finite set of elements of X occurring in the expressions of the generators of ρF as linear combinations of elements of X. By hypothesis, we can partition X as $X = X_1 \sqcup X_2 \sqcup X_3$ with X_1 and X_2 in bijective correspondence such that corresponding elements have the same images under ϵ. Write $F_i = \operatorname{Fr}_R(X_i)$, so that $L \subseteq F_1$ and $F = F_1 \oplus F_2 \oplus F_3$, and let $\pi_1 : F \to F_1$ denote projection to the first summand. The bijection $X_1 \to X_2$ defines an injective homomorphism $\sigma : F_1 \to F_2 \hookrightarrow F$ with $\pi_1\sigma = 0$ and $\epsilon\sigma = \epsilon$. Define

$$\tau = id + \sigma(\rho - id) : F_1 \to F,$$

with image $L' = \tau F_1$. (Roughly speaking, $\tau - id$ is just $\rho - id$ shifted away from F_1.) Evidently, $\pi_1\tau = id$, so that the inclusion of L' in F is split by

$\tau\pi_1 : F \to \tau F_1$. Because the restriction of τ to L is the identity map, $L \subseteq L'$. Finally, since $\epsilon\sigma = \epsilon$ and $\epsilon\rho = 0$, we have $L' \subseteq \operatorname{Ker}\epsilon$. □

3.2.14 A pairing on $\mathcal{M}_{OD_R}$

As we remarked following (3.1.10), the tensor product $-\otimes_R-$ can be regarded as a bifunctor from $\mathcal{M}_{OD_R} \times {}_R\mathcal{M}_{OD}$ to $\mathcal{A}_B$. For a general ring R and arbitrary modules M_R and ${}_RN$, there is no natural method by which $M \otimes_R N$ can be endowed with an R-module structure. However, if both M and N belong to the category ${}_R\mathcal{B}_{IMOD_R}$ of R-R-bimodules (and R-R-bimodule homomorphisms), then $M \otimes_R N$ is also in ${}_R\mathcal{B}_{IMOD_R}$ (see (3.1.3)).

This observation permits us to use the tensor product to define a binary operation in $\mathcal{M}_{OD_R}$ when the ring R is commutative. An R-R-bimodule is said to be *balanced* if $rm = mr$ for all m in M and r in R. Clearly, any right (or left) R-module M can be viewed as a balanced R-R-bimodule by setting $rm = mr$ for all r in R and m in M. Thus we can choose to identify both $\mathcal{M}_{OD_R}$ and ${}_R\mathcal{M}_{OD}$ with the category $\mathcal{B}_{AL_R}$ of balanced R-bimodules.

Using this identification, we can regard $-\otimes_R-$ as a bifunctor from $\mathcal{M}_{OD_R} \times \mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_R}$. Such a bifunctor which is additive in both variables is sometimes called a *pairing* on $\mathcal{M}_{OD_R}$. It is clear that this pairing induces a pairing on $\mathcal{M}_R$, the category of finitely generated R-modules.

The next result shows that this pairing is, in essence, commutative. The proof is an easy exercise.

3.2.15 Proposition

Let M and N be balanced bimodules over the commutative ring R. Then there is a natural isomorphism $M \otimes_R N \cong N \otimes_R M$ of balanced R-bimodules. □

The above pairing can be viewed as a generalization of the product of fractional ideals of a commutative domain $\mathcal{O}$. Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$, and recall that a fractional ideal of $\mathcal{O}$ is a finitely generated (balanced) $\mathcal{O}$-submodule of $\mathcal{K}$, and that the product $\mathfrak{ab}$ of fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ is the submodule of $\mathcal{K}$ generated by all products ab with $a \in \mathfrak{a}, b \in \mathfrak{b}$ (2.3.20). Then $\mathfrak{ab}$ is also a fractional ideal. The connection between the pairing and the product is given by the following useful computation.

3.2.16 Proposition

Let $\mathcal{O}$ be a commutative domain with field of fractions $\mathcal{K}$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of $\mathcal{O}$.

Then there is an isomorphism $\mathfrak{a} \otimes_{\mathcal{O}} \mathfrak{b} \to \mathfrak{a}\mathfrak{b} \subseteq \mathcal{K}$, *given by* $a \otimes b \mapsto ab$.

Proof

Clearly, the map $a \otimes b \mapsto ab$ is a well-defined surjection, so we need only check injectivity. Suppose first that $\mathfrak{b} = \mathcal{O}x$ is principal. Then $\mathfrak{b}$ is projective and so flat as an $\mathcal{O}$-module, and the natural inclusion $\mathfrak{a} \to \mathcal{K}$ gives an injective homomorphism $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}x \to \mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}x$. Arguing as in (3.1.4), we have $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}x \cong \mathfrak{a}x$ and $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}x \cong \mathcal{K}x = \mathcal{K}$ via $a \otimes x \mapsto ax$.

In general, suppose that $a_1, \ldots, a_k$ are in $\mathfrak{a}$ and $b_1, \ldots, b_k$ are in $\mathfrak{b}$, and that $\sum_i a_i \otimes b_i \mapsto 0$ in $\mathcal{K}$. Finding a common denominator $x \in \mathcal{K}$, we can write $b_i = c_i x$ with $c_i \in \mathcal{O}$ for all i. Then

$$\sum_i a_i \otimes b_i = \sum_i a_i c_i \otimes x \longmapsto 0,$$

so $\sum_i a_i c_i = 0$ in $\mathfrak{a}$. Thus $\sum_i a_i \otimes c_i = 0$ in $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}$ and, after multiplication by x, $\sum_i a_i \otimes b_i = 0$ in $\mathfrak{a} \otimes \mathfrak{b}$. □

Exercises

3.2.1 Let R be the ring of dual numbers $A[\epsilon]$ over a ring A, and recall that A is both a left and right R-module, with ϵ acting as 0 on A. By considering the short exact sequence of right R-modules

$$0 \longrightarrow \epsilon A[\epsilon] \xrightarrow{\alpha} A[\epsilon] \xrightarrow{\beta} A \longrightarrow 0,$$

where α is the inclusion and β the obvious surjection, show that A is not R-flat.

3.2.2 Let $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$ be a direct sum of left R-modules, where Λ is any ordered set. Show that N is flat if and only if each N_λ is flat.

3.2.3 An R-module M is said to be *finitely related* if there is a short exact sequence of the form

$$0 \longrightarrow R^m \longrightarrow R^\Lambda \longrightarrow M \longrightarrow 0.$$

Show that if also M is flat, then M is projective.

3.2.4 Given a subring $\mathcal{O}$ of a ring R, we introduce two *$\mathcal{O}$-relative G-exact categories* $\mathcal{M}_{R,\mathcal{O}}$ and $\mathcal{P}_{R,\mathcal{O}}$ as follows.

The category $\mathcal{M}_{R,\mathcal{O}}$ has as underlying category $\mathcal{M}_R$ and we take the set $\mathrm{Ex}(\mathcal{M}_{R,\mathcal{O}})$ of admissible short exact sequences to be all the short exact sequences in $\mathcal{M}_R$ which are split as exact sequences of $\mathcal{O}$-modules. The underlying category of $\mathcal{P}_{R,\mathcal{O}}$ is the full subcategory

of $\mathcal{M}_R$ whose objects are in $\mathcal{P}_\mathcal{O}$, the class $\mathrm{Ex}(\mathcal{P}_{R,\mathcal{O}})$ consisting of all short exact sequences with terms in $\mathcal{P}_\mathcal{O}$ (so $\mathcal{P}_{R,\mathcal{O}}$ is repletely exact).

Show that $- \otimes_R N$ is exact on $\mathcal{M}_{R,\mathcal{O}}$ and $\mathcal{P}_{R,\mathcal{O}}$ for any left R-module N.

(In applications, $\mathcal{M}_{R,\mathcal{O}}$ is useful when $\mathcal{O}$ is a commutative domain and R is an $\mathcal{O}$-order, while $\mathcal{P}_{R,\mathcal{O}}$ is useful when $\mathcal{O}$ is arbitrary and R is the group ring $\mathcal{O}G$ for a suitable group G ([Berrick 1982] p. 94).)

3.2.5 **Firm modules over nonunital rings**

For a nonunital ring R with unitalization $\overline{R}$, recall from (1.3.2)(iv) that an R-module is just the same thing as an $\overline{R}$-module. In particular, R itself is an R-bimodule. We define a right R-module V to be *firm* if multiplication induces a right R-module isomorphism $V \otimes_{\overline{R}} R \to V$; similarly for firm left R-modules. The ring R itself is called a *firm nonunital ring* if it is firm as a right R-module. Show that this is equivalent to R being firm as a left R-module, and to there being a canonical (nonunital) ring isomorphism $R \otimes_{\overline{R}} R \to R$, where $R \otimes_{\overline{R}} R$ has ring structure defined by

$$(r_1 \otimes r_2)(r_1' \otimes r_2') = r_1(r_2 r_1') \otimes r_2' = r_1 \otimes (r_2 r_1')r_2'.$$

Now suppose that R is a firm nonunital ring, and let V be a firm right R-module and W a firm left R-module equipped with a surjective R-bimodule homomorphism $\sigma : W \otimes_\mathbb{Z} V \to R$. Define a multiplication on the abelian group $S = V \otimes_R W$ by

$$(v \otimes w)(v' \otimes w') = v \cdot \sigma(w, v') \otimes w' = v \otimes \sigma(w, v') \cdot w'.$$

Show that S is thereby also a firm nonunital ring.

3.2.6 Let $\mathcal{O}$ be any commutative ring. Using (3.2.14), show that the tensor product induces biadditive functors from

(i) $\mathcal{M}_\mathcal{O} \times \mathcal{M}_\mathcal{O}$ to $\mathcal{M}_\mathcal{O}$,
(ii) $\mathcal{F}_\mathcal{O} \times \mathcal{F}_\mathcal{O}$ to $\mathcal{F}_\mathcal{O}$,
(iii) $\mathcal{P}_\mathcal{O} \times \mathcal{P}_\mathcal{O}$ to $\mathcal{P}_\mathcal{O}$,
and
(iv) $\mathcal{M}_\mathcal{O} \times \mathcal{T}_{\mathcal{O}R\mathcal{O}}$ to $\mathcal{T}_{\mathcal{O}R\mathcal{O}}$.

3.2.7 **Calculations with** Tor

For this exercise, we assume the properties of the functor Tor as outlined in (3.2.6).

(i) Let M be an abelian group (that is, a $\mathbb{Z}$-module.) Show that, for any positive integer a,

$$\mathrm{Tor}_1^\mathbb{Z}(M, \mathbb{Z}/\mathbb{Z}a) = \{m \in M \mid ma = 0\}$$

and that

$$\mathrm{Tor}_n^{\mathbb{Z}}(M,-) \equiv 0 \text{ for } n > 1.$$

(ii) Let $R = A[\epsilon]$ be the ring of dual numbers over a coefficient ring A and let M be a right R-module. After Exercise 3.2.1, there is a short exact sequence (of left or right R-modules)

$$0 \longrightarrow A \longrightarrow R \longrightarrow A \longrightarrow 0$$

in which the inclusion is multiplication by ϵ and the surjection is the natural one.
Show that $M \otimes_R A \cong M/M\epsilon$ and that $\mathrm{Tor}_1^R(M,A) \cong M_\epsilon/M\epsilon$ where $M_\epsilon = \{m \in M \mid m\epsilon = 0\}$. Deduce that $\mathrm{Tor}_1^R(A,A) \cong A$, as a group.
Show also that $\mathrm{Tor}_{n-1}^R(M,A) \cong \mathrm{Tor}_n^R(M,A)$ for $n > 1$.

(iii) Prove (3.2.10) using the long exact sequence.

3.2.8 Let

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

be any short exact sequence of left R-modules, with N'' flat, and let M be any right R-module.

(a) Choose a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F flat, and construct a 3×3 commutative diagram as in (3.2.10). By chasing the diagram, show that $M \otimes_R N'$ maps injectively into $M \otimes_R N$.

(b) From the case where N is free, observe that $\mathrm{Tor}_1^R(M,N'') = 0$.

(c) Also deduce from (a) that if N is flat, then so is N', a converse to (3.2.10).

3.3 CHANGE OF SCALARS

Now that the tensor product and the language of category theory are available to us, we can analyse in detail the relationships between various categories of modules that arise from homomorphisms between rings.

Given a ring homomorphism

$$f : R \longrightarrow S,$$

where R and S are arbitrary rings, we define two functors

$$f^{\#} : \mathcal{M}\mathrm{od}_S \longrightarrow \mathcal{M}\mathrm{od}_R$$

and

$$f_{\#} : \mathcal{M}\mathrm{od}_R \longrightarrow \mathcal{M}\mathrm{od}_S,$$

which are called respectively restriction and extension of scalars. These functors generalize the notions of restriction and extension of scalars between, for example, real and complex vector spaces (where f is the inclusion map of $\mathbb{R}$ in $\mathbb{C}$), to modules over arbitrary rings.

An important special case occurs when the ring homomorphism is an automorphism of R, which we prefer to write as α rather than f. In this situation, an R-module M can be 'twisted' to obtain a new R-module M^α (3.3.22), which turns out to be the restriction $\alpha^{\#}M$ (3.3.25). Twisted modules are encountered frequently in the theory of skew polynomial rings.

The functorial analysis of change of scalars through the tensor product was initiated in [Cartan & Eilenberg 1956], Ch. II §6. Our emphasis on the preservation of exact sequences and the twisting of module structures anticipates the requirements of K-theory.

As usual, we concentrate on right modules. It will be obvious that our definitions and results have analogues for left modules (see (3.3.21)). We do not always state these analogues separately. Note that we always write ring homomorphisms on the left of their arguments.

The reader is warned that there is considerable variation in the notation and terminology for restriction and extension. For example, [Quillen 1973] §4 (5), uses f_* where we use $f^{\#}$, and f^* for our $f_{\#}$. The variation stems from the convention that covariant behaviour is usually indicated by a subscript and contravariant by a superscript. In applications to geometry and topology, the interest is in the action of restriction and extension on categories of varieties, schemes or topological spaces rather than modules [Fulton & Lang 1985], II §1. However, the functors that link rings to these objects are themselves contravariant [Hartshorne 1997], I (3.8), II (2.3), with a resulting switch in the variance of restriction and extension. We mention the alternative terminologies from time to time.

3.3.1 Restriction

Let M be a right S-module and suppose that there is a ring homomorphism $f : R \to S$.

Given elements r of R and m of M, define

$$m \cdot r = m \cdot (fr).$$

It is clear that this rule gives M the structure of a right R-module, which we

denote by $f^{\#}M$. It is evident that a homomorphism

$$\alpha : M' \longrightarrow M$$

of right S-modules is also a homomorphism

$$f^{\#}\alpha : f^{\#}M' \longrightarrow f^{\#}M$$

of right R-modules and that

$$f^{\#} : \mathcal{M}\text{OD}_S \longrightarrow \mathcal{M}\text{OD}_R$$

is an additive covariant functor, which we call *restriction* or *restriction of scalars*

The terminology is most appropriate when f is an injection, because we are then 'restricting the scalars' in the literal sense. When f is surjective, the reader may meet the terms *pull-back* to describe $f^{\#}$ and *coinduced module* to describe $f^{\#}M$. Finally, in K-theory $f^{\#}$ gives rise to various homomorphisms of K-groups; many of these homomorphisms are called *transfer* maps.

Here are some elementary properties of the restriction functor.

3.3.2 Lemma

(i) $(id_R)^{\#}$ *is the identity functor* $\mathrm{Id}_R : \mathcal{M}\text{OD}_R \to \mathcal{M}\text{OD}_R$.

(ii) *Given ring homomorphisms* $f : R \to S$ *and* $g : S \to T$, *there is a natural isomorphism between the functors* $(gf)^{\#}$ *and* $f^{\#}g^{\#}$. □

If we put $(R)^{\#} = \mathcal{M}\text{OD}_R$, then $(-)^{\#}$ is a contravariant functor from the category $\mathcal{R}\text{ING}$ of all rings to the category $\mathcal{C}\text{AT}$ of all categories.

Next, we consider the behaviour of the restriction functor on various subcategories of $\mathcal{M}\text{OD}_S$. To save notation, we use the same symbol $f^{\#}$ to denote a number of functors arising from $f^{\#}$.

3.3.3 Proposition

(i) *The functor* $f^{\#}$ *induces a functor*

$$f^{\#} : \mathcal{P}\text{ROJ}_S \longrightarrow \mathcal{P}\text{ROJ}_R$$

if and only if $f^{\#}S$ *is a projective right* R*-module.*

(ii) *The functor* $f^{\#}$ *induces a functor*

$$f^{\#} : \mathcal{M}_S \longrightarrow \mathcal{M}_R$$

if and only if $f^{\#}S$ *is a finitely generated right* R*-module.*

(iii) *The functor $f^{\#}$ induces a functor*

$$f^{\#} : \mathcal{P}_S \longrightarrow \mathcal{P}_R$$

if and only if $f^{\#}S$ is a finitely generated projective right R-module.

(iv) *The functor $f^{\#}$ induces a functor*

$$f^{\#} : \mathcal{F}_S \longrightarrow \mathcal{F}_R$$

if and only if $f^{\#}S$ is a free right R-module of finite rank.

In each case, if the functor $f^{\#}$ exists, it is exact.

Proof

Since all the subcategories that occur are full subcategories of $\mathcal{M}od_R$ or $\mathcal{M}od_S$, we need only verify that $f^{\#}$ has the desired action on modules.

(i). Suppose first that $f^{\#}S$ is R-projective. If M is in $\mathcal{P}_{\mathcal{R}oj_S}$, then $M \oplus N \cong S^{\Lambda}$ for some module N and index set Λ ([BK: IRM] Theorem 2.5.8), and clearly $f^{\#}M \oplus f^{\#}N \cong (f^{\#}S)^{\Lambda}$ also, which shows that $f^{\#}M$ is projective ([BK: IRM] Theorem 2.5.5).

The converse is obvious since S is in $\mathcal{P}_{\mathcal{R}oj_S}$.

(ii) – (iv). Similar to (i); note that if an S-module M has a finite set of generators $\{m_1, \ldots, m_\ell\}$ and S has a finite set of generators $\{s_1, \ldots, s_k\}$ as a right R-module, then the set of products $\{m_i s_j\}$ is a finite set of generators of the R-module $f^{\#}M$.

The final assertion follows from the observation that restriction must preserve exact sequences of modules since it has no effect on the underlying abelian groups. Note also that all the categories mentioned are repletely G-exact subcategories of $\mathcal{M}od_R$ or $\mathcal{M}od_S$. □

3.3.4 Extension

Again, let $f : R \to S$ be a ring homomorphism. To define the extension functor $f_{\#} : \mathcal{M}od_R \to \mathcal{M}od_S$, we first consider S as a *left* R-module by the rule $r \cdot s = (fr)s$, so that S becomes an R-S-bimodule. (Strictly speaking, this amounts to replacing S by its image $f^{\#}(S)$ under the restriction functor for left modules – see (3.3.21).)

Then we put

$$f_{\#} = - \otimes_R S : \mathcal{M}od_R \longrightarrow \mathcal{M}od_S;$$

explicitly, for a right R-module M,

$$f_{\#}(M) = M \otimes_R S$$

and for a homomorphism

$$\alpha : M' \longrightarrow M$$

of right R-modules,

$$f_\#(\alpha) = \alpha \otimes id_S : f_\#(M') \longrightarrow f_\#(M).$$

By (3.1.7), $f_\#$ is an additive covariant functor from $\mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_S}$.

The functor $f_\#$ is called *extension of scalars* and the module $f_\#(M)$ the *extended module.* In particular, for a right ideal $\mathfrak{a}$ of R, $f_\#\mathfrak{a}$ is easily seen to be naturally isomorphic to the right ideal of S generated by the image $f(\mathfrak{a})$ of $\mathfrak{a}$; $f_\#\mathfrak{a}$ is commonly known as the *extension* or *extended ideal* of $\mathfrak{a}$.

As with restriction, the terminology is perhaps most appropriate when f is an injection. In special cases, alternative terms may be used. For example, when $S = R_\Sigma$ is a localization of R, $f_\#$ is also called *localization*, which we discuss in detail in Chapter 6. When H is a subgroup of a group G and $f : AH \to AG$ is the obvious inclusion of group rings (see Exercise 3.3.8), $f_\#$ is called *induction.* Here A can be any coefficient ring.

Here are some of the basic properties of the extension functor.

3.3.5 Lemma

(i) $(id_R)_\#$ *is the identity functor* $\mathrm{Id}_R : \mathcal{M}_{OD_R} \to \mathcal{M}_{OD_R}$.

(ii) *Given ring homomorphisms* $f : R \to S$ *and* $g : S \to T$, *then there is a natural isomorphism*

$$g_\# f_\# \simeq (gf)_\#.$$

Proof

The first assertion is obvious. The second follows from the fact that, for any R-module M, there is an associativity isomorphism from $M \otimes_R (S \otimes_S T)$ to $(M \otimes_R S) \otimes_S T$ (3.1.5), which is easily seen to be natural in M. □

3.3.6 Corollary

In (ii) *above, suppose that* S *is flat when viewed as a left* R*-module by restriction and that* T *is flat as a left* S*-module. Then* T *is flat as a left* R*-module.*

Proof

We appeal to (3.2.8). Since ${}_RS$ and ${}_ST$ are flat, $f_\#$ and $g_\#$ are exact functors. Thus $(gf)_\#$ is an exact functor and so ${}_RT$ is flat. □

If we write $f_\#(R) = \mathcal{M}_{OD_R}$ and put $g_\# f_\# = (gf)_\#$, then $(-)_\#$ is a covariant functor from $\mathcal{R}_{ING}$ to $\mathcal{C}_{AT}$.

As with restriction, we use the same symbol $f_\#$ for several functors arising from $f_\#$.

3.3.7 Proposition

(i) *For any index set* Λ, $f_\#(R^\Lambda) \cong S^\Lambda$.

(ii) $f_\#$ *induces additive functors*

$$f_\# : \mathcal{P}_{ROJ_R} \longrightarrow \mathcal{P}_{ROJ_S},$$

$$f_\# : \mathcal{M}_R \longrightarrow \mathcal{M}_S,$$

$$f_\# : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

and

$$f_\# : \mathcal{F}_R \longrightarrow \mathcal{F}_S.$$

Proof

The functor $f_\#(-) = - \otimes_R S$ is additive (3.1.7) and has $f_\#(R) \cong S$ by (ii) of (3.1.4), so (i) follows from the left-handed version of (3.1.8). Explicitly, the isomorphism is defined by the map

$$(r_\lambda) \otimes s \longmapsto (f(r_\lambda)s),$$

which has inverse

$$(s_\lambda) \longmapsto \sum (e_\lambda \otimes s_\lambda),$$

where e_λ has entry 1 in the λth place and zero elsewhere (recall that only a finite number of entries of either R^Λ or S^Λ can be nonzero).

The remaining assertions in (ii) follow from (3.1.17). □

3.3.8 Exactness

The functors

$$f_\# : \mathcal{P}_{ROJ_R} \longrightarrow \mathcal{P}_{ROJ_S},$$

$$f_\# : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

and

$$f_\# : \mathcal{F}_R \longrightarrow \mathcal{F}_S$$

are all exact, since in each case the domain is a split G-exact category. In general,

$$f_{\#} : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

will not be exact, which can be seen by taking $f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ to be the standard surjection and considering the effect of $f_{\#}$ on the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as in (3.2.1).

3.3.9 An identification

Suppose that R is a subring of S and that $f : R \to S$ is the inclusion. Given a right R-module M, it is tempting to view the extended module $f_{\#}M$ simply as the right S-module MS generated by the members of M. This temptation must usually be resisted, since there may be no S-module that contains the elements of M. For example, take R to be $\mathbb{Z}$, S to be $\mathbb{Q}$ and M to be $\mathbb{Z}/2\mathbb{Z}$; then $f_{\#}(M) = 0$.

However, it is legitimate to identify $f_{\#}M$ with MS when M is a flat R-module, since f then induces an injection

$$M \xrightarrow{\cong} M \otimes_R R \xrightarrow{id \otimes f} M \otimes_R S;$$

more properly, we identify each $m \in M$ with its image $m \otimes 1 \in M \otimes_R S$, so that

$$M \otimes_R S = \{\sum_{i=1}^{k} m_i s_i \mid m_i \in M, s_i \in S, k \geq 1\} = MS.$$

Likewise, if N is a flat left R-module, we can write $f_{\#}N = S \otimes_R N$ as SN.

3.3.10 The quotient functor

When the ring homomorphism $f : R \to S$ is a surjection, $f_{\#}$ is called the *quotient functor*, which we now describe in more detail.

Let $\mathfrak{a}$ be the kernel of f. Given an R-module M, write

$$f_Q(M) = M/M\mathfrak{a},$$

where $M\mathfrak{a}$ is the submodule of M generated by all products mx with $x \in \mathfrak{a}$ and $m \in M$.

It is clear that $f_Q(M)$ is an S-module under the rule

$$\overline{m} \cdot fr = \overline{mr},$$

and that an R-module homomorphism $\alpha : M \to M''$ defines an S-module homomorphism $f_Q(\alpha) : f_Q(M) \to f_Q(M'')$ by the rule

$$f_Q(\alpha)(\overline{m}) = (\overline{\alpha m}).$$

Thus we have an additive covariant functor

$$f_Q : \mathcal{M}od_R \longrightarrow \mathcal{M}od_S.$$

3.3.11 Proposition

The functors $f_\#$ and f_Q are naturally isomorphic.

Proof

Let M be a right R-module. By (3.2.3), there is an exact sequence

$$M \otimes_R \mathfrak{a} \longrightarrow M \otimes_R R \longrightarrow M \otimes_R S \longrightarrow 0.$$

By (3.1.4), the obvious map $\eta_M : m \otimes r \mapsto mr$ from $M \otimes_R R$ to M is an isomorphism of right R-modules, and this isomorphism clearly maps the image of $M \otimes_R \mathfrak{a}$ in $M \otimes_R R$ onto $M\mathfrak{a}$ in M.

Thus η_M induces an isomorphism between $f_\# M$ and $f_Q M$, which can be verified directly or, more eruditely, by appealing to the Five Lemma (2.3.23). Since the construction is natural in M, we have a natural isomorphism between $f_\#$ and f_Q. □

The relationship between the extension functor and the restriction functor in general is given by the following result.

3.3.12 Proposition

There are natural transformations

(i)
$$\eta : f_\# f^\# \longrightarrow \mathrm{Id}_S$$

(ii) *and*
$$\nu : \mathrm{Id}_R \longrightarrow f^\# f_\#,$$

where Id_S and Id_R are the identity functors on $\mathcal{M}od_S$ and $\mathcal{M}od_R$ respectively.

(iii) *For any right S-module N, we have*

$$\eta_N \nu_{f^\# N} = id_N.$$

Proof

(i) Let N be a right S-module. Then $f_\# f^\# N = N \otimes_R S$ and we define

$$\eta_N : N \otimes_R S \longrightarrow N$$

by

$$\eta_N(n \otimes s) = ns;$$

as in (3.1.4), this is a well-defined homomorphism which is clearly natural in N.

(ii) On the other hand, given the right R-module M, we define the homomorphism

$$\nu_M : M \longrightarrow M \otimes_R S$$

by

$$\nu_M m = m \otimes 1,$$

which is again natural.

(iii) This is immediate from the above formulas. □

There are also relations between these functors and tensor products, when the rings act on the appropriate sides of the modules. The first is called a *projection* (or *reciprocity*) *formula*. (See Exercise 3.3.12 for the usage of 'reciprocity' in the representation theory of groups.) Its proof follows easily from the naturality of the various constructions involved, using (3.1.11) and Exercise 3.1.4.

3.3.13 Proposition

For a ring homomorphism $f : R \to S$ and modules M in $\mathcal{M}\text{OD}_R$ and N in ${}_S\mathcal{B}\text{IMOD}_S$ there is a natural isomorphism

$$f^\#(f_\# M \otimes_S N) \cong M \otimes_R f^\# N$$

of right R-modules.

In particular, when $N = S$,

$$f^\# f_\# M \cong M \otimes_R f^\# S.$$ □

The next result follows from (3.2.15), again using the naturality results in (3.1.11) and Exercise 3.1.4.

3.3.14 Proposition

For a homomorphism $f : R \to S$ of commutative rings and modules M in $\mathcal{M}\text{OD}_R$ and N in $\mathcal{B}\text{AL}_R$, there is a natural isomorphism

$$(f_\# M) \otimes_S (f_\# N) \cong f_\#(M \otimes_R N)$$

of right S-modules. □

A further relationship between restriction and extension can be most neatly expressed in the language of category theory.

3.3.15 Proposition

The extension functor is left adjoint to the restriction functor. In other words, given a ring homomorphism $f : R \to S$ and modules M in $\mathcal{M}\text{OD}_R$ and N in $\mathcal{M}\text{OD}_S$, then there is a natural isomorphism

$$\text{Hom}_S(f_\# M, N) \cong \text{Hom}_R(M, f^\# N).$$

Proof

Recall from the Adjointness Theorem (3.1.19) that for arbitrary rings A, B and C and bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated, there is a natural isomorphism

$$\eta : \text{Hom}_{A-C}(L \otimes_B M, N) \longrightarrow \text{Hom}_{A-B}(L, \text{Hom}(M_C, N_C))$$

of trifunctors from ${}_A\mathcal{B}\text{IMOD}_B \times {}_B\mathcal{B}\text{IMOD}_C \times {}_A\mathcal{B}\text{IMOD}_C$ to $\mathcal{A}\text{B}$.

Take $A = \mathbb{Z}$, $B = R$ and $C = S$, and let M be S viewed as a (fixed) R-S-bimodule in the usual way. We then obtain a natural isomorphism

$$\eta : \text{Hom}_S(L \otimes_R S, N) \longrightarrow \text{Hom}_R(L, \text{Hom}(S_S, N_S))$$

of bifunctors from $\mathcal{M}\text{OD}_R \times \mathcal{M}\text{OD}_S$ to $\mathcal{A}\text{B}$.

The functor $N_S \mapsto \text{Hom}(S_S, N_S)$ is naturally isomorphic to the identity functor on $\mathcal{M}\text{OD}_S$ (Exercise 2.1.3), so the result follows on noting that $f_\# L = L \otimes_R S$ and changing notation. □

***3.3.16* $\mathbb{R}$ *and* $\mathbb{C}$**

To illustrate what has gone before, we consider the special case in which $f : \mathbb{R} \to \mathbb{C}$ is the usual inclusion of the field of real numbers in the complex numbers.

Let V be a real vector space, say of finite dimension n, with basis

$$\{e_1, \ldots, e_n\}.$$

Then $f_\# V$ is a complex vector space of the same dimension, with basis

$$\{e_1 \otimes 1, \ldots, e_n \otimes 1\},$$

and $f^\# f_\# V$ is the real space of dimension $2n$ with basis

$$\{e_1 \otimes 1, e_1 \otimes i, \ldots, e_n \otimes 1, e_n \otimes i\},$$

where $i = \sqrt{-1}$.

In the other direction, a complex vector space W of dimension n becomes a real space $f^\# W$ of dimension $2n$ and then a complex space $f_\# f^\# W$ of dimension $2n$. These matters are developed further in [Adams 1969], Chapter 3.

The relationship generalizes in the following way.

3.3.17 Proposition

Suppose that a ring homomorphism $f : R \to S$ is an injection which gives S the structure of a free right R-module R^k of finite rank, with basis $\{s_1, \ldots, s_k\}$.

(i) *If $M \cong R^h$ is a free right R-module, with basis $\{m_1, \ldots, m_h\}$, then $f^\# f_\# M$ is a free right R-module of rank hk, with basis $\{m_i \otimes s_j\}$.*

(ii) *If $N \cong S^n$, then $f^\# N \cong R^{nk}$ and $f_\# f^\# N \cong S^{nk}$.* □

3.3.18 Skew fields unbalanced

The structure of S as a left R-module is irrelevant in the above result. It is worth remarking at this point that the left R-module structure of S may well differ from the right module structure. For example, [Schofield 1985] gives examples of inclusions of skew fields $\mathcal{D} \to \mathcal{D}'$ such that the left and right dimensions of $\mathcal{D}'$ over $\mathcal{D}$ take any preassigned pair of values, provided neither is 1.

Next, we consider the relationship between the functors which arise when we have two ring homomorphisms f and g from R to S.

3.3.19 Theorem

Let f and $g : R \to S$ be ring homomorphisms. Then the following statements are equivalent.

(i) *There is an element λ of S such that*

$$f(r)\lambda = \lambda g(r)$$

for all elements r of R.

(ii) *There is a natural transformation of restriction functors*

$$\eta : g^{\#} \longrightarrow f^{\#}.$$

(iii) *There is a natural transformation of extension functors*

$$\nu : g_{\#} \longrightarrow f_{\#}.$$

Furthermore, λ is a unit of R if and only if either one (and hence both) of η and ν is an isomorphism.

Proof

(ii) $\Rightarrow$ (i). We know that there is an R-module homomorphism

$$\eta_S : g^{\#}S \longrightarrow f^{\#}S.$$

Write $\lambda = \eta_S(1_S)$.

Now, for any element n of an S-module N, there is a homomorphism

$$\rho(n) : S \longrightarrow N$$

given by

$$\rho(n)s = ns,$$

and because η is a natural transformation, there is a commutative diagram

$$\begin{array}{ccc} g^{\#}S & \xrightarrow{\eta_S} & f^{\#}S \\ {\scriptstyle g^{\#}\rho(n)}\downarrow & & \downarrow{\scriptstyle f^{\#}\rho(n)} \\ g^{\#}N & \xrightarrow{\eta_S} & f^{\#}N \end{array}$$

which means that $\eta_N(\rho(n)s) = \rho(n)(\eta_S(s))$.

With $s = 1_S$, we see that

$$\eta_N n = n\lambda, \tag{3.1}$$

so that the natural transformation η is completely determined by the element λ of S.

Now, the fact that η_S respects the R-module structures on S means that $\eta_S(1_S \cdot r) = \eta_S(1_S) \cdot r$, that is,

$$g(r)\lambda = \lambda f(r).$$

To establish the converse implication (i) $\Rightarrow$ (ii), it is enough to verify that, for every right R-module N, the formula

$$\eta_N^{\lambda} n = n\lambda \text{ for } n \text{ in } N,$$

defines a natural transformation $\eta^\lambda : g^\# \to f^\#$ when λ satisfies the given condition (i). This is left to the reader, as is the equivalence of (i) and (iii).

For the final assertion, recall from Equation 3.1 above that any η as in (ii) is $\eta = \eta^\lambda$ for some $\lambda \in S$. The claim now follows from the observations that $\eta^\mu \eta^\lambda = \eta^{\lambda\mu}$ and $\eta^{(1_S)} = \mathrm{Id}_S$. □

We note an important consequence of the theorem. Recall that a ring endomorphism α of R is an inner automorphism if $\alpha r = \lambda r \lambda^{-1}$ for some unit λ of R.

3.3.20 Corollary

For an endomorphism α of a ring R, the following statements are equivalent.

(i) *α is an inner automorphism of R.*
(ii) *$\alpha^\#$ is naturally isomorphic to Id_R.*
(iii) *$\alpha_\#$ is naturally isomorphic to Id_R.* □

3.3.21 The definitions for left modules

For the convenience of the reader, we quickly review the definitions of restriction and extension for left modules.

Given a ring homomorphism $f : R \to S$ and a left S-module N, the restriction of N is the left R-module $f^\# N$, with action given by the rule

$$r \cdot n = (fr) \cdot n.$$

With right-handed restriction, the ring S is itself an S-R-bimodule, and the extended module $f_\# N$ is defined to be the left S-module $S \otimes_R N$.

It is clear that all our results for the right-handed restriction and extension functors have left-handed versions.

3.3.22 The twisting of modules

When α is an automorphism of a ring R, we may view the extension and restriction functors in a different light. Given a right R-module M, we define the *twisted module* M^α in $\mathcal{M}_{OD_R}$ as follows. Elements of M^α are symbols m^α, addition is given by

$$(m+n)^\alpha = m^\alpha + n^\alpha,$$

and the action of R is given by

$$m^\alpha \cdot r = (m \cdot \alpha r)^\alpha.$$

Similarly, a left R-module N can be twisted to obtain the left R-module ${}^{\alpha}N$, in which $r \cdot {}^{\alpha}n = {}^{\alpha}(\alpha r \cdot n)$.

Applying these operations to R itself, we obtain new R-R-bimodules R^{α} and ${}^{\alpha}R$.

3.3.23 Lemma

(i) *The map $\zeta : x \mapsto (\alpha x)^{\alpha}$ from R to R^{α} is an isomorphism of right R-modules.*

(ii) *The map $\xi : x \mapsto {}^{\alpha}(\alpha x)$ from R to ${}^{\alpha}R$ is an isomorphism of left R-modules.*

Proof

For r in R, $\zeta(xr) = (\alpha x \cdot \alpha r)^{\alpha}$, which is $(\alpha x)^{\alpha} \cdot r$ by the definition. Thus $\zeta(xr) = \zeta(x) \cdot r$. Clearly, ζ is an additive bijection, so (i) is proven. The argument for (ii) is much the same. □

The above result generalizes to a description of the effect of twisting on projective modules. We use some results on the relationship between projective modules and idempotent matrices that are considered in detail in [BK: IRM] (2.5.9)ff.

3.3.24 Lemma

Let $\alpha : R \to R$ be a ring automorphism, and let α_k denote the induced automorphism of the matrix ring $M_k(R)$ for each $k \geq 1$ (thus $\alpha_1 = \alpha$). Suppose that P is a finitely generated projective right R-module. Then the following hold.

(i) *$P \cong \eta R^k$ for some $k \times k$ idempotent matrix η over R.*

(ii) *There is a right R-module isomorphism $P^{\alpha} \cong \alpha_k^{-1}(\eta)R^k$.*

(iii) *P^{α} is also a finitely generated projective right R-module.*

Proof

(i) There is a surjective R-module homomorphism $\pi : R^k \to P$ for some k which is split by an R-module homomorphism $\sigma : P \to R^k$. Then $\sigma\pi$ is an idempotent endomorphism of R^k which can be represented by a $k \times k$ idempotent matrix η (1.3.4), and clearly $P \cong \eta R^k$.

(ii) Let $\alpha_{\oplus k} = \alpha \oplus \cdots \oplus \alpha : R^k \to R^k$ be the R-module homomorphism induced by α, and define

$$\theta : P^{\alpha} \longrightarrow \alpha_k^{-1}(\eta)R^k$$

by

$$\theta((\eta x)^\alpha) = \alpha_{\oplus k}^{-1}(\eta x) = \alpha_k^{-1}(\eta)\alpha_{\oplus k}^{-1}(x).$$

It is clear that θ is a well-defined additive bijection. Moreover, for any r in R, we have

$$\begin{aligned}\theta((\eta x)^\alpha \cdot r) &= \theta((\eta x \cdot \alpha(r))^\alpha) \\ &= \alpha_{\oplus k}^{-1}(\eta x \cdot \alpha(r)) \\ &= \alpha_{\oplus k}^{-1}(\eta x) \cdot r \\ &= \theta((\eta x)^\alpha) \cdot r,\end{aligned}$$

showing θ to be an R-module homomorphism as desired.

(iii) Since $\alpha_k^{-1}(\eta)$ is again idempotent, $\alpha_k^{-1}(\eta)R^k$ is a direct summand of R^k and hence projective. □

The next result shows that a twisted module can be viewed equally as being a restricted or an extended module.

3.3.25 Proposition

(a) *Let M be a right R-module. Then there are isomorphisms, natural in M, between the following right R-modules:*

  (i) *the twisted module M^α,*
  (ii) *the restricted module $\alpha^\# M$, and*
  (iii) *the extended module $(\alpha^{-1})_\# M$.*

(b) *Let N be a left R-module. Then there are isomorphisms, natural in N, between the following left R-modules:*

  (i) *the twisted module ${}^\alpha N$,*
  (ii) *the restricted module $\alpha^\# N$, and*
  (iii) *the extended module $(\alpha^{-1})_\# N$.*

Proof

We give the argument for (a) only. We first consider the relation between the twisted module and the restricted module.

For clarity, we write a typical element of $\alpha^\# M$ as ${}^\# m$, so that the R-action on $\alpha^\# M$ is given by ${}^\# m \cdot r = {}^\#(m \cdot \alpha r)$. Then the function $\alpha^\# M \to M^\alpha$ sending ${}^\# m$ to m^α is clearly an isomorphism.

Next, to compare M^α with the extended module $(\alpha^{-1})_\# M$, recall that the extended module is generated by elements of the form $m \otimes t$, where $ms \otimes t = m \otimes (\alpha^{-1}s)t$. Because α is an automorphism, it follows that each element

of $(\alpha^{-1})_{\#}M$ may in fact be written in the form $m \otimes 1$. Then the function $M^{\alpha} \to (\alpha^{-1})_{\#}M$ given by $m^{\alpha} \mapsto m \otimes 1$ is evidently a bijection. The R-actions are respectively

$$m^{\alpha} \cdot r = (m \cdot \alpha r)^{\alpha} \text{ on } M^{\alpha},$$

and

$$(m \otimes 1) \cdot r = m \otimes r = m \otimes \alpha^{-1}(\alpha r) = m \cdot \alpha r \otimes 1 \text{ on } (\alpha^{-1})_{\#}M,$$

so that the map is also a homomorphism. □

3.3.26 Group rings

The representation theory of groups plays an important role in K-theory, both as a tool for use in constructing the theory, and as a source of problems within the theory. We therefore give a brief summary of the terminology and some special constructions which arise when we apply 'change of rings' to modules over group rings.

Let A be a commutative coefficient ring and let G be a group, usually but not necessarily finite. The group ring AG is the free (left) A-module generated by the set G, with multiplication inherited from G. Thus an element $x \in AG$ looks like $x = \sum_{g \in G} x_g g$ with only a finite set of nonzero coefficients x_g, and multiplication is given by

$$\Big(\sum_{g \in G} x_g g\Big)\Big(\sum_{h \in G} y_h h\Big) = \sum_{gh \in G} x_g y_h gh.$$

In particular, AG is a balanced A-module, and the identity for the multiplication is $1_A 1_G$.

Any right AG-module will be a right A-module, which we always take to be a balanced A-module.

A group homomorphism $f : H \to G$ gives an evident ring homomorphism $f : AH \to AG$ (of the same name). In this context, the extension functor $f_{\#} : M_{AH} \to (M \otimes_{AH} AG)_{AG}$ is called the *induction functor*, $M \otimes_{AH} AG$ being the *induced module*. The terminology for restriction is unchanged. Both the terms induction and restriction are most appropriate when the group homomorphism f is an inclusion, but their usage is extended to general homomorphisms.

Given a right AH-module M, the induced AG-module is more often written M^G, the ring A being understood. Similarly, the restriction of a right AG-module N is written N_H.

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defines a natural transformation $\eta^\lambda : g^\# \to f^\#$ when λ satisfies the given condition (i). This is left to the reader, as is the equivalence of (i) and (iii).

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For an endomorphism α of a ring R, the following statements are equivalent.

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3.3.21 The definitions for left modules

For the convenience of the reader, we quickly review the definitions of restriction and extension for left modules.

Given a ring homomorphism $f : R \to S$ and a left S-module N, the restriction of N is the left R-module $f^\# N$, with action given by the rule

$$r \cdot n = (fr) \cdot n.$$

With right-handed restriction, the ring S is itself an S-R-bimodule, and the extended module $f_\# N$ is defined to be the left S-module $S \otimes_R N$.

It is clear that all our results for the right-handed restriction and extension functors have left-handed versions.

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(ii) $\Rightarrow$ (i). We know that there is an R-module homomorphism

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Write $\lambda = \eta_S(1_S)$.

Now, for any element n of an S-module N, there is a homomorphism

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and because η is a natural transformation, there is a commutative diagram

$$\begin{array}{ccc} g^{\#}S & \xrightarrow{\eta_S} & f^{\#}S \\ {\scriptstyle g^{\#}\rho(n)}\downarrow & & \downarrow{\scriptstyle f^{\#}\rho(n)} \\ g^{\#}N & \xrightarrow{\eta_S} & f^{\#}N \end{array}$$

which means that $\eta_N(\rho(n)s) = \rho(n)(\eta_S(s))$.

With $s = 1_S$, we see that

$$\eta_N n = n\lambda, \tag{3.1}$$

so that the natural transformation η is completely determined by the element λ of S.

Now, the fact that η_S respects the R-module structures on S means that $\eta_S(1_S \cdot r) = \eta_S(1_S) \cdot r$, that is,

$$g(r)\lambda = \lambda f(r).$$

To establish the converse implication (i) $\Rightarrow$ (ii), it is enough to verify that, for every right R-module N, the formula

$$\eta_N^{\lambda} n = n\lambda \text{ for } n \text{ in } N,$$

Then $f_{\#}V$ is a complex vector space of the same dimension, with basis

$$\{e_1 \otimes 1, \ldots, e_n \otimes 1\},$$

and $f^{\#}f_{\#}V$ is the real space of dimension $2n$ with basis

$$\{e_1 \otimes 1, e_1 \otimes i, \ldots, e_n \otimes 1, e_n \otimes i\},$$

where $i = \sqrt{-1}$.

In the other direction, a complex vector space W of dimension n becomes a real space $f^{\#}W$ of dimension $2n$ and then a complex space $f_{\#}f^{\#}W$ of dimension $2n$. These matters are developed further in [Adams 1969], Chapter 3.

The relationship generalizes in the following way.

3.3.17 Proposition

Suppose that a ring homomorphism $f : R \to S$ is an injection which gives S the structure of a free right R-module R^k of finite rank, with basis $\{s_1, \ldots, s_k\}$.

(i) *If $M \cong R^h$ is a free right R-module, with basis $\{m_1, \ldots, m_h\}$, then $f^{\#}f_{\#}M$ is a free right R-module of rank hk, with basis $\{m_i \otimes s_j\}$.*

(ii) *If $N \cong S^n$, then $f^{\#}N \cong R^{nk}$ and $f_{\#}f^{\#}N \cong S^{nk}$.* □

3.3.18 Skew fields unbalanced

The structure of S as a left R-module is irrelevant in the above result. It is worth remarking at this point that the left R-module structure of S may well differ from the right module structure. For example, [Schofield 1985] gives examples of inclusions of skew fields $\mathcal{D} \to \mathcal{D}'$ such that the left and right dimensions of $\mathcal{D}'$ over $\mathcal{D}$ take any preassigned pair of values, provided neither is 1.

Next, we consider the relationship between the functors which arise when we have two ring homomorphisms f and g from R to S.

3.3.19 Theorem

Let f and $g : R \to S$ be ring homomorphisms. Then the following statements are equivalent.

(i) *There is an element λ of S such that*

$$f(r)\lambda = \lambda g(r)$$

for all elements r of R.

3.3.14 Proposition

For a homomorphism $f : R \to S$ of commutative rings and modules M in $\mathcal{M}\text{od}_R$ and N in $\mathcal{B}\text{al}_R$, there is a natural isomorphism

$$(f_\# M) \otimes_S (f_\# N) \cong f_\#(M \otimes_R N)$$

of right S-modules. □

A further relationship between restriction and extension can be most neatly expressed in the language of category theory.

3.3.15 Proposition

The extension functor is left adjoint to the restriction functor. In other words, given a ring homomorphism $f : R \to S$ and modules M in $\mathcal{M}\text{od}_R$ and N in $\mathcal{M}\text{od}_S$, then there is a natural isomorphism

$$\text{Hom}_S(f_\# M, N) \cong \text{Hom}_R(M, f^\# N).$$

Proof

Recall from the Adjointness Theorem (3.1.19) that for arbitrary rings A, B and C and bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated, there is a natural isomorphism

$$\eta : \text{Hom}_{A\text{–}C}(L \otimes_B M, N) \longrightarrow \text{Hom}_{A\text{–}B}(L, \text{Hom}(M_C, N_C))$$

of trifunctors from ${}_A\mathcal{B}\text{imod}_B \times {}_B\mathcal{B}\text{imod}_C \times {}_A\mathcal{B}\text{imod}_C$ to $\mathcal{A}\text{b}$.

Take $A = \mathbb{Z}$, $B = R$ and $C = S$, and let M be S viewed as a (fixed) R-S-bimodule in the usual way. We then obtain a natural isomorphism

$$\eta : \text{Hom}_S(L \otimes_R S, N) \longrightarrow \text{Hom}_R(L, \text{Hom}(S_S, N_S))$$

of bifunctors from $\mathcal{M}\text{od}_R \times \mathcal{M}\text{od}_S$ to $\mathcal{A}\text{b}$.

The functor $N_S \mapsto \text{Hom}(S_S, N_S)$ is naturally isomorphic to the identity functor on $\mathcal{M}\text{od}_S$ (Exercise 2.1.3), so the result follows on noting that $f_\# L = L \otimes_R S$ and changing notation. □

***3.3.16* $\mathbb{R}$ *and* $\mathbb{C}$**

To illustrate what has gone before, we consider the special case in which $f : \mathbb{R} \to \mathbb{C}$ is the usual inclusion of the field of real numbers in the complex numbers.

Let V be a real vector space, say of finite dimension n, with basis

$$\{e_1, \ldots, e_n\}.$$

Proof

(i) Let N be a right S-module. Then $f_\# f^\# N = N \otimes_R S$ and we define

$$\eta_N : N \otimes_R S \longrightarrow N$$

by

$$\eta_N(n \otimes s) = ns;$$

as in (3.1.4), this is a well-defined homomorphism which is clearly natural in N.

(ii) On the other hand, given the right R-module M, we define the homomorphism

$$\nu_M : M \longrightarrow M \otimes_R S$$

by

$$\nu_M m = m \otimes 1,$$

which is again natural.

(iii) This is immediate from the above formulas. □

There are also relations between these functors and tensor products, when the rings act on the appropriate sides of the modules. The first is called a *projection* (or *reciprocity*) *formula*. (See Exercise 3.3.12 for the usage of 'reciprocity' in the representation theory of groups.) Its proof follows easily from the naturality of the various constructions involved, using (3.1.11) and Exercise 3.1.4.

3.3.13 Proposition

For a ring homomorphism $f : R \to S$ and modules M in $\mathcal{M}_{\mathcal{OD}_R}$ and N in ${}_S\mathcal{B}_{\mathcal{IMOD}_S}$ there is a natural isomorphism

$$f^\#(f_\# M \otimes_S N) \cong M \otimes_R f^\# N$$

of right R-modules.

In particular, when $N = S$,

$$f^\# f_\# M \cong M \otimes_R f^\# S.$$ □

The next result follows from (3.2.15), again using the naturality results in (3.1.11) and Exercise 3.1.4.

and that an R-module homomorphism $\alpha : M \to M''$ defines an S-module homomorphism $f_Q(\alpha) : f_Q(M) \to f_Q(M'')$ by the rule

$$f_Q(\alpha)(\overline{m}) = (\overline{\alpha m}).$$

Thus we have an additive covariant functor

$$f_Q : \mathcal{M}\text{od}_R \longrightarrow \mathcal{M}\text{od}_S.$$

3.3.11 Proposition

The functors $f_\#$ and f_Q are naturally isomorphic.

Proof

Let M be a right R-module. By (3.2.3), there is an exact sequence

$$M \otimes_R \mathfrak{a} \longrightarrow M \otimes_R R \longrightarrow M \otimes_R S \longrightarrow 0.$$

By (3.1.4), the obvious map $\eta_M : m \otimes r \mapsto mr$ from $M \otimes_R R$ to M is an isomorphism of right R-modules, and this isomorphism clearly maps the image of $M \otimes_R \mathfrak{a}$ in $M \otimes_R R$ onto $M\mathfrak{a}$ in M.

Thus η_M induces an isomorphism between $f_\# M$ and $f_Q M$, which can be verified directly or, more eruditely, by appealing to the Five Lemma (2.3.23). Since the construction is natural in M, we have a natural isomorphism between $f_\#$ and f_Q. □

The relationship between the extension functor and the restriction functor in general is given by the following result.

3.3.12 Proposition

There are natural transformations

(i)
$$\eta : f_\# f^\# \longrightarrow \mathrm{Id}_S$$

(ii) *and*
$$\nu : \mathrm{Id}_R \longrightarrow f^\# f_\#,$$

where Id_S and Id_R are the identity functors on $\mathcal{M}\text{od}_S$ and $\mathcal{M}\text{od}_R$ respectively.

(iii) *For any right S-module N, we have*

$$\eta_N \nu_{f^\# N} = id_N.$$

are all exact, since in each case the domain is a split G-exact category. In general,

$$f_{\#} : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

will not be exact, which can be seen by taking $f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ to be the standard surjection and considering the effect of $f_{\#}$ on the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as in (3.2.1).

3.3.9 An identification

Suppose that R is a subring of S and that $f : R \to S$ is the inclusion. Given a right R-module M, it is tempting to view the extended module $f_{\#}M$ simply as the right S-module MS generated by the members of M. This temptation must usually be resisted, since there may be no S-module that contains the elements of M. For example, take R to be $\mathbb{Z}$, S to be $\mathbb{Q}$ and M to be $\mathbb{Z}/2\mathbb{Z}$; then $f_{\#}(M) = 0$.

However, it is legitimate to identify $f_{\#}M$ with MS when M is a flat R-module, since f then induces an injection

$$M \xrightarrow{\cong} M \otimes_R R \xrightarrow{id \otimes f} M \otimes_R S;$$

more properly, we identify each $m \in M$ with its image $m \otimes 1 \in M \otimes_R S$, so that

$$M \otimes_R S = \{\sum_{i=1}^{k} m_i s_i \mid m_i \in M, s_i \in S, k \geq 1\} = MS.$$

Likewise, if N is a flat left R-module, we can write $f_{\#}N = S \otimes_R N$ as SN.

3.3.10 The quotient functor

When the ring homomorphism $f : R \to S$ is a surjection, $f_{\#}$ is called the *quotient functor*, which we now describe in more detail.

Let $\mathfrak{a}$ be the kernel of f. Given an R-module M, write

$$f_Q(M) = M/M\mathfrak{a},$$

where $M\mathfrak{a}$ is the submodule of M generated by all products mx with $x \in \mathfrak{a}$ and $m \in M$.

It is clear that $f_Q(M)$ is an S-module under the rule

$$\overline{m} \cdot fr = \overline{mr},$$

If we write $f_\#(R) = \mathcal{M}_{OD_R}$ and put $g_\# f_\# = (gf)_\#$, then $(-)_\#$ is a covariant functor from $\mathcal{R}_{ING}$ to $\mathcal{C}_{AT}$.

As with restriction, we use the same symbol $f_\#$ for several functors arising from $f_\#$.

3.3.7 Proposition

(i) *For any index set* Λ, $f_\#(R^\Lambda) \cong S^\Lambda$.

(ii) $f_\#$ *induces additive functors*

$$f_\# : \mathcal{P}_{ROJ_R} \longrightarrow \mathcal{P}_{ROJ_S},$$

$$f_\# : \mathcal{M}_R \longrightarrow \mathcal{M}_S,$$

$$f_\# : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

and

$$f_\# : \mathcal{F}_R \longrightarrow \mathcal{F}_S.$$

Proof

The functor $f_\#(-) = - \otimes_R S$ is additive (3.1.7) and has $f_\#(R) \cong S$ by (ii) of (3.1.4), so (i) follows from the left-handed version of (3.1.8). Explicitly, the isomorphism is defined by the map

$$(r_\lambda) \otimes s \longmapsto (f(r_\lambda)s),$$

which has inverse

$$(s_\lambda) \longmapsto \sum (e_\lambda \otimes s_\lambda),$$

where e_λ has entry 1 in the λth place and zero elsewhere (recall that only a finite number of entries of either R^Λ or S^Λ can be nonzero).

The remaining assertions in (ii) follow from (3.1.17). □

3.3.8 Exactness

The functors

$$f_\# : \mathcal{P}_{ROJ_R} \longrightarrow \mathcal{P}_{ROJ_S},$$

$$f_\# : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

and

$$f_\# : \mathcal{F}_R \longrightarrow \mathcal{F}_S$$

and for a homomorphism

$$\alpha : M' \longrightarrow M$$

of right R-modules,

$$f_\#(\alpha) = \alpha \otimes id_S : f_\#(M') \longrightarrow f_\#(M).$$

By (3.1.7), $f_\#$ is an additive covariant functor from $\mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_S}$.

The functor $f_\#$ is called *extension of scalars* and the module $f_\#(M)$ the *extended module.* In particular, for a right ideal $\mathfrak{a}$ of R, $f_\#\mathfrak{a}$ is easily seen to be naturally isomorphic to the right ideal of S generated by the image $f(\mathfrak{a})$ of $\mathfrak{a}$; $f_\#\mathfrak{a}$ is commonly known as the *extension* or *extended ideal* of $\mathfrak{a}$.

As with restriction, the terminology is perhaps most appropriate when f is an injection. In special cases, alternative terms may be used. For example, when $S = R_\Sigma$ is a localization of R, $f_\#$ is also called *localization*, which we discuss in detail in Chapter 6. When H is a subgroup of a group G and $f : AH \to AG$ is the obvious inclusion of group rings (see Exercise 3.3.8), $f_\#$ is called *induction.* Here A can be any coefficient ring.

Here are some of the basic properties of the extension functor.

3.3.5 Lemma

(i) $(id_R)_\#$ *is the identity functor* $\mathrm{Id}_R : \mathcal{M}_{OD_R} \to \mathcal{M}_{OD_R}$.

(ii) *Given ring homomorphisms* $f : R \to S$ *and* $g : S \to T$, *then there is a natural isomorphism*

$$g_\# f_\# \simeq (gf)_\#.$$

Proof

The first assertion is obvious. The second follows from the fact that, for any R-module M, there is an associativity isomorphism from $M \otimes_R (S \otimes_S T)$ to $(M \otimes_R S) \otimes_S T$ (3.1.5), which is easily seen to be natural in M. □

3.3.6 Corollary

In (ii) *above, suppose that* S *is flat when viewed as a left* R*-module by restriction and that* T *is flat as a left* S*-module. Then* T *is flat as a left* R*-module.*

Proof

We appeal to (3.2.8). Since ${}_RS$ and ${}_ST$ are flat, $f_\#$ and $g_\#$ are exact functors. Thus $(gf)_\#$ is an exact functor and so ${}_RT$ is flat. □

(iii) *The functor $f^{\#}$ induces a functor*

$$f^{\#} : \mathcal{P}_S \longrightarrow \mathcal{P}_R$$

if and only if $f^{\#}S$ is a finitely generated projective right R-module.

(iv) *The functor $f^{\#}$ induces a functor*

$$f^{\#} : \mathcal{F}_S \longrightarrow \mathcal{F}_R$$

if and only if $f^{\#}S$ is a free right R-module of finite rank.

In each case, if the functor $f^{\#}$ exists, it is exact.

Proof

Since all the subcategories that occur are full subcategories of $\mathcal{M}_{OD_R}$ or $\mathcal{M}_{OD_S}$, we need only verify that $f^{\#}$ has the desired action on modules.

(i). Suppose first that $f^{\#}S$ is R-projective. If M is in $\mathcal{P}_{R\mathcal{O}J_S}$, then $M \oplus N \cong S^{\Lambda}$ for some module N and index set Λ ([BK: IRM] Theorem 2.5.8), and clearly $f^{\#}M \oplus f^{\#}N \cong (f^{\#}S)^{\Lambda}$ also, which shows that $f^{\#}M$ is projective ([BK: IRM] Theorem 2.5.5).

The converse is obvious since S is in $\mathcal{P}_{R\mathcal{O}J_S}$.

(ii) – (iv). Similar to (i); note that if an S-module M has a finite set of generators $\{m_1, \ldots, m_\ell\}$ and S has a finite set of generators $\{s_1, \ldots, s_k\}$ as a right R-module, then the set of products $\{m_i s_j\}$ is a finite set of generators of the R-module $f^{\#}M$.

The final assertion follows from the observation that restriction must preserve exact sequences of modules since it has no effect on the underlying abelian groups. Note also that all the categories mentioned are repletely G-exact subcategories of $\mathcal{M}_{OD_R}$ or $\mathcal{M}_{OD_S}$. □

3.3.4 Extension

Again, let $f : R \to S$ be a ring homomorphism. To define the extension functor $f_{\#} : \mathcal{M}_{OD_R} \to \mathcal{M}_{OD_S}$, we first consider S as a *left* R-module by the rule $r \cdot s = (fr)s$, so that S becomes an R-S-bimodule. (Strictly speaking, this amounts to replacing S by its image $f^{\#}(S)$ under the restriction functor for left modules – see (3.3.21).)

Then we put

$$f_{\#} = - \otimes_R S : \mathcal{M}_{OD_R} \longrightarrow \mathcal{M}_{OD_S};$$

explicitly, for a right R-module M,

$$f_{\#}(M) = M \otimes_R S$$

denote by $f^{\#}M$. It is evident that a homomorphism

$$\alpha : M' \longrightarrow M$$

of right S-modules is also a homomorphism

$$f^{\#}\alpha : f^{\#}M' \longrightarrow f^{\#}M$$

of right R-modules and that

$$f^{\#} : \mathcal{M}\text{od}_S \longrightarrow \mathcal{M}\text{od}_R$$

is an additive covariant functor, which we call *restriction* or *restriction of scalars*

The terminology is most appropriate when f is an injection, because we are then 'restricting the scalars' in the literal sense. When f is surjective, the reader may meet the terms *pull-back* to describe $f^{\#}$ and *coinduced module* to describe $f^{\#}M$. Finally, in K-theory $f^{\#}$ gives rise to various homomorphisms of K-groups; many of these homomorphisms are called *transfer* maps.

Here are some elementary properties of the restriction functor.

3.3.2 Lemma

(i) $(id_R)^{\#}$ *is the identity functor* $\mathrm{Id}_R : \mathcal{M}\text{od}_R \to \mathcal{M}\text{od}_R$.

(ii) *Given ring homomorphisms* $f : R \to S$ *and* $g : S \to T$, *there is a natural isomorphism between the functors* $(gf)^{\#}$ *and* $f^{\#}g^{\#}$. □

If we put $(R)^{\#} = \mathcal{M}\text{od}_R$, then $(-)^{\#}$ is a contravariant functor from the category $\mathcal{R}\text{ing}$ of all rings to the category $\mathcal{C}\text{at}$ of all categories.

Next, we consider the behaviour of the restriction functor on various subcategories of $\mathcal{M}\text{od}_S$. To save notation, we use the same symbol $f^{\#}$ to denote a number of functors arising from $f^{\#}$.

3.3.3 Proposition

(i) *The functor* $f^{\#}$ *induces a functor*

$$f^{\#} : \mathcal{P}\text{roj}_S \longrightarrow \mathcal{P}\text{roj}_R$$

if and only if $f^{\#}S$ *is a projective right* R*-module.*

(ii) *The functor* $f^{\#}$ *induces a functor*

$$f^{\#} : \mathcal{M}_S \longrightarrow \mathcal{M}_R$$

if and only if $f^{\#}S$ *is a finitely generated right* R*-module.*

and

$$f_{\#} : \mathcal{M}\text{od}_R \longrightarrow \mathcal{M}\text{od}_S,$$

which are called respectively restriction and extension of scalars. These functors generalize the notions of restriction and extension of scalars between, for example, real and complex vector spaces (where f is the inclusion map of $\mathbb{R}$ in $\mathbb{C}$), to modules over arbitrary rings.

An important special case occurs when the ring homomorphism is an automorphism of R, which we prefer to write as α rather than f. In this situation, an R-module M can be 'twisted' to obtain a new R-module M^{α} (3.3.22), which turns out to be the restriction $\alpha^{\#}M$ (3.3.25). Twisted modules are encountered frequently in the theory of skew polynomial rings.

The functorial analysis of change of scalars through the tensor product was initiated in [Cartan & Eilenberg 1956], Ch. II §6. Our emphasis on the preservation of exact sequences and the twisting of module structures anticipates the requirements of K-theory.

As usual, we concentrate on right modules. It will be obvious that our definitions and results have analogues for left modules (see (3.3.21)). We do not always state these analogues separately. Note that we always write ring homomorphisms on the left of their arguments.

The reader is warned that there is considerable variation in the notation and terminology for restriction and extension. For example, [Quillen 1973] §4 (5), uses f_* where we use $f^{\#}$, and f^* for our $f_{\#}$. The variation stems from the convention that covariant behaviour is usually indicated by a subscript and contravariant by a superscript. In applications to geometry and topology, the interest is in the action of restriction and extension on categories of varieties, schemes or topological spaces rather than modules [Fulton & Lang 1985], II §1. However, the functors that link rings to these objects are themselves contravariant [Hartshorne 1997], I (3.8), II (2.3), with a resulting switch in the variance of restriction and extension. We mention the alternative terminologies from time to time.

3.3.1 Restriction

Let M be a right S-module and suppose that there is a ring homomorphism $f : R \to S$.

Given elements r of R and m of M, define

$$m \cdot r = m \cdot (fr).$$

It is clear that this rule gives M the structure of a right R-module, which we

and that

$$\mathrm{Tor}_n^{\mathbb{Z}}(M,-) \equiv 0 \text{ for } n > 1.$$

(ii) Let $R = A[\epsilon]$ be the ring of dual numbers over a coefficient ring A and let M be a right R-module. After Exercise 3.2.1, there is a short exact sequence (of left or right R-modules)

$$0 \longrightarrow A \longrightarrow R \longrightarrow A \longrightarrow 0$$

in which the inclusion is multiplication by ϵ and the surjection is the natural one.
Show that $M \otimes_R A \cong M/M\epsilon$ and that $\mathrm{Tor}_1^R(M,A) \cong M_\epsilon/M\epsilon$ where $M_\epsilon = \{m \in M \mid m\epsilon = 0\}$. Deduce that $\mathrm{Tor}_1^R(A,A) \cong A$, as a group.
Show also that $\mathrm{Tor}_{n-1}^R(M,A) \cong \mathrm{Tor}_n^R(M,A)$ for $n > 1$.

(iii) Prove (3.2.10) using the long exact sequence.

3.2.8 Let

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

be any short exact sequence of left R-modules, with N'' flat, and let M be any right R-module.

(a) Choose a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F flat, and construct a 3×3 commutative diagram as in (3.2.10). By chasing the diagram, show that $M \otimes_R N'$ maps injectively into $M \otimes_R N$.

(b) From the case where N is free, observe that $\mathrm{Tor}_1^R(M,N'') = 0$.

(c) Also deduce from (a) that if N is flat, then so is N', a converse to (3.2.10).

3.3 CHANGE OF SCALARS

Now that the tensor product and the language of category theory are available to us, we can analyse in detail the relationships between various categories of modules that arise from homomorphisms between rings.

Given a ring homomorphism

$$f : R \longrightarrow S,$$

where R and S are arbitrary rings, we define two functors

$$f^{\#} : \mathcal{M}\mathit{od}_S \longrightarrow \mathcal{M}\mathit{od}_R$$

of $\mathcal{M}_R$ whose objects are in $\mathcal{P}_\mathcal{O}$, the class $\mathrm{Ex}(\mathcal{P}_{R,\mathcal{O}})$ consisting of all short exact sequences with terms in $\mathcal{P}_\mathcal{O}$ (so $\mathcal{P}_{R,\mathcal{O}}$ is repletely exact).

Show that $- \otimes_R N$ is exact on $\mathcal{M}_{R,\mathcal{O}}$ and $\mathcal{P}_{R,\mathcal{O}}$ for any left R-module N.

(In applications, $\mathcal{M}_{R,\mathcal{O}}$ is useful when $\mathcal{O}$ is a commutative domain and R is an $\mathcal{O}$-order, while $\mathcal{P}_{R,\mathcal{O}}$ is useful when $\mathcal{O}$ is arbitrary and R is the group ring $\mathcal{O}G$ for a suitable group G ([Berrick 1982] p. 94).)

3.2.5 **Firm modules over nonunital rings**

For a nonunital ring R with unitalization $\overline{R}$, recall from (1.3.2)(iv) that an R-module is just the same thing as an $\overline{R}$-module. In particular, R itself is an R-bimodule. We define a right R-module V to be *firm* if multiplication induces a right R-module isomorphism $V \otimes_{\overline{R}} R \to V$; similarly for firm left R-modules. The ring R itself is called a *firm nonunital ring* if it is firm as a right R-module. Show that this is equivalent to R being firm as a left R-module, and to there being a canonical (nonunital) ring isomorphism $R \otimes_{\overline{R}} R \to R$, where $R \otimes_{\overline{R}} R$ has ring structure defined by

$$(r_1 \otimes r_2)(r_1' \otimes r_2') = r_1(r_2 r_1') \otimes r_2' = r_1 \otimes (r_2 r_1') r_2'.$$

Now suppose that R is a firm nonunital ring, and let V be a firm right R-module and W a firm left R-module equipped with a surjective R-bimodule homomorphism $\sigma : W \otimes_\mathbb{Z} V \to R$. Define a multiplication on the abelian group $S = V \otimes_R W$ by

$$(v \otimes w)(v' \otimes w') = v \cdot \sigma(w, v') \otimes w' = v \otimes \sigma(w, v') \cdot w'.$$

Show that S is thereby also a firm nonunital ring.

3.2.6 Let $\mathcal{O}$ be any commutative ring. Using (3.2.14), show that the tensor product induces biadditive functors from

(i) $\mathcal{M}_\mathcal{O} \times \mathcal{M}_\mathcal{O}$ to $\mathcal{M}_\mathcal{O}$,
(ii) $\mathcal{F}_\mathcal{O} \times \mathcal{F}_\mathcal{O}$ to $\mathcal{F}_\mathcal{O}$,
(iii) $\mathcal{P}_\mathcal{O} \times \mathcal{P}_\mathcal{O}$ to $\mathcal{P}_\mathcal{O}$,
and
(iv) $\mathcal{M}_\mathcal{O} \times \mathcal{T}_{\mathcal{O}R\mathcal{O}}$ to $\mathcal{T}_{\mathcal{O}R\mathcal{O}}$.

3.2.7 **Calculations with** Tor

For this exercise, we assume the properties of the functor Tor as outlined in (3.2.6).

(i) Let M be an abelian group (that is, a $\mathbb{Z}$-module.) Show that, for any positive integer a,

$$\mathrm{Tor}_1^\mathbb{Z}(M, \mathbb{Z}/\mathbb{Z}a) = \{m \in M \mid ma = 0\}$$

Then there is an isomorphism $\mathfrak{a} \otimes_{\mathcal{O}} \mathfrak{b} \to \mathfrak{ab} \subseteq \mathcal{K}$, *given by* $a \otimes b \mapsto ab$.

Proof

Clearly, the map $a \otimes b \mapsto ab$ is a well-defined surjection, so we need only check injectivity. Suppose first that $\mathfrak{b} = \mathcal{O}x$ is principal. Then $\mathfrak{b}$ is projective and so flat as an $\mathcal{O}$-module, and the natural inclusion $\mathfrak{a} \to \mathcal{K}$ gives an injective homomorphism $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}x \to \mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}x$. Arguing as in (3.1.4), we have $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}x \cong \mathfrak{a}x$ and $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}x \cong \mathcal{K}x = \mathcal{K}$ via $a \otimes x \mapsto ax$.

In general, suppose that $a_1, \ldots, a_k$ are in $\mathfrak{a}$ and $b_1, \ldots, b_k$ are in $\mathfrak{b}$, and that $\sum_i a_i \otimes b_i \mapsto 0$ in $\mathcal{K}$. Finding a common denominator $x \in \mathcal{K}$, we can write $b_i = c_i x$ with $c_i \in \mathcal{O}$ for all i. Then

$$\sum_i a_i \otimes b_i = \sum_i a_i c_i \otimes x \longmapsto 0,$$

so $\sum_i a_i c_i = 0$ in $\mathfrak{a}$. Thus $\sum_i a_i \otimes c_i = 0$ in $\mathfrak{a} \otimes_{\mathcal{O}} \mathcal{O}$ and, after multiplication by x, $\sum_i a_i \otimes b_i = 0$ in $\mathfrak{a} \otimes \mathfrak{b}$. □

Exercises

3.2.1 Let R be the ring of dual numbers $A[\epsilon]$ over a ring A, and recall that A is both a left and right R-module, with ϵ acting as 0 on A. By considering the short exact sequence of right R-modules

$$0 \longrightarrow \epsilon A[\epsilon] \xrightarrow{\alpha} A[\epsilon] \xrightarrow{\beta} A \longrightarrow 0,$$

where α is the inclusion and β the obvious surjection, show that A is not R-flat.

3.2.2 Let $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$ be a direct sum of left R-modules, where Λ is any ordered set. Show that N is flat if and only if each N_λ is flat.

3.2.3 An R-module M is said to be *finitely related* if there is a short exact sequence of the form

$$0 \longrightarrow R^m \longrightarrow R^\Lambda \longrightarrow M \longrightarrow 0.$$

Show that if also M is flat, then M is projective.

3.2.4 Given a subring $\mathcal{O}$ of a ring R, we introduce two *$\mathcal{O}$-relative G-exact categories* $\mathcal{M}_{R,\mathcal{O}}$ and $\mathcal{P}_{R,\mathcal{O}}$ as follows.

The category $\mathcal{M}_{R,\mathcal{O}}$ has as underlying category $\mathcal{M}_R$ and we take the set $\mathrm{Ex}(\mathcal{M}_{R,\mathcal{O}})$ of admissible short exact sequences to be all the short exact sequences in $\mathcal{M}_R$ which are split as exact sequences of $\mathcal{O}$-modules. The underlying category of $\mathcal{P}_{R,\mathcal{O}}$ is the full subcategory

$\tau\pi_1 : F \to \tau F_1$. Because the restriction of τ to L is the identity map, $L \subseteq L'$. Finally, since $\epsilon\sigma = \epsilon$ and $\epsilon\rho = 0$, we have $L' \subseteq \operatorname{Ker}\epsilon$. □

3.2.14 A pairing on $\mathcal{M}_{OD_R}$

As we remarked following (3.1.10), the tensor product $-\otimes_R-$ can be regarded as a bifunctor from $\mathcal{M}_{OD_R} \times {}_R\mathcal{M}_{OD}$ to $\mathcal{A}_B$. For a general ring R and arbitrary modules M_R and ${}_RN$, there is no natural method by which $M \otimes_R N$ can be endowed with an R-module structure. However, if both M and N belong to the category ${}_R\mathcal{B}_{IMOD_R}$ of R-R-bimodules (and R-R-bimodule homomorphisms), then $M \otimes_R N$ is also in ${}_R\mathcal{B}_{IMOD_R}$ (see (3.1.3)).

This observation permits us to use the tensor product to define a binary operation in $\mathcal{M}_{OD_R}$ when the ring R is commutative. An R-R-bimodule is said to be *balanced* if $rm = mr$ for all m in M and r in R. Clearly, any right (or left) R-module M can be viewed as a balanced R-R-bimodule by setting $rm = mr$ for all r in R and m in M. Thus we can choose to identify both $\mathcal{M}_{OD_R}$ and ${}_R\mathcal{M}_{OD}$ with the category $\mathcal{B}_{AL_R}$ of balanced R-bimodules.

Using this identification, we can regard $-\otimes_R-$ as a bifunctor from $\mathcal{M}_{OD_R} \times \mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_R}$. Such a bifunctor which is additive in both variables is sometimes called a *pairing* on $\mathcal{M}_{OD_R}$. It is clear that this pairing induces a pairing on $\mathcal{M}_R$, the category of finitely generated R-modules.

The next result shows that this pairing is, in essence, commutative. The proof is an easy exercise.

3.2.15 Proposition

Let M and N be balanced bimodules over the commutative ring R. Then there is a natural isomorphism $M \otimes_R N \cong N \otimes_R M$ of balanced R-bimodules. □

The above pairing can be viewed as a generalization of the product of fractional ideals of a commutative domain $\mathcal{O}$. Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$, and recall that a fractional ideal of $\mathcal{O}$ is a finitely generated (balanced) $\mathcal{O}$-submodule of $\mathcal{K}$, and that the product $\mathfrak{ab}$ of fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ is the submodule of $\mathcal{K}$ generated by all products ab with $a \in \mathfrak{a}, b \in \mathfrak{b}$ (2.3.20). Then $\mathfrak{ab}$ is also a fractional ideal. The connection between the pairing and the product is given by the following useful computation.

3.2.16 Proposition

Let $\mathcal{O}$ be a commutative domain with field of fractions $\mathcal{K}$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of $\mathcal{O}$.

and

$$\sum_{i=1}^{k} \rho(x_i)a_i = \rho(\sum_{i=1}^{k} x_i a_i) = \sum_{i=1}^{k} x_i a_i,$$

as required. □

Note that here L is not quite a direct summand of F. We now show that in favourable circumstances we do obtain a direct summand. The first result is obvious, the next rather deeper.

3.2.12 Corollary

Suppose that $\epsilon : F \twoheadrightarrow M$ is surjective, with F free and M flat. If $\operatorname{Ker}\epsilon$ *is finitely generated, then $F \cong \operatorname{Ker}\epsilon \oplus M$.* □

Although the finitely generated submodule L in Villamayor's Lemma need not be a summand of F, the next result shows that it is contained in a finitely generated summand provided that F is 'big enough'. This result is used in (5.2.7) to help in the derivation of another important characterization of flatness.

3.2.13 Lemma

Suppose that (ϵ, L) is a pair as in Villamayor's Lemma, and also that M is flat and that $F = \mathrm{Fr}_R(X)$, where for each $x \in X$ there exist infinitely many $y \in X$ such that $\epsilon y = \epsilon x$. Then there is a finitely generated direct summand L' of F with

$$L \subseteq L' \subseteq \operatorname{Ker}\epsilon.$$

Proof

With $\rho : F \to \operatorname{Ker}\epsilon$ as above, let X_1 be the finite set of elements of X occurring in the expressions of the generators of ρF as linear combinations of elements of X. By hypothesis, we can partition X as $X = X_1 \sqcup X_2 \sqcup X_3$ with X_1 and X_2 in bijective correspondence such that corresponding elements have the same images under ϵ. Write $F_i = \mathrm{Fr}_R(X_i)$, so that $L \subseteq F_1$ and $F = F_1 \oplus F_2 \oplus F_3$, and let $\pi_1 : F \to F_1$ denote projection to the first summand. The bijection $X_1 \to X_2$ defines an injective homomorphism $\sigma : F_1 \to F_2 \hookrightarrow F$ with $\pi_1\sigma = 0$ and $\epsilon\sigma = \epsilon$. Define

$$\tau = id + \sigma(\rho - id) : F_1 \to F,$$

with image $L' = \tau F_1$. (Roughly speaking, $\tau - id$ is just $\rho - id$ shifted away from F_1.) Evidently, $\pi_1\tau = id$, so that the inclusion of L' in F is split by

the obvious map $M \otimes_R \mathfrak{a} \to M \otimes_R R \cong M$ is an injection. So in the following commutative diagram with exact rows, the right vertical arrow is injective.

$$\begin{array}{ccccccccc} & & (\operatorname{Ker}\epsilon)\otimes_R \mathfrak{a} & \longrightarrow & F\otimes_R \mathfrak{a} & \longrightarrow & M\otimes_R \mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\epsilon & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since u_1 is the image of the element $\sum_{i=1}^{k} x_i \otimes r_i$ of $F \otimes_R \mathfrak{a}$, a diagram chase shows that

$$\sum_{i=1}^{k} x_i \otimes r_i = \operatorname{Im}(\sum_{j=1}^{h} y_j \otimes s_j)$$

for elements $y_j \in \operatorname{Ker}\epsilon$ and $s_j \in \mathfrak{a}$. Write $s_j = \sum_{i=1}^{k} a_{ij} r_i$ for $j = 1, \ldots, h$, with each a_{ij} in R, and put $v_i = \sum_{j=1}^{h} y_j a_{ij}$ for each i. Then

$$u_1 = \operatorname{Im}(\sum_{i=1}^{k} v_i \otimes r_i) = \sum_{i=1}^{k} v_i r_i \text{ with each } v_i \in \operatorname{Ker}\epsilon.$$

The R-homomorphism $\rho_1 : F \to \operatorname{Ker}\epsilon$, sending x_i to v_i for $i = 1, \ldots, k$ and each other generator of F to zero, has finitely generated image and $\rho_1 u_1 = u_1$.

$n > 1$. Let $\rho_1 : F \to \operatorname{Ker}\epsilon$ be as above, and let $\rho' : F \to \operatorname{Ker}\epsilon$ be the homomorphism obtained by applying the induction hypothesis to the submodule of $\operatorname{Ker}\epsilon$ generated by $u_2 - \rho_1 u_2, \ldots, u_n - \rho_1 u_n$. Then the R-homomorphism

$$\rho = \rho_1 + \rho'(id - \rho_1) : F \longrightarrow \operatorname{Ker}\epsilon$$

has the desired property.

For the converse, we wish to apply the Flat Test. Therefore let $\mathfrak{a}$ be a left ideal of R (with inclusion $\iota : \mathfrak{a} \to R$) and let $\epsilon : F \to M$ be some surjection from a free module F. We obtain the same diagram as above, but this time seek to establish the injectivity of the right-hand vertical arrow.

Since the free module F is flat by the left-handed version of (3.2.5), $id \otimes \iota : F \otimes_R \mathfrak{a} \to F \otimes_R R \cong F$ is injective. Thus, by an easy diagram chase, the injectivity of the right-hand arrow will follow once we can show that if $\sum_{i=1}^{k} x_i \otimes a_i$ is in $F \otimes_R \mathfrak{a}$ and $\epsilon(\sum_{i=1}^{k} x_i a_i) = 0$, then $\sum_{i=1}^{k} x_i a_i$ lies in $(id \otimes \iota)((\operatorname{Ker}\epsilon) \otimes_R \mathfrak{a})$.

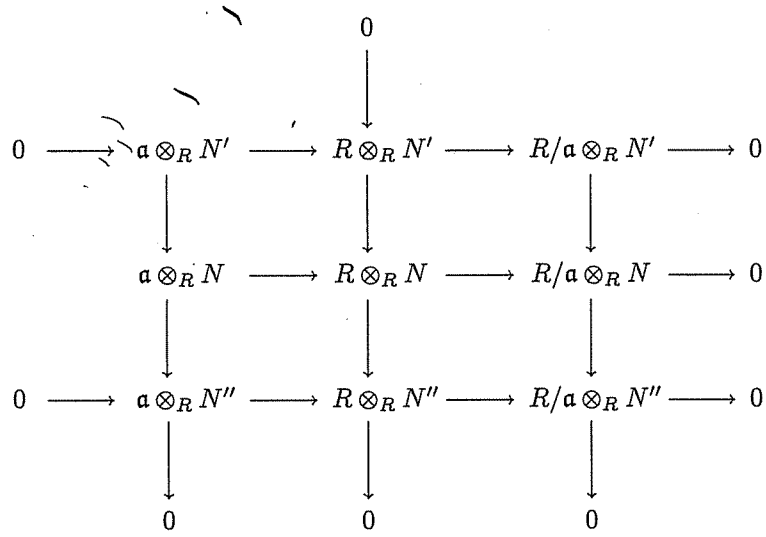
Now by hypothesis there exists an R-homomorphism $\rho : F \to \operatorname{Ker}\epsilon$ which restricts to the identity map on $\sum_{i=1}^{k} x_i a_i$. Then $\sum_{i=1}^{k} \rho x_i \otimes a_i \in (\operatorname{Ker}\epsilon) \otimes_R \mathfrak{a}$

3.2.10 Corollary

Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of left R-modules. If N' and N'' are flat, then so too is N.

Proof

Considering any inclusion $\mathfrak{a} \to R$ of a right ideal in R, simply chase the following commutative diagram of exact rows and columns to show that the middle row commences with an injection.



□

3.2.11 Villamayor's Lemma

Let M be a right R-module. Then the following assertions are equivalent.

(i) *M is flat.*

(ii) *Given any pair (ϵ, L), where $\epsilon : F \to M$ is a surjective homomorphism of right R-modules with F free, and L is a finitely generated submodule of* $\operatorname{Ker} \epsilon$, *then there is an R-homomorphism $\rho : F \to \operatorname{Ker} \epsilon$ which restricts to the identity map on L and which has finitely generated image ρF.*

Proof

First assume that M is flat. We argue by induction on the number of generators $u_1, \ldots, u_n$ of the submodule L.

$n = 1$. Write $u_1 = \sum_{i=1}^{k} x_i r_i$ with $x_1, \ldots, x_n$ elements of a free generating set for F. Let $\mathfrak{a}$ be the left ideal of R generated by $r_1, \ldots, r_k$. Since M is flat,

induction on n to show that $-\otimes_R N$ respects injectivity for $\mu : M \to R^n$, the case $n = 1$ having just been settled.

For the inductive step, we use the standard embedding of R^{n-1} in R^n. Put $M'' = M/(M \cap R^{n-1})$. Since $M'' \cong (M + R^{n-1})/R^{n-1}$ and $R^n/R^{n-1} \cong R$, there is an injective homomorphism from M'' to R. So there is a commuting diagram with exact rows, the lower of which is split:

$$\begin{array}{ccccccccc} & & (M \cap R^{n-1}) \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & M'' \otimes_R N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R^{n-1} \otimes_R N & \longrightarrow & R^n \otimes_R N & \longrightarrow & R \otimes_R N & \longrightarrow & 0 \end{array}$$

Then the injectivity of the two outside vertical arrows implies that of the middle one.

The next step is to show that $-\otimes_R N$ respects the injectivity of a monomorphism $\mu : M \to F$ where F is an arbitrary free module. Again, it is enough to check the case where the domain M is finitely generated. But then, if $(\mu \otimes id)x = 0$ in $F \otimes_R M$, there is a finitely generated free submodule G of the codomain F such that $(\mu \otimes id)x = 0$ in $G \otimes_R M$ already, since any member of the relation group $B(F, M)$ (3.1.2) can involve only finitely many generators of F. By the previous step, $x = 0$.

For the final step, let $\mu : M' \to M$ be an arbitrary injective homomorphism of right R-modules, and let F be a free right R-module with $\beta : F \to M$ an R-homomorphism onto M, having kernel K, say. Then, in the pull-back

$$\begin{array}{ccc} \overline{F} & \xrightarrow{\overline{\beta}} & M' \\ {\scriptstyle \overline{\mu}}\downarrow & & \downarrow{\scriptstyle \mu} \\ F & \xrightarrow{\beta} & M \end{array}$$

$\overline{\beta}$ also has kernel K, and so there is a commuting diagram with exact rows

$$\begin{array}{ccccccc} K \otimes_R N & \longrightarrow & \overline{F} \otimes_R N & \longrightarrow & M' \otimes_R N & \longrightarrow & 0 \\ {\scriptstyle id}\downarrow & & {\scriptstyle \overline{\mu} \otimes id}\downarrow & & {\scriptstyle \mu \otimes id}\downarrow & & \\ K \otimes_R N & \longrightarrow & F \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & 0 \end{array}$$

Since $\overline{\mu} \otimes id$ is now known to be injective, so too is $\mu \otimes id$, and the proof is complete. □

3.2.8 Proposition

(a) *For a left R-module N, the following assertions are equivalent.*

(i) ${}_RN$ *is flat.*

(ii) *The functor $-\otimes_R N$ respects the injectivity of any R-module monomorphism $\alpha : M'_R \to M_R$ with M'_R finitely generated. That is,*

$$\alpha \otimes id : M' \otimes_R N \longrightarrow M \otimes_R N$$

is also injective.

(iii) *The functor $-\otimes_R N$ respects the injectivity of any R-module monomorphism $M'_R \rightarrowtail M_R$.*

(iv) *The functor $- \otimes_R N : \mathcal{M}_{OD_R} \to \mathcal{A}_B$ is exact.*

(b) *For a right R-module M, the following assertions are equivalent.*

(i) M_R *is flat.*

(ii) *The functor $M\otimes_R -$ respects the injectivity of any R-module monomorphism ${}_RN' \to {}_RN$ with ${}_RN'$ finitely generated.*

(iii) *The functor $M\otimes_R -$ respects the injectivity of any R-module monomorphism ${}_RN' \to {}_RN$.*

(iv) *The functor $M \otimes_R - : {}_R\mathcal{M}_{OD} \to \mathcal{A}_B$ is exact.*

Proof

The only outstanding points to check are the implications (ii) $\Rightarrow$ (iii); we deal with (a) only. Consider an injective homomorphism $\alpha : M'_R \to M_R$ with M'_R not necessarily finitely generated, and let $x = \sum_{i=1}^{k}(m'_i \otimes n_i)$ be in $\mathrm{Ker}(\alpha \otimes id)$. Then x also lies in $\mathrm{Ker}(\alpha' \otimes id)$, where α' is the restriction of α to the submodule of M generated by $m'_1, \ldots, m'_k$. This forces $x = 0$, so that $\mathrm{Ker}(\alpha \otimes id) = 0$ as required. □

The following criterion is designed to simplify one's workload in establishing that a given module is flat.

3.2.9 The Flat Test

A left R-module N is flat if and only if $-\otimes_R N$ respects the injectivity of all inclusions $\mu : \mathfrak{a} \to R$ of finitely generated right ideals in R. The corresponding statement holds with left and right interchanged.

Proof

Necessity of the condition is immediate from the preceding proposition. Sufficiency is proved in four stages. First, we note that the argument just given to prove (ii) $\Rightarrow$ (iii) above also shows that $-\otimes_R N$ respects the injectivity of inclusions $\mu : \mathfrak{a} \to R$ of arbitrary right ideals in R. Second, we argue by

of left R-modules with F flat (for example, projective), and define

$$\mathrm{Tor}_1^R(M,N) = \mathrm{Ker}(M \otimes_R L \to M \otimes_R F)$$

and

$$\mathrm{Tor}_n^R(M,N) = \mathrm{Tor}_{n-1}^R(M,L) \text{ for } n > 1.$$

It can be shown that the abelian groups $\mathrm{Tor}_n^R(M,N)$ are independent of the choice of the short exact sequence for N, and that they could equally be defined by taking a similar short exact sequence for M and considering the kernel after tensoring with N. Further, $\mathrm{Tor}_n^R(M,N)$ is a covariant additive functor in each of its arguments, and, given a short exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

of left R-modules, there is a long exact sequence

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & \mathrm{Tor}_{n+1}^R(M,N'') \longrightarrow \\
\longrightarrow & \mathrm{Tor}_n^R(M,N') & \longrightarrow & \mathrm{Tor}_n^R(M,N) & \longrightarrow & \mathrm{Tor}_n^R(M,N'') & \longrightarrow \\
\longrightarrow & \cdots & & & & & \\
 & & & & \cdots & \longrightarrow & \mathrm{Tor}_1^R(M,N'') \longrightarrow \\
\longrightarrow & M \otimes_R N' & \longrightarrow & M \otimes_R N & \longrightarrow & M \otimes_R N'' & \longrightarrow 0,
\end{array}$$

and likewise for the other variable.

Much analysis is required to verify these claims; a good account is given in [Rotman 1979].

Note that the flat left R-modules can be characterized as those modules F for which the functors $\mathrm{Tor}_n^R(-,F)$ $(n \geq 1)$ are identically zero (and correspondingly for right flat modules). Some very elementary calculations with the Tor functors are indicated in Exercise 3.2.7 below, and in Exercise 6.2.1, which justifies the notation 'Tor'. Flat modules made a reticent entrance into the mathematical landscape in Exercise 3 of Ch. VI of [Cartan & Eilenberg 1956], where they are defined in terms of the vanishing of the functors Tor_n. The pivotal role that flat modules now play will become clear in subsequent chapters of this text.

3.2.7 Criteria for flatness and Villamayor's Lemma

We now obtain some useful criteria for flatness and thence Villamayor's Lemma. For the next few results and proofs, it is convenient to say that a functor F *respects the injectivity* of a module monomorphism μ if $F\mu$ is also a monomorphism.

by

$$\mu(m'', n) = m \otimes n + I,$$

where m is any element of M such that $\beta m = m''$.

If also $\beta m_1 = m''$, then exactness at M implies that $m - m_1$ is in $\operatorname{Im}\alpha$ and hence

$$m \otimes n + I = m_1 \otimes n + I.$$

Thus μ is well-defined. Since μ is trivial on $B(M'', N)$, it induces a homomorphism $M'' \otimes_R N \longrightarrow (M \otimes_R N)/I$ which is evidently inverse to λ.

The proof of (b) is analogous. □

3.2.4 Corollary

A left R-module N is flat if and only if for each injective right R-module homomorphism $\alpha : M' \to M$, the group homomorphism

$$\alpha \otimes id_N : M' \otimes_R N \longrightarrow M \otimes_R N,$$

is also an injection. □

The existence of flat modules is guaranteed by the next result.

3.2.5 Corollary

Suppose that N is a projective left R-module. Then N is flat.

Proof

That any free module R^Λ is flat follows from (3.1.14), since any injective right R-homomorphism $M' \to M$ induces an injection $M'^\Lambda \to M^\Lambda$. Now, for the module N, we have $N \oplus Q \cong {}^\Lambda R$ for some left R-module Q and free module ${}^\Lambda R$. Therefore, by (3.1.13),

$$- \otimes_R N \oplus - \otimes_R Q \simeq - \otimes_R {}^\Lambda R.$$

So $- \otimes_R N$ is exact by Exercise 2.4.6. □

3.2.6 The functors Tor_n^R

The failure of the tensor product to be an exact functor can be measured by a sequence of derived functors $\operatorname{Tor}_n^R(-,-)$ for $n \geq 1$. These may be constructed as follows.

Given a right R-module M and a left R-module N, where R is an arbitrary ring, we take a short exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow N \longrightarrow 0$$

is exact. □

Before we make a detailed analysis of the properties of flat modules, we discuss the extent to which exactness is preserved by modules in general.

3.2.3 Proposition

(a) *Let N be a left R-module and let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of right R-modules. Then the sequence

$$M' \otimes_R N \xrightarrow{\alpha \otimes id} M \otimes_R N \xrightarrow{\beta \otimes id} M'' \otimes_R N \longrightarrow 0$$

is exact.

(b) *Let M be a right R-module and let*

$$0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$$

be a short exact sequence of left R-modules. Then the sequence

$$M \otimes_R N' \xrightarrow{id \otimes \alpha} M \otimes_R N \xrightarrow{id \otimes \beta} M \otimes_R N'' \longrightarrow 0$$

is exact.

Proof

(a) The fact that $M'' \otimes N$ is generated by symbols $m'' \otimes n$ shows immediately that $\beta \otimes id$ is surjective. Since

$$0 = \beta\alpha \otimes id = (\beta \otimes id)(\alpha \otimes id),$$

we have

$$\mathrm{Im}(\alpha \otimes id) \subseteq \mathrm{Ker}(\beta \otimes id).$$

We must verify that this inclusion is equality. Write $I = \mathrm{Im}(\alpha \otimes id)$ and let

$$\lambda : (M \otimes_R N)/I \longrightarrow M'' \otimes_R N$$

be the homomorphism induced by β. It suffices to show that λ is an isomorphism, which we do by constructing an inverse.

Recall that $M'' \otimes_R N = M''\#N/B(M'', N)$ (3.1.2). Define

$$\mu : M''\#N \longrightarrow (M \otimes_R N)/I$$

We also show that, for R commutative, the tensor product defines a binary operation on the category of R-modules.

Unless otherwise stated, the ring R is arbitrary. Our main object of attention is a left module, since we wish to investigate operators on categories of right modules.

3.2.1 *Flat modules*

Let $\mathcal{C}$ be a G-exact category of right R-modules, that is, $\mathcal{C}$ is a subcategory of $\mathcal{M}_{OD_R}$ together with a specified set $\mathrm{Ex}(\mathcal{C})$ of short exact sequences as in (2.4.1), and let N be an R-S-bimodule. The additive functor $-\otimes_R N$ may or may not be an exact functor on $\mathcal{C}$ as defined in (2.4.6). It will be so if $\mathcal{C}$ is split exact, but it is not in general, as the following example for $\mathcal{M}_{OD_{\mathbb{Z}}}$ shows.

For any nonzero integer n, there is an exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

in which the first map is multiplication by n. Tensoring with $\mathbb{Z}/n\mathbb{Z}$ (as a $\mathbb{Z}$-module) gives the sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{0} \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

which evidently fails to be exact at the first term $\mathbb{Z}/n\mathbb{Z}$.

A left R-module N for which $-\otimes_R N : \mathcal{M}_{OD_R} \to \mathcal{M}_{OD_{\mathbb{Z}}}$ is an exact functor is, by definition, a *flat* module. Thus, the left R-module N is flat if for any short exact sequence

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

of right R-modules the sequence

$$0 \longrightarrow M' \otimes_R N \xrightarrow{\alpha \otimes id} M \otimes_R N \xrightarrow{\beta \otimes id} M'' \otimes_R N \longrightarrow 0$$

is also a short exact sequence. The definition of a flat right R-module is analogous.

We have an almost tautologous lemma.

3.2.2 Lemma

Suppose that the R-S-bimodule N is flat as a left R-module and finitely generated as a right S-module (where R and S are arbitrary rings). Then the functor

$$-\otimes_R N : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

3.1.4 **Associativity of the tensor product**

Let R and S be rings and let L_R, ${}_RM_S$ and ${}_SN$ be modules as indicated.

Show that the isomorphism

$$(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N)$$

of (3.1.5) is natural in each variable.

Deduce that the functors $(- \otimes_R -) \otimes_S -$ and $- \otimes_R (- \otimes_S -)$ from $\mathcal{M}od_R \times {}_R\mathcal{B}i\mathcal{M}od_S \times {}_S\mathcal{M}od$ to $\mathcal{A}b$ are naturally isomorphic.

3.1.5 Let A, B, and C be rings with bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated.

Using the Adjointness Theorem (3.1.19) together with the characterizations of projective and injective modules in terms of the Hom functor given in (2.1.8), show that

(i) if L_B and M_C are both projective, then $L \otimes_B M$ is projective as a right C-module (here, take $A = \mathbb{Z}$);

(ii) if M_C is projective and ${}_AN$ is injective, then $\mathrm{Hom}(M_C, N_C)$ is left A-injective (here, take $B = \mathbb{Z}$.)

3.1.6 (Tricky!)

Let T be the ring of upper triangular 2×2-matrices over R. By (3.1.21), there is an equivalence between the categories $\mathcal{M}od_T$ and $\mathcal{M}od(\mathcal{M}od_R)$. Show that the projective T-modules correspond to split surjections $\alpha : P \to Q$ with both P and Q R-projective.

Dualize this statement to one about injectives.

3.2 EXACTNESS OF THE TENSOR PRODUCT

We next investigate whether or not a tensor product functor $- \otimes_R N$ is an exact functor. The answer depends both on the left R-module N that is kept fixed and on the category of right R-modules on which it operates. The modules that always give exact functors form an important class of modules, namely, the flat modules. We give some criteria for a module to be flat, and we prove Villamayor's Lemma, a fundamental result which shows that a flat module enjoys a weak form of the splitting property that defines projective modules. We also give a summary of the properties of the functors Tor_n^R that are used to repair the non-exactness of the tensor product, and we show that the tensor product induces a pairing on $\mathcal{M}od_R$ for a commutative ring R.

The novelty of our proofs is that they are 'elementary' in that they avoid the use of the machinery of homological algebra.

Proof

As the details of the argument are very similar to those of the proof of (1.3.18), we merely indicate the correspondence between right T-modules L and homomorphisms $\alpha : M \otimes_R \vec{W} \to N$.

Suppose that L is a right T-module. Since the direct product $R \times S$ is a subring of T, L can be decomposed into a direct sum $M \oplus N$ with M a right R-module and N a right S-module. For m in M and w in W, we can write

$$(m, 0)\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = (0, \alpha(m, w))$$

with $\alpha(m, w) \in N$. It is easy to check that α is biadditive and R-balanced, and so gives rise to a homomorphism $\beta : M \otimes_R W \to N$.

Conversely, given such a homomorphism β, we can define a T-module structure on $M \oplus N$. The remaining details are for the reader. □

Exercises

3.1.1 Let $\beta : M \to M''$ and $\xi : N \to N''$ be surjective homomorphisms of right and left R-modules respectively. Show that $\operatorname{Ker}\beta \otimes \xi$ is the subgroup of $M \otimes N$ generated by the set

$$\{k \otimes n, m \otimes \ell \mid k \in \operatorname{Ker}\beta,\ \ell \in \operatorname{Ker}\xi,\ m \in M,\ n \in N\}.$$

3.1.2 Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of the ring R. Show that $R/\mathfrak{a} \otimes_R R/\mathfrak{b} \cong R/(\mathfrak{a}+\mathfrak{b})$. (The preceding exercise is relevant.)

Hence compute $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z}$ for various integers a and b.

3.1.3 **Dedekind domains**

Let $\mathcal{O}$ be a Dedekind domain. This exercise requires some facts about $\mathcal{O}$-modules which are summarized in (2.3.20).

(a) Let P be a finitely generated projective $\mathcal{O}$-module. Then, by Steinitz' Theorem, P is isomorphic to a projective module in the standard form $P \cong \mathcal{O}^{r-1} \oplus \mathfrak{a}$ where the rank r and the ideal class $\{\mathfrak{a}\}$ are uniquely determined by P. Compute $P \otimes_{\mathcal{O}} Q$ for any finitely generated projective modules P, Q.

(b) Using the Primary Decomposition Theorem and Exercise 3.1.2 above, describe $M \otimes_{\mathcal{O}} N$ for any pair M and N of finitely generated torsion $\mathcal{O}$-modules.

(c) Using the fact that any finitely generated $\mathcal{O}$-module has the form $T \oplus P$ with T torsion and P projective, describe $M \otimes_{\mathcal{O}} N$ for any pair M and N of finitely generated $\mathcal{O}$-modules.

For any $m \in M$,

$$\begin{aligned}((\eta\alpha)(a\ell b))(m) &= \alpha(a\ell b \otimes m) = \alpha(a(\ell \otimes bm)) \\ &= a(\alpha(\ell \otimes bm)) = a(\eta\alpha(\ell))(bm) = (a((\eta\alpha)(\ell))b)(m),\end{aligned}$$

which is the desired result.

We define the inverse τ of η as follows. Let $\gamma \in \mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C))$, and put $(\tau\gamma)(\ell \otimes m) = \gamma(\ell)(m)$. Routine checking confirms that $\tau\gamma$ arises from a B-balanced biadditive map on $L \times M$ in the expected way and that $\tau\gamma$ is an A-C-bimodule homomorphism.

It is clear that η and τ are mutually inverse isomorphisms for each triple L, M and N, and further routine verification shows them to be natural in each variable. □

3.1.20 An equivalence of categories

Finally, we use the tensor product to generalize the characterization of triangular matrix rings that was given in (1.3.18).

Let R and S be arbitrary rings and let W be an R-S-bimodule. We define the *morphism category relative to* W (a variant of the fibre category of (1.2.9)) to be the category $\mathcal{M}_{OR}(W; R, S)$ whose objects are triples (α, M, N) in which M is a right R-module, N is a right S-module and $\alpha : M \otimes_R W \to N$ is a homomorphism of right S-modules.

A morphism from (α', M', N') to (α, M, N) is a pair (μ, ν) of homomorphisms, $\mu : M' \to M$ and $\nu : N' \to N$, such that $\nu\alpha' = \alpha(\mu \otimes id)$, which means that the following diagram must commute:

$$\begin{array}{ccc} M' \otimes_R W & \xrightarrow{\alpha'} & N' \\ {\scriptstyle \mu \otimes id}\downarrow & & \downarrow{\scriptstyle \nu \otimes id} \\ M \otimes_R W & \xrightarrow{\alpha} & N \end{array}$$

Thus with $W = R = S$, we regain the morphism category $\mathcal{M}_{OR}(\mathcal{M}_{OD_R})$.

The rings R and S and the bimodule W also define the generalized triangular matrix ring $T = \begin{pmatrix} R & W \\ 0 & S \end{pmatrix}$.

3.1.21 Theorem

The categories $\mathcal{M}_{OD_T}$ and $\mathcal{M}_{OR}(W; R, S)$ are equivalent.

The Adjointness Theorem tells us that all three general constructions yield essentially the same trifunctor. More precisely, the following result holds.

3.1.19 The Adjointness Theorem for Hom and $\otimes$

Let A, B and C be arbitrary rings. Then there are natural isomorphisms

$$\eta : \mathrm{Hom}_{A-C}(L \otimes_B M, N) \longrightarrow \mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C)),$$

and

$$\psi : \mathrm{Hom}_{A-C}(L \otimes_B M, N) \longrightarrow \mathrm{Hom}_{B-C}(M, \mathrm{Hom}({}_AL, {}_AN))$$

of trifunctors from ${}_A\mathcal{B}\mathit{IMOD}_B \times {}_B\mathcal{B}\mathit{IMOD}_C \times {}_A\mathcal{B}\mathit{IMOD}_C$ to $\mathcal{A}_B$.

In particular, for M in ${}_B\mathcal{B}\mathit{IMOD}_C$, the functor

$$- \otimes_B M : {}_A\mathcal{B}\mathit{IMOD}_B \longrightarrow {}_A\mathcal{B}\mathit{IMOD}_C$$

is left adjoint to

$$\mathrm{Hom}(M_C, -) : {}_A\mathcal{B}\mathit{IMOD}_C \longrightarrow {}_A\mathcal{B}\mathit{IMOD}_B,$$

and for L in ${}_A\mathcal{B}\mathit{IMOD}_B$, the functor

$$L \otimes_B - : {}_B\mathcal{B}\mathit{IMOD}_C \longrightarrow {}_A\mathcal{B}\mathit{IMOD}_C$$

is left adjoint to

$$\mathrm{Hom}({}_AL, -) : {}_A\mathcal{B}\mathit{IMOD}_C \longrightarrow {}_B\mathcal{B}\mathit{IMOD}_C.$$

Proof

We establish the first assertion only, the argument for the second being similar.

Take bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ and let α be in $\mathrm{Hom}_{A-C}(L \otimes_B M, N)$. Then a map $\eta\alpha : L \to \mathrm{Hom}(M_C, N_C)$ is defined by setting

$$((\eta\alpha)(\ell))(m) = \alpha(\ell \otimes m) \text{ for each } \ell \in L \text{ and } m \in M.$$

Since C acts on $L \otimes M$ by $(\ell \otimes m)c = \ell \otimes mc$ and α is a right C-module homomorphism (and the tensor product is linear in each variable), every $(\eta\alpha)(\ell)$ is a right C-module homomorphism. It is also clear that $\eta\alpha$ is an additive map on L.

The verification that $\eta\alpha$ is an A-B-bimodule homomorphism is a matter of careful book-keeping. We need to check that for all $\ell \in L$, $a \in A$ and $b \in B$ we have

$$(\eta\alpha)(a\ell b) = a((\eta\alpha)(\ell))b.$$

if and only if N is finitely generated and projective as a right S-module.

Proof

Necessity is clear in both cases, since $N \cong R \otimes_R N$ must be in the target category. On the other hand, if N has a finite set $\{n_1, \ldots, n_k\}$ of generators as an S-module, and M has generators $\{m_1, \ldots, m_\ell\}$ as an R-module, then $\{m_1 \otimes n_1, \ldots, m_\ell \otimes n_k\}$ generates $M \otimes_R N$, which gives (i).

For (ii), note that any module P in $\mathcal{P}_R$ is a direct summand of R^n for some integer n. But then $P \otimes_R N$ is a direct summand of $R^n \otimes_R N \cong N^n$, which is projective precisely when N is, by a standard result ([BK: IRM] Theorem 2.5.5). □

3.1.18 The adjointness of the functors Hom *and* $\otimes$

The homomorphism functors and the tensor product are connected by an adjointness relation, the existence of which serves to explain and simplify some calculations, especially in the Morita theory.

It will be convenient to depart from our customary labels for rings in order to avoid some confusing substitutions in applications. Let A, B and C be arbitrary rings, and take bimodules ${}_AL_B$, ${}_BM_C$ and ${}_AN_C$ as indicated.

Then $L \otimes_B M$ is an A-C-bimodule, and we can form the abelian group $\mathrm{Hom}_{A-C}(L \otimes_B M, N)$ of A-C-bimodule homomorphisms from $L \otimes_B M$ to N. It is easy to verify that this group is natural with respect to bimodule homomorphisms of each variable, being contravariant in the first two variables and covariant in the third. Thus we have a trifunctor

$$\mathrm{Hom}_{A-C}(- \otimes_B -, -) : {}_A\mathcal{B}_{IMOD_B} \times {}_B\mathcal{B}_{IMOD_C} \times {}_A\mathcal{B}_{IMOD_C} \longrightarrow \mathcal{A}_B.$$

We can also give $\mathrm{Hom}(M_C, N_C)$ the structure of an A-B-bimodule by the usual rule:

$$(a\mu b)(m) = a(\mu(bm)) \text{ for } m \in M,\ \mu \in \mathrm{Hom}(M_C, N_C).$$

We can then form the group $\mathrm{Hom}_{A-B}(L, \mathrm{Hom}(M_C, N_C))$, so obtaining a trifunctor

$$\mathrm{Hom}_{A-B}(-, \mathrm{Hom}(-_C, -_C)) : {}_A\mathcal{B}_{IMOD_B} \times {}_B\mathcal{B}_{IMOD_C} \times {}_A\mathcal{B}_{IMOD_C} \longrightarrow \mathcal{A}_B.$$

A third variation on this theme is to view $\mathrm{Hom}({}_AL, {}_AN)$ as a B-C-bimodule and form the group $\mathrm{Hom}_{B-C}(M, \mathrm{Hom}({}_AL, {}_AN)$. Notice that the order of the bimodules L, M and N is changed. Despite this alteration, we again have a trifunctor

$$\mathrm{Hom}_{B-C}(-, \mathrm{Hom}({}_A-, {}_A-)) : {}_A\mathcal{B}_{IMOD_B} \times {}_B\mathcal{B}_{IMOD_C} \times {}_A\mathcal{B}_{IMOD_C} \longrightarrow \mathcal{A}_B.$$

are naturally isomorphic.

Proof

It suffices to discuss (a). The key point is that the sequences (m_λ), with each $m_\lambda \in M$, $\lambda \in \Lambda$, that make up M^Λ, have only finitely many terms m_λ nonzero. Writing the standard generators of ${}^\Lambda R$ as e_λ, we map $M \otimes_R {}^\Lambda R$ to M^Λ by sending $\sum_i (\sum_\lambda m_i \otimes r_{i\lambda} e_\lambda)$ to $(\sum_i m_i r_{i\lambda})_\lambda$. Clearly this defines a surjective natural transformation, whose injectivity follows from the interchangeability of the finite summations:

$$\sum_i (\sum_\lambda m_i \otimes r_{i\lambda} e_\lambda) = \sum_\lambda (\sum_i m_i r_{i\lambda}) \otimes e_\lambda. \qquad \square$$

We also note an important special case of the above proposition. Let $M_{m,n}(R)$ denote the additive group of $m \times n$ matrices with entries in the ring R. Clearly, $M_{m,n}(R)$ is an $M_m(R)$-$M_n(R)$-bimodule, and we know that R^m is an $M_m(R)$-R-bimodule and that the standard free left R-module ${}^n R$ is an R-$M_n(R)$-bimodule. A direct verification gives the following result.

3.1.15 Corollary

$R^m \otimes_R {}^n R \cong M_{m,n}(R)$ as an $M_m(R)$-$M_n(R)$-bimodule. $\square$

3.1.16 Return of the dyads

In classical tensor analysis, an $m \times n$ matrix that can be expressed as the matrix product of a column vector by a row vector (that is, an $m \times 1$ matrix by a $1 \times n$ matrix) is called a *dyad*, especially when $m = n$. Thus the above result tells us that any matrix is a sum of dyads.

We next investigate the effect of a tensor product when it acts on subcategories of $\mathcal{M}_{\mathcal{OD}_R}$. Recall that $\mathcal{M}_R$ is the category of finitely generated right R-modules and that $\mathcal{P}_R$ is the category of finitely generated projective right R-modules.

3.1.17 Lemma

Let N be an R-S-bimodule.

(i) *The functor $- \otimes_R N$ from $\mathcal{M}_{\mathcal{OD}_R}$ to $\mathcal{M}_{\mathcal{OD}_S}$ induces a functor*

$$- \otimes_R N : \mathcal{M}_R \longrightarrow \mathcal{M}_S$$

if and only if N is finitely generated as a right S-module.

(ii) *The functor $- \otimes_R N$ from $\mathcal{M}_{\mathcal{OD}_R}$ to $\mathcal{M}_{\mathcal{OD}_S}$ induces a functor*

$$- \otimes_R N : \mathcal{P}_R \longrightarrow \mathcal{P}_S$$

3.1.13 Proposition

(a) *Suppose that* $M \cong M' \oplus M''$ *as a right* R*-module. Then there is a natural isomorphism*

$$M \otimes_R - \simeq M' \otimes_R - \oplus M'' \otimes_R -$$

of functors from ${}_R\mathcal{M}\mathrm{od}$ *to* $\mathcal{A}\mathrm{b}$.

(b) *Suppose that* $N \cong N' \oplus N''$ *as a left* R*-module. Then there is a natural isomorphism*

$$- \otimes_R N \simeq - \otimes_R N' \oplus - \otimes_R N''$$

of functors from $\mathcal{M}\mathrm{od}_R$ *to* $\mathcal{A}\mathrm{b}$. □

The above proposition easily leads to the next, in the case of finite direct sums. However, we also need to consider direct sums based on an infinite ordered set Λ, and to introduce some notation. Recall that we write the standard free right R-module on Λ as R^Λ. The *standard free left* R-module is written as ${}^\Lambda R$, and we view ${}^\Lambda R$ as the 'space of row vectors' over ${}^\Lambda R$ indexed by Λ. When the index set is the finite set $\{1, \ldots, n\}$, we use the more familiar notations R^n and ${}^n R$ for the free right and left modules respectively. The direct sum of copies of a general R-module M is written M^Λ or M^n, regardless of chirality.

3.1.14 Proposition

(a) *The two functors*

$$\mathcal{M}\mathrm{od}_R \times \mathcal{O}_{\mathcal{R}D} \longrightarrow \mathcal{M}\mathrm{od}_R$$

given respectively by

$$(M, \Lambda) \longmapsto M \otimes_R {}^\Lambda R$$

and

$$(M, \Lambda) \longmapsto M^\Lambda$$

are naturally isomorphic.

(b) *The two functors*

$${}_R\mathcal{M}\mathrm{od} \times \mathcal{O}_{\mathcal{R}D} \longrightarrow {}_R\mathcal{M}\mathrm{od}$$

given respectively by

$$(N, \Lambda) \longmapsto R^\Lambda \otimes_R N$$

and

$$(N, \Lambda) \longmapsto N^\Lambda$$

We now state a refined version of (3.1.4).

3.1.11 Proposition

(i) *The functor* $- \otimes_R R$ *is naturally isomorphic to the identity functor on* $\mathcal{M}_{OD_R}$.

(ii) *The functor* $R \otimes_R -$ *is naturally isomorphic to the identity functor on* ${}_R\mathcal{M}_{OD}$.

Proof

For each right R-module M, let

$$\eta_M : M \otimes_R R \xrightarrow{\cong} M, \qquad \eta_M(m \otimes r) = mr,$$

be the isomorphism of right R-modules defined in (3.1.4). If $\alpha : M' \to M$ is a homomorphism of right R-modules, then the formula $\alpha(m'r) = \alpha(m')r$ gives

$$\alpha\eta_{M'} = \eta_M(\alpha \otimes id_R);$$

thus η is a natural transformation of functors. □

As $M \otimes_R -$ and $- \otimes_R N$ are additive functors, the next result is immediate from (2.2.20).

3.1.12 Proposition

(a) *Suppose that* $M \cong M' \oplus M''$ *as a right* R*-module and that* N *is an* R-S*-bimodule. Then*

$$M \otimes_R N \cong (M' \otimes_R N) \oplus (M'' \otimes_R N)$$

as a right S*-module.*

(b) *Suppose that* $N \cong N' \oplus N''$ *as a left* R*-module and that* M *is a* T-R*-bimodule. Then*

$$M \otimes_R N \cong (M \otimes_R N') \oplus (M \otimes_R N'')$$

as a left T*-module.* □

Since the isomorphisms in parts (a) and (b) of the above proposition are evidently natural in N and M respectively, the definitions of a direct sum of functors (2.2.21) and of a natural isomorphism (1.3.3) lead to the following functorial interpretation of the proposition.

more convenient to compose it with the mirror functor Mir : ${}_{\mathbb{Z}}\mathcal{M}_{OD} \to \mathcal{M}_{OD\mathbb{Z}}$ and view $M \otimes_R -$ as having values in $\mathcal{A}_B = \mathcal{M}_{OD\mathbb{Z}}$.

Thus, for a homomorphism $\zeta : N' \to N$ of left R-modules, $id_M \otimes \zeta$ is to be viewed as a left operator on $M \otimes_R N'$, which is the convention most commonly met in the literature. With this notation, $M \otimes_R -$ is a covariant functor from ${}_R\mathcal{M}_{OD}$ to $\mathcal{M}_{OD\mathbb{Z}}$ that appears to 'reverse products'; in the terminology of (1.2.6), it is a contrachiral functor.

This approach is especially handy when we have both a right R-module homomorphism $\alpha : M' \to M$ and a left R-module homomorphism $\zeta : N' \to N$, in which case the composite

$$\alpha \otimes \zeta = (\alpha \otimes id_N)(id_{M'} \otimes \zeta) : M' \otimes_R N' \longrightarrow M \otimes_R N$$

cannot in general be regarded as a module homomorphism.

We use the convention that $id_M \otimes \zeta$ is to be considered as a left operator for the remainder of the present chapter.

Next, we note some naturality properties of the functors $M \otimes_R -$ and $- \otimes_R N$.

3.1.10 Proposition

Let T, R and S be arbitrary rings.

(i) *Let $\zeta : N' \to N$ be a homomorphism of left R-modules or of R-S-bimodules. Then ζ induces a natural transformation*

$$- \otimes_R \zeta : - \otimes_R N' \longrightarrow - \otimes_R N.$$

(ii) *Let $\alpha : M' \longrightarrow M$ be a homomorphism of right R-modules or of T-R-bimodules. Then α induces a natural transformation*

$$\alpha \otimes_R - : M' \otimes_R - \longrightarrow M \otimes_R -.$$

Proof

The argument is a matter of interpreting the requirements for a natural transformation (1.3.1) in terms of the notation of the tensor product. Fix ζ, and define η_M for each M in $\mathcal{M}_{OD_R}$ by

$$\eta_M = id_M \otimes \zeta : M \otimes_R N' \longrightarrow M \otimes_R N.$$

Given a homomorphism $\alpha : M' \to M$ of right R-modules, we need to check that $(\alpha \otimes N)\eta_{M'} = \eta_M(\alpha \otimes N')$. But both sides are simply $\alpha \otimes \zeta$. □

If we allow both M and N to vary simultaneously, the preceding result can be interpreted as saying that $- \otimes_R -$ is a bifunctor from $\mathcal{M}_{OD_R} \times {}_R\mathcal{M}_{OD}$ to the category $\mathcal{A}_B$ of abelian groups.

Proof

When Λ is finite, the assertion follows immediately from the preceding result, since additive functors preserve finite direct sums (2.2.20).

Now suppose that Λ is infinite. An element of either $M \otimes N$ or of the direct sum $\bigoplus_{\lambda\in\Lambda}(M_\lambda \otimes_R N)$ involves nonzero terms $m_\lambda \in M_\lambda$ for only a finite subset Σ of Λ, which makes it clear that there is a homomorphism

$$\phi : M \otimes_R N \longrightarrow \bigoplus_{\lambda\in\Lambda}(M_\lambda \otimes_R N)$$

and that ϕ is surjective.

Suppose that $\phi(x) = 0$. Then the preimage of $\phi(x)$ in $\bigoplus_\Sigma(M_\sigma \# N)$ belongs to the direct sum of relation subgroups $\bigoplus_\Sigma B(M_\sigma, N)$. Thus the preimage of x is already 0 in $(\bigoplus_\Sigma M_\sigma) \otimes N$, and hence $x = 0$. □

Any left R-module N is automatically an R-$\mathbb{Z}$-bimodule, so our discussion includes the case when N has not been attributed any particular structure as a right module. The functor $- \otimes_R N$ is then a functor from $\mathcal{M}\mathit{od}_R$ to $\mathcal{A}_B$, the category of abelian groups.

3.1.9 *Fixing the first argument*

We now review the results and notation that arise when the first module of a tensor product is considered to be an operator on the second.

Let M be a T-R-bimodule for some ring T. Then there is an additive functor

$$M \otimes_R - : {}_R\mathcal{M}\mathit{od} \longrightarrow {}_T\mathcal{M}\mathit{od}.$$

On modules,

$$(M \otimes_R -)(N) = M \otimes_R N,$$

with the T-module structure given by

$$t(m \otimes n) = tm \otimes n,$$

and for a homomorphism $\zeta : N' \to N$ of left R-modules,

$$M \otimes \zeta = (id_M \otimes \zeta) : m \otimes n' \longmapsto m \otimes n'\zeta.$$

When $id_M \otimes \zeta$ is to be considered as a left T-module homomorphism, as happens in (5.1.19) for example, we write $id_M \otimes \zeta$ as a right operator, in accord with our convention (1.1.4).

However, sometimes we are given only that M is a right R-module, and we wish to consider M to be a $\mathbb{Z}$-R-bimodule in the natural way. In this case, the functor $M \otimes_R -$ takes values in ${}_\mathbb{Z}\mathcal{M}\mathit{od}$, but, for applications, it is often

noted above, for any right R-module M the tensor product $M \otimes_R N$ is a right S-module under the rule

$$(m \otimes n)s = m \otimes ns, \ s \in S.$$

Thus, for any homomorphism $\alpha : M' \to M$ of right R-modules, the induced homomorphism

$$\alpha \otimes id_N : M' \otimes_R N \longrightarrow M \otimes_R N,$$

defined by

$$(\alpha \otimes id_N)(m' \otimes n) = \alpha m' \otimes n,$$

is a homomorphism of right S-modules.

Write $\alpha \otimes_R N = \alpha \otimes id_N$, so that the symbol $- \otimes_R N$ represents both a mapping from the objects of $\mathcal{M}_{OD_R}$ to those of $\mathcal{M}_{OD_S}$ and a collection of mappings

$$- \otimes_R N : \mathrm{Hom}_R(M', M) \longrightarrow \mathrm{Hom}_S(M' \otimes_R N, M \otimes_R N),$$

one for each pair of R-modules M', M. This information, together with the obvious properties

$$id_M \otimes id_N = id_{M \otimes N}$$

and

$$(\beta\alpha) \otimes id_N = (\beta \otimes id_N)(\alpha \otimes id_N),$$

defines a functor from $\mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_S}$, which we again denote $- \otimes_R N$. Moreover, $- \otimes_R N$ is additive because if $\alpha : M' \to M$ and $\alpha' : M' \to M$ are right R-module homomorphisms, then

$$(\alpha + \alpha') \otimes id_N = \alpha \otimes id_N + \alpha' \otimes id_N.$$

In summary, we have the following result.

3.1.7 Lemma

$- \otimes_R N$ is an additive covariant functor from $\mathcal{M}_{OD_R}$ to $\mathcal{M}_{OD_S}$. □

3.1.8 Proposition

Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ be a direct sum of right R-modules, where Λ is any ordered set. Then

$$M \otimes_R N \cong \bigoplus_{\lambda \in \Lambda} (M_\lambda \otimes_R N).$$

The next fact can be interpreted as saying that the tensor product is associative.

3.1.5 Proposition

Let R and S be rings and let L_R, ${}_RM_S$ and ${}_SN$ be modules as indicated. Then there is an isomorphism

$$(L \otimes_R M) \otimes_S N \cong L \otimes_R (M \otimes_S N).$$

Proof

The expected isomorphism is that under which $(\ell \otimes m) \otimes n$ corresponds to $\ell \otimes (m \otimes n)$. We must check that this correspondence does in fact give a well-defined homomorphism.

Take a fixed element n of N. The map

$$\beta(n) : L \times M \to L \otimes_R (M \otimes_S N),$$

$$(\ell, m) \longmapsto \ell \otimes (m \otimes n),$$

is biadditive and R-balanced (since $r(m \otimes n) = rm \otimes n$), and so it induces a homomorphism

$$\beta'(n) : L \otimes_R M \longrightarrow L \otimes_R (M \otimes_S N).$$

Allowing n to vary, we see that the maps $\beta'(n)$ define a biadditive S-balanced map $\beta' : (L \otimes_R M) \times N \to L \otimes_R (M \otimes_S N)$, which induces a homomorphism $\theta : (L \otimes_R M) \otimes_S N \to L \otimes_R (M \otimes_S N)$ as desired.

Reversing this construction, and appealing to the universal property of the tensor product, we confirm that θ has the expected inverse. □

3.1.6 Functorial properties of tensor products

We wish to see how the tensor product behaves as a functor when one of the modules M_R or ${}_RN$ is kept fixed and the other varies through ${}_R\mathcal{M}_D$ or $\mathcal{M}_{D_R}$. As we are mainly concerned with functors on categories of right modules, we give the arguments in the case where the second module in the tensor product is regarded as being constant. The parallel results where the first module is viewed as constant are noted from time to time, the details usually being left to the reader, who will easily get them right.

Let S be an arbitrary ring and suppose that N is an R-S-bimodule. As

It was given for abelian groups in [Whitney 1938], and subsequently for arbitrary rings and modules in [Bourbaki 1948]. It should be remarked that the tensor product is not in fact a product or coproduct as defined in (1.4.11) and (1.4.12), but a 'coend'. For a definition of this term and a category-theoretic analysis of the tensor product, see [Mac Lane 1971] p. 222.

3.1.3 Bimodule structures

Commonly, $M \otimes_R N$ is rather more than just an abelian group. For example, when N is an R-S-bimodule, then $M \otimes_R N$ becomes an S-module via

$$(m \otimes n)s = m \otimes ns \text{ for } m \in M,\ n \in N \text{ and } s \in S.$$

To verify this assertion, note that $M\#N$ can be made into a right S-module by setting $(m,n)s = (m,ns)$. Then $B(M,N)$ is an S-submodule of $M\#N$ and so the quotient abelian group $M \otimes_R N$ inherits a right S-module structure as indicated.

This method of constructing actions or mappings on $M \otimes_R N$ can be described informally as specifying what happens to elements of the form $m \otimes n$ and then extending the effect to linear combinations of such elements 'by linearity'. Properly speaking, the action or mapping must first be defined on the generators of $M\#N$, and then it must be verified that $B(M,N)$ is invariant, so that there is a corresponding induced action or mapping on $M \otimes_R N$. However, such verifications are usually trivial.

If M is a T-R-bimodule for some ring T, then $M \otimes_R N$ is a left T-module, and if M is a T-R-bimodule and N an R-S-bimodule, then $M \otimes_R N$ is a T-S-bimodule.

Our first computation is the following result, wherein R is viewed as an R-R-bimodule.

3.1.4 Proposition

Let M_R be a right R-module and let ${}_RN$ be a left R-module. Then

(i) *$M \otimes_R R \cong M$ as right R-modules*
and
(ii) *$R \otimes_R N \cong N$ as left R-modules.*

Proof

(i) Define homomorphisms $\eta : M \otimes_R R \to M$ and $\lambda : M \to M \otimes_R R$, by $m \otimes r \mapsto mr$ and $m \mapsto m \otimes 1$ respectively. Since $m \otimes r = mr \otimes 1$ (and $m1 = m$), η and λ are mutually inverse R-module homomorphisms.

(ii) Similar. □

3.1.2 The construction

To see that the tensor product exists, start with the free abelian group $M\#N$ generated by the cartesian product $M \times N$; elements of $M\#N$ can be uniquely expressed as sums of the form

$$z_1(m_1, n_1) + \cdots + z_k(m_k, n_k),$$

where $z_i \in \mathbb{Z}$, $m_i \in M$, $n_i \in N$ for $i = 1, \ldots, k$, and $k \geq 0$ (the vacuous sum with $k = 0$ is to be read as 0).

Next, let $B(M, N)$ be the subgroup of $M\#N$ which is generated by all expressions of the form

$$(m + m', n) - (m, n) - (m', n),$$

$$(m, n + n') - (m, n) - (m, n'),$$

and

$$(mr, n) - (m, rn),$$

with m, m' in M, n, n' in N and r in R.

We write

$$M \otimes_R N = (M\#N)/B(M, N),$$

and put $\tau(m, n) = m \otimes n$, where $\tau : M \times N \to M \otimes_R N$ is the canonical homomorphism. (Where it is felt safe to do so, we occasionally write $M \otimes N$, omitting the ring R.)

A typical element of the tensor product is then a sum of the form

$$z_1(m_1 \otimes n_1) + \cdots + z_k(m_k \otimes n_k),$$

and the identities

$$(m + m') \otimes n = m \otimes n + m' \otimes n,$$

$$m \otimes (n + n') = m \otimes n + m \otimes n',$$

and

$$mr \otimes n = m \otimes rn$$

hold among the generators of $M \otimes_R N$.

Thus τ is biadditive R-balanced as desired. It is straightforward to check that τ is universal among such maps in the sense described previously and thus that $M \otimes_R N$ is the tensor product.

Although applied mathematicians and physicists have long computed with tensors, the formal definition of the tensor product is comparatively recent.

Finally, we show that the category of modules over a generalized triangular matrix ring can be realized as a morphism category.

3.1.1 The definition

Let R be an arbitrary ring, let M_R be a right R-module and let ${}_RN$ be a left R-module.

A *biadditive R-balanced map* on the pair M, N is a function β from the cartesian product $M \times N$ to some abelian group A, with the following properties:

$$\beta(m + m', n) = \beta(m, n) + \beta(m', n),$$

$$\beta(m, n + n') = \beta(m, n) + \beta(m, n')$$

and

$$\beta(mr, n) = \beta(m, rn)$$

for all m, m' in M, n, n' in N and r in R.

The *tensor product* $M \otimes_R N$ is defined to be the universal object with respect to such maps (see (1.4.2)). This means that $M \otimes_R N$ is an abelian group and that there is a biadditive R-balanced map

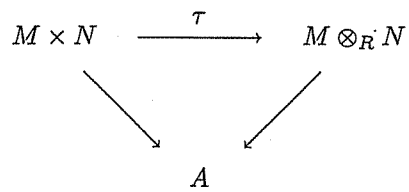
$$\tau : M \times N \longrightarrow M \otimes_R N$$

with the property that, given any β as above, there is a unique homomorphism

$$\alpha : M \otimes_R N \longrightarrow A$$

with $\alpha\tau = \beta$.

For a category-theoretic description, consider the category whose objects are the biadditive R-balanced maps $\beta : M \times N \to A$ on the pair M, N, a morphism from $\beta : M \times N \to A$ to $\beta' : M \times N \to A'$ being a homomorphism of abelian groups $\alpha : A \to A'$ with $\alpha\beta = \beta'$. Then $\tau : M \times N \to M \otimes_R N$ is the initial object in this category. The associated diagram is



Such a definition does not guarantee the existence of the required universal object, which we now establish.

3
CHANGE OF RINGS

In this chapter, we introduce a family of constructions that show off the functorial approach to great advantage. The common theme is that they allow us to pass from a category of modules over one ring to a category of modules over another ring. These constructions are based on the tensor product, which we introduce in the first section. Given a ring R, a right R-module M and a left R-module N, their tensor product $M \otimes_R N$ is an abelian group that has a certain universal property. If also M or N is a bimodule, then $M \otimes_R N$ inherits a module structure. Thus, by holding one of M and N fixed and varying the other, we obtain functors between module categories.

An important matter is whether or not these functors are exact; this leads to the concept of a flat module, which always gives an exact functor. Some fundamental examples and properties of flat modules are given in section 2.

One way of changing rings is by means of a ring homomorphism from one ring to another. Classically, such homomorphisms are inclusions and the functors are extension and restriction of scalars. In the third section, we look at these functors for a ring homomorphism in general.

3.1 THE TENSOR PRODUCT

The tensor product is a fundamental tool for constructing functors between various categories of modules. In this section, we give the definition and basic properties of the tensor product $M \otimes_R N$ of a right R-module M and a left R-module N, where R is an arbitrary ring. We then establish the functorial properties of the tensor product, which can be viewed variously as a functor of the second term N, with M being kept fixed, or of the first term M, with N kept fixed, or as a bifunctor. We discuss how the tensor product behaves on various module categories, and we investigate the adjointness between the tensor product and the Hom functor.

and a further relation

$$[M] = [N]$$

for each weak equivalence $M \to N$ in $\mathcal{C}$.

Show that when $\mathcal{C}$ is a Q-exact category considered as a category with cofibrations, and with weak equivalences chosen to be the isomorphisms of $\mathcal{C}$, then the group just defined coincides with the Grothendieck group of $\mathcal{C}$ as defined previously (2.4.9).

2.4.11 **Q-exact is strictly stronger than G-exact**

Let $\mathcal{C}$ be the full subcategory of $\mathcal{A}_{\mathcal{B}}$ whose objects are the finite abelian 2-groups. To choose $\mathrm{Ex}(\mathcal{C})$ such that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a G-exact category, let $\mathrm{Ex}(\mathcal{C})$ consist of all direct sums (as in GE3) of short exact sequences that are either split or have cyclic middle term. Thus, for example, the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sigma} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0,$$

$$\sigma : (a, b) \mapsto (a, 4b) \qquad \pi : (c, d) \mapsto d$$

(using representative integers to indicate residues) is admissible, while

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0,$$

$$\alpha : x \mapsto (x, 2x) \qquad \beta : (y, z) \mapsto 2y - z$$

is not. Show that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is not a Q-exact category, because it falls foul, for example, of QE3.

Let $\phi : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$, $a \mapsto 2a$. Show that $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is isomorphic to the push-out of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ over $\mathbb{Z}/2\mathbb{Z}$ with respect to ϕ, σ.

Deduce that $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ also violates $\mathrm{QE2}^{\mathrm{op}}$. Thus, the admissible monomorphisms of $\mathrm{Ex}(\mathcal{C})$ fail even to make $\mathcal{C}$ a category with cofibrations.

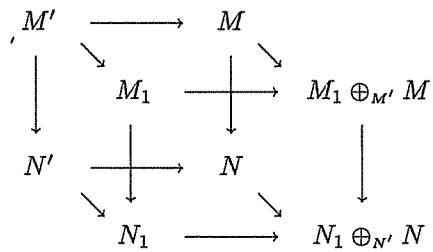
are *exact* in the sense that they preserve zero objects, cofibrations and push-out diagrams.

2.4.10 **Waldhausen categories**

(a) A *Waldhausen category* or *category of weak equivalences* is a category with cofibrations $\mathcal{C}$, together with a specified subcategory $w\mathcal{C}$. The morphisms in $w\mathcal{C}$ are the *weak equivalences*. The following axioms hold.

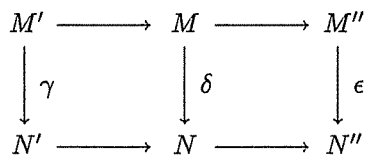
Weq1. Every isomorphism of $\mathcal{C}$ is a morphism of $w\mathcal{C}$.

Weq2. Glueing Axiom. For any commutative diagram in $\mathcal{C}$



in which each horizontal morphism is a cofibration and the three left-most vertical morphisms are weak equivalences, the fourth vertical morphism is also a weak equivalence.

Deduce that if



is a map of cofibration sequences in which γ and δ are weak equivalences, then ϵ is also a weak equivalence.

(b) Define the *Grothendieck group* $K_0(\mathcal{C})$ (which, strictly speaking, should be written $K_0(\mathcal{C}, \mathrm{co}\mathcal{C}, w\mathcal{C})$) of the Waldhausen category $\mathcal{C}$ to be the abelian group generated by an element $[M]$ for each object M of $\mathcal{C}$ and subject to a relation

$$[M] = [M'] + [M'']$$

for each cofibration sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'',$$

(c) Show that, by taking the admissible monomorphisms to be the cofibrations, any Q-exact category may be considered to be a category with cofibrations.

2.4.9 **Categories with cofibrations: constructions.**

Let $\mathcal{C}$ be a category with cofibrations. For each natural number n, define $\mathcal{F}_n\mathcal{C}$ to be the category for which an object is a sequence of cofibrations

$$M_0 \longrightarrow M_1 \longrightarrow \quad \cdots \quad \longrightarrow M_n$$

in $\mathcal{C}$, and for which a morphism is a natural transformation of such diagrams. One can also form a category $\mathcal{F}_n^+\mathcal{C}$ equivalent to $\mathcal{F}_n\mathcal{C}$ by taking for an object, in addition to the above data, a quotient map for each cofibration $M_i \to M_j$, $0 \leq i < j \leq m$. The following assertions generalize readily to $n > 1$.

(a) Show that $\mathcal{F}_1\mathcal{C}$ is a category with cofibrations, where a cofibration from $\alpha : M' \to M$ to $\beta : N' \to N$ in $\mathrm{co}\mathcal{F}_1\mathcal{C}$ is, by definition, any pair γ, δ of morphisms in $\mathcal{C}$ for which

(i) there is a commutative square

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ N' & \xrightarrow{\beta} & N \end{array}$$

(ii) both γ and the induced morphism $N' \oplus_{M'} M \to N$ are cofibrations in $\mathcal{C}$.

Deduce that δ is a cofibration.

(b) Show that an object of $\mathcal{F}_1^+\mathcal{C}$ is a cofibration sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

in which M'' is a specified cokernel of α.

Use (a) and the equivalence $\mathcal{F}_1^+\mathcal{C} \to \mathcal{F}_1\mathcal{C}$ to regard $\mathcal{F}_1^+\mathcal{C}$ as a category with cofibrations. Show that the three functors

$$s, t, q : \mathcal{F}_1^+\mathcal{C} \longrightarrow \mathcal{C}$$

which map a cofibration sequence to its respective terms

$$M', \qquad M, \qquad M''$$

The cofibrations are, by definition, the morphisms of a subcategory $\mathrm{co}\mathcal{C}$ of $\mathcal{C}$ which has the same objects as $\mathcal{C}$ and is subject to the following three axioms.

Cof1. Every isomorphism of $\mathcal{C}$ is in $\mathrm{co}\mathcal{C}$.

Cof2. Every morphism $0 \to C$ in $\mathcal{C}$ belongs to $\mathrm{co}\mathcal{C}$.

Cof3. Cofibrations are closed under cobase change, meaning that, for any cofibration $M' \xrightarrow{\alpha} M$ and any morphism $\phi : M' \to N$ in $\mathcal{C}$, the bottom row of the push-out diagram

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & M \\ \downarrow & & \downarrow \\ N & \xrightarrow{\overline{\alpha}} & N \oplus_{M'} M \end{array}$$

is also in $\mathrm{co}\mathcal{C}$. (This implies that $N \oplus_{M'} M$ is an object of $\mathcal{C}$.)

With these axioms, the pair $(\mathcal{C}, \mathrm{co}\mathcal{C})$ is then called a *category with cofibrations*.

(a) Show that the push-out $0 \oplus_{M'} M$ of a cofibration $\alpha : M' \to M$ and the zero morphism $0 : M' \to 0$ is (isomorphic to) the cokernel $\mathrm{Cok}\,\alpha$ of α. Thus, by Cof3, every cofibration is part of a cofibration sequence

$$M' \xrightarrow{\alpha} M \longrightarrow \mathrm{Cok}\,\alpha.$$

The canonical morphism $M \to \mathrm{Cok}\,\alpha$ is known as a *quotient map*.

(b) Show that cofibration sequences are closed under cobase change, meaning that, for any cofibration sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

and any morphism $\phi : M' \to N$ in $\mathcal{C}$, there is a push-out diagram

$$\begin{array}{ccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\ \downarrow & & \downarrow & & \downarrow \\ N & \xrightarrow{\overline{\alpha}} & N \oplus_{M'} M & \xrightarrow{\overline{\beta}} & \mathrm{Cok}\,\overline{\alpha} \end{array}$$

in which the lower row is also a cofibration sequence and the morphism from M'' to $\mathrm{Cok}\,\overline{\alpha}$ is an isomorphism. (Compare with Axiom $\mathrm{QE2}^{\mathrm{op}}$, (2.4.10).)

2.4.5 Let $\mathcal{O}$ be a commutative domain. A ring R is said to be an *$\mathcal{O}$-order* if there is an injective ring homomorphism from $\mathcal{O}$ into the centre $Z(R)$ of R, and if, as both a left and right $\mathcal{O}$-module, R is finitely generated over $\mathcal{O}$ and torsion-free. The ring $\mathcal{O}$ is called the coefficient ring. If the coefficient ring can be taken as granted, we often refer to R simply as an *order*.

Given an $\mathcal{O}$-order R, let $\mathcal{T}_{or\mathcal{O},R}$, $\mathcal{T}_{\mathcal{O},R}$ and $\mathcal{TF}_{\mathcal{O},R}$ denote the full subcategories of $\mathcal{M}_{od_R}$ whose objects are respectively the $\mathcal{O}$-torsion, finitely generated $\mathcal{O}$-torsion and finitely generated $\mathcal{O}$-torsion-free modules.

Show that all these categories are repletely G-exact categories. In each case, show how to form an $\mathcal{O}$-relative exact category by taking as short exact sequences those which are $\mathcal{O}$-split.

2.4.6 Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between G-exact categories. Show that F and G are both exact if and only if $F \oplus G : \mathcal{C} \to \mathcal{D}$ is exact, and that

$$K_0(F \oplus G) = K_0(F) + K_0(G) : K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D}).$$

2.4.7 Let $\mathcal{C}$ and $\mathcal{D}$ be G-exact categories. Show that

$$K_0(\mathcal{C} \times \mathcal{D}) \cong K_0(\mathcal{C}) \oplus K_0(\mathcal{D}).$$

Generalize this result to an arbitrary finite direct product of categories, and to infinite direct sums of categories (2.3.19).

2.4.8 **Categories with cofibrations: axioms**

In [Waldhausen 1985], Quillen's conditions QE1-QE3 (2.4.10) on the admissible exact sequences of a Q-exact category are replaced by weaker requirements which still suffice for the construction of (higher) K-theory, with the advantage that a wider range of categories can be used. This and the subsequent exercises outline the definitions and some initial calculations.

Let $\mathcal{C}$ be a category with zero object 0. The notion of an admissible monomorphism is replaced by that of a *cofibration* (which need not in fact be a monomorphism), and the counterpart of an admissible exact sequence is a *cofibration sequence*

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

in which α is a cofibration and (M'', β) is a cokernel of α (as defined in (1.4.15)).

Generalize these statements for arbitrary finite direct products of G-exact categories, and for infinite direct sums of categories as in (2.3.19).

Let $\mathcal{K}$ be a field. Show that $\mathcal{M}_{\mathcal{OR}}(\mathcal{M}_{\mathcal{K}})$ is not split G-exact, even though $\mathcal{M}_{\mathcal{K}}$ is.

2.4.3 Let $\mathcal{C}$ be a G-exact category, and let $N_1 \to N_2$ be an isomorphism in $\mathcal{C}$. Show that if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\mathrm{Ex}(\mathcal{C})$, then the short exact sequences

$$0 \longrightarrow M' \oplus N_1 \longrightarrow M \oplus N_2 \longrightarrow M'' \longrightarrow 0$$

and

$$0 \longrightarrow M' \longrightarrow M \oplus N_1 \longrightarrow M'' \oplus N_2 \longrightarrow 0$$

are also in $\mathrm{Ex}(\mathcal{C})$.

2.4.4 Show that axioms QE1 and QE2 together imply axiom GE3, as follows. Take short exact sequences

(1) $$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and

(2) $$0 \longrightarrow N' \longrightarrow N \xrightarrow{\nu} N'' \longrightarrow 0$$

in $\mathrm{Ex}(\mathcal{C})$, construct firstly the pull-back of (1) by the admissible epimorphism

$$(id, 0) : M'' \oplus N'' \longrightarrow M'',$$

and then the pull-back of the resulting sequence by the admissible epimorphism

$$id \oplus \nu : M'' \oplus N \longrightarrow M'' \oplus N''$$

to obtain a short exact sequence in $\mathrm{Ex}(\mathcal{C})$

(3) $$0 \longrightarrow M' \longrightarrow W \xrightarrow{\omega} M'' \oplus N \longrightarrow 0.$$

Show that $\theta = (id \oplus \nu)\omega : W \to M'' \oplus N''$ is admissible, and that the corresponding short exact sequence

$$0 \longrightarrow L \longrightarrow W \xrightarrow{\theta} M'' \oplus N'' \longrightarrow 0$$

is isomorphic to the direct sum of (1) and (2).

[Quillen 1973], §2, observes that the above corollary has a converse: if a Q-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is (skeletally) small, then there exists a repletely exact abelian category in which $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is sub-Q-exact. This is shown by mimicking the proofs of the earlier embedding theorems for additive categories, which were outlined in (2.3.22). The corollary and its converse enable us to define a Q-exact category as a sub-Q-exact category of a repletely exact abelian category. The new definition has the advantage that its axioms are relatively simple.

Observe that a subcategory $\mathcal{C}$ of a Q-exact category $\mathcal{A}$ may be Q-exact without being a sub-Q-exact category of $\mathcal{A}$. For example, the category $\mathcal{ASS}_R$ of Artinian semisimple modules over a ring R is an abelian category, and Q-exact; it is in fact both split and repletely exact. The inclusion functor from $\mathcal{ASS}_R$ to $\mathcal{M}_R$ is exact, but $\mathcal{ASS}_R$ is not a sub-Q-exact category of $\mathcal{M}_R$, unless R is Artinian semisimple (which makes Condition SubQ1 hold.)

The reader may wish to verify the accuracy of the following listing of Q-exact categories. Here, Q-exactness follows immediately from the corollary above.

2.4.16 Theorem

For any ring R, $\mathcal{M}\mathit{OD}_R$ is a repletely exact abelian category, and the following are (repletely) sub-Q-exact categories and therefore Q-exact.

(i) *$\mathcal{P}_R$, the category of finitely generated projective modules.*
(ii) *$\mathcal{F}_R$, the category of free modules of finite rank.*
(iii) *$\mathcal{A}\mathit{RT}_R$, the category of Artinian modules.*
(iv) *$\mathcal{N}\mathit{OETH}_R$, the category of Noetherian modules.*
(v) *$\mathcal{M}_R$, the category of finitely generated modules.*
(vi) *$\mathcal{M}\mathit{OD}_T$, for any extension ring T of R.* □

Exercises

Note. These exercises refer (mostly) to G-exact categories. The enthusiastic reader may wish to extend them to Q-exact categories where appropriate.

2.4.1 Let $\mathcal{C}$ be a G-exact category. Show that $\mathcal{C}^{\mathrm{op}}$ is also a G-exact category.

2.4.2 Let $\mathcal{C}$ and $\mathcal{D}$ be G-exact categories. Show that there are canonical ways to make $\mathcal{C} \times \mathcal{D}$ and $\mathcal{M}\mathit{OR}\,\mathcal{C}$ into G-exact categories.

Prove that $\mathcal{C} \times \mathcal{D}$ is split or repletely G-exact if and only if both $\mathcal{C}$ and $\mathcal{D}$ are split or repletely G-exact respectively.

Although the conditions for a Q-exact category appear formidable, they are usually fairly easy to check in any given instance. For example, the checking of the claims in the following lemma is straightforward.

2.4.12 Lemma

(a) *Any split G-exact category is Q-exact.*

(b) *Let $\mathcal{C}$ be an abelian category and let $\mathrm{Ex}(\mathcal{C})$ be the class of all short exact sequences of $\mathcal{C}$. Then the repletely G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is Q-exact.*

□

Because of (b) above, we use the term *repletely exact abelian* to describe a repletely G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ with $\mathcal{C}$ abelian.

2.4.13 Exact subcategories

To see when a subcategory of a Q-exact category is also Q-exact, we need a preliminary definition. Let $(\mathcal{A}, \mathrm{Ex}(\mathcal{A}))$ be a Q-exact category. Then $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a *sub-Q-exact category* of $(\mathcal{A}, \mathrm{Ex}(\mathcal{A}))$ if the following conditions are satisfied.

SubQ1. $\mathcal{C}$ is a full additive subcategory of $\mathcal{A}$ with the property that whenever

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

is a short exact sequence in $\mathcal{A}$ such that A' and A'' are isomorphic to objects of $\mathcal{C}$, then A is also isomorphic to an object of $\mathcal{C}$.

SubQ2. $\mathrm{Ex}(\mathcal{C})$ is the class of all short exact sequences in $\mathrm{Ex}(\mathcal{A})$ whose terms belong to $\mathcal{C}$.

Condition SubQ1 is often rephrased as: $\mathcal{C}$ is *closed under extensions*, or *extension closed.* Condition SubQ2 implies that the inclusion functor from $\mathcal{C}$ to $\mathcal{A}$ is exact.

In practice, it is usually obvious from the nature of $\mathcal{C}$ that the first condition is satisfied, while the canonical choice of $\mathrm{Ex}(\mathcal{C})$ is that suggested by the second condition. Thus the following theorem gives a widely applicable condition for Q-exactness. The proof is by routine verification of the axioms.

2.4.14 Theorem

Any sub-Q-exact category of a Q-exact category is again Q-exact. □

2.4.15 Corollary

Any sub-Q-exact category of a repletely exact abelian category is Q-exact.

□

QE2$^{\mathrm{op}}$. Suppose that the short exact sequence

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\mu} M \xrightarrow{\lambda} M'' \longrightarrow 0$$

is in $\mathrm{Ex}(\mathcal{C})$ and that $\phi : M' \to N'$ is a morphism in $\mathcal{C}$. Then the push-out exact sequence

$$\phi_*\mathbf{E} \qquad 0 \longrightarrow N' \xrightarrow{\alpha} N' \oplus_{L'} M \xrightarrow{\beta} M'' \longrightarrow 0$$

also belongs to $\mathrm{Ex}(\mathcal{C})$. (That is, $\mathrm{Ex}(\mathcal{C})$ is closed under cobase change.)

QE3. Suppose that the morphism $\beta : M \to M''$ in $\mathcal{C}$ has a kernel in $\mathcal{C}$, and that there is a morphism $\varphi : L \to M$ such that $\beta\varphi : L \to M''$ is an admissible epimorphism. Then β is already an admissible epimorphism.

QE3$^{\mathrm{op}}$. Suppose that the morphism $\alpha : M' \to M$ in $\mathcal{C}$ has a cokernel in $\mathcal{C}$, and that there is a morphism $\psi : M \to N$ such that $\psi\alpha : M' \to N$ is an admissible monomorphism. Then α is already an admissible monomorphism.

2.4.11 Comments on the axioms

(i) Pull-backs and push-outs in an abstract category are defined by their universal properties, which are given for modules in ([BK: IRM] Proposition 2.4.9), and in Exercises 1.4.10 and 1.4.11 for categories in general. The definitions of the pull-back sequence $\theta^*\mathbf{E}$ and the push-out sequence $\phi_*\mathbf{E}$ follow those given for modules in Exercise 2.1.7.

(ii) In the statement of the axioms, we take for granted the fact that various constructions on short exact sequences yield short exact sequences. This is easily verified in a module category. In an abstract category, the verification can be made by using universal properties, or, more conveniently, by invoking the Embedding Theorems (2.3.22).

(iii) Since the underlying category $\mathcal{C}$ is assumed to be additive, rather than abelian, part of the requirement imposed by the axioms is that the various pull-backs and push-outs are objects in $\mathcal{C}$ (note Exercises 2.3.6 and 2.3.7).

(iv) It is not difficult to show that these conditions make the 'elementary' condition GE3 redundant – see Exercise 2.4.4. The fact that the Q-conditions are stronger than the G-conditions is illustrated in Exercise 2.4.11.

We call a Q-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ *repletely Q-exact* when $\mathrm{Ex}(\mathcal{C})$ comprises all short exact sequences in $\mathcal{C}$, and *split Q-exact* when $\mathrm{Ex}(\mathcal{C})$ consists only of the split exact sequences in $\mathcal{C}$.

quick introduction), this is a convenient place to list the additional conditions on the class $\mathrm{Ex}(\mathcal{C})$ that are required by Quillen's theory. We take our definition directly from [Quillen 1973], §2.

A *Q-exact category* is a G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ in which the class $\mathrm{Ex}(\mathcal{C})$ of admissible exact sequences satisfies the following requirements in addition to GE1–3. Recall that a monomorphism or epimorphism in $\mathcal{C}$ is admissible if it occurs in a short exact sequence that belongs to $\mathrm{Ex}(\mathcal{C})$.

QE1. $\mathrm{Ex}(\mathcal{C})$ is *closed under composition of admissible epimorphisms* : if

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

and

$$0 \longrightarrow N' \xrightarrow{\gamma} M'' \xrightarrow{\delta} N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so is the short exact sequence

$$0 \longrightarrow M \times_{M''} N' \longrightarrow M \xrightarrow{\delta\beta} N'' \longrightarrow 0.$$

QE1$^{\mathrm{op}}$. $\mathrm{Ex}(\mathcal{C})$ is *closed under composition of admissible monomorphisms* : if

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

and

$$0 \longrightarrow M \xrightarrow{\gamma} N \xrightarrow{\delta} N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so is the short exact sequence

$$0 \longrightarrow M' \xrightarrow{\gamma\alpha} N \longrightarrow M'' \oplus_M N \longrightarrow 0.$$

QE2. Suppose that the short exact sequence

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is in $\mathrm{Ex}(\mathcal{C})$ and that $\theta : L'' \to M''$ is a morphism in $\mathcal{C}$. Then the pull-back exact sequence (cf. Exercise 1.4.11 and [BK: IRM] (2.4.10))

$$\theta^*\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\mu} M \times_{M''} L'' \xrightarrow{\lambda} L'' \longrightarrow 0$$

also belongs to $\mathrm{Ex}(\mathcal{C})$. (That is, $\mathrm{Ex}(\mathcal{C})$ is closed under base change.)

initially applied (in the notation $K(X)$) to the category of coherent sheaves on an algebraic variety X and to the category of vector bundles over a topological space X. Grothendieck chose the symbol K (for the German 'Klasse') to avoid $C(X)$, already in use as the ring of functions on X. For further history, see [Bak 1987].

Although we do not give any computations of Grothendieck groups in this text, we can give a plausibility argument to illustrate how the choice of the class $\mathrm{Ex}(\mathcal{C})$ influences $K_0(\mathcal{C})$.

Consider the category $\mathcal{FA}_B$ of finite abelian groups, which is $\mathcal{A}_{RT\mathbb{Z}}$ in disguise. The Primary Decomposition Theorem (2.3.20 – F) shows that the irreducible modules in $\mathcal{FA}_B$ are the cyclic groups $\mathbb{Z}/p\mathbb{Z}$ of prime order p. Since any finite abelian group A has a composition series with factors of the form $\mathbb{Z}/p\mathbb{Z}$ for various primes p, the symbol $[A]$ in $K_0(\mathcal{FA}_B)$ can be written as a sum of symbols $[\mathbb{Z}/p\mathbb{Z}]$, and it can be shown that $K_0(\mathcal{FA}_B)$ is the free abelian group on the set

$$\{[\mathbb{Z}/p\mathbb{Z}] \mid p \text{ prime}\}.$$

However, if we work in the Grothendieck group $K_0(\mathcal{FA}_B{}^{\oplus})$ of the G-exact category $\mathcal{FA}_B{}^{\oplus}$, in which only split exact sequences are permitted, then the symbol $[A]$ can be reduced to symbols corresponding to indecomposable finite abelian groups, which have the form $\mathbb{Z}/p^k\mathbb{Z}$ for p prime, $k \geq 1$, but no further reduction is possible. As might be anticipated, $K_0(\mathcal{FA}_B{}^{\oplus})$ is the free abelian group on the set

$$\{[\mathbb{Z}/p^k\mathbb{Z}] \mid p \text{ prime}, k \geq 1\}.$$

Granted these computations, we can see that the identity functor on the additive category $\mathcal{FA}_B$ induces an exact functor I from the split G-exact category $\mathcal{FA}_B{}^{\oplus}$ to the repletely G-exact category $\mathcal{FA}_B$, but that

$$K_0(I) : K_0(\mathcal{FA}_B{}^{\oplus}) \longrightarrow K_0(\mathcal{FA}_B)$$

is not injective, since

$$K_0(I)([\mathbb{Z}/p^k\mathbb{Z}]) = k[\mathbb{Z}/p\mathbb{Z}] = K_0(I)([(\mathbb{Z}/p\mathbb{Z})^k]).$$

2.4.10 Q-exact categories

In a fundamental paper, [Quillen 1973], Quillen has shown how to construct higher dimensional analogues, $K_1(\mathcal{C})$, $K_2(\mathcal{C}), \ldots$, of the Grothendieck group of a suitably defined exact category. Although we cannot give any indication of Quillen's construction in this text (see [Rosenberg 1996] pp. 289–297 for a

2.4.8 Proposition

Suppose that $\mathcal{C}$ is a split G-exact category and that $\mathcal{D}$ is a G-exact category. Then any additive functor from $\mathcal{C}$ to $\mathcal{D}$ is exact. □

The above result shows that, when $\mathcal{C}$ is split G-exact, condition EFun1 implies EFun2. More surprisingly, when $\mathcal{C}$ and $\mathcal{D}$ are abelian, EFun2 always implies EFun1, as can be seen from Exercise 2.3.11.

2.4.9 The Grothendieck group

We now trespass into K-theory to give the construction of the Grothendieck group of a G-exact category. This will at least serve to motivate our definition of a G-exact category.

Let $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ be a G-exact category, and assume that $\mathcal{C}$ is small, or that $\mathcal{C}$ has a small skeleton (1.3.15). For an object C of $\mathcal{C}$, let $\langle C\rangle$ denote the isomorphism class of C, so that $\langle C\rangle = \langle C'\rangle$ precisely when $C \cong C'$. Then the isomorphism classes of objects of $\mathcal{C}$ constitute a set $\mathrm{Is}(\mathcal{C})$, and we can form the free abelian group $\mathrm{Fr}(\mathcal{C})$ on $\mathrm{Is}(\mathcal{C})$. Let $\mathrm{Rel}(\mathcal{C})$ be the subgroup of $\mathrm{Fr}(\mathcal{C})$ generated by all expressions of the form

$$\langle C\rangle - \langle C'\rangle - \langle C''\rangle\,,$$

one for each short exact sequence

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

belonging to $\mathrm{Ex}(\mathcal{C})$.

The *Grothendieck group* $K_0(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ of $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is the quotient group $\mathrm{Fr}(\mathcal{C})/\mathrm{Rel}(\mathcal{C})$. It is usual to omit to mention the class of exact sequences and write simply $K_0(\mathcal{C})$.

Clearly, $K_0(\mathcal{C})$ is an abelian group because $\mathrm{Fr}(\mathcal{C})$ is. The image of $\langle C\rangle$ in $K_0(\mathcal{C})$ is denoted $[C]$. By construction, a short exact sequence as above gives the equality

$$[C] = [C'] + [C''] \text{ in } K_0(\mathcal{C}).$$

It is evident that an exact functor $F : \mathcal{C} \to \mathcal{D}$ between G-exact categories gives a homomorphism

$$K_0(F) : K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D})$$

of abelian groups.

The Grothendieck group K_0 was introduced by A. Grothendieck in 1957 in letters to J.-P. Serre that formed the basis of [Borel & Serre 1958]. It was

the use of 'semisimple' suggests that results about semisimple modules ought to hold in the category. However, there are many results, such as the Artinian Splitting Theorem ([BK: IRM] (4.1.17)), which hold in $\mathcal{ASS}_R$ but do not necessarily hold in a split exact category such as $\mathcal{P}_R$.

2.4.6 Exact functors

Let $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ and $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ be G-exact categories. An *exact functor* from $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ to $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which satisfies the following conditions.

EFun1. F is an additive functor.
EFun2. If

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is a short exact sequence belonging to $\mathrm{Ex}(\mathcal{C})$, then

$$F\mathbf{E} \qquad 0 \longrightarrow FM' \xrightarrow{F\alpha} FM \xrightarrow{F\beta} FM'' \longrightarrow 0$$

must belong to $\mathrm{Ex}(\mathcal{D})$.

Notice that requirement EFun2 is, in effect, two conditions – firstly, $F\mathbf{E}$ must be a short exact sequence in $\mathcal{D}$, and then it must also belong to the distinguished class $\mathrm{Ex}(\mathcal{D})$. In the literature, $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ and $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ are commonly both repletely G-exact. One then speaks of F simply as being an exact functor from $\mathcal{C}$ to $\mathcal{D}$.

2.4.7 Examples

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are G-exact categories and that both have the same underlying additive category $\mathcal{B}$. Then the identity functor on $\mathcal{B}$ will induce an exact functor from $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ to $(\mathcal{D}, \mathrm{Ex}(\mathcal{D}))$ if and only if $\mathrm{Ex}(\mathcal{C})$ is a subclass of $\mathrm{Ex}(\mathcal{D})$. In particular, the identity functor fails to be an exact functor from $\mathcal{M}_R$ to $\mathcal{M}_R^{\oplus}$ (unless R is Artinian semisimple – see (2.3.9) and (2.4.3)).

Conversely, an exact functor between G-exact categories need not preserve all short exact sequences; examples are given by the homomorphism functors $\mathrm{Hom}(L, -)$ on $\mathcal{M}\mathit{od}_R^{\oplus}$ with L not projective (2.1.8). However, any additive functor preserves split exact sequences (2.2.20) and so will act as an exact functor on a split G-exact category. We record this fact for reference.

2.4.3 Theorem

For any ring R, the following are repletely G-exact categories of right R-modules.

(i) $\mathcal{M}_{OD_R}$ *itself.*
(ii) $\mathcal{P}_R$, *the category of finitely generated projective modules.*
(iii) $\mathcal{F}_R$, *the category of free modules of finite rank.*
(iv) $\mathcal{A}_{RT_R}$, *the category of Artinian modules.*
(v) $\mathcal{N}_{OETH_R}$, *the category of Noetherian modules.*
(vi) $\mathcal{M}_R$, *the category of finitely generated modules.*
(vii) $\mathcal{ASS}_R$, *the category of Artinian semisimple modules.*

The categories $\mathcal{P}_R$, $\mathcal{F}_R$ and $\mathcal{ASS}_R$ are also split G-exact.

Proof

By (2.2.14), $\mathcal{P}_R$, $\mathcal{F}_R$ and $\mathcal{M}_R$ are additive categories, and by (2.3.8), $\mathcal{N}_{OETH_R}$, $\mathcal{A}_{RT_R}$ and $\mathcal{ASS}_R$ are also additive (in fact, abelian). Assertions (i) to (vii) follow by trivial verifications.

The definition of a projective module shows that any short exact sequence in $\mathcal{P}_R$, and hence in $\mathcal{F}_R$, must be split, while any short exact sequence in $\mathcal{ASS}_R$ is split by the Artinian Splitting Theorem ([BK: IRM] (4.1.17)). □

2.4.4 Relative exact categories

We next give some illustrations of more subtle choices for $\mathcal{C}$ or $\mathrm{Ex}(\mathcal{C})$.

Suppose that S is a subring of R. One can then define a G-exact category $\mathcal{M}_{R,S}$, the *S-relative category*, to be the additive category $\mathcal{M}_R$, but with $\mathrm{Ex}(\mathcal{M}_{R,S})$ consisting of all those short exact sequences in $\mathcal{M}_R$ that are split when regarded as short exact sequences of S-modules. For examples, see Exercise 2.4.5.

Similarly, it is sometimes useful to consider the full subcategory $\mathcal{D}$ of $\mathcal{M}_R$ whose objects are projective as S-modules; then all short exact sequences in $\mathcal{D}$ are split over S.

The common generalization of both these examples is given by an additive subcategory $\mathcal{C}$ of an additive category $\mathcal{A}$ ($\mathcal{M}_S$ in the preceding examples), with $\mathrm{Ex}(\mathcal{C})$ comprising all those short exact sequences in $\mathcal{C}$ that are split in $\mathcal{A}$.

2.4.5 On terminology

The term 'semisimple category' is sometimes used by K-theorists instead of 'split exact category'. However, we feel that 'split' is more descriptive, since

in which all the vertical arrows are isomorphisms in $\mathcal{C}$ and the top sequence is in $\mathrm{Ex}(\mathcal{C})$, then so also is the bottom sequence.

GE3. $\mathrm{Ex}(\mathcal{C})$ is *closed under direct sums*: if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

are in $\mathrm{Ex}(\mathcal{C})$, so also is

$$0 \longrightarrow M' \oplus N' \longrightarrow M \oplus N \longrightarrow M'' \oplus N'' \longrightarrow 0.$$

The *admissible exact sequences* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ are the short exact sequences belonging to $\mathrm{Ex}(\mathcal{C})$. An *admissible monomorphism* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is a monomorphism $\alpha : M' \to M$ in $\mathcal{C}$ that occurs in some short exact sequence belonging to $\mathrm{Ex}(\mathcal{C})$, and an *admissible epimorphism* in $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is similarly one occurring in an admissible short exact sequence. Thus any admissible monomorphism has a cokernel, which must be an admissible epimorphism, and dually.

In many contexts where there is a standard choice for the class $\mathrm{Ex}(\mathcal{C})$, a G-exact category $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ is referred to simply as $\mathcal{C}$.

2.4.2 Split and repletely G-exact categories

Given an additive category $\mathcal{C}$, there are two extreme choices of $\mathrm{Ex}(\mathcal{C})$ that make $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ into a G-exact category.

On the one hand, we can take $\mathrm{Ex}(\mathcal{C})$ to be the class of all short exact sequences in $\mathcal{C}$. In this case, we say that $\mathcal{C}$ is a *repletely G-exact category*. As we frequently wish to consider an additive category $\mathcal{C}$ to be a repletely G-exact category, we use the same symbol for both and omit $\mathrm{Ex}(\mathcal{C})$ from the notation.

At the other extreme, we can take $\mathrm{Ex}(\mathcal{C})$ to be the class of all split short exact sequences with terms in $\mathcal{C}$. By condition GE1, this is the smallest possible choice for $\mathrm{Ex}(\mathcal{C})$, and a series of routine verifications establishes that axioms GE2 and GE3 hold. In this case, we say that the category is *split G-exact* and we use the notation $\mathcal{C}^{\oplus}$ to indicate that we view $\mathcal{C}$ as a G-exact category in this way.

In some categories, notably the categories $\mathcal{P}_R$ and $\mathcal{F}_R$, these extremal choices coincide, since any short exact sequence in them must be split.

We note next that many familiar module categories are G-exact.

suffice for many purposes, in particular, the construction of the Grothendieck group. The more refined advanced properties are designed to ensure that Quillen's Q-construction can be carried out, as needed for a definition of higher K-theory. We do not discuss Quillen's construction in this text.

Several authors have formulated definitions of a 'category with admissible exact sequences', sometimes from the point of view of homological algebra, sometimes from that of K-theory. [Mac Lane 1975], Ch. XII §4, introduces *proper* classes of exact sequences in an abelian category. Such a class satisfies conditions GE1–3 of (2.4.1) together with QE1, QE1^{op}, QE2 and QE2^{op} of (2.4.10). Other variations can be found in [Buchsbaum 1959], [Buchsbaum 1960] and [Heller 1965]. The definition of a 'K-theoretic' exact category that we adopt here appears in [Quillen 1973].

To aid the exposition, we use the term 'G-exact' for a category which satisfies the elementary conditions that allow the construction of the Grothendieck group, and 'Q-exact' for a category satisfying Quillen's conditions. The reason for these neologisms is that the term 'exact category' is already used in category theory with a different meaning ([Mitchell 1965], p. 18), and we feel it useful to preserve a distinction in an expository text. On the other hand, some authors of papers on K-theory use 'exact' to mean 'Q-exact' as in this text.

As usual, we discuss right categories only; it is clear that everything works also for left categories.

2.4.1 *G-exact categories*

A *G-exact category* is a pair $(\mathcal{C}, \mathrm{Ex}(\mathcal{C}))$ consisting of an additive category $\mathcal{C}$ together with a specified class $\mathrm{Ex}(\mathcal{C})$ of short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

all of whose terms M', M and M'' are in $\mathcal{C}$. The class $\mathrm{Ex}(\mathcal{C})$ must satisfy the following requirements.

GE1. Every split exact sequence in $\mathcal{C}$ belongs to $\mathrm{Ex}(\mathcal{C})$.

GE2. $\mathrm{Ex}(\mathcal{C})$ is *closed under isomorphism*: if we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\mu'} & M & \xrightarrow{\mu} & M'' & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & N' & \xrightarrow{\nu'} & N & \xrightarrow{\nu} & N'' & \longrightarrow & 0 \end{array}$$

objects, together with the corresponding monomorphisms and epimorphisms, then F is additive.

2.3.12 **The Three Lemma**

Extend the Three Lemma ([BK: IRM] Exercise 2.4.1) from modules to an arbitrary abelian category $\mathcal{A}$: if $\alpha : L \to M$ and $\beta : M \to N$ are any morphisms in $\mathcal{A}$, there is an induced exact sequence

$$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta\alpha \to \operatorname{Ker}\beta \to \operatorname{Cok}\alpha \to \operatorname{Cok}\beta\alpha \to \operatorname{Cok}\beta \to 0.$$

2.3.13 **The Snake Lemma**

Extend the Snake Lemma ([BK: IRM] Exercise 2.4.2) from modules to an arbitrary abelian category $\mathcal{A}$: given a commuting diagram of objects and morphisms in $\mathcal{A}$ with both rows exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{\mu'} & M & \xrightarrow{\mu} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\
0 & \longrightarrow & N' & \xrightarrow{\nu'} & N & \xrightarrow{\nu} & N'' & \longrightarrow & 0
\end{array}$$

construct a *connecting morphism* $\delta : \operatorname{Ker}\alpha'' \to \operatorname{Cok}\alpha'$ which fits into an exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Ker}\alpha' & \longrightarrow & \operatorname{Ker}\alpha & \longrightarrow & \operatorname{Ker}\alpha'' \\
 & & & & & \overset{\delta}{\searrow} & \\
 & & \operatorname{Cok}\alpha' & \longrightarrow & \operatorname{Cok}\alpha & \longrightarrow & \operatorname{Cok}\alpha'' \longrightarrow 0
\end{array}$$

2.4 EXACT CATEGORIES

We now introduce a structure that plays a fundamental role in K-theory, namely, that of an exact category. This is an additive category $\mathcal{C}$ together with a specified class of 'admissible' exact sequences chosen from the class of all short exact sequences in $\mathcal{C}$. The definitions, and properties, of the K-groups $K_0(\mathcal{C}), K_1(\mathcal{C}), \ldots$ depend not only on the category $\mathcal{C}$ but also on the class $\operatorname{Ex}(\mathcal{C})$. To illustrate this phenomenon, we discuss a few basic properties of the Grothendieck group $K_0(\mathcal{C})$. We state the properties required of $\operatorname{Ex}(\mathcal{C})$ in two parts, one 'elementary', the other 'advanced'. The elementary properties

Theorem ([BK: IRM] (4.1.17)), any short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with term M in $\mathcal{ASS}_R$ must be split. However, this need not be so when L and N are viewed as modules in $\mathcal{M}_{OD_R}$ and M can lie outside $\mathcal{ASS}_R$.

For an explicit example, take $R = \mathcal{K}[\epsilon]$, the ring of dual numbers over a field $\mathcal{K}$. (This is the polynomial ring whose indeterminate ϵ has $\epsilon^2 = 0$.)

Show that the Jacobson radical of R is $\mathrm{rad}(R) = \epsilon\mathcal{K}[\epsilon]$ and that $\mathcal{ASS}_R = \mathcal{M}_{\mathcal{K}}$, with ϵ acting as 0 on $\mathcal{K}$-spaces (see (2.3.9)(iv)). Now consider the exact sequence of R-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}[\epsilon] \longrightarrow \mathcal{K} \longrightarrow 0.$$

2.3.11 Let A be an object of an additive category $\mathcal{C}$. Write $\sigma_j : A \to A \oplus A$, $\pi_j : A \oplus A \to A$ for the usual monomorphisms and epimorphisms, $j = 1, 2$. Then the *diagonal map* $\Delta_A : A \to A \oplus A$ is defined by universality to be the unique morphism with $\pi_j \Delta_A = id_A$, $j = 1, 2$. Dually, the *codiagonal map* $\nabla_A : A \oplus A \to A$ is defined to be the unique map such that $\nabla_A \sigma_j = id_A$, $j = 1, 2$.

Show that, when $\mathcal{C}$ is abelian, $\Delta_A : A \to A \oplus A$ is the kernel of $\pi_1 - \pi_2 : A \oplus A \to A$, while $\nabla_A : A \oplus A \to A$ is the cokernel of $\sigma_1 - \sigma_2 : A \to A \oplus A$.

Confirm also that these definitions extend those given for modules in Exercise 2.1.7.

Show too that, for morphisms $\alpha : A \to C$ and $\beta : B \to D$ in $\mathcal{C}$,

$$\alpha \oplus \beta : A \oplus B \longrightarrow C \oplus D,$$

as defined in (2.2.16), is the unique morphism such that

$$(\alpha \oplus \beta)\sigma_A = \sigma_C \alpha \text{ and } (\alpha \oplus \beta)\sigma_B = \sigma_D \beta.$$

Further, when $A = B$ and $C = D$, $\alpha + \beta : A \to C$ is uniquely defined by

$$\alpha + \beta = \nabla_C (\alpha \oplus \beta) \Delta_A.$$

[Pareigis 1970] 4.1 develops this line of argument to give an alternative treatment of additive categories.

Finally, prove the converse of (2.2.20): if $F : \mathcal{C} \to \mathcal{D}$ is a functor between additive categories, which preserves finite direct sums of

2.3.7 Let $\mathcal{C}$ be a preadditive category. Show that if $\mathcal{C}$ contains all possible pull-backs and push-outs of its objects and morphisms, then $\mathcal{C}$ is abelian.

2.3.8 Here, for a ring T and T-module homomorphism λ, we denote by $\mathrm{Ker}_T \lambda$ and $\mathrm{Cok}_T \lambda$ the kernel and cokernel of λ in the category $\mathcal{M}_{OD_T}$.

By the Embedding Theorems (2.3.22), given an abelian category $\mathcal{C}$, there is a ring S and embedding of $\mathcal{C}$ in $\mathcal{M}_{OD_S}$ such that for any morphism λ in $\mathcal{C}$ both $\mathrm{Ker}_S \lambda$ and $\mathrm{Cok}_S \lambda$ lie in $\mathcal{C}$. Deduce that if $\mathcal{C}$ is also embedded in $\mathcal{M}_{OD_R}$ for another ring R, then $\mathrm{Ker}_R \lambda \cong \mathrm{Ker}_S \lambda$ and $\mathrm{Cok}_R \lambda \cong \mathrm{Cok}_S \lambda$ as abelian groups, and hence $\mathrm{Ker}_R \lambda$ and $\mathrm{Cok}_R \lambda$ lie in $\mathcal{C}$ (at least to within a canonical isomorphism). This proves the converse to (2.3.5).

2.3.9 Let $\mathcal{C}$ be a full subcategory of $\mathcal{M}_{OD_R}$ containing the zero module. Show that the following are equivalent.

(a) $\mathcal{C}$ is abelian.

(b) For any short exact sequence of right R-modules

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

both the following hold:

(i) if $M \in \mathcal{C}$, then both M' and M'' are in $\mathcal{C}$, and

(ii) when the sequence is split, if M' and M'' are in $\mathcal{C}$, then $M \in \mathcal{C}$.

2.3.10 **Extensions depend on the category**

Generalizing (2.3.5), we may define an *abelian subcategory* $\mathcal{C}$ of an abelian category $\mathcal{A}$ to be an additive subcategory with the further property that for any morphism $\gamma : C \to D$ in $\mathcal{C}$, then the kernel and cokernel of γ in $\mathcal{A}$ are (isomorphic to) objects in $\mathcal{C}$. More precisely, if (K, κ) is a kernel for γ in $\mathcal{A}$, then there is an object K' in $\mathcal{C}$, a morphism $\kappa' : K' \to C$ in $\mathcal{C}$ and an isomorphism $\theta : K \to K'$ (in $\mathcal{A}$) such that $\kappa = \kappa'\theta$, and similarly for cokernels.

Verify that an abelian subcategory is indeed abelian.

By the discussion in (2.3.7), for any ring R, the category of Artinian semisimple modules $\mathcal{ASS}_R$ is abelian. Verify that $\mathcal{ASS}_R$ is an abelian subcategory of $\mathcal{M}_{OD_R}$.

Now suppose that L, N lie in $\mathcal{ASS}_R$. By the Artinian Splitting

rise to a commuting diagram of the following kind.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\kappa} & L & \xrightarrow{\tau} & I & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longleftarrow & N & \xleftarrow{\chi} & M & \xleftarrow{\nu} & I' & \longleftarrow & 0 \end{array}$$

Here the horizontal sequences are both short exact, θ is an isomorphism, and $\kappa = \operatorname{Ker}\alpha$ and $\chi = \operatorname{Cok}\alpha$. Thus $I = \operatorname{Cok}\operatorname{Ker}\alpha$ is the image $\operatorname{Im}\alpha$ of α.

Generalizing (2.1.2) from module categories to arbitrary abelian categories, we say that a sequence

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\alpha_{i-1}} A_i \xrightarrow{\alpha_i} A_{i+1} \longrightarrow \cdots$$

in $\mathcal{C}$ is *exact at* A_i if $\operatorname{Ker}\alpha_i = \operatorname{Im}\alpha_{i-1}$, and *exact* if it is exact at each term A_i in the sequence. Show that a short exact sequence (as defined in (2.2.7)) is an exact five-term sequence which begins and ends in the zero object 0 of $\mathcal{C}$.

2.3.4 Suppose that an additive category $\mathcal{C}$ is the direct sum of its subcategories $\mathcal{C}_1$ and $\mathcal{C}_2$, and that P_1, P_2 are the projection functors as in (2.3.17). Show that for any pair of objects M, N of $\mathcal{C}$, there is an isomorphism of abelian groups

$$\operatorname{Mor}_{\mathcal{C}}(M,N) \cong \operatorname{Mor}_{\mathcal{C}_1}(P_1M, P_1N) \oplus \operatorname{Mor}_{\mathcal{C}_2}(P_2M, P_2N)$$

which is natural in each term M, N.

Show that, for any nonzero integer a,

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/a\mathbb{Z}) \cong \mathbb{Z}/a\mathbb{Z}$$

and deduce that $\mathcal{M}_{\mathbb{Z}}$ is *not* the direct sum of $T_{\mathbb{Z}}$ and $\mathcal{T}F_{\mathbb{Z}}$ – see (2.3.21).

2.3.5 Let $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$ be a set of categories, where the index set Λ may be infinite. Show that if each $\mathcal{C}_\lambda$ is additive (or abelian), then the direct product $\prod_\Lambda \mathcal{C}_\lambda$ (1.1.13) is again additive (or abelian).

Describe the projective objects in the direct product $\prod_\Lambda \mathcal{C}_\lambda$ and direct sum $\bigoplus_\Lambda \mathcal{C}_\lambda$ – see (2.3.19).

2.3.6 Let $\mathcal{C}$ be an abelian category. Show that a pull-back in $\mathcal{C}$ can be obtained as the kernel of a suitably defined morphism, and that a push-out can be obtained as a cokernel.

Deduce that $\mathcal{C}$ contains the pull-back of any pull-back diagram of objects and morphisms in $\mathcal{C}$ and likewise for push-outs.

Then, by embedding the category in $\mathcal{A}_B$, we may work with elements in the various objects. For example:

> *Suppose that γ_2 and γ_4 are epimorphisms, and that γ_5 is a monomorphism. Then γ_3 is also an epimorphism.*

To prove this, let $y_3 \in N_3$. Taking $x_4 \in M_4$ with $\gamma_4 x_4 = \beta_3 y_3 \in N_4$, we have

$$\gamma_5 \alpha_4 x_4 = \beta_4 \beta_3 y_3 = 0.$$

Thus $x_4 \in \operatorname{Ker} \alpha_4 = \operatorname{Im} \alpha_3$, and so we can find $x_3 \in M_3$ with $\alpha_3 x_3 = x_4$; then

$$y_3 - \gamma_3 x_3 \in \operatorname{Ker} \beta_3 = \operatorname{Im} \beta_2 = \operatorname{Im} \beta_2 \gamma_2 = \operatorname{Im} \gamma_3 \alpha_2.$$

So $y_3 \in \operatorname{Im} \gamma_3$ as required.

With slightly less rigmarole, one can prove the dual statement for γ_3 to be a monomorphism. In combination, these diagram chases yield the following highly useful result.

> *If γ_1, γ_2, γ_4 and γ_5 are isomorphisms, then so also is γ_3.*

Other, less widely used, diagram chase theorems to be found in the literature have such evocative names as the Snake Lemma, the Horseshoe Lemma, the Windmill Lemma, the 3×3 Lemma, the Nine Lemma, etc.

Exercises

2.3.1 Show that the opposite category of an abelian category is also abelian.

2.3.2 Let $\alpha : L \to M$ and $\beta : M \to N$ be morphisms in an abelian category. Show (for example, by (2.3.2)), that the following statements are equivalent.

(i) $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is a short exact sequence;

(ii) α is a monomorphism and $\beta = \operatorname{Cok} \alpha$;

(iii) β is an epimorphism and $\alpha = \operatorname{Ker} \beta$.

Recall that our example $0 \to \mathbb{Q} \to \mathbb{R} \to 0 \to 0$ in $\mathcal{HA}_B$ shows that (ii) need not imply (i) when the category is additive but not abelian.

2.3.3 Show that in an abelian category $\mathcal{C}$ any morphism $\alpha : L \to M$ gives

2.3.22 The Embedding Theorems

There are several results to the effect that an abstract additive or abelian category can be viewed as embedded as a subcategory in a concrete abelian category, such as $\mathcal{A}_{B}$, or $\mathcal{M}_{OD_R}$ for some ring R.

If $\mathcal{C}$ is a small additive category, then $\mathcal{C}$ can be embedded contravariantly as an additive subcategory in the functor category $[\mathcal{C}, \mathcal{A}_{B}]$ by associating with an object C the morphism functor $\mathrm{Mor}_{\mathcal{C}}(C, -)$ ([Mitchell 1965], IV (2.3)). Since the opposite of an additive category is also additive, this result gives a covariant embedding of $\mathcal{C}$ in the additive category $[\mathcal{C}^{\mathrm{op}}, \mathcal{A}_{B}]$.

This result applies also to abelian categories, and extends to show that a small additive or abelian category can be embedded in $\mathcal{A}_{B}$ (*ibid.* IV (2.6)), which result is sometimes known as the Lubkin-Heron-Freyd Representation Theorem, although it is also attributed to Mitchell. (The discoverers of this circle of results are generous with their attributions.)

In [Pareigis 1970] (4.14) Theorem 3, we find the more refined result that, given a small abelian category $\mathcal{C}$, there is a ring R and a covariant full faithful functor (1.3.13) from $\mathcal{C}$ to the category $\mathcal{M}_{OD_R}$. Anticipating the next section, we remark that this functor is also 'exact' in that it preserves exact sequences.

The philosophical consequence of these results is that it suffices to verify any sufficiently general statement about abelian categories in the category $\mathcal{A}_{B}$; this point of view is discussed in [Mitchell 1965], IV. However, it is usually more natural to work directly in a given additive or abelian category.

2.3.23 Example: The Famous Five Lemma

This lemma† is one of a number of useful results which hold in an abstract abelian category, but which are proved more readily by a 'diagram chase' in a module category.

Suppose that we have a commuting diagram, with both rows exact, in an additive category:

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & M_4 & \xrightarrow{\alpha_4} & M_5 \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_5} \\
N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 & \xrightarrow{\beta_3} & N_4 & \xrightarrow{\beta_4} & N_5
\end{array}$$

† Possibly so-named because it leads to the Smuggler's Top Theorem, a truly marvellous result which this footnote is unfortunately too small to contain ([Blyton 1950]).

2.3.21 Module categories over Dedekind domains

We can now give the promised direct sum decompositions of categories that arise from the structure theory of modules over a Dedekind domain $\mathcal{O}$.

Since $\mathcal{O}$ is a Noetherian ring, any finitely generated (right) $\mathcal{O}$-module M is also Noetherian, and in particular, any ascending chain of submodules of M has a maximal member, which is again finitely generated ([BK: IRM] (3.1.6)). Thus the category $\mathcal{M}_{\mathcal{O}}$ of finitely generated (right) $\mathcal{O}$-modules is abelian (2.3.6).

Let $\mathcal{T}_{\mathcal{O}}$ be the full subcategory of $\mathcal{M}_{\mathcal{O}}$ whose objects are all the finitely generated torsion $\mathcal{O}$-modules. Since the direct sum of two modules belonging to $\mathcal{T}_{\mathcal{O}}$ is again a module in $\mathcal{T}_{\mathcal{O}}$, we see that $\mathcal{T}_{\mathcal{O}}$ is an additive subcategory of $\mathcal{M}_{\mathcal{O}}$. As a submodule or quotient module of a module in $\mathcal{T}_{\mathcal{O}}$ again belongs to $\mathcal{T}_{\mathcal{O}}$, the kernel and cokernel of a homomorphism in $\mathcal{T}_{\mathcal{O}}$ is also in $\mathcal{T}_{\mathcal{O}}$, so that $\mathcal{T}_{\mathcal{O}}$ is an abelian category (2.3.5).

Write $\mathcal{TF}_{\mathcal{O}}$ for the category of finitely generated torsion-free (right) $\mathcal{O}$-modules. By (2.3.20 - B), $\mathcal{TF}_{\mathcal{O}}$ is the same as $\mathcal{P}_{\mathcal{O}}$, the category of finitely generated projective $\mathcal{O}$-modules.

Let M and N be finitely generated $\mathcal{O}$-modules. Clearly, an $\mathcal{O}$-module homomorphism $\lambda : M \to N$ induces a homomorphism $T(\lambda) : T(M) \to T(N)$. Thus we have a functor

$$T : \mathcal{M}_{\mathcal{O}} \longrightarrow \mathcal{T}_{\mathcal{O}}.$$

There is also a functor $F : \mathcal{M}_{\mathcal{O}} \to \mathcal{P}_{\mathcal{O}}$, which sends M to $M/T(M)$. Now, any module M has a direct sum decomposition $M = T(M) \oplus M'$ with $M' \cong M/T(M)$ projective, but there is no direct sum decomposition of the category $\mathcal{M}_{\mathcal{O}}$ in terms of $\mathcal{P}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}}$, essentially because there is no canonical choice of M' – see Exercise 2.3.4.

For each nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ let $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$ denote the full subcategory of $\mathcal{T}_{\mathcal{O}}$ given by the $\mathfrak{p}$-primary modules. Again, $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$ is an abelian category, and we have functors

$$T_{\mathfrak{p}} : \mathcal{T}_{\mathcal{O}} \longrightarrow \mathcal{T}_{\mathfrak{p},\mathcal{O}}$$

for each $\mathfrak{p}$.

By (2.3.20 - F), there is a direct sum decomposition of

$$\mathcal{T}_{\mathcal{O}} = \bigoplus_{\mathfrak{p}\in\mathbf{P}} \mathcal{T}_{\mathfrak{p},\mathcal{O}}$$

of the category $\mathcal{T}_{\mathcal{O}}$ into its subcategories $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$.

E Corollary

Let M be a finitely generated module over a Dedekind domain $\mathcal{O}$. Then

$$M \cong T(M) \oplus M/T(M)$$

with $M/T(M)$ projective. ○

To complete the picture, we describe the structure of torsion modules over a Dedekind domain $\mathcal{O}$. A finitely generated torsion $\mathcal{O}$-module M has *annihilator*

$$\mathrm{Ann}(M) = \{x \in \mathcal{O} \mid mx = 0 \text{ for all } m \in M\},$$

which is a nonzero ideal of $\mathcal{O}$. If $\mathrm{Ann}(M) = \mathfrak{p}^n$ for some nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ and natural number n, then M is said to be $\mathfrak{p}$-*primary*.

Since $\mathcal{O}$ is Noetherian, any set of submodules of a finitely generated module M has maximal members, each of which is finitely generated ([BK: IRM], (3.1.6)). However, the sum of any two $\mathfrak{p}$-primary submodules of M is again $\mathfrak{p}$-primary, so that M has a unique maximal $\mathfrak{p}$-primary submodule, $T_{\mathfrak{p}}(M)$. The submodule $T_{\mathfrak{p}}(M)$ is the $\mathfrak{p}$-*primary component* of M; it is again finitely generated.

Then we have the following result ([BK: IRM] (6.3.15), (6.3.20)).

F The Primary Decomposition Theorem

Let M be a finitely generated torsion module over a Dedekind domain $\mathcal{O}$. Then:

(i) *there is a direct sum decomposition*

$$M = \bigoplus\nolimits_{\mathfrak{p}} T_{\mathfrak{p}}(M),$$

where almost all (that is, all except a finite number) of the $\mathfrak{p}$-primary components of M are zero;

(ii) *for each prime ideal $\mathfrak{p}$ of $\mathcal{O}$,*

$$T_{\mathfrak{p}}(M) = \mathcal{O}/\mathfrak{p}^{\delta(\mathfrak{p},1)} \oplus \cdots \oplus \mathcal{O}/\mathfrak{p}^{\delta(\mathfrak{p},\ell_{\mathfrak{p}})}$$

with $\delta(\mathfrak{p},1) \leq \cdots \leq \delta(\mathfrak{p},\ell_{\mathfrak{p}})$ and $\ell_{\mathfrak{p}} \geq 0$;

(iii) *the collection of integers $\delta(\mathfrak{p},1), \ldots, \delta(\mathfrak{p},\ell_{\mathfrak{p}})$, where $\mathfrak{p}$ ranges over all the nonzero prime ideals of $\mathcal{O}$, is uniquely determined by M and in turn determines M to within isomorphism.* ○

are written, and the exponents are uniquely determined once an order has been fixed. ○

Let $\mathcal{O}$ again be an arbitrary commutative domain, and let M be an $\mathcal{O}$-module. The *torsion submodule* $T(M)$ is

$$T(M) = \{m \in M \mid mx = 0 \text{ for some } x \in \mathcal{O},\ x \neq 0\}.$$

We say that M is a *torsion module* if $M = T(M)$ and that M is *torsion-free* if $T(M) = 0$. (These definitions anticipate the more general definitions that will be made in section 6.2.)

Here are the fundamental results for projective modules over a Dedekind domain.

B Theorem

Let $\mathcal{O}$ be a Dedekind domain and let M be a finitely generated $\mathcal{O}$-module. Then M is projective if and only if M is torsion-free (see ([BK: IRM] (6.3.4)).○

C Theorem

Let $\mathcal{O}$ be a Dedekind domain and let M be a finitely generated projective $\mathcal{O}$-module. Then

$$M \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$$

for some fractional ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ of $\mathcal{O}$ ([BK: IRM] (6.1.2)). ○

The next result gives the criterion for two projective modules to be isomorphic ([BK: IRM] (6.1.6)).

D Steinitz' Theorem

Let $P \cong \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_r$ be a projective module over a Dedekind domain $\mathcal{O}$. Then

$$P \cong \mathcal{O}^{r-1} \oplus \mathfrak{a}_1 \cdots ; \mathfrak{a}_r.$$

If $Q \cong \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s$ is also a projective $\mathcal{O}$-module, then the following statements are equivalent.

(i) $P \cong Q$;
(ii) $r = s$ *and* $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\} = \{\mathfrak{b}_1 \cdots \mathfrak{b}_s\}$ *in* $\mathrm{Cl}(\mathcal{O})$. ○

The integer r is called the *rank* of P and the ideal class $\{\mathfrak{a}_1 \cdots \mathfrak{a}_r\}$ is called the *ideal class* of P. A projective module is said to be in *standard form* if $P = \mathcal{O}^{r-1} \oplus \mathfrak{a}$.

The next result is immediate from the observation that $M/T(M)$ is torsion-free.

The reader is invited to characterize infinite direct sums in terms of inclusion and projection functors.

2.3.20 *Dedekind domains: a review*

Some interesting direct sum decompositions of categories arise from the structure theory of modules over a Dedekind domain. As we use this structure theory in several places in this text, we turn aside to review its basic definitions and results, which are developed in detail in Chapters 5 and 6 of [BK: IRM].

We start with some ideal theory. Let $\mathcal{O}$ be an arbitrary commutative domain with field of fractions $\mathcal{K}$. A *fractional ideal* of $\mathcal{O}$ is a finitely generated nonzero $\mathcal{O}$-submodule $\mathfrak{a}$ of $\mathcal{K}$. If $\mathfrak{b}$ is also a fractional ideal of $\mathcal{O}$, so is their *product*

$$\mathfrak{ab} = \{a_1b_1 + \cdots + a_kb_k \mid a_1, \ldots, a_k \in \mathfrak{a},\ b_1, \ldots, b_k \in \mathfrak{b},\ \ k \geq 1\}.$$

A fractional ideal $\mathfrak{a}$ is *invertible* if

$$\mathfrak{ab} = \mathcal{O}$$

for some $\mathfrak{b}$.

Our definition of a Dedekind domain ([BK: IRM] (5.1.10)) is that it is a commutative domain $\mathcal{O}$ all of whose fractional ideals are invertible. It can then be shown that $\mathcal{O}$ is a Noetherian ring ([BK: IRM] Exercise 5.1.2). Clearly, the fractional ideals of a Dedekind domain form a group $\mathrm{Frac}(\mathcal{O})$, and the set of nonzero principal ideals $\mathrm{Pr}(\mathcal{O})$ is a subgroup of $\mathrm{Frac}(\mathcal{O})$. The *ideal class group* of $\mathcal{O}$ is the quotient group

$$\mathrm{Cl}(\mathcal{O}) = \mathrm{Frac}(\mathcal{O})/\mathrm{Pr}(\mathcal{O}).$$

The image of $\mathfrak{a}$ in $\mathrm{Cl}(\mathcal{O})$ is written $\{\mathfrak{a}\}$.

It is obvious that a commutative principal ideal domain is the same thing as a Dedekind domain with trivial ideal class group.

We record [BK: IRM] (5.1.19).

A The Unique Factorization Theorem for Ideals

Let $\mathfrak{a}$ be a fractional ideal of a Dedekind domain $\mathcal{O}$. Then there is a set of distinct nonzero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ and a set of nonzero integers $n_1, \ldots, n_k$ such that

$$\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_k^{n_k}.$$

The prime ideals are uniquely determined apart from the order in which they

$G(\alpha_1, 0)$, and similarly for the other variable. The assertion follows, on noting that

$$(C_1, C_2) \cong (C_1, 0) \oplus (0, C_2) \text{ in } C_1 \times C_2.$$

Conversely, if (ii) holds, define $FC = (P_1C, P_2C)$ and $G(C_1, C_2) = I_1C_1 \oplus I_2C_2$, and similarly for morphisms. □

For completeness, we record the extension of the above result to a direct product of any finite set of additive categories. The result is analogous to that for modules (see [BK: IRM], (2.1.7)).

2.3.18 Corollary

Let $\mathcal{C}$ and $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be additive categories. Then the following are equivalent.

(i) *There is an equivalence of additive categories*

$$\mathcal{C} \simeq \mathcal{C}_1 \times \cdots \times \mathcal{C}_n.$$

(ii) *There are additive functors $I_i : \mathcal{C}_i \to \mathcal{C}$ and $P_i : \mathcal{C} \to \mathcal{C}_i$, for $i = 1, \ldots, n$, with natural isomorphisms*

$$I_1P_1 \oplus \cdots \oplus I_nP_n \simeq \mathrm{Id}_{\mathcal{C}}$$

and

$$P_iI_i \simeq \mathrm{Id}_{\mathcal{C}_i} \quad \text{for all } i,$$

and further

$$P_jI_i = 0 \quad \text{for } i \neq j.$$ □

2.3.19 Infinite direct sums of categories

Since an additive category has a zero object, it is sensible to speak of the *direct sum* $\bigoplus_\Lambda \mathcal{C}_\lambda$ of an infinite set $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$ of categories, where Λ is an ordered set. The objects of the direct sum are all sequences of the form

$$C_\Lambda = (C_\lambda \mid C_\lambda \in \mathcal{C}_\lambda, \text{ almost all } C_\lambda = 0),$$

and a morphism $\alpha_\Lambda : C'_\lambda \to C_\Lambda$ is a sequence $(\alpha_\lambda : C' \to C_\lambda)$ with each α_λ a morphism in $\mathcal{C}_\lambda$.

It is straightforward to verify that $\bigoplus_\Lambda \mathcal{C}_\lambda$ is again an additive category under the usual componentwise definitions. Thus the zero object is $0_\Lambda = (0)$, and the direct sum in $\bigoplus_\Lambda \mathcal{C}_\lambda$ is given by

$$C'_\Lambda \oplus C''_\Lambda = (C'_\lambda \oplus C''_\lambda).$$

2.3.15 Theorem

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be an abelian category. Then the functor category $[\mathcal{C}, \mathcal{D}]$ is an abelian category. □

2.3.16 Direct sums of categories

Since we are able to define the direct sum of two, or more, functors between additive categories, we can interpret the direct product of two additive categories as a *direct sum*, because we can mimic the description of a direct sum of modules in terms of injections and projections (2.2.11).

Throughout the sequel we demand that an *equivalence* between additive categories must be given by an additive functor.

2.3.17 Theorem

Let $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be additive categories. Then the following statements are equivalent.

(i) *There is an equivalence of additive categories*

$$\mathcal{C} \simeq \mathcal{C}_1 \times \mathcal{C}_2.$$

(ii) *There are additive functors*

$$I_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}, \quad I_2 : \mathcal{C}_2 \longrightarrow \mathcal{C}, \quad P_1 : \mathcal{C} \longrightarrow \mathcal{C}_1, \quad P_2 : \mathcal{C} \longrightarrow \mathcal{C}_2$$

such that there are natural isomorphisms

$$I_1 P_1 \oplus I_2 P_2 \simeq \mathrm{Id}_{\mathcal{C}}$$

and

$$P_1 I_1 \simeq \mathrm{Id}_{\mathcal{C}_1}, \quad P_2 I_2 \simeq \mathrm{Id}_{\mathcal{C}_2},$$

and also

$$P_2 I_1 = 0, \quad P_1 I_2 = 0.$$

(I and P stand for inclusion and projection.)

Proof

Suppose that (i) holds, and let $F : \mathcal{C} \to \mathcal{C}_1 \times \mathcal{C}_2$ and $G : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ be the (additive) functors that give the equivalence. If C is an object of $\mathcal{C}$, we have $FC = (C_1, C_2)$ for objects C_1 of $\mathcal{C}_1$ and C_2 of $\mathcal{C}_2$. Similarly, for a morphism α we have $F\alpha = (\alpha_1, \alpha_2)$. Define $P_1 C = C_1$ and $P_1 \alpha = \alpha_1$, and likewise for P_2.

For an object C_1 and morphism α_1 in $\mathcal{C}_1$, let $I_1 C_1 = G(C_1, 0)$ and $I_1 \alpha_1 =$

The reader should not have much difficulty in verifying that these approaches give essentially the same objects, to within isomorphisms.

The kernels and cokernels in the (homo)morphism categories $\mathcal{M}or(\mathcal{M}od_R)$, $\mathcal{E}nd(\mathcal{M}od_R)$, $\mathcal{I}so(\mathcal{M}od_R)$ and $\mathcal{A}ut(\mathcal{M}od_R)$ are those in $\mathcal{M}od_{R\times R}$.

2.3.14 Functor categories

We can also give pointwise descriptions of kernels, cokernels, and so on, in a functor category $[\mathcal{C},\mathcal{D}]$ provided the appropriate objects exist in $\mathcal{D}$. For simplicity we suppose that $\mathcal{D}$ is abelian.

For example, the kernel $(\operatorname{Ker}\eta,\kappa)$ of a morphism η from F to G, that is, of a natural transformation $\eta \in \operatorname{Nat}(F,G)$, is obtained as follows.

For each object C of $\mathcal{C}$, write $(\operatorname{Ker}\eta_C,\kappa_C)$ for the kernel of $\eta_C : FC \to GC$. Then the object $\operatorname{Ker}\eta$ in $[\mathcal{C},\mathcal{D}]$ is the functor from $\mathcal{C}$ to $\mathcal{D}$ given by

$$(\operatorname{Ker}\eta)(C) = \operatorname{Ker}(\eta_C : FC \longrightarrow GC)$$

for each object C of $\mathcal{C}$.

To see that $\operatorname{Ker}\eta$ is a functor, note that a morphism $\alpha : C' \to C$ gives a commutative diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

which, together with the definition of a kernel as a universal object (1.4.15), shows that there is an induced morphism

$$(\operatorname{Ker}\eta)(\alpha) : (\operatorname{Ker}\eta)(C') \longrightarrow (\operatorname{Ker}\eta)(C).$$

There is a natural transformation

$$\kappa : \operatorname{Ker}\eta \longrightarrow F$$

given by

$$\kappa_C : (\operatorname{Ker}\eta)(C) \longrightarrow F(C),$$

κ_C as above, and it is straightforward to verify that the pair $(\operatorname{Ker}\eta,\kappa)$ is a kernel for η.

Similarly, cokernels, monomorphisms, epimorphisms, projective and injective objects in $[\mathcal{C},\mathcal{D}]$ are given pointwise by the corresponding objects or morphisms in $\mathcal{D}$.

We record the result for future reference.

in $\mathcal{C} \times \mathcal{D}$ is a short exact sequence if and only both component sequences

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

are short exact sequences. □

Similar arguments are used for the morphism categories attached to the category $\mathcal{C}$. In the morphism category $\mathcal{M}or_R\mathcal{C}$, the kernel of

$$(\alpha, \beta) : (C, D, \gamma) \longrightarrow (C', D', \delta)$$

is $(\operatorname{Ker}\alpha, \operatorname{Ker}\beta, \overline{\gamma})$, where $\overline{\gamma}$ is the restriction of γ to $\operatorname{Ker}\alpha$, and the cokernel is $(\operatorname{Cok}\alpha, \operatorname{Cok}\beta, \overline{\delta})$, where $\overline{\delta}$ is induced by δ. Images in $\mathcal{M}or_R\mathcal{C}$ are defined in the expected way. The result is as follows.

2.3.12 Proposition

Let $\mathcal{C}$ be an abelian category. Then the categories $\mathcal{M}or_R\mathcal{C}$, $\mathcal{E}nd\mathcal{C}$, $\mathcal{I}so\mathcal{C}$ and $\mathcal{A}ut\,\mathcal{C}$ are also abelian.

In particular, a sequence

$$0 \longrightarrow (C', D', \gamma') \longrightarrow (C, D, \gamma) \longrightarrow (C'', D'', \gamma'') \longrightarrow 0$$

in $\mathcal{M}or_R\mathcal{C}$ is a short exact sequence if and only both component sequences

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

are short exact sequences in $\mathcal{C}$.

Note that in $\mathcal{E}nd\,\mathcal{C}$ and $\mathcal{A}ut\,\mathcal{C}$, the two component sequences are identical.

□

2.3.13 Module categories

Let R and S be rings. As we saw in (1.3.16), the product category is equivalent to the category $\mathcal{M}od_{R\times S}$, since every object (M, N) in the product category can be regarded as a module over $R \times S$ and every morphism is an $R \times S$-module homomorphism (1.1.12). Thus there are two ways in which, for instance, a kernel can be defined in $\mathcal{M}od_R \times \mathcal{M}od_S$, either by using the abstract machinery or by taking an equivalent object in $\mathcal{M}od_{R\times S}$.

R-module is both semisimple and projective ([BK: IRM] (4.2.6)). Conversely, if the equalities hold, then R is Artinian semisimple by definition.

(iv) By [BK: IRM] (4.3.15), $R/\operatorname{rad}(R)$ is Artinian semisimple, and, by [BK: IRM] (4.3.16), a right R-module M is semisimple if and only if $M \cdot \operatorname{rad}(R) = 0$. Thus a semisimple R-module can be regarded as an $R/\operatorname{rad}(R)$-module by the scalar multiplication

$$m \cdot \overline{r} = m \cdot r \quad \text{for } r \in R,$$

and conversely, this rule can be used to make any $R/\operatorname{rad}(R)$-module into a semisimple R-module. □

2.3.10 Product and morphism categories

Next, we show that product and morphism categories of abelian categories are also abelian. As we already know that such categories are additive, it suffices to verify the axioms Ab 1, Ab 2 and Ab 2^{op}, which will follow at once from explicit descriptions of kernels, cokernels and so forth.

Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A morphism in $\mathcal{C} \times \mathcal{D}$ is a pair

$$(\alpha, \beta) : (C, D) \longrightarrow (C', D'),$$

with α a morphism in $\mathcal{C}$ and β a morphism in $\mathcal{D}$. Thus

$$\operatorname{Mor}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) = \operatorname{Mor}_{\mathcal{C}}(C, C') \times \operatorname{Mor}_{\mathcal{D}}(D, D'),$$

and any assertion about a morphism (α, β) in $\mathcal{C} \times \mathcal{D}$ can be verified on its components α, β separately. Thus (α, β) is a monomorphism precisely when α and β are both monomorphisms, and likewise for epimorphisms.

The kernel of (α, β) is $((\operatorname{Ker}\alpha, \operatorname{Ker}\beta), (\kappa, \psi))$, where $(\operatorname{Ker}\alpha, \kappa)$ and $(\operatorname{Ker}\beta, \psi)$ are the respective kernels of α and β, and the cokernel is $((\operatorname{Cok}\alpha, \operatorname{Cok}\beta), (\chi, \zeta))$. In practice, we omit the maps κ and ψ, and write

$$\operatorname{Ker}(\alpha, \alpha') = (\operatorname{Ker}\alpha, \operatorname{Ker}\alpha'),$$

and similarly for the cokernel.

The image is the pair $\operatorname{Im}(\alpha, \beta) = (\operatorname{Im}\alpha, \operatorname{Im}\beta)$, with the obvious implicit morphisms. We summarize as follows.

2.3.11 Proposition

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are abelian categories. Then the category $\mathcal{C} \times \mathcal{D}$ is also abelian.

In particular, a sequence

$$0 \longrightarrow (C', D') \longrightarrow (C, D) \longrightarrow (C'', D'') \longrightarrow 0$$

since it contains any submodule or quotient module of any of its members – see [BK: IRM], Proposition 3.1.2, Proposition 4.1.5 and the Artinian Splitting Theorem (4.1.17) respectively. We record our assertion formally.

2.3.8 Theorem

The categories $\mathcal{N}_{OETH_R}$, $\mathcal{A}_{RT_R}$ and $\mathcal{ASS}_R$ are abelian. □

By [BK: IRM] Corollary 4.1.18, there is an inclusion of categories

$$\mathcal{ASS}_R \subseteq \mathcal{A}_{RT_R}$$

and, by the same result together with the definition of Noetherian, there are inclusions

$$\mathcal{ASS}_R' \subseteq \mathcal{N}_{OETH_R} \subseteq \mathcal{M}_R.$$

In general, $\mathcal{A}_{RT_R}$ is not a subcategory of $\mathcal{N}_{OETH_R}$. An example (due to P. M. Cohn) of a cyclic Artinian, non-Noetherian module is exhibited in Exercise 4.1.7 of [BK: IRM].

Next, we give the conditions for equality between these various categories.

2.3.9 Proposition

(i) *$\mathcal{N}_{OETH_R} = \mathcal{M}_R$ if and only if R is right Noetherian.*
(ii) *$\mathcal{A}_{RT_R} = \mathcal{M}_R$ if and only if R is right Artinian.*
(iii) *$\mathcal{ASS}_R = \mathcal{M}_R$ if and only if R is an Artinian semisimple ring (in which case $\mathcal{ASS}_R = \mathcal{P}_R$, the category of finitely generated projective right R-modules).*
(iv) *Suppose that R is right Artinian. Then the ring $R/\operatorname{rad}(R)$ is Artinian semisimple, where $\operatorname{rad}(R)$ is the Jacobson radical, and*

$$\mathcal{ASS}_R = \mathcal{M}_{R/\operatorname{rad}(R)}.$$

Proof

(i) Since R itself is in $\mathcal{M}_R$, the equality of categories makes R right Noetherian. Conversely, if R is right Noetherian then R^k is Noetherian for any natural number k and hence any finitely generated module M is also Noetherian – see [BK: IRM] (3.1.4).

(ii) The argument is similar to the proof of (i); we use the fact that if R is right Artinian, then so is any finitely generated right R-module ([BK: IRM] (4.1.7)).

(iii) Suppose that R is an Artinian semisimple ring. By the Wedderburn-Artin Theorem ([BK: IRM] (4.2.3)), R is a direct product of matrix rings over division rings, and so the categories are equal since every finitely generated

2.3.5 Theorem

Let R be a ring and let $\mathcal{C}$ be a full additive subcategory of $\mathcal{M}_{OD_R}$ such that, for any homomorphism $\lambda : L \to M$ of R-modules in $\mathcal{C}$, the R-modules $\operatorname{Ker}\lambda$ *and* $\operatorname{Cok}\lambda$ *are both in $\mathcal{C}$.*

Then $\mathcal{C}$ is abelian.

Proof

The R-modules $\operatorname{Ker}\lambda$ and $\operatorname{Cok}\lambda$ evidently satisfy the universal properties needed to make them the kernel and cokernel of λ in $\mathcal{C}$. So Ab 1 holds. To see that Ab 2 holds, consider a monomorphism $\kappa : K \to L$ in $\mathcal{C}$. As we remarked in (2.2.3), κ is an injective mapping. Now $\mathcal{C}$ must also contain $\operatorname{Cok}\kappa = L/\operatorname{Im}\kappa$, and so K is the kernel of the canonical module epimorphism $\lambda : L \to L/\operatorname{Im}\kappa$, which is in $\mathcal{C}$. The proof of Ab 2^{op} is similar. □

For the next result, recall that, given a ring R, a right R-module M is *Noetherian* if every R-submodule of M is finitely generated. The ring R itself is *right Noetherian* if every right ideal of R is finitely generated. An account of Noetherian rings and modules is given in [BK: IRM], §3.1.

2.3.6 Corollary

Let R be a ring. Then the category $\mathcal{M}_R$ of finitely generated R-modules is an additive subcategory of $\mathcal{M}_{OD_R}$, and $\mathcal{M}_R$ is abelian precisely when the ring R is right Noetherian.

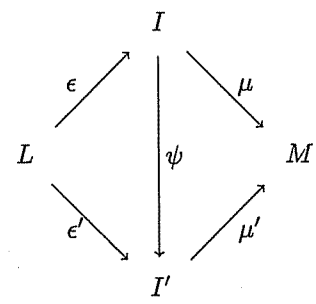
Proof

The sole obstruction to $\mathcal{M}_R$ being abelian is that it might not contain the kernel of some homomorphism, that is, there may be a finitely generated module M having a submodule which is not finitely generated. By Corollary 3.1.7 of [BK: IRM], this cannot happen if R is right Noetherian. In the reverse direction, if $\mathcal{M}_R$ is abelian, then every ideal of R is in $\mathcal{M}_R$. □

2.3.7 More examples

The category $\mathcal{HA}_B$ of Hausdorff abelian groups is an example of a full additive subcategory of $\mathcal{M}_{OD_{\mathbb{Z}}}$ which is not abelian. Observe that the cokernel (in $\mathcal{TOPA}_B$) of the inclusion monomorphism $\mathbb{Q} \to \mathbb{R}$ does not lie in $\mathcal{HA}_B$.

There are some subcategories of $\mathcal{M}_{OD_R}$ that are abelian for an arbitrary ring R. Let $\mathcal{NOETH}_R$ denote the full subcategory of $\mathcal{M}_{OD_R}$ whose objects are the Noetherian right R-modules, let $\mathcal{ART}_R$ denote the full subcategory whose objects are the Artinian right R-modules and let $\mathcal{ASS}_R$ be the full subcategory given by the Artinian semisimple modules. Each of these categories is abelian



commute; moreover, ψ is an isomorphism. ○

The isomorphic objects $\operatorname{Ker}\chi$ and I of this theorem are commonly referred to as the image $\operatorname{Im}\lambda$ of λ.

2.3.3 Module-like behaviour

In essence, the above result tells us that the Induced Mapping Theorem ([BK: IRM] (1.2.11)) must hold in an abelian category. It is then possible to translate many general results about modules into the context of an abstract abelian category. For example, the Isomorphism Theorems ([BK: IRM] Exercise 1.2.2) can be reproduced. However, the proofs are more delicate since it is not possible to work with elements. Instead, the universal properties of kernels, cokernels, etc. must be used. Arguments of this type are developed in [Mac Lane 1975], Chapter IX and [Mac Lane 1971], Chapter VIII.

On the other hand, there are some features of module categories which need not be reproducible in an abelian category. These usually relate to the existence of an object with some specific property. For instance, an abelian category need not contain any projective object, as is the case with the category $\mathcal{FA}_{\mathcal{B}}$ of finite abelian groups (see Exercise 2.2.10).

2.3.4 Example

For an example of an additive category that satisfies axiom Ab 1 but not Ab 2, consider the full additive subcategory of $\mathcal{A}_{\mathcal{B}}$ whose objects comprise all finite direct sums of the form $\mathbb{Z}^m \oplus \mathbb{Q}^n$ with $m, n \geq 0$. Here the inclusion monomorphism $\mathbb{Z} \to \mathbb{Q}$ is not a kernel.

We now look for abelian categories of modules. The first result simplifies the checking in many cases of interest. (Its converse, while also true, is less elementary – see Exercise 2.3.8 for the details.)

The requirement that a category should contain kernels and cokernels leads to the next definition.

Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is an *abelian category* if the following conditions are satisfied.

Ab 1. If $\lambda : L \to M$ is a morphism in $\mathcal{C}$, then $\mathcal{C}$ contains a kernel $(\operatorname{Ker}\lambda, \kappa)$ and a cokernel $(\operatorname{Cok}\lambda, \chi)$ of λ. (This means not only that the objects $\operatorname{Ker}\lambda$ and $\operatorname{Cok}\lambda$ are in $\mathcal{C}$, but also that the morphisms

$$\kappa : \operatorname{Ker}\lambda \longrightarrow L \quad \text{and} \quad \chi : M \longrightarrow \operatorname{Cok}\lambda$$

belong to $\mathcal{C}$.)

Ab 2. If $\kappa : K \to L$ is a monomorphism in $\mathcal{C}$, then (K, κ) is a kernel of some morphism $\lambda : L \to M$ in $\mathcal{C}$.

Ab 2^{op}. If $\chi : M \to C$ is an epimorphism in $\mathcal{C}$, then (C, χ) is a cokernel of some morphism $\lambda : L \to M$ in $\mathcal{C}$.

Statement Ab 2 is often phrased as 'every monomorphism is normal'. Alternative axiom sets abound in the literature. For example, see [Freyd 1964] or [Herrlich & Strecker 1979] for the following.

2.3.2 Theorem

Suppose that $\mathcal{C}$ is an additive category satisfying Ab 1. Then the following statements are equivalent.

(i) *Ab 2 and Ab 2^{op} hold in $\mathcal{C}$.*

(ii) *For any morphism $\lambda : L \to M$ with kernel $(\operatorname{Ker}\lambda, \kappa)$ and cokernel $(\operatorname{Cok}\lambda, \chi)$, the morphism*

$$\psi : \operatorname{Cok}\kappa \longrightarrow \operatorname{Ker}\chi$$

(which exists by the universal properties of kernels and cokernels) is an isomorphism.

(iii) *Every morphism $\lambda : L \to M$ has a unique (epi, mono) factorization; that is, there exists an epimorphism $\epsilon : L \to I$ and monomorphism $\mu : I \to M$ with $\mu\epsilon = \lambda$ such that, if $\epsilon' : L \to I'$, $\mu' : I' \to M$ is another (epi, mono) factorization of λ, then there is a unique morphism $\psi : I \to I'$ making*

category $\mathcal{C}^{\mathrm{op}}$, where addition of morphisms is given by $\alpha^{\mathrm{op}} + \beta^{\mathrm{op}} = (\alpha + \beta)^{\mathrm{op}}$ as in (2.2.5).

Show that C^{op} is a projective object in $\mathcal{C}^{\mathrm{op}}$ if and only if C is an injective object in $\mathcal{C}$, and vice-versa.

Give an example of an additive category which contains nontrivial injective objects but only the zero projective object.

2.2.12 Just as a monoid can be regarded as the same thing as a category with one object (Exercise 1.1.4), a ring may be taken to be the same as a preadditive category with one nonzero object. Viewing a ring R in this way, show that right R-modules correspond to contravariant, cochiral additive functors from R to $\mathcal{M}_{OD_{\mathbb{Z}}}$, while left R-modules correspond to covariant, contrachiral additive functors from R to ${}_{\mathbb{Z}}\mathcal{M}_{OD}$. Show also that R-module homomorphisms correspond to natural transformations.

2.3 ABELIAN CATEGORIES

For the next stage in our examination of categories whose objects in some way resemble modules, we consider abelian categories. Such categories must contain the kernel and cokernel of each of their morphisms, and most constructions that can be carried out in the full module categories $\mathcal{M}_{OD_R}$ are reproducible in an abelian category. Indeed, there are Embedding Theorems, which show that an arbitrary abelian category can be realized as a subcategory of $\mathcal{M}_{OD_R}$ for some ring R. However, an abelian category may lack nontrivial examples of projective or injective objects.

We show that the various morphism categories associated to abelian categories are again abelian, as are the functor and product categories. We also introduce the idea of a direct sum of categories, which leads to an interesting categorical interpretation of the module theory of a Dedekind domain.

2.3.1 The definition

Although we have defined kernels and cokernels in an arbitrary category with zero object, and thus in an additive category, a morphism in an additive category need not have a kernel or cokernel in that category. Indeed, the category $\mathcal{P}_{A[\epsilon]}$ of finitely generated projective modules over a ring of dual numbers $A[\epsilon]$ (where $\epsilon^2 = 0$) fails to contain either the kernel or the cokernel of the multiplication homomorphism $\epsilon : A[\epsilon] \to A[\epsilon]$, since each is isomorphic to A, which is not projective over $A[\epsilon]$.

Show that the inclusion map $\iota : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the (nonadditive) category $\mathcal{R}_{ING}$ of rings with identity.

2.2.5 Show that $C \oplus 0 \cong C$ for any object C of a preadditive category $\mathcal{C}$.

2.2.6 Let C, C' and C'' be objects of a preadditive category $\mathcal{C}$. Verify that the following statements are equivalent, as asserted in (2.2.11).

(i) C is the product of C' and C''.
(ii) C is the coproduct of C' and C''.
(iii) C is the direct sum of C' and C''.

Generalize this result to an arbitrary finite set of objects in $\mathcal{C}$.

2.2.7 Let $\mathcal{C}$ be a preadditive category, with $\alpha : A \to B$, $\beta : B \to C$ and $\gamma : C \to D$ composable morphisms in $\mathcal{C}$.

Show that if β and γ are epimorphisms, then $\gamma\beta$ is an epimorphism, and if $\gamma\beta$ is an epimorphism, then γ is an epimorphism.

Prove also that if α and β are monomorphisms, so is $\beta\alpha$, and if $\beta\alpha$ is a monomorphism, so also is α.

Give counterexamples to the claims that if $\gamma\beta$ is an epimorphism, then β is an epimorphism, and if $\beta\alpha$ is a monomorphism, so also is β.

2.2.8 Let $A_1, \ldots, A_k$ be objects of an additive category $\mathcal{A}$. Show that $A_1 \oplus \cdots \oplus A_k$ is a projective object in $\mathcal{A}$ if and only $A_1, \ldots, A_k$ are each projective objects in $\mathcal{A}$. (This generalizes a well-known result for modules (Theorem 2.5.5 of [BK: IRM]).)

2.2.9 **Projectives depend on the category**

Generalize (2.2.10) to show that, for any ring R, every module in $\mathcal{ASS}_R$ is projective as an object in that category.

Suppose that R is right Artinian but not semisimple. Find a module in $\mathcal{ASS}_R$ which is not projective in $\mathcal{M}_{OD_R}$.

Hint. [BK: IRM], section 4.3 is releyant, particularly Exercise 4.3.7.

2.2.10 **A category without projectives**

Let $\mathcal{FA}_B$ be the category of finite abelian groups. It is well known that any finite abelian group is the direct sum of finite cyclic groups (see Theorem 6.3.24 of [BK: IRM], for example). So if $\mathcal{FA}_B$ contains a nonzero projective, there must be a nontrivial finite cyclic projective, by Exercise 2.2.8 above. However, this is easily seen to be impossible – consider the natural surjection of $\mathbb{Z}/a^2\mathbb{Z}$ to $\mathbb{Z}/a\mathbb{Z}$ for any positive integer a.

2.2.11 **The opposite category**

Show that if $\mathcal{C}$ is an additive category, so likewise is its opposite

Exercises

2.2.1 Let $\mathcal{D}$ be a preadditive subcategory of the preadditive subcategory $\mathcal{C}$. Show that $\mathrm{End}_{\mathcal{D}}(X)$ is a subring of $\mathrm{End}_{\mathcal{C}}(X)$ for every object X in $\mathcal{D}$.

Given an object X of $\mathcal{C}$ and a subring D of $\mathrm{End}_{\mathcal{C}}(X)$, define a minimal preadditive subcategory $\mathcal{D}$ of $\mathcal{C}$ with $\mathrm{End}_{\mathcal{D}}(X) = D$.

2.2.2 Given a nonzero object X of an additive category $\mathcal{C}$ and a subring D of $\mathrm{Mor}_{\mathcal{C}}(X, X)$, define a minimal additive subcategory $\mathcal{D}$ of $\mathcal{C}$ with $\mathrm{End}_{\mathcal{D}}(X) = D$.

Show that the morphisms in $\mathcal{D}$ can be represented as sets $M_{m,n}(D)$ of $m \times n$ matrices over D.

Hint. Exercise 2.1.6′ of [BK: IRM] helps.

2.2.3 There seems to be a curious asymmetry in the exact sequences of abelian groups obtained in (2.2.3) above, in that they begin, but need not end, with the zero group. Show that this is inevitable for the following reason.

In a preadditive (right) category, the following statements are equivalent.

(i) $\varphi : X \to Y$ has a right inverse;
(ii) for all $L \in \mathcal{C}$, $\mathrm{Mor}(L, X) \xrightarrow{\varphi_*} \mathrm{Mor}(L, Y) \to 0$ is exact;
(iii) for all $L \in \mathcal{C}$, $\varphi_* : \mathrm{Mor}(L, X) \to \mathrm{Mor}(L, Y)$ has a right inverse.

There is an obvious dualization of the above.

Deduce that the following are equivalent.

(i) φ is an isomorphism;
(ii) $\mathrm{Mor}(-, \varphi)$ is always an isomorphism in $\mathcal{A}_{\mathcal{B}}$;
(iii) $\mathrm{Mor}(\varphi, -)$ is always an isomorphism in $\mathcal{A}_{\mathcal{B}}$.

2.2.4 **Epimorphisms and monomorphisms in arbitrary categories**

Let $\mathcal{C}$ be a (right) category. A morphism $\gamma : C \to D$ in $\mathcal{C}$ is defined to be a *monomorphism* if the following holds.

If β, $\beta' : B \to C$ are two morphisms with $\gamma\beta = \gamma\beta'$, then $\beta = \beta'$.

Verify that a monomorphism in a preadditive category is again a monomorphism in this sense. Observe too that if γ has a left inverse, then γ is a monomorphism.

Give a definition of an epimorphism by duality, that is, γ is an epimorphism in $\mathcal{C}$ if and only if γ^{op} is a monomorphism in $\mathcal{C}^{\mathrm{op}}$.

Verify directly that the two definitions of epimorphism coincide in a preadditive category, and that any morphism with a right inverse is an epimorphism.

so that any property or condition on $\mathrm{Nat}(F,G)$ can be verified, or imposed, separately in each $\mathrm{Mor}_{\mathcal{D}}(F(C),G(C))$.

Suppose that $\mathcal{D}$ is additive. If η and ζ are in $\mathrm{Nat}(F,G)$, the pointwise definition of their sum $\eta+\zeta$ in $\mathrm{Nat}(F,G)$ is

$$(\eta+\zeta)_C = \eta_C + \zeta_C \text{ for all } C \in \mathcal{C},$$

and the zero natural transformation 0 in $\mathrm{Nat}(F,G)$ is given by

$$0_C = 0 : FC \longrightarrow GC.$$

Thus $\mathrm{Nat}(F,G)$ is an abelian group for any pair of functors.

Since $\mathcal{D}$ has a zero object, we can define the *zero functor*

$$0 : \mathcal{C} \longrightarrow \mathcal{D}$$

by

$$0(C) = 0 \text{ for all } C \text{ in } \mathcal{C},$$

which is a zero object in $[\mathcal{C},\mathcal{D}]$.

We can also define the *direct sum* of two functors F and G in $[\mathcal{C},\mathcal{D}]$ in the obvious way:

$$(F \oplus G)(C) = F(C) \oplus G(C) \text{ for all } C \text{ in } \mathcal{C}.$$

To verify that this actually gives a direct sum, we again use the characterization of direct sums in a preadditive category which we gave in (2.2.11). For each object C, there are morphisms

$$\alpha_C : FC \longrightarrow FC \oplus GC, \quad \beta_C : FC \oplus GC \longrightarrow GC,$$

$$\gamma_C : FC \oplus GC \longrightarrow FC \text{ and } \delta_C : GC \longrightarrow FC \oplus GC$$

with

$$\beta_C\alpha_C = 0,\ \gamma_C\delta_C = 0,\ \beta_C\delta_C = id_{GC},\ \gamma_C\alpha_C = id_{FC}$$

and

$$\alpha_C\gamma_C + \delta_C\beta_C = id_{FC\oplus GC},$$

which gives a set $\alpha,\beta,\gamma,\delta$ of natural transformations that exhibit $F \oplus G$ as a direct sum in $[\mathcal{C},\mathcal{D}]$.

Thus we have the following result.

2.2.22 Proposition

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be an additive category. Then the functor category $[\mathcal{C},\mathcal{D}]$ is an additive category. □

functor. If 0 *is a zero morphism or object in* $\mathcal{C}$*, then* $F0$ *is a zero morphism or object in* $\mathcal{D}$*.*

Proof

Temporarily, we indicate a zero object in $\mathcal{C}$ by Z to distinguish it from the various zero maps that we use.

If $0 : C' \to C$ is any zero map in $\mathcal{C}$, then $F0$ must be the zero map from FC' to FC in $\mathcal{D}$, since F is additive. As $id_Z = 0$ on Z itself, $id_{FZ} = F(id_Z) = 0$ on FZ. Hence for a zero object Z' in $\mathcal{D}$, the unique maps $FZ \to Z'$ and $Z' \to FZ$ are mutually inverse isomorphisms. □

2.2.20 Theorem

Let $\mathcal{C}$ *and* $\mathcal{D}$ *be additive categories and let* $F : \mathcal{C} \to \mathcal{D}$ *be an additive functor. Then for any two objects* C' *and* C'' *of* $\mathcal{C}$*,*

$$F(C' \oplus C'') \cong F(C') \oplus F(C'') \text{ in } \mathcal{D}.$$

Proof

Recall from (2.2.11) that the direct sum of the pair of objects C' and C'' in $\mathcal{C}$ can be characterized as an object C of $\mathcal{C}$ for which there are morphisms

$$\sigma' : C' \longrightarrow C,\ \sigma'' : C'' \longrightarrow C,\ \pi' : C \longrightarrow C' \text{ and } \pi'' : C \longrightarrow C''$$

satisfying the relations

$$\pi'\sigma' = id_{C'},\ \pi''\sigma'' = id_{C''} \text{ and } \sigma'\pi' + \sigma''\pi'' = id_C.$$

Since F preserves sums and products, and converts identities to identities and zeroes to zeroes, the maps $F\sigma'$, $F\sigma''$, $F\pi'$ and $F\pi''$ satisfy the equations that show FC to be a direct sum also. □

2.2.21 Functor categories

Recall from (1.3.9) that, given a small category $\mathcal{C}$ and an arbitrary category $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ has as objects the covariant functors from $\mathcal{C}$ to $\mathcal{D}$, the morphisms from F to G in $[\mathcal{C}, \mathcal{D}]$ comprising the set $\mathrm{Nat}(F, G)$ of all natural transformations from F to G.

Our aim now is to show that properties of $\mathcal{D}$ can be carried over to $[\mathcal{C}, \mathcal{D}]$ by making *pointwise* definitions; the category $\mathcal{C}$ can be completely arbitrary.

The essential observation is that a set $\mathrm{Nat}(F, G)$ of morphisms in $[\mathcal{C}, \mathcal{D}]$ is in fact a disjoint union of sets $\mathrm{Mor}_{\mathcal{D}}(F(C), G(C))$, one for each object C of $\mathcal{C}$,

where

$$\gamma \oplus \gamma' = \sigma_D \gamma \pi_C + \sigma_{D'} \gamma' \pi_{C'} : C \oplus C' \to D \oplus D',$$

π_C and $\pi_{C'}$ being the split epimorphisms associated with $C \oplus C'$, and σ_D and $\sigma_{D'}$ being the split monomorphisms associated with $D \oplus D'$. In a module category, this becomes the familiar formula

$$(\gamma \oplus \gamma')(c, c') = (\gamma c, \gamma' c') \text{ for all } c \in C, c' \in C'.$$

We record the consequences of our discussion.

2.2.17 Lemma

Suppose that $\mathcal{C}$ is an additive category. Then the categories $\mathcal{M}or_{\mathcal{C}}$, $\mathcal{E}nd_{\mathcal{C}}$, $\mathcal{I}so_{\mathcal{C}}$ and $\mathcal{A}ut_{\mathcal{C}}$ are additive categories. □

2.2.18 Additive functors

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. For each pair of objects C', C in $\mathcal{C}$, the set $\mathrm{Mor}_{\mathcal{C}}(C', C)$ is an additive group, as is $\mathrm{Mor}_{\mathcal{D}}(D', D)$ for each pair D', D in $\mathcal{D}$. In this circumstance, it is natural to consider functors between $\mathcal{C}$ and $\mathcal{D}$ that respect the additive structures on the sets of morphisms.

A covariant functor F from $\mathcal{C}$ to $\mathcal{D}$ is said to be *additive* if the map

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC)$$

is a group homomorphism for all objects C' and C in $\mathcal{C}$:

$$F(\alpha + \beta) = F\alpha + F\beta \quad \text{always.}$$

Additive contravariant functors are defined similarly, as are additive multifunctors on products of additive categories.

First examples are the morphism functors themselves. For a fixed object C of an additive category, $\mathrm{Mor}_{\mathcal{C}}(C, -)$ is a covariant additive functor from $\mathcal{C}$ to $\mathcal{A}_{\mathcal{B}}$ and $\mathrm{Mor}_{\mathcal{C}}(-, C)$ is a contravariant additive functor from $\mathcal{C}$ to $\mathcal{A}_{\mathcal{B}}$.

It is clear that if $\mathcal{C}'$ is an additive subcategory of the additive category $\mathcal{C}$, then the inclusion functor is additive. If F is an additive functor from $\mathcal{C}$ to $\mathcal{D}$, then the restriction of F to $\mathcal{C}'$ will also be additive.

An important property of an additive functor $F : \mathcal{C} \to \mathcal{D}$ is that it preserves zero morphisms, zero objects and direct sums.

2.2.19 Lemma

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories, and let $F : \mathcal{C} \to \mathcal{D}$ be an additive

2.1.8). Also, by Lemma 2.5.7 of [BK: IRM], $M' \oplus M''$ is finitely generated if and only if its summands are finitely generated, which equivalence is easy to check directly.

We record the results as follows.

2.2.14 Lemma

The categories $\mathcal{P}_{ROJR}$, $\mathcal{F}_{REER}$, $\mathcal{M}_R$, $\mathcal{F}_R$ and $\mathcal{P}_R$ are additive subcategories of $\mathcal{M}_{ODR}$. If S is a subring of R, then $\mathcal{M}_{ODR}$ is an additive subcategory of $\mathcal{M}_{ODS}$. □

Next, we consider product categories.

2.2.15 Lemma

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. Then $\mathcal{C} \times \mathcal{D}$ is an additive category.

Proof

The sum of morphisms (α, β) and (α', β') from (C, D) to (C', D') in $\mathcal{C} \times \mathcal{D}$ is given componentwise:

$$(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta').$$

In effect, there is an identity

$$\mathrm{Mor}_{\mathcal{C}}(C, C') \times \mathrm{Mor}_{\mathcal{D}}(D, D') = \mathrm{Mor}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D'))$$

which gives us the abelian group structure that we require in $\mathcal{C} \times \mathcal{D}$. Note that the zero is the pair $(0, 0)$ and the direct sum is

$$(C, D) \oplus (C', D') = (C \oplus C', D \oplus D').$$ □

2.2.16 Morphism categories

Equally straightforward is the extension of the above result to morphism categories. Let $\mathcal{C}$ be an additive category. If (α, β) and (α', β') are both morphisms from (C, D, γ) to (C', D', γ') in $\mathcal{M}_{OR\mathcal{C}}$, then so is their sum. Composition of morphisms is distributive over addition because this is already true in the product category $\mathcal{C} \times \mathcal{C}$, and the zero is $(0, 0, id_0)$.

The direct sum in $\mathcal{M}_{OR\mathcal{C}}$ is given by

$$(C, D, \gamma) \oplus (C', D', \gamma') = (C \oplus C', D \oplus D', \gamma \oplus \gamma'),$$

We also note in passing that when the direct sum $M' \oplus M''$ exists, so also does $M'' \oplus M'$, which is isomorphic to $M' \oplus M''$ via a unique isomorphism. Thus one may speak of the direct sum of an unordered pair of objects.

Note also that it is possible to characterize direct sums in terms of *split exact sequences*, again mimicking the result for modules.

A short exact sequence

$$0 \longrightarrow M' \xrightarrow{\sigma'} M \xrightarrow{\pi'} M'' \longrightarrow 0$$

in a preadditive category is *split* if there are morphisms σ'' and π'' as above. As with modules, it is enough to know that one of the splitting morphisms σ'' or π'' exists, since the other can then be constructed, and we then find that $M \cong M' \oplus M''$.

2.2.12 Additive subcategories

A subcategory $\mathcal{D}$ of an additive category $\mathcal{C}$ is an *additive subcategory* if, as well as being a preadditive subcategory, $\mathcal{D}$ is additive.

Let M', M'' be a pair of objects in $\mathcal{D}$. Any direct sum of M' and M'' in $\mathcal{D}$ is necessarily their direct sum in $\mathcal{C}$ also, but the favoured choice $M' \oplus M''$ of the direct sum in $\mathcal{C}$ need not itself be an object of $\mathcal{D}$. However, there is to be some choice, necessarily isomorphic to this, which, together with its associated inclusions and projections $\sigma', \sigma'', \pi', \pi''$, does lie in $\mathcal{D}$.

As with preadditive categories, we can define a minimal additive subcategory $\mathcal{A}_{\mathcal{DD}}(X)$ of an additive category $\mathcal{C}$ in terms of a nonzero object X of $\mathcal{C}$ by taking $\mathcal{A}_{\mathcal{DD}}(X)$ to be the full subcategory of $\mathcal{C}$ with objects the finite direct sums X^n, $n \geq 1$, together with the zero object.

2.2.13 Examples

Before developing the theory of additive categories, we give some more examples.

All the module categories introduced in (1.1.9), namely, $\mathcal{P}_{\mathcal{ROJ}R}$, $\mathcal{F}_{\mathcal{REE}R}$, $\mathcal{M}_R$, $\mathcal{F}_R$ and $\mathcal{P}_R$, are additive subcategories of $\mathcal{M}_{\mathcal{OD}R}$; the verifications rely on simple facts about direct sums which are mostly immediate consequences of the definitions. For projective modules, we use the fact that $M' \oplus M''$ is projective if and only if both summands are projective ([BK: IRM] Corollary 2.5.6); and for free modules, we note that the zero module is free on the empty set, and that if M and N are free on sets X and Y respectively, then $M \oplus N$ is isomorphic to the free module on the disjoint union $X \sqcup Y$ ([BK: IRM] Exercise

Theorem† ([BK: IRM] (4.1.17)) tells us that, for any module M in $\mathcal{ASS}_{\mathbb{Z}}$, any submodule or quotient module of M must be a direct summand, and so every module in $\mathcal{ASS}_{\mathbb{Z}}$ is both projective and injective as an object of that category. In contrast, no such (nonzero) module can be projective or injective in the category $\mathcal{FA}_{\mathcal{B}}$.

Although the concept of exactness was introduced in [Hurewicz 1941], in the context of homology groups of topological spaces, the term 'exact sequence' did not appear until [Kelley & Pitcher 1947]. The crucial role of exact sequences in contemporary module theory and homology theory is evident from almost any modern text on these subjects.

2.2.11 Additive categories

As we remarked in (1.4.10), the direct sum $M' \oplus M''$ of two modules M' and M'' is characterized up to isomorphism by the existence of homomorphisms

$$\sigma' : M' \longrightarrow M,\ \sigma'' : M'' \longrightarrow M,\ \pi' : M \longrightarrow M' \text{ and } \pi'' : M \longrightarrow M''$$

satisfying the relations

$$\pi'\sigma' = id_{M'},\ \pi''\sigma'' = id_{M''} \text{ and } \sigma'\pi' + \sigma''\pi'' = id_M.$$

These relations make sense in any preadditive category $\mathcal{C}$, and so we may define the *direct sum* of objects M' and M'' of $\mathcal{C}$ to be an object M of $\mathcal{C}$ which satisfies these relations. However, a preadditive category need not contain the direct sum of a pair of objects; an easy example is given by $\mathcal{P}_{READD}(\mathbb{Z})$ in $\mathcal{A}_{\mathcal{B}}$. These observations lead to the following definition.

A category $\mathcal{C}$ is said to be *additive* if it is a preadditive category satisfying the next condition.

Ad 4. Any two objects of $\mathcal{C}$ have a direct sum in $\mathcal{C}$.

As in the case of modules (the same arguments apply), such a direct sum is both a product and a coproduct and so it is unique to within unique isomorphism. The notation $M' \oplus M''$ is used to indicate any convenient choice of the direct sum – in most contexts, there is an obvious candidate for this choice.

It can be shown (Exercise 2.2.6) that a product in a preadditive category must be a direct sum, as must be a coproduct. Conversely, when products and coproducts exist and they are the same as direct sums in a category $\mathcal{C}$ satisfying Ad 3, then Ad 1 and Ad 2 hold, so that $\mathcal{C}$ is additive. (See [Gabriel 1962] for details.)

† The Artinian Splitting Theorem holds over any ring R.

Proj 1. The generalization of our basic definition for modules (1.1.7): an object P of $\mathcal{C}$ is *projective* (in $\mathcal{C}$) if every short exact sequence

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} P \longrightarrow 0$$

in $\mathcal{C}$ *splits* at P, meaning that there exists $\sigma : P \to M$ in $\mathcal{C}$ with $\pi\sigma = id_P$.

Proj 2. P is projective if and only if, whenever the row is exact, the following diagram can be completed (so as to provide a commuting triangle):

$$\begin{array}{ccccc} & & P & & \\ & \overset{\beta}{\swarrow} & \downarrow{\scriptstyle \gamma} & & \\ Y & \xrightarrow[\eta]{} & Z & \longrightarrow & 0 \end{array}$$

(The equivalence of Proj 1 and Proj 2 for modules is shown in Theorem 2.5.4 of [BK: IRM].)

Proj 3. P is projective if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Mor}\,(P, X) \xrightarrow{\xi_*} \mathrm{Mor}\,(P, Y) \xrightarrow{\eta_*} \mathrm{Mor}\,(P, Z) \longrightarrow 0$$

is exact for every short exact sequence

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

in $\mathcal{C}$ (see (2.1.8) for the module version).

The proof of the fact that these definitions are mutually equivalent is much the same as that for modules, and is left to the reader. Likewise, the reader should have no problem in formulating and verifying the equivalence of three definitions for an injective object.

However, there is no reason why a preadditive category should contain any nontrivial projective or injective object. For example, the category $\mathcal{FA}_{\mathcal{B}}$ of finite abelian groups (which can also be described as the category of finitely generated torsion $\mathbb{Z}$-modules or the category of Artinian $\mathbb{Z}$-modules) has no nonzero projectives – this can be seen as in Exercise 2.2.10 below.

More seriously, it may happen that a projective object in a subcategory is not projective in a larger category. For an illustration, consider the category $\mathcal{ASS}_{\mathbb{Z}}$ of Artinian semisimple $\mathbb{Z}$-modules. Such a module is a finite sum of irreducible $\mathbb{Z}$-modules, that is, abelian groups $\mathbb{Z}/p\mathbb{Z}$ of prime order for various primes p (see (2.3.20 - F) below). A fundamental result, the Artinian Splitting

(ii) $\gamma : B \to C$ *is a cokernel for* α *if and only if, for any* X *in* $\mathcal{C}$,

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, X) \xrightarrow{\gamma^*} \mathrm{Mor}_{\mathcal{C}}(B, X) \xrightarrow{\alpha^*} \mathrm{Mor}_{\mathcal{C}}(A, X)$$

is an exact sequence of abelian groups. Thus, for any choice of $\mathrm{Cok}\,\alpha$,

$$\mathrm{Mor}_{\mathcal{C}}(\mathrm{Cok}\,\alpha, X) = \mathrm{Ker}\,\mathrm{Mor}_{\mathcal{C}}(\alpha, X),$$

(*where* α^* *is an alternative notation for* $\mathrm{Mor}_{\mathcal{C}}(\alpha, X)$). □

An immediate consequence is that a morphism is a monomorphism precisely when its kernel is the zero object, and an epimorphism precisely when its cokernel is the zero object.

2.2.9 *Short exact sequences*

A *short exact sequence* in a preadditive category is a sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

in which α is a kernel for β and β is a cokernel for α. Thus, in particular, α is a monomorphism and β is an epimorphism.

An easy exercise is that β is an isomorphism if and only if A is the zero object, while α is an isomorphism just when $C = 0$.

The distinction between an epimorphism and a surjection means that some caution must be exercised when working with short exact sequences in a general preadditive category. The unexpected phenomena that can arise are illustrated by the inclusion map $\iota : \mathbb{Q} \to \mathbb{R}$.

As we noted in (2.2.6), ι is both an epimorphism and monomorphism in the category $\mathcal{HA}_{\mathcal{B}}$ of Hausdorff abelian topological groups, but it is not an isomorphism. Thus the sequence

$$0 \longrightarrow \mathbb{Q} \xrightarrow{\iota} \mathbb{R} \xrightarrow{0} 0 \longrightarrow 0$$

is not short exact. In fact, $\mathbb{Q}$ cannot be the kernel of $\mathbb{R} \to 0$ because the identity map on $\mathbb{R}$ cannot be factored through $\mathbb{Q}$.

2.2.10 *Projective and injective objects*

There are three equivalent definitions for a projective object in an arbitrary preadditive category $\mathcal{C}$, which generalize well-known characterizations of projective modules.

Let $\mathcal{HA}_B$ be the category of Hausdorff topological abelian groups. Thus an object A of $\mathcal{HA}_B$ is a Hausdorff topological space which is also an abelian group in such a way that the group operations are continuous maps, and a morphism from A to B in $\mathcal{HA}_B$ is a continuous map which is also a group homomorphism. We exploit the following result *ibid,* (13.14): if $f, g : X \to Y$ are continuous maps of topological spaces with Y Hausdorff, and f and g agree on a dense subset of X, then $f = g$.

Now the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are both objects of $\mathcal{HA}_B$ with their customary additions and topologies, and moreover $\mathbb{Q}$ is dense in $\mathbb{R}$. The result quoted above shows that the inclusion map $\iota : \mathbb{Q} \to \mathbb{R}$ is an epimorphism in $\mathcal{HA}_B$, although not a surjection. Note that ι is also a monomorphism and, of course, an injection.

In particular, we have an example of a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.

The quotient group $\mathbb{R}/\mathbb{Q}$ can be given a topology (indiscrete) so that it becomes the cokernel of ι in the category $\mathcal{TOPA}_B$ of all topological abelian groups, but it is not Hausdorff (see [Higgins 1974] Chapter II, §3 and Proposition 5).

We also note that $\mathcal{HA}_B$ is a preadditive subcategory of $\mathcal{TOPA}_B$ but that ι is no longer an epimorphism in $\mathcal{TOPA}_B$.

2.2.7 Kernel and cokernel

Before we define short exact sequences in a preadditive category, we record the next, easily verified lemma, which shows that the concepts of kernel and cokernel in a preadditive category correspond to their counterparts in $\mathcal{A}_B$ via the morphism functors.

2.2.8 Lemma

Let $\alpha : A \to B$ be a morphism in a preadditive category C. Then the following assertions hold.

(i) *$\kappa : K \to A$ is a kernel for α if and only if, for any L in C,*

$$0 \longrightarrow \mathrm{Mor}_C(L, K) \xrightarrow{\kappa_*} \mathrm{Mor}_C(L, A) \xrightarrow{\alpha_*} \mathrm{Mor}_C(L, B)$$

is an exact sequence of abelian groups. Thus, for any choice of $\mathrm{Ker}\,\alpha$,

$$\mathrm{Mor}_C(L, \mathrm{Ker}\,\alpha) = \mathrm{Ker}\,\mathrm{Mor}_C(L, \alpha).$$

(*Note that α_* is an alternative notation for* $\mathrm{Mor}_C(L, \alpha)$.)

The most blatant examples in which a monomorphism cannot be an injection are provided by categories whose objects are not sets. More seriously, it may happen that the objects of a preadditive category are sets, but that a monomorphism need not be an injection or an epimorphism need not be a surjection. Such pathologies are somewhat tricky to find.

We give examples of both phenomena.

2.2.5 *Example: the opposite category*

Recall from (1.1.6) that the opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ has objects C^{op} in bijective correspondence with the objects C of $\mathcal{C}$, and that for each pair of objects C^{op}, D^{op} of $\mathcal{C}^{\mathrm{op}}$, there is a bijection between the morphisms α in $\mathrm{Mor}_{\mathcal{C}}(C, D)$ and the morphisms α^{op} in $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(D^{\mathrm{op}}, C^{\mathrm{op}})$.

The composition in $\mathcal{C}^{\mathrm{op}}$ is given by the rule

$$(\alpha^{\mathrm{op}})(\beta^{\mathrm{op}}) = (\beta\alpha)^{\mathrm{op}}.$$

If $\mathcal{C}$ is preadditive, then so also is $\mathcal{C}^{\mathrm{op}}$, under the natural definition

$$\alpha^{\mathrm{op}} + \beta^{\mathrm{op}} = (\alpha + \beta)^{\mathrm{op}}.$$

A morphism $\xi : X \to Y$ in $\mathcal{C}$ induces

$$\xi_* : \mathrm{Mor}_{\mathcal{C}}(L, X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L, Y),$$

while its opposite counterpart $\xi^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ gives

$$(\xi^{\mathrm{op}})^* : \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, L^{\mathrm{op}}) \longrightarrow \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(Y^{\mathrm{op}}, L^{\mathrm{op}}).$$

For any $\lambda : L \to X$, we have $(\xi^{\mathrm{op}})^*(\lambda^{\mathrm{op}}) = (\xi_*(\lambda))^{\mathrm{op}}$, so that $(\xi^{\mathrm{op}})^*$ and ξ_* are either both injective or both not injective.

Thus ξ^{op} is an epimorphism exactly when ξ is a monomorphism. Similarly, ξ^{op} is a monomorphism precisely when ξ is an epimorphism.

Thus, if $\mathcal{C}$ contains both monomorphisms and epimorphisms, so does $\mathcal{C}^{\mathrm{op}}$. However, the objects of $\mathcal{C}^{\mathrm{op}}$ are not sets, so its monomorphisms cannot be injective mappings, nor can its epimorphisms be surjections.

2.2.6 *Example: topological abelian groups*

This example relies on some elementary topology which we do not develop in this text. (The definitions of terms below can be found in [Willard 1970] §13.) As well as providing an epimorphism which is not a surjection, it shows that an epimorphism in a subcategory may cease to be an epimorphism in a larger category.

functors on a preadditive category take values in $\mathcal{A}_B$ (rather than $\mathcal{S}_{ET}$) is crucial in what follows.

We start with monomorphisms and epimorphisms, which are the analogues of injective and surjective homomorphisms – the reason for the change in terminology will be explained shortly.

By analogy with (2.1.3), we make the following definitions in a preadditive category C.

A morphism $\mu : X \to Y$ in C is said to be a *monomorphism* if, for any object L of C,

$$0 \longrightarrow \mathrm{Mor}_C(L, X) \xrightarrow{\mu_*} \mathrm{Mor}_C(L, Y)$$

is an exact sequence of abelian groups.

A morphism $\epsilon : M \to N$ in C is said to be an *epimorphism* if, for any object X of C,

$$0 \longrightarrow \mathrm{Mor}_C(N, X) \xrightarrow{\epsilon^*} \mathrm{Mor}_C(M, X)$$

is an exact sequence of abelian groups.

Important examples of monomorphisms and epimorphisms are provided by the following result.

2.2.4 Lemma

If $\mu : X \to Y$ has a left inverse $\rho : Y \to X$ (that is, $\rho\mu = id_X$), then μ is a monomorphism.

If $\epsilon : M \to N$ has a right inverse $\sigma : N \to M$ (that is, $\epsilon\sigma = id_N$), then ϵ is an epimorphism.

Proof

By functoriality of $\mathrm{Mor}_C(L, -)$, $\rho_*\mu_*$ is just the identity homomorphism on the group $\mathrm{Mor}_C(L, X)$. So μ_* is injective. Similarly, ϵ_* is surjective. □

A morphism with a left inverse is often called a split monomorphism, while a morphism with a right inverse is known as a split epimorphism.

Warning. These descriptions are valid only for right categories! The reader may wish to formulate their left category counterparts. (The definitions of epimorphism and monomorphism can be extended to arbitrary categories; see Exercise 2.2.4.)

In a module category $\mathcal{M}_{OD_R}$, the basic result (2.1.3) shows that a monomorphism is the same thing as an injective morphism while an epimorphism is just a surjective morphism. However, these coincidences need not happen in an abstract category, which explains the need for new terminology.

Thus $\mathrm{Mor}_{\mathcal{C}}(-,X)$ and $\mathrm{Mor}_{\mathcal{C}}(L,-)$ are functors on $\mathcal{C}$, contravariant and covariant respectively, with values in the category $\mathcal{A}_{\mathcal{B}}$ of abelian groups. As with the Hom functor, in a preadditive category we have equalities

$$(\lambda+\lambda')^* = \lambda^* + \lambda'^* \text{ and } (\xi+\xi')_* = \xi_* + \xi'_*$$

for pairs of morphisms $\lambda, \lambda' : L \to M$ and $\xi, \xi' : X \to Y$. Each pair of morphisms $\lambda : L \to M$ and $\xi : X \to Y$ gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}(M,X) & \xrightarrow{\xi_*} & \mathrm{Mor}(M,Y) \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda^*} \\ \mathrm{Mor}(L,X) & \xrightarrow{\xi_*} & \mathrm{Mor}(L,Y) \end{array}$$

showing that the homomorphisms λ^* define a natural transformation from the functor $\mathrm{Mor}\,(M,-)$ to the functor $\mathrm{Mor}\,(L,-)$ and that the homomorphisms ξ_* give a natural transformation from $\mathrm{Mor}\,(-,X)$ to $\mathrm{Mor}\,(-,Y)$.

2.2.2 *Preadditive subcategories*

Next we define a preadditive subcategory of a preadditive category. There is no general agreement as to the proper definition, so we have chosen one suited to our immediate purposes.

A subcategory $\mathcal{D}$ of a preadditive category $\mathcal{C}$ is said to be a *preadditive subcategory* if

PAS 1. $\mathrm{Mor}_{\mathcal{D}}(L,X)$ is a subgroup of $\mathrm{Mor}_{\mathcal{C}}(L,X)$ for any pair of objects L and X of $\mathcal{D}$, and

PAS 2. $\mathcal{D}$ contains a zero object of $\mathcal{C}$.

Evidently, such a subcategory $\mathcal{D}$ is itself a preadditive category.

The zero object of a preadditive category can be viewed as defining a preadditive subcategory with one object. Less trivially, a nonzero object X of $\mathcal{C}$ gives a minimal full preadditive subcategory $\mathcal{P}_{\mathcal{PREADD}}(X)$ of $\mathcal{C}$; the set of objects of $\mathcal{P}_{\mathcal{PREADD}}(X)$ is $\{0,X\}$ and the set of morphisms is $\mathrm{End}_{\mathcal{C}}(X)$ together with the requisite zero morphisms.

2.2.3 *Monomorphisms and epimorphisms*

We now use the Mor functors to transcribe definitions and results from module categories $\mathcal{M}_{\mathcal{OD}_R}$ to arbitrary preadditive categories. The fact that the Mor

there is a corresponding discussion for left categories. Modules are right modules over an arbitrary ring R, unless otherwise specified.

To make the comparison and transcription of results easier, we often use the same symbols – $L, M, \ldots, X, Y, \ldots$ – for objects of an abstract category $\mathcal{C}$ as we used for modules in the preceding section.

2.2.1 Preadditive categories

A category $\mathcal{C}$ is called *preadditive* if the following conditions are satisfied.

Ad 1. For each pair of objects M and N of $\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(M,N)$ is an abelian group.

Ad 2. If $\lambda, \lambda' : L \to M$ and $\mu, \mu' : M \to N$ are morphisms in $\mathcal{C}$, then

$$\mu(\lambda + \lambda') = \mu\lambda + \mu\lambda' \text{ and } (\mu + \mu')\lambda = \mu\lambda + \mu'\lambda.$$

Ad 3. $\mathcal{C}$ contains a zero object 0 such that $\mathrm{Mor}_{\mathcal{C}}(0,N) = 0$ and $\mathrm{Mor}_{\mathcal{C}}(M,0) = 0$, the trivial group, for all objects M and N.

Here, a zero object is as defined in (1.4.14). There is (usually) more than one zero object, but any two zero objects are isomorphic by a unique isomorphism, so we often refer to 'the' zero object of $\mathcal{C}$.

Note that the set $\mathrm{End}_{\mathcal{C}}(M)$ of endomorphisms of any object M in $\mathcal{C}$ must be a ring.

Of course, $\mathcal{A}_{\mathcal{B}}$ is a preadditive category, and so is the category $\mathcal{R}_{\mathcal{NG}}$ of nonunital rings. However, the category $\mathcal{R}_{\mathcal{ING}}$ of rings is *not* preadditive because the zero mapping is not in general a ring homomorphism. It is clear that $\mathcal{M}_{\mathcal{OD}_R}$ is preadditive for any ring R, and it is easy to see that the basic functorial properties of $\mathrm{Hom}_R(-,-)$, given in (2.1.1), transcribe to $\mathrm{Mor}_{\mathcal{C}}(-,-)$. We list them for convenience.

A morphism $\lambda : L \to M$ in $\mathcal{C}$ defines a map

$$\lambda^* : \mathrm{Mor}_{\mathcal{C}}(M,X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L,X) \text{ by } \lambda^*(\xi) = \xi\lambda.$$

From Ad 2, λ^* is a homomorphism of abelian groups. For another morphism $\mu : M \to N$,

$$(\mu\lambda)^* = \lambda^*\mu^* : \mathrm{Mor}_{\mathcal{C}}(N,X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L,X).$$

In the other variable, a morphism $\xi : X \to Y$ gives a homomorphism of abelian groups

$$\xi_* : \mathrm{Mor}_{\mathcal{C}}(L,X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(L,Y) \text{ by } \xi_*(\alpha) = \xi\alpha,$$

and $(\eta\xi)_* = \eta_*\xi_*$ for a morphism $\eta : Y \to Z$.

It is also possible to make a definition in terms of the second variable by taking a short exact sequence

$$0 \longrightarrow X \longrightarrow J \longrightarrow C \longrightarrow 0$$

with J injective (see Exercise 2.1.5 above). Much hard work reveals that the two definitions coincide, or, more properly, that the functors obtained are naturally isomorphic. Full details are given in [Mac Lane 1975] and [Rotman 1979].

2.2 ADDITIVE CATEGORIES

When [Eilenberg & Mac Lane 1945] had given the abstract definition of a category, a natural next step was to seek the conditions on the morphism sets of an abstract category which enable the definitions and constructions of module theory to be developed in that category. The exploration led to the definitions of additive, exact (in the non-K-theory sense) [Buchsbaum 1955] and abelian categories [Grothendieck 1957]. These abstractions were motivated by the desire to apply homological methods in categories that arise in algebraic geometry, such as categories of sheaves and of vector bundles (which we do not encounter in this text), as well as intellectual curiosity. The widest class of categories that share some properties of modules comprises the preadditive categories. In such a category $\mathcal{C}$, each morphism set $\mathrm{Mor}_{\mathcal{C}}(L, X)$ is an abelian group and the functors arising from morphisms must respect addition. These requirements suffice to permit the recognition of direct sums, exact sequences, kernels and cokernels, and projective and injective objects in $\mathcal{C}$. However, there need not be any nontrivial examples of such things within the category.

The next step is the definition of an additive category, which is required to contain a direct sum of any two of its objects. Many interesting module categories are additive categories, for example, the categories $\mathcal{M}_R$ of finitely generated modules and $\mathcal{P}_R$ of finitely generated projectives. Also, many of the categories of interest in K-theory are additive categories.

There is some variation in the literature as to the precise definition of a preadditive category, as there is with the definitions of (pre)additive subcategories of (pre)additive categories. Indeed, some authorities do not make formal definitions. Even where terms like additive and abelian categories have standard meanings, there is a complex web of interacting axioms woven through the literature. Our own choice of axioms has been guided by intuitive appeal rather than minimality.

Throughout this section, we work with right categories, it being clear that

(v) Let $\phi, \psi : L'' \to M''$. Verify the following chain of equivalences:

$$\begin{aligned}\phi^*\mathbf{E} + \psi^*\mathbf{E} &\equiv (\Sigma')_*(\Delta'')^*(\phi^*\mathbf{E} \oplus \psi^*\mathbf{E}) \\ &\equiv (\Sigma')_*(\Delta'')^*(\phi \oplus \psi)^*(\mathbf{E} \oplus \mathbf{E}) \\ &\equiv (\Sigma''(\phi \oplus \psi)\Delta'')^*\mathbf{E} \\ &\equiv (\phi + \psi)^*\mathbf{E}.\end{aligned}$$

Deduce that $\mathrm{Ext}^1_R(-, M')$ is a contravariant functor. Show also that $\mathrm{Ext}^1_R(M'', -)$ is a covariant functor and that $\mathrm{Ext}^1_R(-,-)$ is a bifunctor.

(vi) Show that $\mathrm{Ext}^1_R(P, X) = 0$ for all R-modules X if and only if P is a projective R-module.

(vii) Let

$$\mathbf{Y} \qquad 0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

be an exact sequence of right R-modules. For $\lambda \in \mathrm{Hom}(L, Z)$, define

$$\partial(\lambda) = [\lambda^*\mathbf{Y}] \in \mathrm{Ext}^1_R(Z, X),$$

and, anticipating the Snake Lemma (Exercise 2.3.13), verify that

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}_R(L, X) &\longrightarrow \mathrm{Hom}_R(L, Y) \longrightarrow \mathrm{Hom}_R(L, Z) \\ \xrightarrow{\partial} \mathrm{Ext}^1_R(L, X) &\longrightarrow \mathrm{Ext}^1_R(L, Y) \longrightarrow \mathrm{Ext}^1_R(L, Z)\end{aligned}$$

is an exact sequence of abelian groups.

(viii) Derive the corresponding exact sequence with fixed second variable, and show that $\mathrm{Ext}^1_R(L, J) = 0$ for all R-modules L if and only if J is an injective R-module.

(ix) Here is an inductive definition of $\mathrm{Ext}^n_R(L, X)$ for $n \geq 2$. Choose a short exact sequence

$$0 \longrightarrow S \longrightarrow P \longrightarrow L \longrightarrow 0$$

with P projective (such a sequence exists; we may take P to be free by Lemma 2.5.7 of [BK: IRM]), and put

$$\mathrm{Ext}^n_R(L, X) = \mathrm{Ext}^{n-1}_R(S, L) \text{ for } n \geq 2.$$

It can be shown that the right-hand side is independent of the choice of the short exact sequence above.

$$\mathbf{E}_1 \qquad 0 \longrightarrow M' \xrightarrow{\alpha_1} M_1 \xrightarrow{\beta_1} M'' \longrightarrow 0$$

and

$$\mathbf{E}_2 \qquad 0 \longrightarrow M' \xrightarrow{\alpha_2} M_2 \xrightarrow{\beta_2} M'' \longrightarrow 0$$

are *equivalent* if there is an isomorphism $\chi : M_1 \to M_2$ with $\chi\alpha_1 = \alpha_2$ and $\beta_2\chi = \beta_1$. The notation is $\mathbf{E}_1 \equiv \mathbf{E}_2$.

For fixed M' and M'', equivalence of short exact sequences is an eqivalence relation on the set of all short exact sequences $\mathbf{E}$, and it is compatible with pull-backs and push-outs.

We write $[\mathbf{E}]$ for the class of $\mathbf{E}$ and we define $\mathrm{Ext}^1_R(M'', M')$ to be the set of all such equivalence classes.

The direct sum $\mathbf{E}_1 \oplus \mathbf{E}_2$ is defined to be the short exact sequence

$$0 \longrightarrow M' \oplus M' \xrightarrow{\alpha_1 \oplus \alpha_2} M_1 \oplus M_2 \xrightarrow{\beta_1 \oplus \beta_2} M'' \oplus M'' \longrightarrow 0.$$

Define an operation, the *Baer sum*, on $\mathrm{Ext}^1_R(M'', M')$, by

$$[\mathbf{E}_1] + [\mathbf{E}_2] = [(\Sigma')_*(\Delta'')^*(\mathbf{E}_1 \oplus \mathbf{E}_2)],$$

where the diagonal homomorphism

$$\Delta'' : M'' \longrightarrow M'' \oplus M'' \text{ has } \Delta'' m'' = (m'', m'')$$

and the codiagonal homomorphism

$$\Sigma' : M' \oplus M' \longrightarrow M' \text{ has } \Sigma'(m', n') = m' + n'.$$

(i) Verify that the Baer sum depends only on the equivalence classes of $\mathbf{E}_1$ and $\mathbf{E}_2$, so that it is a well-defined law of composition on $\mathrm{Ext}^1_R(M'', M')$.

(ii) Confirm that the middle term of $\mathbf{E}_1 + \mathbf{E}_2$ is

$$(M_1 \times_{M''} M_2)/\{(\alpha_1 m', -\alpha_2 m') \mid m' \in M'\}.$$

(iii) Verify that the composition is associative and commutative.

(iv) Deduce that $\mathrm{Ext}^1_R(M'', M')$ is an abelian group, with zero element corresponding to the standard split exact sequence, and

$$-[\mathbf{E}] = [(-id)_*\mathbf{E}]$$

where $-id$ is the negative of the identity map on M'.

any other injective R-module homomorphism $\eta : M \to I$ with I an injective module, there is an injective homomorphism $\rho : E(M) \to I$ with $\eta = \rho\epsilon$. As a universal object, the module $E(M)$ is unique up to isomorphism and is known as the *injective hull* of M. The construction can be found in most texts on ring theory, for instance [Rotman 1979], §3.

2.1.6 Let Q be a divisible abelian group. For a left R-module N, let $\Delta(N) = \mathrm{Hom}(N_{\mathbb{Z}}, Q_{\mathbb{Z}})$. Verify that $\Delta(N)$ is a right R-module, with $\Delta(R) = \widehat{R}$ as in the previous exercise.

Confirm that Δ is a contravariant functor from ${}_R\mathcal{M}\mathit{od}$ to $\mathcal{M}\mathit{od}_R$ and that Δ restricts to a functor from ${}_R\mathcal{P}\mathit{roj}$ to $\mathcal{I}\mathit{nj}_R$, the category of injective right R-modules.

2.1.7 **The group Ext^1_R**

This substantial exercise gives one construction, using the Baer sum, of the groups $\mathrm{Ext}^1_R(M'', M')$. Our treatment is based on Chapter III, §2, of [Mac Lane 1975]. Alternative definitions, and further details, can be found in most texts on homological algebra (see, for example, [Weibel 1995]).

Our arguments depend on some basic methods of manipulating short exact sequences that are described in more detail in section 2.4 of [BK: IRM]. Let

$$\mathbf{E} \qquad 0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence and let $\theta : L'' \to M''$ be a homomorphism. Then the pull-back of M and L'' over M'' gives an exact sequence

$$\theta^*\mathbf{E} \quad 0 \longrightarrow M' \xrightarrow{\mu} M \times_{M''} L'' \xrightarrow{\overline{\beta}} L'' \longrightarrow 0,$$

the *pull-back* of $\mathbf{E}$ along θ. (An alternative expression is that $\theta^*\mathbf{E}$ is obtained from $\mathbf{E}$ by *base change*.)

Similarly, given a homomorphism $\phi : M' \to N'$, the push-out of N' and M over M' gives a short exact sequence

$$\phi_*\mathbf{E} \quad 0 \longrightarrow N' \xrightarrow{\overline{\alpha}} N' \oplus_{M'} M \xrightarrow{\nu} M'' \longrightarrow 0,$$

the *push-out* of $\mathbf{F}$ along ϕ. (Alternatively, $\phi_*\mathbf{E}$ is said to be obtained from $\mathbf{E}$ by *cobase change*.)

Two short exact sequences

far from straightforward, since it requires some notion of convergence. A related question is treated in Exercise 2.3.1 of [BK: IRM].

2.1.3 Regard the ring R as an R-R-bimodule, and let M be a right R-module. Define a map $\epsilon : M_R \to \mathrm{Hom}(R_R, M_R)$ by $\epsilon(m)(r) = mr$. Show that ϵ is an isomorphism of right R-modules, and deduce that the functor $\mathrm{Hom}(R_R, -) = H^R(-)$ is naturally isomorphic to the identity functor on $\mathcal{M}_{OD_R}$.

Verify the left-handed version of this result.

(The map ϵ is sometimes called *left multiplication by* m; its inverse is *evaluation at* 1.)

2.1.4 Show that an arbitrary direct product of injective R-modules is again injective.

2.1.5 **Injectives exist**

This longish exercise outlines the proof that, for any ring R and right R-module M, there is an injective right R-module N and an injective R-module homomorphism from M to N.

(i) Let Q be any divisible $\mathbb{Z}$-module. Then it is known that Q is an injective $\mathbb{Z}$-module (see Exercise 2.5.11 of [BK: IRM]). Put $\widehat{R} = \mathrm{Hom}(R_\mathbb{Z}, Q_\mathbb{Z})$. Verify that $\widehat{R}$ is a right R-module by $(\rho r)(x) = \rho(rx)$, for ρ in $\widehat{R}$, r, x in R.

(ii) Let $\mathfrak{a}$ be a right ideal of R, let $\iota : \mathfrak{a} \to R$ be the inclusion map, and let $\varphi : \mathfrak{a} \to \widehat{R}$ be an R-module homomorphism. It is a fact that $\widehat{R}$ is injective if for any such φ there is a homomorphism $\psi : R \to \widehat{R}$ with $\psi\iota = \varphi$ – see Exercise 2.5.8 in [BK: IRM].
Let $a \in \mathfrak{a}$. Verify that $\gamma : \mathfrak{a} \to Q$, $\gamma(a) = (\varphi a)(1_R)$, is a homomorphism of $\mathbb{Z}$-modules. Choose $\delta : R_\mathbb{Z} \to Q_\mathbb{Z}$ such that $\delta\iota = \gamma$ and put $(\psi r)(x) = \delta(rx)$. Verify that $\psi r \in \widehat{R}$ and that ψ is as required.

(iii) Let M be any right R-module. Using the fact that there is an isomorphism of $\mathbb{Z}$-modules $M \cong \mathbb{Z}^\Lambda/A$ for some free abelian group $\mathbb{Z}^\Lambda$ and subgroup A of $\mathbb{Z}^\Lambda$, and the divisibility of the group $\mathbb{Q}$ of additive rationals, show that M is a subgroup of a divisible abelian group Q.

(iv) Confirm that there is a chain of injective R-module homomorphisms

$$M \rightarrowtail \mathrm{Hom}(R, M) \rightarrowtail \mathrm{Hom}(R, Q) = \widehat{R}.$$

Remark. It can be shown that an R-module M can be embedded in a 'minimal' injective R-module $E(M)$; this means that there is an injective R-module homomorphism $\epsilon : M \to E(M)$ such that for

There are then long exact sequences of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(L,X) & \longrightarrow & \mathrm{Hom}(L,Y) & \longrightarrow & \mathrm{Hom}(L,Z) & \longrightarrow \\ & \longrightarrow & \mathrm{Ext}^1(L,X) & \longrightarrow & \mathrm{Ext}^1(L,Y) & \longrightarrow & \mathrm{Ext}^1(L,Z) & \longrightarrow \\ & \longrightarrow & \mathrm{Ext}^2(L,X) & \longrightarrow & \cdots & & & \end{array}$$

which may continue for ever to the right, depending on the nature of the ring R and the particular modules involved. There are analogous long exact sequences in which the roles of the variables are exchanged.

The term 'extension functor' is used since the elements of $\mathrm{Ext}^1_R(L,X)$ describe all the possible extensions of L by X, that is, all short exact sequences

$$0 \longrightarrow X \longrightarrow M \longrightarrow L \longrightarrow 0$$

with M a right R-module, under a suitable equivalence relation. The higher extension groups $\mathrm{Ext}^n_R(L,X)$ have a similar interpretation.

The construction of these functors is outlined in Exercise 2.1.7 below.

Exercises

2.1.1 Let ${}_RM$ and ${}_RX$ be left R-modules. Following (2.1.1), define functors $\mathrm{Hom}({}_RM,-) : {}_R\mathcal{M}_{OD} \to \mathcal{A}_B$ and $\mathrm{Hom}(-,{}_RX) : {}_R\mathcal{M}_{OD} \to \mathcal{A}_B$, and determine their chiralities.

Following (2.1.9), show that a bimodule ${}_RM_S$ defines a functor $H^M(-) : {}_R\mathcal{M}_{OD} \to {}_S\mathcal{M}_{OD}$ and that a bimodule ${}_RX_T$ defines a functor $H_X(-) : {}_R\mathcal{M}_{OD} \to \mathcal{M}_{ODT}$. Give the chiralities of these functors.

Thus $\mathrm{Hom}(-,-)$ affords an example of a bifunctor of mixed variance and chirality.

2.1.2 Let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a set of right R-modules, where the index set Λ may be infinite. Show that, for any right R-modules L and X, there are isomorphisms of abelian groups

$$\mathrm{Hom}(\textstyle\bigoplus_\Lambda M_\lambda, X) \cong \prod_\Lambda \mathrm{Hom}(M_\lambda, X)$$

and

$$\mathrm{Hom}(L, \textstyle\prod_\Lambda M_\lambda) \cong \prod_\Lambda \mathrm{Hom}(L, M_\lambda)$$

Show that the inverse of the second of the above isomorphisms restricts to an injective homomorphism

$$\textstyle\bigoplus_\Lambda \mathrm{Hom}(L, M_\lambda) \longrightarrow \mathrm{Hom}(L, \bigoplus_\Lambda M_\lambda),$$

which is in fact an isomorphism when L is finitely generated.

Warning. The description of the fourth term $\mathrm{Hom}(\prod_\Lambda M_\lambda, X)$ is

If N is another right R-module and $\mu : M \to N$ is an R-module homomorphism, we define

$$H_X(\mu) : H_X(N) \longrightarrow H_X(M) \text{ by } \alpha H_X(\mu) = \alpha\mu.$$

In the language of (1.2.6), H_X is a contravariant and contrachiral functor from $\mathcal{M}\text{od}_R$ to ${}_S\mathcal{M}\text{od}$, with $H_X(\mu\nu) = H_X(\mu)H_X(\nu)$ for composable μ and ν.

Now assume that M is a T-R-bimodule for some ring T. Then $\text{Hom}(M_R, X_R)$ becomes a right T-module for any right R-module X, under the rule $(\alpha t)m = \alpha(tm)$ for $\alpha \in \text{Hom}(M_R, X_R), t \in T, m \in M$.

We write $H^M(-) = \text{Hom}(M_R, -)$. The action of this functor on morphisms is given by $H^M(\xi)\alpha = \xi\alpha$, where $\alpha \in \text{Hom}(M_R, X_R)$ and $\xi \in \text{Hom}(X_R, Y_R)$, and Y is a right R-module. We see that $H^M(\eta)H^M(\xi) = H^M(\eta\xi)$ whenever the composition is defined, and that $H^M(-)$ is a covariant, cochiral functor from $\mathcal{M}\text{od}_R$ to $\mathcal{M}\text{od}_T$.

Note that these constructions include the case that X or M is simply an R-module, since any right R-module can be viewed as a $\mathbb{Z}$-R-bimodule. The functor H^M is then the same as the functor $\text{Hom}(M_R, -)$, both having values in the right category $\mathcal{A}\text{b}$, which is $\mathcal{M}\text{od}_{\mathbb{Z}}$ under another name. However, the functor H_X takes values in the mirror category ${}_{\mathbb{Z}}\mathcal{M}\text{od}$ of $\mathcal{A}\text{b}$, so that $\text{Mor}(-, X_R) = \text{Mir} \circ H_X(-)$: compare with (1.1.5) and (1.2.6).

We can also regard both M and X simultaneously as being variables for Hom, which is then a bifunctor from $\mathcal{M}\text{od}_R \times \mathcal{M}\text{od}_R$ to $\mathcal{A}\text{b}$. More generally, Hom gives rise to bifunctors from $\mathcal{M}\text{od}_R \times {}_S\mathcal{M}\text{od}_R$ to ${}_S\mathcal{M}\text{od}$ and from ${}_T\mathcal{M}\text{od}_R \times \mathcal{M}\text{od}_R$ to $\mathcal{M}\text{od}_T$. These will have mixed variance and chirality.

2.1.10 The extension functors

As we saw in (2.1.4), an exact sequence

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

of right R-modules gives rise to an exact sequence of abelian groups

$$0 \longrightarrow \text{Hom}(L, X) \xrightarrow{\xi_*} \text{Hom}(L, Y) \xrightarrow{\eta_*} \text{Hom}(L, Z)$$

for any R-module L. By the example in (2.1.5), the Hom sequence cannot always be extended on the right to form a short exact sequence.

This fault can be repaired by introducing the *extension functors* $\text{Ext}^n_R(-,-)$, $n \geq 1$, which, like $\text{Hom}_R(-,-)$, are bifunctors from $\mathcal{M}\text{od}_R \times \mathcal{M}\text{od}_R$ to $\mathcal{A}\text{b}$.

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N,X) \xrightarrow{\mu^*} \mathrm{Hom}(M,X) \xrightarrow{\lambda^*} \mathrm{Hom}(L,X) \longrightarrow 0$$

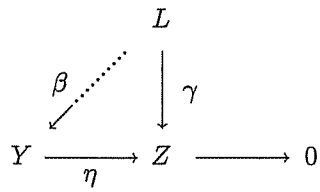
is exact for every exact sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

of right R-modules.

Proof

For (a), we note that the surjectivity of η_* is the same as the assertion that, for any homomorphism $\gamma : L \to Z$, there is a homomorphism $\beta : L \to Y$ which makes the following diagram commute:



But this is a standard criterion for L to be projective (see [BK: IRM] Theorem 2.5.4).

For $(a)^{\mathrm{op}}$, use the dual characterization of injective modules (Exercise 2.5.7 of [BK: IRM]). □

2.1.9 Homomorphism functors arising from bimodules

So far, we have regarded $\mathrm{Hom}(M_R, X_R)$ simply as an abelian group, so that both the functors $\mathrm{Hom}(-, X_R)$ and $\mathrm{Hom}(M_R, -)$ take values in the right category $\mathcal{A}_{\mathcal{B}}$. However, it sometimes happens that we wish to take account of a further module structure on $\mathrm{Hom}(M_R, X_R)$ under which it may be most naturally viewed as a left module, and then our functors will take values in a left category. We therefore introduce a separate notation to alert the reader to our changed point of view.

Suppose that ${}_SX_R$ is an S-R-bimodule for some rings R and S. Then $\mathrm{Hom}(M_R, X_R)$ has a natural left S-module structure given by

$$(s\alpha)(m) = s(\alpha m), \quad s \in S,\ m \in M,\ \alpha : M \to X.$$

Thus $\mathrm{Hom}(-, X_R)$ is a contravariant functor from $\mathcal{M}_{\mathcal{OD}_R}$ to ${}_S\mathcal{M}_{\mathcal{OD}}$, which we write as $H_X(-)$ to distinguish it from its more common manifestation as a functor with values in $\mathcal{A}_{\mathcal{B}}$.

Proof

We prove (a) only. Suppose that (i) holds, that is,

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

is split exact. Then there are R-module homomorphisms $\vartheta : Y \to X$ and $\zeta : Z \to Y$ such that

$$\vartheta\xi = id_X,\ \eta\zeta = id_Z \text{ and } \xi\vartheta + \zeta\eta = id_Y.$$

Since the identity map id_A of a module induces the identity map $(id_A)_*$ on a homomorphism group $\mathrm{Hom}(L, A)$, we have

$$\vartheta_*\xi_* = id,\ \eta_*\zeta_* = id \text{ and } \xi_*\vartheta_* + \zeta_*\eta_* = id,$$

with the identity maps appropriately interpreted. This shows that (ii) holds.

Trivially (ii) gives (iii). Assume that (iii) holds. Taking the special case $L = Z$, we see that there is some homomorphism $\zeta : Z \to Y$ with $\xi_*(\zeta) = id_Y$, that is, $\xi\zeta = id_Y$, which establishes that the module sequence is split exact. □

2.1.7 Projective and injective modules

An alternative procedure for ensuring that the Hom sequences are short exact is to impose conditions on the 'fixed' term rather than the exact sequence of 'variables'. This leads to a characterization of projective modules and their duals, the injective modules.

2.1.8 Proposition

(a) *Let L be a right R-module. Then the following are equivalent.*

(i) *L is projective.*

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z) \longrightarrow 0$$

is exact for every exact sequence

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

of right R-modules.

(a)$^{\mathrm{op}}$ *Let X be a right R-module. Then the following are equivalent.*

(i) *X is injective.*

existence of only one of the maps γ or δ, the other then being constructible, and that when the sequence is split, we have $M \cong M' \oplus M''$.

Conversely, if we are given a direct sum decomposition of M, then we can reconstruct a split exact sequence as above.

2.1.6 Proposition

(a) *The following statements are equivalent.*

(i) *The sequence*

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \longrightarrow 0$$

of right R-modules is split exact.

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z) \longrightarrow 0$$

is split exact for any right R-module L.

(iii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z) \longrightarrow 0$$

is exact for any right R-module L.

$(a)^{\mathrm{op}}$ *The following statements are equivalent.*

(i) *The sequence*

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

of right R-modules is split exact.

(ii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X) \xrightarrow{\lambda^*} \mathrm{Hom}(L, X) \longrightarrow 0$$

is split exact for any right R-module X.

(iii) *The sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X) \xrightarrow{\lambda^*} \mathrm{Hom}(L, X) \longrightarrow 0$$

is exact for any right R-module X.

2.1.5 Short exact sequences

A *short exact sequence* is a sequence

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

which is exact at each of the terms M', M and M''. There is an evident extension of the statement of the preceding result in which all the sequences are taken to be short exact sequences, but the result then fails, as can be seen from the following example.

Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

in which ι is left multiplication by 2: $z \mapsto 2z$, and π is the canonical surjection. This induces a Hom sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\iota^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0.$$

Now $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ itself under the map which assigns to an endomorphism α of $\mathbb{Z}$ its value $\alpha(1)$ at 1, and it is not hard to see that the endomorphism of $\mathbb{Z}$ induced by ι^* is ι again and hence non-surjective. Thus the Hom sequence is not exact on the right.

For the dual example, we look at

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\iota_*} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi_*} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

The terms $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ are evidently 0, while $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. So again the sequence fails to be exact on the right.

There are several interesting circumstances in which the Hom sequences are guaranteed to be short exact sequences. The first is the most immediate extension of the last proposition. To state the result, we need the following definition. A short exact sequence of R-modules (or of abelian groups)

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is said to be a *split exact sequence* if we can find homomorphisms

$$\gamma : M \longrightarrow M' \text{ and } \delta : M'' \longrightarrow M$$

such that

$$\beta\delta = id_{M''}, \quad \gamma\alpha = id_{M'} \text{ and } \alpha\gamma + \delta\beta = id_M.$$

It can be shown ([BK: IRM] Theorem 2.4.5) that it is enough to know the

is an exact sequence of R-modules tells us that X is isomorphic via ξ to the kernel of η, while the exactness of

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

is equivalent to the fact that N is isomorphic via μ to the cokernel of λ.

2.1.4 Proposition

(i) *A sequence*

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$$

of right R-modules is exact if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y) \xrightarrow{\eta_*} \mathrm{Hom}(L, Z)$$

is exact for any right R-module L.

(i)$^{\mathrm{op}}$ *A sequence*

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0$$

of right R-modules is exact if and only if the sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X) \xrightarrow{\lambda^*} \mathrm{Hom}(L, X)$$

is exact for any right R-module X.

Proof

We prove (i) only. Suppose that $0 \to X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$ is exact. By the previous result, ξ_* is injective, so it is enough to check exactness at $\mathrm{Hom}(L, Y)$. For any $\alpha \in \mathrm{Hom}(L, X)$, we have $\eta_*\xi_*\alpha = \eta\xi\alpha = 0$ since $\eta\xi = 0$. Suppose that $\beta \in \mathrm{Hom}(L, Y)$ and $\eta_*\beta = 0$. Then $\eta\beta = 0$, and so there is a homomorphism $\alpha' : L \to \mathrm{Ker}\,\eta$ such that $\kappa\alpha' = \beta$, where κ is the canonical injection of $\mathrm{Ker}\,\eta$ into Y. But, as we noted above, $\mathrm{Ker}\,\eta$ is isomorphic via ξ to X, so we can construct a homomorphism $\alpha : L \to X$ with $\beta = \xi\alpha = \xi_*\alpha$.

Conversely, suppose that the Hom sequence is exact. Again, the previous result tells us that ξ is injective. Substituting X for L, we see that $0 = \eta_*\xi_*(id_X) = \eta\xi$, that is, $\xi X \subseteq \mathrm{Ker}\,\eta$.

To obtain equality, take an element $y \in \mathrm{Ker}\,\eta$ and put $L = yR$. Let $\beta : L \to Y$ be the inclusion map. Since $\eta_*\beta = 0$, $\beta = \xi_*\alpha = \xi\alpha$ for some $\alpha : L \to X$, which shows that $y \in \xi X$. □

is called an exact sequence if it is exact at each of its *terms* M_i. Such a sequence may have finite length, or it may be infinite. (A more detailed treatment of exact sequences can be found in [BK: IRM], particularly section 2.4.) Thus the sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M$$

is exact precisely when the homomorphism λ is injective, while

$$M \xrightarrow{\mu} N \longrightarrow 0$$

is exact precisely when μ is surjective.

2.1.3 Proposition

(i) *The sequence*

$$0 \longrightarrow X \xrightarrow{\xi} Y$$

is exact in $\mathcal{M}_{OD_R}$ *if and only if*

$$0 \longrightarrow \mathrm{Hom}(L, X) \xrightarrow{\xi_*} \mathrm{Hom}(L, Y)$$

is an exact sequence of abelian groups for any right R*-module* L.

(i)$^{\mathrm{op}}$ *The sequence*

$$M \xrightarrow{\mu} N \longrightarrow 0$$

is exact in $\mathcal{M}_{OD_R}$ *if and only if*

$$0 \longrightarrow \mathrm{Hom}(N, X) \xrightarrow{\mu^*} \mathrm{Hom}(M, X)$$

is an exact sequence of abelian groups for any right R*-module* X.

Proof

We prove (i) only, (i)$^{\mathrm{op}}$ being similar. Suppose that ξ is injective and that $\xi_*\alpha = 0, \alpha : L \to X$. By definition, this means that $\xi\alpha = 0$, and so $\alpha = 0$.

Conversely, suppose that ξ_* is injective. Choose any element $x \in \mathrm{Ker}\,\xi$, put $L = xR$ and let $\alpha : L \to X$ be the inclusion map. Then $\xi_*\alpha = 0$, hence $\alpha = 0$, and so $xR = 0$, which means that $x = 0$. □

Next we examine what happens when we extend the exact sequences by one term. The assertion that

$$0 \longrightarrow X \xrightarrow{\xi} Y \xrightarrow{\eta} Z$$

Each set $\mathrm{Hom}(L_R, X_R)$ is an abelian group, with addition given by the rule

$$(\alpha + \alpha')\ell = \alpha\ell + \alpha'\ell \text{ for } \ell \in L \text{ and } \alpha, \alpha' \in \mathrm{Hom}(L_R, X_R),$$

and it is easy to verify that the maps λ^* and ξ_* are homomorphisms of abelian groups, that is,

$$\lambda^*(\alpha + \alpha') = \lambda^*\alpha + \lambda^*\alpha' \text{ and } \xi_*(\alpha + \alpha') = \xi_*\alpha + \xi_*\alpha'$$

for $\alpha, \alpha' \in \mathrm{Hom}(L_R, X_R)$.

Thus $\mathrm{Hom}(-, X)$ and $\mathrm{Hom}(L, -)$ are functors with values in the category $\mathcal{A}_B$ of abelian groups rather than the category $\mathcal{S}_{ET}$ of sets. Furthermore, the equalities

$$(\lambda + \lambda')^* = \lambda^* + \lambda'^* \text{ and } (\xi + \xi')_* = \xi_* + \xi'_*$$

hold for pairs of homomorphisms $\lambda, \lambda' : L \to M$ and $\xi, \xi' : X \to Y$ and, as in (1.3.2)(i), each pair of homomorphisms $\lambda : L \to M$ and $\xi : X \to Y$ gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(M_R, X_R) & \xrightarrow{\xi_*} & \mathrm{Hom}(M_R, Y_R) \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda^*} \\ \mathrm{Hom}(L_R, X_R) & \xrightarrow{\xi_*} & \mathrm{Hom}(L_R, Y_R) \end{array}$$

which can be interpreted as showing that the homomorphisms λ^* define a natural transformation from the functor $\mathrm{Hom}(M_R, -)$ to the functor $\mathrm{Hom}(L_R, -)$, or, equally, that the homomorphisms ξ_* give a natural transformation from $\mathrm{Hom}(-, X_R)$ to $\mathrm{Hom}(-, Y_R)$.

2.1.2 Exact sequences

We now interpret the exactness of sequences of modules in terms of the Hom functors. By definition, a sequence

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N$$

of right R-modules and right R-module homomorphisms is *exact* if

$$\mathrm{Ker}(\mu) = \mathrm{Im}(\lambda).$$

More generally, a sequence of right R-modules

$$\cdots \longrightarrow M_{i-1} \xrightarrow{\alpha_{i-1}} M_i \xrightarrow{\alpha_i} M_{i+1} \longrightarrow \cdots$$

an R-module homomorphism, or the injectivity or projectivity of a module, can be reinterpreted in terms of the behaviour of the functors $\mathrm{Hom}(M,-)$ and $\mathrm{Hom}(-,N)$. Such results are useful not only for their applications in module theory, but also because they suggest a method whereby concepts such as injectivity and projectivity can be extended to abstract categories, provided the morphism sets in these categories share enough of the properties of $\mathrm{Hom}(-,-)$. This theme will be developed in the next section. The Hom functors are basic to the study of homology, and their introduction in [Eilenberg & Mac Lane 1942] predates category theory.

As usual, we work with categories of right modules, except in a few places where we are obliged to consider left modules. The transcription of the main part of our discussion to left modules is not quite immediate, although straightforward. The complication arises since we usually view $\mathrm{Hom}(M,N)$ as a member of the right category $\mathcal{A}_B$, regardless of the chirality of the modules M and N. There are also some awkward but essentially trivial variations in notation which arise when we need to regard $\bar{\mathrm{H}}\mathrm{om}(M,N)$ as a left module over some ring.

2.1.1 *Basic properties*

We start with a discussion of the functorial properties of the sets of homomorphisms between modules.

Let R be a ring and let L_R and X_R be right R-modules. Since $\mathrm{Hom}(L_R,X_R)$ is simply the set $\mathrm{Mor}_{\mathcal{C}}(L,X)$ where the category $\mathcal{C}$ is the category $\mathcal{M}\mathrm{od}_R$ of right R-modules, the discussion in (1.2.7) shows that Hom defines a pair of functors $\mathrm{Hom}(-,X)$ and $\mathrm{Hom}(L,-)$, which are respectively contravariant and covariant.

Explicitly, given a right R-module homomorphism $\lambda : L \to M$, the map

$$\lambda^* : \mathrm{Hom}(M_R,X_R) \longrightarrow \mathrm{Hom}(L_R,X_R)$$

is given by $\lambda^*(\xi) = \xi\lambda$. If $\mu : M \to N$ is another right R-module homomorphism, then

$$(\mu\lambda)^* = (\lambda)^*(\mu)^* : \mathrm{Hom}(N_R,X_R) \longrightarrow \mathrm{Hom}(L_R,X_R).$$

On the other hand, for any R-module homomorphism $\xi : X \to Y$, the map

$$\xi_* : \mathrm{Hom}(L_R,X_R) \longrightarrow \mathrm{Hom}(L_R,Y_R)$$

is given by $\xi_*(\alpha) = \xi\alpha$, and $(\eta\xi)_* = \eta_*\xi_*$ for an R-module homomorphism $\eta : Y \to Z$.

2
CATEGORIES AND EXACT SEQENCES

The definition of an abstract category imposes few conditions on the objects and morphisms of a category, with the consequence that a great diversity of mathematical entities can be regarded as categories. In contrast, a category of modules has a richer structure than the minimum required for the definition of a category. The set of homomorphisms $\mathrm{Hom}(M, N)$ between two modules is always an abelian group, and we can recognise whether or not a homomorphism is an injection or surjection. Furthermore, the category may or may not contain such things as direct sums, kernels, projective modules, etc., depending on the nature of the modules that belong to it.

Our aim in this chapter is to analyse the extra structure that is available to module categories, and thereby to formulate sets of conditions which ensure that an abstract category behaves as a module category. We start by investigating the functorial properties of the abelian groups $\mathrm{Hom}(M, N)$ and we see how to reformulate the basic properties of modules and homomorphisms in terms of these functorial properties. We then use these interpretations to define a series of types of abstract category – preadditive, additive, abelian – which increasingly resemble module categories. Ultimately, we arrive at the notion of an exact category, which is a category that has enough structure for the purposes of algebraic K-theory.

2.1 THE HOMOMORPHISM FUNCTORS

In this section, we take a break from the theory of categories in general and return to the consideration of module categories. Our aim is to investigate the functorial properties of the group of homomorphisms $\mathrm{Hom}(M_R, N_R)$ as M and N vary through the category $\mathcal{M}od_R$ of right R-modules, where R is a ring.

We find that certain properties, such as the injectivity or surjectivity of

1.4.11 **Pull-backs**

Suppose that, in a category $\mathcal{C}$, we are given a diagram of the form

$$\begin{array}{ccc} & & L'' \\ & & \downarrow{\scriptstyle \phi} \\ M & \xrightarrow{\beta} & M'' \end{array}$$

where β and θ are morphisms.

Arguing by duality, define the *pull-back* $M \times_{M''} L''$ of M and L'' over M'' in terms of a universal property.

Confirm that in the category of right R-modules, the pull-back is given explicitly (as in (2.4.8) of [BK: IRM]) by the formula

$$M \times_{M''} L'' = \{(m, \ell'') \in M \oplus L'' \mid \beta m = \theta \ell''\}.$$

Show also that if $\mathcal{C}$ has a terminal object, then a product is a special case of a pull-back.

1.4.12 Determine the push-outs and pull-backs in the category $\mathcal{S}_{ET}$ (see Exercise 1.4.9 above).

1.4.13 Let $\beta : M \to M''$ and $\delta : M'' \to N''$ be morphisms in a category with a zero object, and let $\alpha : N' \to M''$ be the kernel of δ. By comparing universal properties, show that, whenever the pull-back exists, the canonical morphism $\overline{\alpha} : M \times_{M''} N' \to M$ is the kernel of $\delta\beta : M \to N''$. (See Exercise 2.4.7 of [BK: IRM] for a discussion of this situation in $\mathcal{M}_{OD_R}$.)

1.4.14 For this, one needs some routine facts about products of groups. These can be found in [Scott 1964] Chapter 8, for example.

Show that the pull-back in $\mathcal{G}_P$ is the restricted direct product and that the push-out is the amalgamated free product.

Remark. There are corresponding but more complicated results for rings, where it is usually preferable to work in a category of *A-algebras*: for an arbitrary commutative ring A, an A-algebra is a ring R that has A contained in its centre. Some details can be found in [Rowen 1988], §§1.4, 1.9, and [Eisenbud 1995] A6.3.

1.4.15 Let $\mathcal{C}_{AT}$ be a category of small categories. Show that for categories $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{C}_{AT}$, their direct product $\mathcal{C} \times \mathcal{D}$ is a product in $\mathcal{C}_{AT}$ in the sense of (1.4.11).

Furthermore, there are homomorphisms $\overline{\mu} : M' \to M' \oplus_{L'} L$ and $\overline{\phi} : L \to M' \oplus_{L'} L$ which give a commutative diagram

$$\begin{array}{ccc} L' & \xrightarrow{\mu} & L \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} \\ M' & \xrightarrow{\overline{\mu}} & M' \oplus_{L'} L \end{array}$$

and if

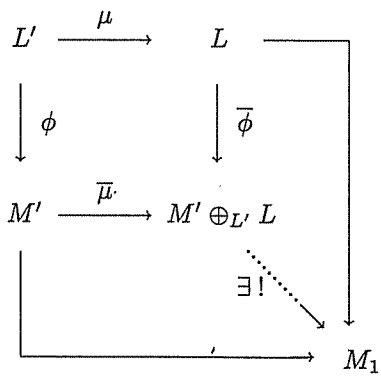
$$\begin{array}{ccc} L' & \xrightarrow{\mu} & L \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_1} \\ M' & \xrightarrow{\mu_1} & M_1 \end{array}$$

is a commutative diagram, then there is a unique homomorphism

$$\xi : M' \oplus_{L'} L \longrightarrow M_1$$

with $\xi\overline{\mu} = \mu_1$ and $\xi\overline{\phi} = \phi_1$.

Thus the universal property may be described by the diagram

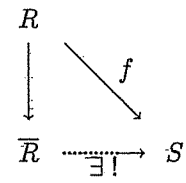


Show that the push-out arises through an initial object in an appropriate category, and hence extend the definition of a push-out to an arbitrary category $\mathcal{C}$.

Prove further that a push-out in $\mathcal{C}$ may be regarded as a coproduct in a suitable right-fibre category, and conversely, that if $\mathcal{C}$ has an initial object, then a coproduct is a special case of a push-out.

is the forgetful functor from $\mathcal{R}_{ING}$ to $\mathcal{R}_{NG}$ which simply ignores the multiplicative identity of a ring.

To describe the associated universal property, let $f : R \to S$ be a nonunital ring homomorphism. Show that if S is a ring, then f factors uniquely through the standard embedding of R in $\overline{R}$. Diagrammatically, we have:



1.4.9 **Sets**

Show that the empty set $\emptyset$ is an initial object in $\mathcal{S}_{ET}$ (note that $\mathrm{Map}(\emptyset, X)$ contains $|X|^0$ members since $0 = |\emptyset|$). Show also that $\mathcal{S}_{ET}$ has no terminal object.

Prove that the product (in $\mathcal{S}_{ET}$) of a set $\{X_\lambda \mid \lambda \in \Lambda\}$ of sets is the cartesian product $\prod_\Lambda X_\lambda$.

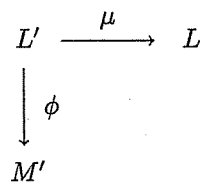
Define the *disjoint union* $\bigsqcup_\Lambda X_\lambda$ to be the union $\bigcup_\Lambda (X_\lambda, \lambda)$, where (X_λ, λ) is the set of pairs (x_λ, λ) with $x_\lambda \in X_\lambda$. (The purpose of this construction is to remove any 'accidental' overlap between the sets X_λ; note that $(X_\lambda, \lambda) \cap (X_\mu, \mu) = \emptyset$ for $\lambda \neq \mu$, even if X_λ and X_μ have common members.)

Show that the coproduct of the set of sets $\{X_\lambda\}$ is $\bigsqcup_\Lambda X_\lambda$.

1.4.10 **Push-outs**

This and the following exercises indicate how the definitions of push-outs and pull-backs can be generalized from modules to arbitrary categories. The constructions for modules are given in detail in [BK: IRM] (2.4.8)ff.

Given a diagram



of right R-modules, the push-out (of M' and L over L') is

$$M' \oplus_{L'} L = (M' \oplus L)/\{(\phi\ell', -m\ell') \mid \ell' \in L'\}.$$

Define a functor $\gamma_* : \mathcal{C}/C \to \mathcal{C}/D$ by $\gamma_*(B,\beta) = (B,\gamma\beta)$ and, for any morphism $\xi : (B',\beta') \to (B,\beta)$ in $\mathcal{C}/C$ (that is, $\beta\xi = \beta'$),

$$\gamma_*\xi = \xi : (B',\gamma\beta') \longrightarrow (B,\gamma\beta).$$

Show that the left-fibre category $\gamma_*/(0 \to D)$, where $(0 \to D)$ is the initial object of the category $\mathcal{C}/D$, has as objects commuting squares of the form

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \gamma \\ 0 & \longrightarrow & D \end{array}$$

Deduce that $\mathcal{K}\varepsilon\mathcal{R}\,\gamma$ is isomorphic to $\gamma_*/(0 \to D)$.

Provide the dual description of $\mathcal{C}o\mathcal{K}\,\gamma$.

1.4.5 What is the universal property enjoyed by the abelianization G_{ab} of a group G?

1.4.6 Let $\gamma : G \to F$ be a homomorphism of groups. Show that the usual definition of $\operatorname{Ker}\gamma$ does give a kernel in the category $\mathcal{G}_{\mathcal{P}}$.

Suppose that H is a normal subgroup of G, and let η be the inclusion homomorphism. Show that $\operatorname{Cok}\eta = G/H$.

Let H be an arbitrary subgroup of G. Then the *normal closure* H^G of H is the smallest normal subgroup of G which contains H. Verify that the inclusion homomorphism $\eta : H \to G$ has $\operatorname{Cok}\eta = G/H^G$.

Deduce that for an arbitrary homomorphism $\eta : H \to G$ in $\mathcal{G}_{\mathcal{P}}$, $\operatorname{Cok}\eta = G/(\operatorname{Im}\eta)^G$.

1.4.7 In the category $\mathcal{R}_{\mathcal{N}G}$ of nonunital rings (that is, 'rings' which need not have an identity element and homomorphisms which need not preserve it), show that the zero ring 0 is a zero object.

Let $\theta : R \to S$ be a homomorphism of nonunital rings. Show that the usual kernel $\operatorname{Ker}\theta$ is a kernel in $\mathcal{R}_{\mathcal{N}G}$, and that $\operatorname{Ker}\theta$ is an ideal of R (where we extend the definition of ideals to nonunital rings in the obvious way).

Show that if I is an ideal of S, then there is a residue nonunital ring R/I which is a cokernel for the inclusion homomorphism of I in R.

Given a sub-nonunital ring S' of S, let $I(S')$ be the smallest ideal of S which contains S'. Show that $\operatorname{Cok}\theta = S/I(\operatorname{Im}\theta)$.

1.4.8 Consider the functor from $\mathcal{R}_{\mathcal{N}G}$ to $\mathcal{R}_{\mathcal{I}\mathcal{N}G}$ which takes a nonunital ring R to its enveloping ring $\overline{R}$ (1.3.2)(iv). Show that its right adjoint

aim by introducing the notion of a 'functor', as they called a function F that not only operates on a set C (with extra structure) to give a new set $F(C)$, but also on a (structure-preserving) function $\gamma : C \to D$ to give a function $F(\gamma) : F(C) \to F(D)$. Then the idea of a natural transformation can be made precise, as in (1.3).

Subsequently, in [Eilenberg & Mac Lane 1945], the concept of a category was introduced to formalize the viewpoint that objects that share a common structure should not be considered in isolation, but together with the mappings that respect this structure. Once this level of abstraction had been attained, it became possible to introduce categories, such as opposite categories, morphism categories, etc., whose objects are no longer 'sets-with-structure'. In turn, the freedom to construct categories led to the characterization of universal constructions as initial or terminal objects in a suitably manufactured category. The notion of 'universality' first appears in [Samuel 1948], while the description of universal constructions in terms of adjoint functors (1.4.8) is due to [Kan 1958].

Exercises

1.4.1 Let $\mathcal{D}$ be the category with one object X and one morphism f other than the identity id_X, subject to $f \circ f = f$, say. Let $\mathcal{C}$ be the subcategory obtained by removing the morphism f, and let $G : \mathcal{D} \to \mathcal{C}$ be the constant functor (in fact, the only possible functor). Show that the right-fibre category $X\backslash G$ is isomorphic to $\mathcal{D}$. Evidently $\mathcal{D}$ lacks an initial object, so the same is true of $X\backslash G$.

The reader is invited to construct other right-fibre categories which lack an initial object. However, initial objects are surprisingly common in any choices of right-fibre category that one is likely to make.

1.4.2 Show that if (K, κ) is the kernel of a morphism γ in a category $\mathcal{C}$, then $(K^{\mathrm{op}}, \kappa^{\mathrm{op}})$ is the cokernel of γ^{op} in $\mathcal{C}^{\mathrm{op}}$ (and conversely).

Show also that if C, D are objects of $\mathcal{C}$ with product $C \prod D$ in $\mathcal{C}$, then $(C \prod D)^{\mathrm{op}} = C^{\mathrm{op}} \coprod D^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$ (and conversely).

1.4.3 Let R and S be rings. Show that the direct product $R \times S$ is a product in $\mathcal{R}_{ING}$ but not a coproduct.

1.4.4 Sometimes one has to be quite ingenious to present a universal construction as an initial or terminal object in a right- or left-fibre category. To discuss kernels and cokernels in this light, let $\gamma : C \to D$ be a morphism in a category $\mathcal{C}$ which has zero object 0. As noted, the category $\mathcal{K}_{ER}\gamma$ is a subcategory of the left-fibre category $Id_C/C = \mathcal{C}/C$.

(Here, $\operatorname{Ker}\eta_C$ is any fixed choice from among the mutually isomorphic objects in $\mathcal{D}$ which represent the kernel of η_C.)

To define $(\operatorname{Ker}\eta)(\alpha)$ for a morphism $\alpha : C' \to C$, we first recall that the diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

is commutative. Thus the definition of a kernel as a terminal object (1.4.15) shows that there is an induced morphism

$$(\operatorname{Ker}\eta)(\alpha) : (\operatorname{Ker}\eta)(C') \longrightarrow (\operatorname{Ker}\eta)(C)$$

which fits into the commutative diagram

$$\begin{array}{ccccc} (\operatorname{Ker}\eta)(C') & \xrightarrow{\kappa(\eta_{C'})} & FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle (\operatorname{Ker}\eta)(\alpha)}\vdots & & {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ (\operatorname{Ker}\eta)(C) & \xrightarrow{\kappa(\eta_C)} & FC & \xrightarrow{\eta_C} & GC \end{array}$$

It is now easy to confirm that, with the above choice of $(\operatorname{Ker}\eta)(\alpha)$, $\operatorname{Ker}\eta$ is indeed a functor.

The natural transformation

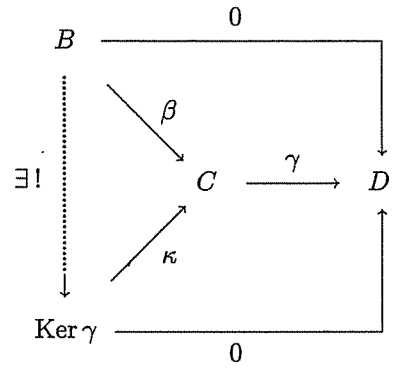
$$\kappa : \operatorname{Ker}\eta \longrightarrow F$$

is given by $\kappa_C = \kappa(\eta_C) : (\operatorname{Ker}\eta)(C) \to F(C)$. A routine verification shows that we have a kernel for η, as claimed.

1.4.19 Some history

A nice description of the origin of category theory is given in Chapter 1, §5E of [Dieudonné 1989], which we summarize. In the early 1940s, Eilenberg and Mac Lane were seeking to make precise the idea of a 'natural' transformation between various constructions in algebraic topology. Typically, such constructions start with a topological space and yield a group (fundamental group, homology group, etc.). The authors sought a framework for saying that a map of topological spaces gives rise to a homomorphism of groups in a well-regulated manner. In [Eilenberg & Mac Lane 1942], they achieved this

described by means of a diagram. Such a diagram typically highlights the morphism whose existence and uniqueness determines the universal object. For example, one may say that the kernel of a morphism $\gamma : C \to D$ enjoys the universal property



that makes the above diagram commutative. Here, the existence and uniqueness of the vertical arrow from B to $\operatorname{Ker}\gamma$ which makes the above diagram commutative expresses the fact that the pair $(\operatorname{Ker}\gamma, \kappa)$ is terminal in the category $\mathcal{K}_{\mathcal{E}R}(\gamma)$.

1.4.17 Pleasure versus guilt: the universal dilemma

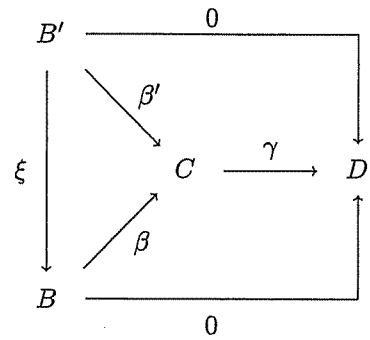
Being natural, functors have their emotional side too. Typically, as we have just seen, a universal property is pleasantly depicted by a commuting diagram (leading to the existence of a unique morphism). It is said that the universal object 'enjoys' the universal property. The existence of an adjoint functor, on the other hand, often involves a tortuous check; a functor is said to 'admit' an adjoint. See Exercise 1.4.8 for an example of this phenomenon.

1.4.18 Kernels of natural transformations

We can also define kernels, cokernels, and so on, in the functor category $[\mathcal{C}, \mathcal{D}]$ provided they exist in $\mathcal{D}$. Suppose, for example, that any morphism α in $\mathcal{D}$ has a kernel $(\operatorname{Ker}\alpha, \kappa(\alpha))$. We construct for each $\eta \in \operatorname{Nat}(F, G)$ a kernel $(\operatorname{Ker}\eta, \kappa)$ as follows.

The object $\operatorname{Ker}\eta$ in $[\mathcal{C}, \mathcal{D}]$ is to be a functor from $\mathcal{C}$ to $\mathcal{D}$. For each object C of $\mathcal{C}$ we take the evident definition

$$(\operatorname{Ker}\eta)(C) = \operatorname{Ker}\eta_C.$$



Evidently, $\mathcal{K}_{ER}(\gamma)$ is a subcategory of the left-fibre category Id_C/C (see Exercise 1.4.4 below).

A *kernel* of γ is a pair $(\mathrm{Ker}\,\gamma, \kappa)$ which is terminal in $\mathcal{K}_{ER}(\gamma)$, if such an object exists. There is some redundancy in this notation, since $\mathrm{Ker}\,\gamma$ is the domain of κ. Thus one may validly refer to the morphism κ as the kernel of γ. Alternatively, given the object $\mathrm{Ker}\,\gamma$, there is often a canonical choice of κ (most commonly, κ is taken to be an inclusion map, which is usually possible when C is a subcategory of $\mathcal{S}_{ET}$). In this case, it is the object $\mathrm{Ker}\,\gamma$ that is called the kernel of γ.

To define the cokernel of γ, we introduce a category $\mathcal{C}_{OK}(\gamma)$ with objects (E, ϵ) where $\epsilon\gamma = 0$, and morphisms $\nu : (E, \epsilon) \to (E'', \epsilon'')$ given by morphisms $\nu : E \to E''$ with $\nu\epsilon = \epsilon''$. A *cokernel* of γ is then an initial object $(\mathrm{Cok}\,\gamma, \chi)$ in this category, if there is one.

It is easy to check that in $\mathcal{M}_{OD_R}$, the usual kernel and cokernel remain such in this abstract sense. Because the maps κ and χ are then always taken to be inclusion and the standard surjection respectively, it is usual to omit them.

Further discussion of the circumstances in which categories may fail to contain kernels and cokernels is given in the next chapter. An immediate (but contrived) example is provided by $\mathcal{I}_{NF}\mathcal{A}_B$, the full subcategory of $\mathcal{A}_B$ whose objects are the infinite abelian groups. Since $\mathcal{A}_B$ is $\mathcal{M}_{OD_\mathbb{Z}}$ under another name, it has kernels and cokernels as expected. Take A to be the multiplicative group of nonzero real numbers, and let σ be the endomorphism $\sigma(x) = x^2$ of A. Then σ has neither kernel nor cokernel in $\mathcal{I}_{NF}\mathcal{A}_B$.

1.4.16 Universal properties

At this point we introduce a third term commonly found in the literature, along with 'universal object' and 'universal construction'. A *universal property* is the property that characterizes a universal object; it is frequently

and evident morphisms, and the *coproduct* $\coprod_\Lambda C_\lambda$ to be the universal object of $\mathcal{C}$ given by an initial object in $\mathcal{C}_{OPRD}(C(\Lambda))$.

When the index set Λ is infinite, the product and coproduct differ even in the category $\mathcal{M}_{OD_R}$. Using the definitions ([BK: IRM] (2.1.11)), the reader should have no difficulty verifying that the product of a set of modules $\{M_\lambda\}$ is the direct product $\prod_\Lambda M_\lambda$, while the coproduct is the direct sum $\bigoplus_\Lambda M_\lambda$. (However, the direct sum and product coincide for any finite index set.)

1.4.14 *Zero objects*

An object 0 of a category $\mathcal{C}$ which is both initial and terminal is said to be a *zero object* of $\mathcal{C}$. In this case, the unique morphisms $0 \to C$ and $C \to 0$ are both denoted 0, so that both $\mathrm{Mor}_\mathcal{C}(0, C) = \{0\}$ and $\mathrm{Mor}_\mathcal{C}(C, 0) = \{0\}$. Clearly, a zero object is unique up to unique isomorphism if it exists.

The zero module is the zero object in the category $\mathcal{M}_{OD_R}$ and the trivial group is the zero object in $\mathcal{G}_P$. On the other hand, $\mathcal{R}_{ING}$ has no zero object, since its initial and terminal objects $\mathbb{Z}$ and 0 are different.

When $\mathcal{C}$ has a zero object 0, we can define the zero morphism between any two objects C and D to be the composite

$$0 : C \longrightarrow 0 \longrightarrow D.$$

This morphism is easily seen to be independent of the choice of zero object, and has the property that

$$0\beta = 0 : B \longrightarrow D \quad \text{and} \quad \delta 0 = 0 : C \longrightarrow E$$

for any morphisms $\beta : B \to C$ and $\delta : D \to E$.

1.4.15 *Kernels and cokernels*

The definition of a kernel and cokernel of a morphism can be extended to any category with a zero object, although at this level of abstraction, a given morphism need have neither a kernel nor cokernel.

Let $\gamma : C \to D$ be a morphism in such a category $\mathcal{C}$, and define a new category $\mathcal{K}_{ER}(\gamma)$ as follows. The objects of $\mathcal{K}_{ER}(\gamma)$ are all the pairs (B, β) where B is an object of $\mathcal{C}$ and $\beta : B \to C$ is a morphism with $\gamma\beta = 0$. A morphism $\xi : (B', \beta') \to (B, \beta)$ is a morphism $\xi : B' \to B$ in $\mathcal{C}$ such that $\beta\xi = \beta'$. The morphisms can be described by commutative diagrams

1.4.12 The coproduct

The analogous definition in terms of inclusions leads to the coproduct. Again, let C' and C'' be two fixed objects of a category $\mathcal{C}$. The category $\mathcal{C}_{OPRD}(C', C'')$ has as objects triples (A, α', α'') where now $\alpha' : C' \to A$ and $\alpha'' : C'' \to A$, and a morphism ξ from (A, α', α'') to (B, β', β'') is given by a morphism $\xi : A \to B$ with $\xi\alpha' = \beta'$ and $\xi\alpha'' = \beta''$. A *coproduct* of C' and C'' is an initial object $(C' \coprod C'', \sigma', \sigma'')$ in this category, or alternatively, the corresponding universal object $C' \coprod C''$ in $\mathcal{C}$.

In $\mathcal{M}_{OD_R}$, the direct sum $M' \oplus M''$ is the coproduct of the modules M' and M'' as well as being their product. On the other hand, the coproduct in the category of groups is the *free product* $G' * G''$ of G' and G'', which assertion is easily seen from the explicit description of the free product. The elements of $G' * G''$ are formal products $h_1 \cdots h_k$ with each h_i in either G' or G'', of arbitrary length $k \geq 1$; the product is given by

$$(h_1 \cdots h_k)(h'_1 \cdots h'_{k'}) = h_1 \cdots h_k h'_1 \cdots h'_{k'}$$

and the inverse by

$$(h_1 \cdots h_k)^{-1} = h_k^{-1} \cdots h_1^{-1}.$$

A detailed account of the construction of the free product can be found in, for example, [Scott 1964], Chapter 8.

Underlying the coincidence of the product and coproduct in $\mathcal{M}_{OD_R}$ is the fact that $\mathrm{Hom}(M, N)$ is an abelian group for any two modules M, N. We look at this property in more detail in the next chapter.

1.4.13 Arbitrary products and coproducts

We next outline briefly the definition of products and coproducts for an arbitrary set of objects.

Let Λ be any ordered set and let $C(\Lambda) = \{C_\lambda \mid \lambda \in \lambda\}$ be a set of objects of some category $\mathcal{C}$ which are indexed by Λ. To define the product, we introduce the category $\mathcal{P}_{RD}(C(\Lambda))$ whose objects are sets $\{\alpha_\lambda : A \to C_\lambda\}$ of morphisms in $\mathcal{C}$, a morphism

$$\xi : \{\alpha_\lambda : A \longrightarrow C_\lambda\} \longrightarrow \{\beta_\lambda : B \longrightarrow C_\lambda\}$$

being given by a morphism $\xi : A \to B$ with $\alpha_\lambda = \beta_\lambda \xi$ for all λ.

Then the *product* $\prod_\Lambda C_\lambda$ is the universal object in $\mathcal{C}$ corresponding to a terminal object in $\mathcal{P}_{RD}(C(\Lambda))$.

Dually, we define the category $\mathcal{C}_{OPRD}(C(\Lambda))$ to have objects $\{\alpha_\lambda : C_\lambda \to A\}$

Conversely, the existence of such a set of homomorphisms for a module M shows that it is isomorphic to the direct sum of M' and M'' (Proposition 2.1.7 of [BK: IRM]).

A further characterization is in terms of the existence of split short exact sequences [BK: IRM] (2.4.5).

In a category where we can speak about sums of maps or about exact sequences, we can investigate whether there are pairs of morphisms analogous to the projections π', π'' and the inclusions σ', σ''. We pursue this line of attack in the next chapter.

In the most general setting, however, sums of maps and exact sequences are not available to us. It is then more productive to focus on the pair of projections, or alternatively the pair of inclusion maps, from the standpoint of universal objects. The former approach gives rise to the general definition of the product of two objects, while the latter leads to the coproduct. These two generalizations of 'direct sum' need not coincide.

1.4.11 The product

Let $\mathcal{C}$ be any category and let C' and C'' be any two objects of $\mathcal{C}$. Define a category $\mathcal{P}_{\mathcal{RD}}(C', C'')$ as follows. The objects of $\mathcal{P}_{\mathcal{RD}}(C', C'')$ are all the triples (A, α', α'') where $\alpha' : A \to C'$ and $\alpha'' : A \to C''$ are morphisms in $\mathcal{C}$, and a morphism $\xi : (A, \alpha', \alpha'') \to (B, \beta', \beta'')$ is given by a morphism $\xi : A \to B$ in $\mathcal{C}$ so that $\beta'\xi = \alpha'$ and $\beta''\xi = \alpha''$. Thus a morphism is given by a commutative diagram

$$\begin{array}{ccccc} C' & \xleftarrow{\alpha'} & A & \xrightarrow{\alpha''} & C'' \\ \| & & \downarrow \xi & & \| \\ C' & \xleftarrow{\beta'} & B & \xrightarrow{\beta''} & C'' \end{array}$$

A *product* of C' and C'' is a terminal object $(C' \prod C'', \pi', \pi'')$ in the category $\mathcal{P}_{\mathcal{RD}}(C', C'')$, if such exists. It is usual in everyday language to refer to the corresponding universal object $C' \prod C''$ of $\mathcal{C}$ as the product, suppressing the morphisms π' and π''.

In $\mathcal{M}_{OD_R}$, the direct sum $M' \oplus M''$ is the product in this sense, while in the category $\mathcal{G}_{\mathcal{P}}$ of groups, the product of G' and G'' is the usual cartesian product $G' \times G''$ with componentwise multiplication.

1.4.9 Terminal objects

Before we give more examples of universal objects, we mention another manner in which they can arise.

A *terminal object* in a category $\mathcal{C}$ is an object T such that $\mathrm{Mor}(C,T)$ contains exactly one morphism $\tau_C : C \to T$ for each C in $\mathcal{C}$. As with initial objects, a terminal object is unique to within unique isomorphism, if one exists at all.

It sometimes happens that the description of a universal object which satisfies some set of conditions is more naturally given in terms of a terminal object of a convenient ancillary category, rather than an initial object as in (1.4.2) above. We therefore extend the definition of a universal object to include those arising in this way. (Some authors prefer to use the term *couniversal object* for such objects.) Notice that the object T in $\mathcal{C}$ is terminal if and only if the corresponding object T^{op} is initial in the opposite category $\mathcal{C}^{\mathrm{op}}$ (1.1.6), so it is possible to avoid the use of terminal objects by changing the ancillary category. However, it would be rather artificial to do this.

1.4.10 The direct sum revisited

Our next examples of universal objects arise by generalizing the definitions of direct sums and products of modules to arbitrary categories. We draw on some well-known basic definitions and results that are discussed in detail in section 2.1 of [BK: IRM].

The most elementary definition of the direct sum $M = M' \oplus M''$ of right R-modules M' and M'' is in terms of ordered pairs:

$$M = \{(m', m'') \mid m' \in M', \ \ m'' \in M''\}$$

with the expected componentwise addition and scalar multiplication. However, this definition does not lend itself to generalization, so we must recast it in terms of homomorphisms. When $M = M' \oplus M''$, there are homomorphisms

$$\sigma' : M' \longrightarrow M \quad \text{and} \quad \sigma'' : M'' \longrightarrow M$$

given by inclusion on the first and second summands respectively, and homomorphisms

$$\pi' : M \longrightarrow M' \quad \text{and} \quad \pi'' : M \longrightarrow M''$$

given by the corresponding projections.

These homomorphisms satisfy the relations

$$\pi'\sigma' = id_{M'}, \ \ \pi''\sigma'' = id_{M''} \ \text{ and } \ \sigma'\pi' + \sigma''\pi'' = id_M.$$

Appealing to uniqueness again, we see that $F(id_C) = id_{F(C)}$ and that $F(\gamma'\gamma) = F(\gamma')F(\gamma)$ for a morphism $\gamma' : C' \to C''$. Thus F is a functor.

By the preceding lemma, the map $\varphi_{(C,D)}$ is a bijection for each pair C, D, and φ is natural in D. The remaining point to check is that it is also natural in C.

By (1.3.2)(i), this amounts to checking that, for any morphism $\gamma : C \to C'$ in $\mathcal{C}$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(F(C), D) & \xrightarrow{\phi_{(C,D)}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D)) \\ \uparrow{\scriptstyle F(\gamma)^*} & & \uparrow{\scriptstyle \gamma^*} \\ \mathrm{Mor}_{\mathcal{D}}(F(C'), D) & \xrightarrow{\phi_{(C',D)}} & \mathrm{Mor}_{\mathcal{C}}(C', G(D)) \end{array}$$

(Note that the vertical arrows point upwards.)

Take any $h \in \mathrm{Mor}_{\mathcal{D}}(F(C'), D)$. Then

$$\begin{aligned} \varphi_{(C,D)}(F(\gamma)^*(h)) &= \varphi_{(C,D)}(h \circ F(\gamma)) \\ &= G(h \circ F(\gamma)) \circ u_C \\ &= G(h) \circ GF(\gamma) \circ u_C, \end{aligned}$$

while

$$\begin{aligned} \gamma^*(\varphi_{(C',D)}(h)) &= \gamma^*(G(h) \circ u_{C'}) \\ &= G(h) \circ u_{C'} \circ \gamma. \end{aligned}$$

However, $u_{C'} \circ \gamma = GF(\gamma) \circ u_C$ by construction; so the result follows. □

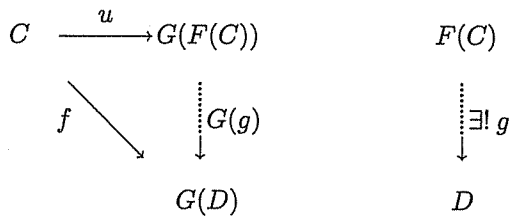
1.4.8 *Universal constructions*

In the light of this theorem we may now make the formal definition that a *universal construction* is a functor which has a right adjoint.

We have already noted in (1.3.8) above that the forgetful functor $\Upsilon : \mathcal{M}od_R \to \mathcal{S}et$ has as its left adjoint the functor $\mathrm{Fr}_R : \mathcal{S}et \to \mathcal{M}od_R$ which associates to each set X the free right R-module $\mathrm{Fr}_R(X)$ on X. Similarly, the free group and free abelian group constructions defined on $\mathcal{S}et$ are examples of universal constructions whose right adjoints are the forgetful functors from $\mathcal{G}p$ and $\mathcal{A}b$ to $\mathcal{S}et$.

An extensive catalogue of examples may be found in [Mac Lane 1971] and [Herrlich & Strecker 1979].

as described by the diagram



(Here the triangle commutes, and the exclamation mark expresses uniqueness rather than astonishment.) □

We now extend the preceding lemma by showing that the existence of an initial object in $C\backslash G$ for every object C can be neatly reformulated in the language of adjoint functors.

1.4.7 Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then the following assertions are equivalent.

(i) *For any object C of $\mathcal{C}$ there exists an object $F(C)$ of $\mathcal{D}$ and a morphism $u_C : C \to G(F(C))$ in $\mathcal{C}$ such that $(F(C), u_C)$ is an initial object in the right-fibre category $C\backslash G$.*

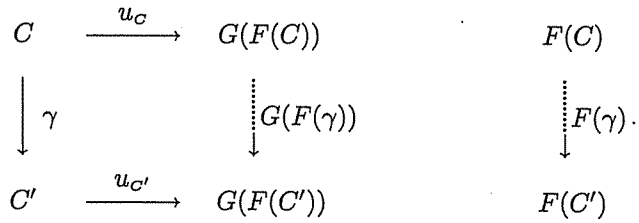
(ii) *G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$.*

Proof

(ii) $\Rightarrow$ (i) follows immediately from the above lemma.

(i) $\Rightarrow$ (ii): As the notation suggests, the effect of the functor F on an object C of $\mathcal{C}$ is to send it to $F(C)$. We must specify the effect of F on morphisms and verify functoriality.

Let $\gamma : C \to C'$ be a morphism in $\mathcal{C}$. The composite $u_{C'} \circ \gamma$ gives an object $(F(C'), u_{C'} \circ \gamma)$ of the right-fibre category $C\backslash G$, so we may define $F(\gamma) : F(C) \to F(C')$ as the unique morphism from $(F(C), u_C)$ to $(F(C'), u_{C'} \circ \gamma)$ in $C\backslash G$; this definition can be encapsulated by the diagram



in which the square is commutative.

1.4.6 Lemma

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, let C be an object of $\mathcal{C}$ and let $F(C)$ be an object of $\mathcal{D}$, with $u : C \to G(F(C))$ a morphism in $\mathcal{C}$. Then the following assertions are equivalent.

(i) *$(F(C), u)$ is an initial object in the right-fibre category $C\backslash G$.*

(ii) *There is a natural isomorphism of functors*

$$\varphi_{(C,-)} : \mathrm{Mor}_{\mathcal{D}}(F(C), -) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(-))$$

such that $u = \varphi_{(C,F(C))}(id_{F(C)})$.

Proof

We first use the preceding lemma to make explicit the effect of the natural transformation φ occurring in (ii) on any $g \in \mathrm{Mor}_{\mathcal{D}}(F(C), D)$. On the one hand, if (ii) holds, then by the lemma,

$$\varphi_{(C,D)}(g) = G(g) \circ \varphi_{(C,F(C))}(id_{F(C)}) = G(g) \circ u.$$

On the other hand, if we start with u, then for each D in $\mathcal{D}$ we may define a mapping

$$\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D))$$

by the equation

$$\varphi_{(C,D)}(g) = G(g) \circ u.$$

Since then

$$\varphi_{(C,F(C))}(id_{F(C)}) = G(id_{F(C)}) \circ u = u,$$

it follows from the lemma that this mapping is a natural transformation with respect to D. So in either event, it suffices to consider the natural transformation $\varphi_{(C,-)}$ defined by

$$\varphi_{(C,D)}(g) = G(g) \circ u.$$

Then it is easy to see that this natural transformation is an isomorphism if and only if (i) holds, since both properties correspond to the existence of a unique morphism g from $(F(C), u)$ to (D, f) for each object (D, f) of $C\backslash G$,

1.4.5 Lemma

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, let C be an object of $\mathcal{C}$ and let $F(C)$ be an object of $\mathcal{D}$. Suppose that for each object D in $\mathcal{D}$ there is a mapping of sets

$$\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{C}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D)).$$

Then the following are equivalent:

(a) *for all $g \in \mathrm{Mor}_{\mathcal{D}}(F(C), D)$,*

$$\varphi_{(C,D)}(g) = G(g) \circ \varphi_{(C,F(C))}(id_{F(C)});$$

(b) *φ is natural in D.*

Proof

(a) $\Rightarrow$ (b). To show that φ_D is natural in D, we have to verify the criterion of (1.3.2)(i), that for any morphism $\delta : D \to D'$ in $\mathcal{D}$, the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(F(C), D) & \xrightarrow{\phi_{(C,D)}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D)) \\ \downarrow{\scriptstyle \delta_*} & & \downarrow{\scriptstyle G(\delta)_*} \\ \mathrm{Mor}_{\mathcal{D}}(F(C), D') & \xrightarrow{\phi_{(C,D')}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D')) \end{array}$$

is commutative. Take any $g : F(C) \to D$. Then, combining (a) with the definitions in (1.2.7), we have

$$\begin{aligned} G(\delta)_*(\varphi_{(C,D)}(g)) &= G(\delta) \circ G(g) \circ \varphi_{(C,F(C))}(id_{F(C)}) \\ &= G(\delta g) \circ \varphi_{(C,F(C))}(id_{F(C)}) \\ &= \varphi_{(C,D')}(\delta g) \\ &= \varphi_{(C,D')}(\delta_*(g)). \end{aligned}$$

(b) $\Rightarrow$ (a). In the above diagram, we set $D = F(C)$ and $\delta = g : F(C) \to D$. Then commutativity yields

$$\begin{aligned} \varphi_{(C,D)}(g) &= \varphi_{(C,D)}(g \circ id_{F(C)}) \\ &= \varphi_{(C,D)}(g_*(id_{F(C)})) \\ &= G(g)_*(\varphi_{(C,F(C))}(id_{F(C)})). \end{aligned}$$

□

instead we subsitute $\mathcal{A}_B$, we obtain the free abelian group on X (see [Cohn 1982], §9). Indeed, for an arbitrary category $\mathcal{C}$, one has a specification for the 'free object in $\mathcal{C}$ on X', whenever this exists.

To describe a universal object in a category $\mathcal{C}$, how should we choose the ancillary category $\mathcal{C}'$ whose initial object defines the desired universal object in $\mathcal{C}$? The above example gives a clue. It suggests that $\mathcal{C}'$ is likely to be chosen as a suitable right-fibre category.

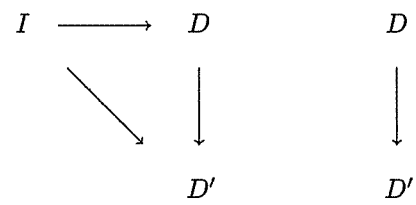
Our first result shows this idea to be formally correct in that an initial object must arise in a right-fibre category. Thus we define a *universal object* to be an initial object in a right-fibre category.

1.4.4 Lemma

If a category $\mathcal{D}$ has an initial object, then it is (isomorphic to) a right-fibre category.

Proof

Let $I \in \mathcal{D}$ be initial. Then $\mathcal{D}$ is isomorphic to the right-fibre category $I\backslash Id_{\mathcal{D}}$, via the correspondence which associates an object D of $\mathcal{D}$ with the object $I \to D$ of $I\backslash Id_{\mathcal{D}}$, and a morphism $D \to D'$ of $\mathcal{D}$ with the morphism



of $I\backslash Id_{\mathcal{D}}$. □

The converse to this lemma is false, as may be seen from Exercise 1.4.1.

In general, the right-fibre categories that we seek will arise from adjoint pairs of functors, as we now proceed to demonstrate.

We start by reminding ourselves of the definition of an adjoint pair (1.3.7). A pair $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ of (covariant) functors between the categories $\mathcal{C}$ and $\mathcal{D}$ is an adjoint pair if there is a natural isomorphism φ between the bifunctors $\mathrm{Mor}_{\mathcal{D}}(F(-), -)$ and $\mathrm{Mor}_{\mathcal{C}}(-, G(-))$.

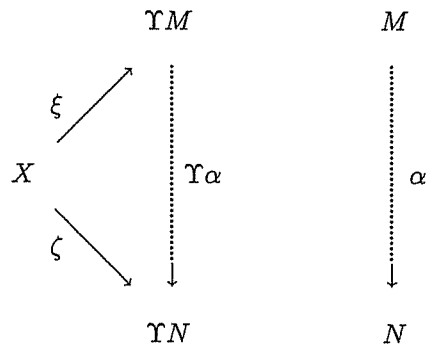
To define such an isomorphism φ, we must exhibit a family of bijections $\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \to \mathrm{Mor}_{\mathcal{C}}(C, G(D))$ which are natural in C and D. We argue in two stages. In the first, we fix C and allow D to vary. After a preliminary lemma to decide when $\varphi_{(C,D)}$ is natural in D, we show that $\varphi_{(C,D)}$ being a bijection for all D corresponds to the existence of an initial object in a suitable right-fibre category.

Given any set $\{n_x \mid x \in X\}$ of elements n_x of any right R-module N, there exists a unique R-module homomorphism $\alpha : M \to N$ with $\alpha m_x = n_x$ for all x in X.

We show that the free right R-module $\mathrm{Fr}_R(X)$ on X (1.2.3) is the universal object in $\mathcal{M}_{OD_R}$ which satisfies this condition.

To construct the appropriate ancillary category, observe that specifying a set $\{n_x\}$ in N amounts to the same thing as defining a mapping of sets $\zeta : X \to N$. We therefore consider X as an object of the category $\mathcal{S}_{ET}$ of sets and their mappings, and use the forgetful functor $\Upsilon : \mathcal{M}_{OD_R} \to \mathcal{S}_{ET}$. If M satisfies our condition, then there must be a map $\xi : X \to M$ and a homomorphism $\alpha : M \to N$ with $\alpha\xi = \zeta$.

We therefore arrive at the category $X\backslash\Upsilon$ of (right) R-modules under X via Υ (1.2.9). The objects of this category are pairs (M, ξ) in which M is an R-module and $\xi : X \to \Upsilon M$ is a mapping (of sets), and a morphism $\alpha : (M, \xi) \to (N, \zeta)$ is given by a commutative triangle



where α is an R-module homomorphism.

The free module $\mathrm{Fr}_R(X)$, together with the map $\xi_0 : X \to \mathrm{Fr}_R(X)$ that simply identifies an element x of X as a free generator of $\mathrm{Fr}_R(X)$, as in (1.2.3), gives an initial element of $X\backslash\Upsilon$. Thus $\mathrm{Fr}_R(X)$, or more properly, the pair $(\mathrm{Fr}_R(X), \xi_0)$, is universal among R-modules which are Υ-under X.

Moreover, any free module F with a set of free generators $\{b_x\}$ labelled by X also gives an initial object (F, ξ) of $X\backslash\Upsilon$, with ξ sending x to b_x. (See [BK: IRM] (2.1.19) ff. for further details.)

Although the map ξ_0 is an essential part of the information, it is often omitted in practice, so that the free module $\mathrm{Fr}_R(X)$ is itself called the universal object. The utility of the notion of a universal object should be apparent. Replacing $\mathcal{M}_{OD_R}$ by $\mathcal{G}_P$, we are led to recover the free group on X, and if

To see this, we note that the morphism ι_I must be the identity morphism on I, and that if $\alpha : C \to C''$ is any morphism in $\mathcal{C}$, then $\alpha\iota_C = \iota_{C''}$. Thus, if I and I' are both initial objects of $\mathcal{C}$, then there are unique morphisms

$$\iota : I \longrightarrow I' \quad \text{and} \quad \iota' : I' \longrightarrow I,$$

which must satisfy the relations

$$\iota'\iota = id_I \quad \text{and} \quad \iota\iota' = id_{I'},$$

that is, ι and ι' are mutually inverse (and unique) isomorphisms.

This type of uniqueness is sometimes summarised in the phrase 'an initial object is *unique up to unique isomorphism*'. Because of it, one often speaks of 'the' initial object rather than 'an' initial object.

In many explicit situations, an initial object will appear to be absolutely unique since there is an obvious choice for one. An example is given by the cokernel $\mathrm{Cok}\,\alpha$ of a homomorphism $\alpha : M \to N$ between R-modules. This is almost invariably defined to be the set of cosets of N modulo $\mathrm{Im}(\alpha)$, given the appropriate module structure. However, it can be thought of as the initial object in the category of homomorphic images N'' of N such that the surjection $\pi : N \twoheadrightarrow N''$ composes with α to yield the zero map $0 = \pi\alpha$.

1.4.2 *Universal objects*

The philosophy underlying the term 'universal object' is as follows.

We wish to specify a type of object in a category $\mathcal{C}$ by its relationships to the totalility of objects of $\mathcal{C}$. Such relationships manifest themselves as a set of conditions which may or may not be satisfied by an object of $\mathcal{C}$. In turn, it is often possible to characterize those objects C of $\mathcal{C}$ which do satisfy the given conditions in terms of associated objects C' of a related or ancillary category $\mathcal{C}'$. If it happens that C' is an initial object in $\mathcal{C}'$, then the corresponding object C of $\mathcal{C}$ is called a universal object of $\mathcal{C}$, for the given conditions.

Since an initial object is unique up to unique isomorphism, the same will be true of a universal object. Thus one often refers to 'the' universal object with some specified properties. An example will illustrate these ideas better than formalism alone. We consider a characterization of free modules.

1.4.3 *Free modules revisited*

Let X be a set and let R be a ring, and consider the following condition on a pair consisting of a right R-module M together with a specified subset $\{m_x\}_{x\in X}$ of M.

a set X and the free R-module on X have different appearances, both share the same universal property, but in the categories $\mathcal{G}_{\mathcal{RO}}$ of groups and $\mathcal{M}_{\mathcal{OD}_R}$ of R-modules respectively. We can therefore define a free object on X in any suitable category $\mathcal{C}$ as an object, in $\mathcal{C}$, with this property, and we can attempt to determine its structure. In such generality, free objects may not exist.

The description of an object in terms of a universal property also provides a means of transferring definitions from one category, in which it is easy to define a particular type of object, to another, where the definition is not immediate. For example, in the module category $\mathcal{M}_{\mathcal{OD}_R}$, the kernel of a homomorphism is defined explicity in terms of the elements of a module. However, a kernel can also be described axiomatically as a universal object in a category associated with $\mathcal{M}_{\mathcal{OD}_R}$, which enables us to extend the definition of a kernel to more general types of category, and to verify whether or not kernels exist in the new setting. This analysis will be very useful in the next chapter.

In fact, many of the definitions in the remainder of this text, and also in K-theory, are best treated as the specification for some universal object.

In this section, we explain the meanings of the terms 'universal object' (in (1.4.2) below), 'universal construction' (1.4.8) and 'universal property' (1.4.16), and present some important examples.

1.4.1 Initial objects

First, we present some basic definitions that underly the theory. An *initial object* of an (abstract) category $\mathcal{C}$ is an object I in $\mathcal{C}$ with the property that there is exactly one morphism

$$\iota_C : I \longrightarrow C$$

for each object C in $\mathcal{C}$; thus

$$\mathrm{Mor}(I, C) = \{\iota_C\} \text{ for each } C.$$

In the category $\mathcal{G}_{\mathcal{RO}}$ of groups, the trivial group is initial, in $\mathcal{M}_{\mathcal{OD}_R}$ the zero module is initial, and in the category $\mathcal{R}_{\mathcal{ING}}$ of rings (with identity), $\mathbb{Z}$ is initial. On the other hand, the infinite cyclic group Z is not initial in $\mathcal{G}_{\mathcal{RO}}$, since there is always more than one homomorphism from Z to a nontrivial group.

An example of a category without an initial object is provided by $\mathcal{F}_{\mathcal{IELD}}$, the full subcategory of $\mathcal{R}_{\mathcal{ING}}$ whose objects are the fields. This follows from the observation that no field can be mapped homomorphically into both $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

An initial object need not be absolutely unique. For example, there are many manifestations of the zero module since the zero element of any module serves. However, an initial object is essentially unique.

Check that the dual basis is indeed a basis, and so there is an isomorphism $\eta_V : (V, \{f_j\}) \xrightarrow{\cong} (V^*, \{f_j^*\})$.

Show that this isomorphism cannot be natural as defined here, because the assignment $\Delta : \mathcal{B}_{ASES_{\mathcal{K}}} \to \mathcal{B}_{ASES_{\mathcal{K}}}$, $\Delta((V, \{f_j\})) = (V^*, \{f_j^*\})$, corresponds to a *contravariant* functor on $\mathcal{B}_{ASES_{\mathcal{K}}}$; it is a morphism functor $\mathrm{Mor}(-, \mathcal{K})$ as in (1.2.7).

One might try to rescue some naturality by hoping that squares of the form

$$\begin{array}{ccc} (V, \{f_j\}) & \xrightarrow{\eta_V} & (V^*, \{f_j^*\}) \\ \theta \downarrow & & \uparrow \theta^* \\ (W, \{g_j\}) & \xrightarrow{\eta_W} & (W^*, \{g_j^*\}) \end{array}$$

commute.

Take $(V, \{f_j\})$ to be $(\mathcal{K}^1, 1)$ and write $\theta(1) = t_1 f_1 + \cdots + t_n f_n$. Verify that $\theta^* \circ \eta_V \circ \theta(1) = (t_1^2 + \cdots + t_n^2)1^*$, and hence that such squares are not commutative in general.

If we go one step further and define $\Delta^2(V) = (V^*)^*$, the *double dual*, we obtain a covariant functor on the category of all finite-dimensional vector spaces over $\mathcal{K}$, (with morphisms the $\mathcal{K}$-linear transformations of such spaces).

Define $\epsilon_V : V \to V^{**}$ by

$$(\epsilon_V(v))(\alpha) = \alpha(v) \text{ for } v \in V,\ \alpha \in V^*.$$

Show that each ϵ_V is an injective linear transformation, and hence an isomorphism by dimension counting, and that the family $\{\epsilon_V\}$ is a natural isomorphism between the identity functor and Δ^2.

1.4 UNIVERSAL OBJECTS

An underlying principle in category theory is that an object should be considered in relation to the other objects in the category, rather than being viewed in isolation. This perspective leads to the idea of a universal object, which, informally, is an object that is defined by the requirement that it should have a specified relationship to the other objects in the category. When such a requirement has been determined, it can often be seen that apparently distinct kinds of object are defined by the same requirement, but in different categories.

An illustration will be provided by free objects. Although the free group on

This apparatus is informally represented as

$$S_0 \;\begin{array}{c}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{array}\; S_1 \;\begin{array}{c}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{array}\; S_2 \cdots$$

where the face operators map from right to left, and the degeneracy operators from left to right. When $\mathcal{C} = \mathcal{S}_{\mathcal{ET}}$, we obtain a simplicial set, when $\mathcal{C} = \mathcal{G}_{\mathcal{P}}$ a simplicial group, etc.

(c) The terminology comes from the following situation. As in [Spanier 1966](3.1), a *simplicial complex* is a set V (the set of *vertices*) together with a distinguished set S (the set of *simplices*) of finite subsets of V, such that

(i) every singleton in V is in S, and
(ii) every nonempty subset of a simplex is a simplex.

Show how an ordering of the vertices gives rise to a simplicial set, and conversely, how a simplicial set describes an ordered simplicial complex, where the face and degeneracy operators are given by

$$d_i(v_0, \ldots, v_n) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n),$$
$$s_i(v_0, \ldots, v_n) = (v_0, \ldots, v_i, v_i, v_{i+1}, \ldots, v_n).$$

(See [May 1967] Ch. 1 for further discussion and topological applications. See also [Hovey 1999] and [Jardine 1996] for highly readable accounts of the relation to Quillen's closed model categories and K-theory.)

1.3.18 **A non-natural isomorphism: duality**

After so many examples of natural behaviour, the reader may begin to wonder if anything is unnatural. Here is an example which anticipates our general discussion of duality in (4.1.1).

Given a finite-dimensional vector space V over a field $\mathcal{K}$, the *dual* of V is $V^* = \mathrm{Hom}_{\mathcal{K}}(V, \mathcal{K})$. Verify that this is also a $\mathcal{K}$-space, with the addition and scalar multiplication :

$$(\alpha + \beta)(v) = \alpha(v) + \beta(v), \quad (k \cdot \alpha)(v) = k(\alpha(v)),$$

where $\alpha, \beta \in V^*$, $v \in V$, $k \in \mathcal{K}$.

Let $\{f_1, \ldots, f_n\}$ be a basis of V. The *dual basis* $\{f_1^*, \ldots, f_n^*\}$ of V^* is defined by

$$f_j^*(f_i) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Show that ${}^*\mathcal{F}_{REER}$ and $\mathcal{O}_{RD}$ are equivalent categories, with $S \circ \mathrm{Fr} = \mathrm{Id}$ but $\mathrm{Fr} \circ S \neq \mathrm{Id}$.

Define a dense and faithful functor $I : \mathcal{O}_{RD} \to \mathcal{S}_{TAN}\mathcal{B}_{ASES}^{\infty}{}_R$.

1.3.16 Prove the following generalization of (1.3.18).

For any ring R, there is a natural isomorphism between the categories $[[n-1], \mathcal{M}_{OD_R}]$ and $\mathcal{M}_{OD_{T_n}}$, where T_n is the ring of upper triangular $n \times n$ matrices over R.

(More results in a similar vein are given by [Mitchell 1965], IX §7ff.)

1.3.17 Let Δ (denoted Δ^* in [May 1967]) be the category whose objects are the categories $[n]$, $n = 0, 1, 2, \ldots$ and whose morphisms are the functors among these categories. For want of a name, we may think of Δ as the *proto-simplicial category* (justification in (c) below).

(a) For $0 \leq i \leq n$, define $\delta_i : [n-1] \to [n]$ and $\sigma_i : [n+1] \to [n]$ by

	$j < i$	$j = i$	$j > i$	**Effect**
$\delta_i(j)$:	j	$j+1$	$j+1$	omit i
$\sigma_i(j)$:	j	j	$j-1$	repeat i

Show that any non-identity morphism $\mu : [n] \to [m]$ in Δ can be factorized uniquely in the following way. Let the objects of $[m]$ that are not in the image of μ be $i_1 > i_2 > \cdots > i_s$, and let $\mu(j) = \mu(j+1)$ precisely when j is one of $j_1 < j_2 < \cdots < j_t$. (Thus $n + s = m + t$.) Then

$$\mu = \delta_{i_1} \cdots \delta_{i_s} \sigma_{j_1} \cdots \sigma_{j_t}.$$

(b) Algebraic topologists define a *simplicial object* in a category $\mathcal{C}$ to be a contravariant functor from Δ to $\mathcal{C}$. Show that this amounts to specifying objects $S_0, S_1, S_2, \ldots$ in $\mathcal{C}$, and two kinds of morphisms, *face operators* $d_i : S_n \to S_{n-1}$ and *degeneracy operators* $s_i : S_n \to S_{n+1}$ $(0 \leq i \leq n)$, and commuting diagrams corresponding to relations

(i)
$$d_i d_j = d_{j-1} d_i \quad \text{for } i < j$$

(ii)
$$s_i s_j = s_{j+1} s_i \quad \text{for } i \leq j$$

(iii)
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ id & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$$

morphism $\alpha : \bullet \to \bullet$, so that $\mathrm{Mor}(\bullet, \bullet) = \{id, \alpha, \alpha^2, \ldots\}$. Show that $[\Omega, \mathcal{D}] = \mathcal{E}_{ND}(\mathcal{D})$ for any category $\mathcal{D}$.

Find a functor $[1] \to \Omega$, where $[1]$ is the category $0 \to 1$, that gives rise to the inclusion of $\mathcal{E}_{ND}(\mathcal{D})$ in $\mathcal{M}_{OR}(\mathcal{D})$.

Devise categories $\mathcal{C}$ such that $[\mathcal{C}, \mathcal{D}]$ is (i) $\mathcal{I}_{SO}\,\mathcal{D}$, (ii) $\mathcal{A}_{UT}\,\mathcal{D}$.

1.3.11 Let $[1]^2 = [1] \times [1]$, where $[1]$ is the category $0 \to 1$, and let $\mathcal{D}$ be an arbitrary category. Show that $[1]^2$ is the 'generic' commutative square

$$\begin{array}{ccc} (0,1) & \to & (1,1) \\ \uparrow & \nearrow & \uparrow \\ (0,0) & \to & (1,0) \end{array}$$

and that $[[1]^2, \mathcal{D}]$ is the category of all commutative squares in $\mathcal{D}$.

Show also that $[[1]^2, \mathcal{D}] = \mathcal{M}_{OR}{}^2(\mathcal{D})$, that is, $\mathcal{M}_{OR}(\mathcal{M}_{OR}(\mathcal{D}))$.

Interpret $[1]^n$ and $[[1]^n, \mathcal{D}]$.

Is the category $[2]^2 = [2] \times [2]$ equivalent to $[4]$?

1.3.12 Show that $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ (1.3.5) is isomorphic to the category whose objects are the natural numbers and whose morphisms from n to m are the $m \times n$ matrices over R.

1.3.13 Let $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ and $\mathcal{B}_{ASES_R}$ be as in (1.3.5) and (1.3.12). Show that $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ is a skeletal subcategory of $\mathcal{B}_{ASES_R}$ if and only if the ring R has invariant basis number.

1.3.14 Given an arbitrary ordered set Λ, the *standard free* right R-module R^Λ is the module of 'column vectors' (r_λ) indexed by the members λ of Λ. The *standard basis* of R^Λ is $\{e_\lambda\}$, where the 'unit vector' e_λ has λ-entry 1 and all other entries 0.

Let $\mathcal{S}_{TAN}\mathcal{B}_{ASES}{}^\infty_R$ and $\mathcal{B}_{ASES}{}^\infty_R$ be as $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ and $\mathcal{B}_{ASES_R}$, but with infinite index sets also allowed. Extend the discussion which we gave in (1.3.5) to $\mathcal{S}_{TAN}\mathcal{B}_{ASES}{}^\infty_R$ and $\mathcal{B}_{ASES}{}^\infty_R$. (The definition and properties of matrices of homomorphisms with respect to infinite bases are fairly straightforward – see (2.2.13) of [BK: IRM].)

1.3.15 Let ${}^*\mathcal{F}_{REE_R}$ denote the category of based free (right) R-modules: an object is a pair (F, B) where F is free and B is a basis, and a morphism from (F', B') to (F, B) is an R-homomorphism from F' to F which restricts to an order-preserving map from B' to B. Thus ${}^*\mathcal{F}_{REE_R}$ is the category of functors from the category $\mathcal{O}_{RD}$ of ordered sets to $\mathcal{F}_{REE_R}$. Let $\mathrm{Fr} : \mathcal{O}_{RD} \to {}^*\mathcal{F}_{REE_R}$ associate with Λ the standard free module R^Λ with basis $\{e_\lambda\}$ and let $S : {}^*\mathcal{F}_{REE_R} \to \mathcal{O}_{RD}$ send (F, B) to the ordered set that indexes B.

and let $\Upsilon : \mathcal{A}_B \to \mathcal{G}_P$ be the inclusion functor. Show that Ab, Υ is an adjoint pair.

1.3.6 (a) Show that if F, G and F', G' are both adjoint pairs, then $F'F$, GG' is also an adjoint pair, whenever the composite functors are defined.

(b) Show that if F, G is an adjoint pair, then each of F and G determines the other to within natural isomorphism.

1.3.7 Show that if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are mutually inverse equivalences, then F, G is an adjoint pair.

1.3.8 Let $A : \mathcal{C}' \to \mathcal{C}$ and $B : \mathcal{D} \to \mathcal{D}''$ be (covariant) functors between small categories. Define functors $[A, -] : [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}', \mathcal{D}]$ and $[-, B] : [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}'']$, and verify that $[-, -]$ is a bifunctor from $\mathcal{C}_{AT} \times \mathcal{C}_{AT}$ to $\mathcal{C}_{AT}$.

1.3.9 (a) In the notation of the preceding exercise (1.3.8) show that if $G : \mathcal{D}'' \to \mathcal{D}$ is right adjoint to $F : \mathcal{D} \to \mathcal{D}''$, then $[\mathcal{C}, F]$, $[\mathcal{C}, G]$ is an adjoint pair.

(b) Each object D in the category $\mathcal{D}$ defines a *constant functor* $\mathrm{Cnst}_D : \mathcal{C} \to \mathcal{D}$ sending each object of $\mathcal{C}$ to D and each morphism of $\mathcal{C}$ to id_D. Show that this assignment defines another constant functor $\mathrm{Cnst}_{\mathcal{D}} : \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$ with the property that, for any functor $G : \mathcal{D}'' \to \mathcal{D}$, the square

$$\begin{array}{ccc} \mathcal{D}'' & \xrightarrow{\mathrm{Cnst}_{\mathcal{D}''}} & [\mathcal{C}, \mathcal{D}''] \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle [\mathcal{C}, G]} \\ \mathcal{D} & \xrightarrow{\mathrm{Cnst}_{\mathcal{D}}} & [\mathcal{C}, \mathcal{D}] \end{array}$$

commutes.

(c) Suppose that $L_{\mathcal{D}} : [\mathcal{C}, \mathcal{D}] \to \mathcal{C}$ is left adjoint to $\mathrm{Cnst}_{\mathcal{D}}$ with $L_{\mathcal{D}''}$ similarly left adjoint to $\mathrm{Cnst}_{\mathcal{D}''}$. By combining Exercise 1.3.6 with the above, show that if $F : \mathcal{D} \to \mathcal{D}''$ has a right adjoint, then $F \circ L_{\mathcal{D}}$ is naturally isomorphic to $L_{\mathcal{D}''} \circ [\mathcal{C}, F]$. In other words, the square obtained from (b) above by replacing the constant functors by their left adjoints also commutes, up to natural isomorphism.

Remark. When $L_{\mathcal{D}} : [\mathcal{C}, \mathcal{D}] \to \mathcal{C}$ is left adjoint to $\mathrm{Cnst}_{\mathcal{D}}$, then, for any functor $G : \mathcal{C} \to \mathcal{D}$, the object $L_{\mathcal{D}}(G)$ is called the *colimit* of G – see (5.1.16) and Exercise 5.1.7.

1.3.10 Let Ω be the *loop* category: Ω has one object $\bullet$, and one generating

by

$$\eta_X(\ell)(\alpha) = F(\alpha)(\ell).$$

Verify that

$$\eta(\ell) = \{\eta_X(\ell) \mid X \text{ an object of } \mathcal{C}\}$$

defines a natural transformation from $\mathrm{Mor}_{\mathcal{C}}(L, -)$ to $F(-)$.

Show that the map $\ell \mapsto \eta(\ell)$ defines a bijection from $F(L)$ to $\mathrm{Nat}(\mathrm{Mor}(L, -), F)$, with inverse given by the map which associates to a natural transformation $\zeta : \mathrm{Mor}_{\mathcal{C}}(L, -) \to F(-)$ the element $\zeta_L(id_L)$ of $F(L)$.

By taking $F(-) = \mathrm{Mor}(M, -)$, show that every natural transformation from $\mathrm{Mor}(L, -)$ to $\mathrm{Mor}(M, -)$ has the form μ^* for some $\mu : M \to L$.

Repeat the exercise for contravariant functors from $\mathcal{C}$ to $\mathcal{S}_{ET}$.

Remark. A covariant functor $F : \mathcal{C} \to \mathcal{S}_{ET}$ is said to be *representable* if it is naturally isomorphic to $\mathrm{Mor}(L, -)$ for some object L in $\mathcal{C}$. The analogous definition is made for contravariant functors.

1.3.3 **Isomorphisms of categories**

Isomorphisms of categories come in four varieties, since the mutually inverse functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ may be either both covariant or contravariant and either both cochiral or contrachiral.

Show that the identity functor, the opposite functor Op, the mirror functor Mir and the composite Mir $\circ$ Op provide examples of all four varieties.

Show also that an arbitrary isomorphism can be converted into a covariant, cochiral isomorphism by composition with one of the isomorphisms listed above.

Now let G and H be groups viewed as one-element categories $\mathcal{B}G$ and $\mathcal{B}H$ (Exercise 1.1.3), and let $F : \mathcal{B}G \to \mathcal{B}H$ be an isomorphism of categories. Show that if F is covariant then F corresponds to an isomorphism of groups in the usual sense, while if F is contravariant, F corresponds to an anti-isomorphism of groups.

1.3.4 We again regard a group G as a category, $\mathcal{B}G$ or $\mathcal{E}G$, in two ways, as in Exercise 1.1.3. Given groups G and H, describe any natural transformations and isomorphisms that there are between a pair of functors between two of the categories $\mathcal{B}G$, $\mathcal{B}H$, $\mathcal{E}G$ and $\mathcal{E}H$.

1.3.5 Let $\mathrm{Ab} : \mathcal{G}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{B}}$ be the abelianization functor between the category $\mathcal{G}_{\mathcal{P}}$ of groups and the category $\mathcal{A}_{\mathcal{B}}$ of abelian groups (Exercise 1.2.3),

1.3.18 Theorem

Let $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ *be the ring of upper triangular* 2×2 *matrices over a ring* R. *Then the categories* $\mathcal{M}_{OD_T}$ *and* $\mathcal{M}_{OR}(\mathcal{M}_{OD_R})$ *are equivalent.*

Proof

Let $\dot{L}$ be a (right) T-module. Since the direct product $R \times R$ is the subring $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ of T, we can write $L = M \oplus N$, where both M and N are R-modules, as in (1.3.16). Using the action of T on $M \oplus N$, define $\alpha : M \to N$ by

$$(m, 0)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0, \alpha m).$$

Then α is easily seen to be a homomorphism of R-modules, and the action of T is given by

$$(*) \qquad (m, n)\begin{pmatrix} r & u \\ 0 & s \end{pmatrix} = (mr, \alpha m u + ns).$$

Let L' be another T-module, let $L' = M' \oplus N'$ be its decomposition and α' the corresponding homomorphism. A T-module homomorphism $\lambda : L' \to L$ induces R-module homomorphisms $\mu : M' \to M$ and $\nu : N' \to N$.

Calculating $\lambda(m', 0)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $m' \in M'$, in two ways, we find that $\nu\alpha' = \alpha\mu$, which shows that we can define a functor $F : \mathcal{M}_{OD_T} \to \mathcal{M}_{OR}(\mathcal{M}_{OD_R})$ by sending L to α.

Conversely, given an R-module homomorphism α from M to N, it is clear how to use the formula $(*)$ to turn $M \oplus N$ into a T-module L, thus defining a functor G in the reverse direction. Evidently, F and G are mutually inverse equivalences. □

Exercises

1.3.1 Let X and Y be sets regarded as discrete categories, and maps between them regarded as functors, as in Exercise 1.2.1.

Show that there is a natural transformation between two maps $f, g : X \to Y$ if and only if $f = g$.

1.3.2 **Yoneda's Lemma**

Let $F : \mathcal{C} \to \mathcal{S}_{ET}$ be a covariant functor, and let L be an object of $\mathcal{C}$ and ℓ an element of $F(L)$. For each object X of $\mathcal{C}$, define

$$\eta_X(\ell) : \mathrm{Mor}_{\mathcal{C}}(L, X) \longrightarrow F(X)$$

as saying that, for a ring R, the category $\mathcal{M}_R$ of finitely generated right R-modules has a small skeleton.

Next we present some equivalences which sharpen the relationships between product and morphism categories on the one hand, and module categories on the other, which we first touched on in (1.1.12) and (1.1.14).

1.3.16 Theorem

Let R and S be rings. Then the direct product category $\mathcal{M}\text{od}_R \times \mathcal{M}\text{od}_S$ is equivalent to $\mathcal{M}\text{od}_{R\times S}$.

Proof

As we noted in (1.1.12), an object (M, N) in the product category can be regarded as an $R \times S$-module by the rule $(m, n) \cdot (r, s) = (mr, ns)$, and a morphism in $\mathcal{M}\text{od}_R \times \mathcal{M}\text{od}_S$ is an $R \times S$-module homomorphism.

On the other hand, any $R \times S$-module is isomorphic to one of the form $M \oplus N$, where M is an R-module and N is an S-module, and the direct sum is the direct sum of abelian groups; furthermore, this decomposition respects homomorphisms (see [BK: IRM] (2.6.6), (2.6.7)).

It is now evident that the equivalence is given by

$$(M, N) \mapsto M \oplus N \text{ and } (\alpha, \beta) \mapsto \alpha \oplus \beta. \qquad \square$$

Recall that $\mathcal{M}_R$ and $\mathcal{P}_R$ are respectively the categories of finitely generated right R-modules and finitely generated projective right R-modules. The components of a finitely generated or projective module will again be finitely generated or projective, as the case may be, since the equivalence given in the previous theorem evidently preserves surjections and direct sums (see Corollary 2.6.9 in [BK: IRM] for more detail). Thus the following result is immediate.

1.3.17 Corollary

There are equivalences of categories

$$\mathcal{M}_R \times \mathcal{M}_S \simeq \mathcal{M}_{R\times S}$$

and

$$\mathcal{P}_R \times \mathcal{P}_S \simeq \mathcal{P}_{R\times S}. \qquad \square$$

Next we describe the morphisms of $\mathcal{M}\text{od}_R$ in terms of modules over a triangular matrix ring.

1.3.12 Example: standard bases

In (1.3.5), we introduced the category $\mathcal{B}_{ASES_R}$ whose objects are free R-modules R^n together with a chosen basis, and its full subcategory $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ of free modules with standard bases. These categories are equivalent since, in our previous notation, we have shown that $T = \mathrm{Inc} \circ S$ is isomorphic to the identity functor on $\mathcal{B}_{ASES_R}$, while $F \circ \mathrm{Inc}$ is clearly the identity functor on $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$.

1.3.13 Faithful, full and dense

In general, a category $\mathcal{C}$ is equivalent to a full subcategory $\mathcal{C}'$ precisely when there is a natural way of associating to each object of $\mathcal{C}$ a representative in $\mathcal{C}'$ of its $\mathcal{C}$-isomorphism class. This idea may be developed as follows.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if the map

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC)$$

is an injection for every pair of objects in $\mathcal{C}$; F is *full* if the above map is a surjection, and F is *dense* or *representative* if for each object D in $\mathcal{D}$, we have $D \cong FC$ for some C in $\mathcal{C}$.

With these definitions, the following result is straightforward. (More details can be found in [Herrlich & Strecker 1979], (14.11), and [Mitchell 1965], II, particularly Proposition 10.1.)

1.3.14 Proposition

The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there is a full, faithful and dense functor from $\mathcal{C}$ to $\mathcal{D}$. □

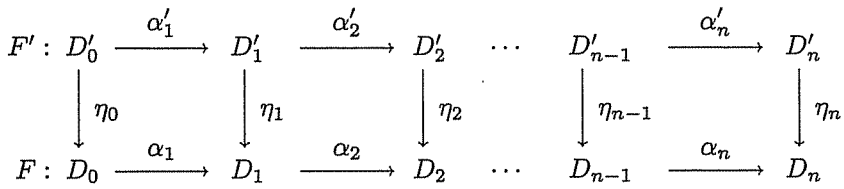
1.3.15 Skeletons

A category $\mathcal{S}$ is called *skeletal* if no two of its objects are isomorphic. Trivial examples are provided by the categories $[n]$, $n \geq 0$.

Given any category $\mathcal{C}$, we can form a full skeletal subcategory $\mathrm{Sk}(\mathcal{C})$ of $\mathcal{C}$ by choosing one object from each isomorphism class of the objects of $\mathcal{C}$; naturally, $\mathrm{Sk}(\mathcal{C})$ is called the *skeleton* of $\mathcal{C}$. The skeleton is not, in general, unique. However, the inclusion functor from $\mathrm{Sk}(\mathcal{C})$ to $\mathcal{C}$ is evidently full, faithful and dense, so $\mathrm{Sk}(\mathcal{C})$ is equivalent to $\mathcal{C}$.

Notice that $\mathrm{Sk}(\mathcal{C})$ is small precisely when the class of isomorphism classes of objects in $\mathcal{C}$ is in fact a set. (This condition suffices for applications in K-theory.) Thus the outcome of the discussion in (1.1.10) can be rephrased

such that the diagram



commutes, that is, each of its squares commutes.

Note that the category $[[0], \mathcal{D}]$ is just $\mathcal{D}$, while $[[1], \mathcal{D}]$ is the morphism category $\mathcal{M}\mathit{OR}\,\mathcal{D}$ under another name. Notice also that an object in $[[2], \mathcal{D}]$ can be interpreted as a commutative triangle.

1.3.11 Equivalence of categories

We now consider the circumstances in which two categories can be regarded as the same.

The most obvious of such circumstances is that the categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*, that is, we have a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with $GF = \mathrm{Id}_{\mathcal{C}}$ and $FG = \mathrm{Id}_{\mathcal{D}}$. This means that F and G are mutually inverse isomorphisms (1.2.8). If we have such a pair of functors, they induce mutually inverse bijections between the classes $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{D})$. But this is a relatively uncommon event, since the definition of the 'value' $F(C)$ of a functor F on an object C very often requires the choice of one object from many isomorphic candidates.

More likely, and still acceptable as a means of comparing categories, is for functors F and G to induce bijections of the collections of isomorphism classes of objects of $\mathcal{C}$ and $\mathcal{D}$.

For example, for a field $\mathcal{K}$, each isomorphism class of objects of $\mathcal{S}\mathit{TAN}\mathcal{B}\mathit{ASES}_{\mathcal{K}}$ is a singleton, comprising only the standard basis of $\mathcal{K}^n$, whereas $\mathcal{B}\mathit{ASES}_{\mathcal{K}}$ has many objects isomorphic to the standard basis of $\mathcal{K}^n$. Yet because *every* object of $\mathcal{B}\mathit{ASES}_{\mathcal{K}}$ is isomorphic to the standard basis, the subcategory $\mathcal{S}\mathit{TAN}\mathcal{B}\mathit{ASES}_{\mathcal{K}}$ is a good working approximation to $\mathcal{B}\mathit{ASES}_{\mathcal{K}}$.

Therefore, the appropriate notion is *equivalence* of categories: two categories $\mathcal{C}$ and $\mathcal{D}$ are (naturally) equivalent if there are (covariant) functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ so that GF is naturally isomorphic to $\mathrm{Id}_{\mathcal{C}}$ and FG is naturally isomorphic to $\mathrm{Id}_{\mathcal{D}}$. The functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are then said to be mutually inverse *equivalences* (of $\mathcal{C}$ and $\mathcal{D}$); the notation is $\mathcal{C} \simeq \mathcal{D}$.

If F and G are contravariant, they are called *dualities*, and the categories are said to be *dual*.

is defined by

$$(\zeta\eta)_C = \zeta_C\eta_C \text{ for each } C \in \mathcal{C}$$

when $\mathcal{D}$ is a right category.

The category $[\mathcal{C}^{\mathrm{op}}, \mathcal{D}]$ is the category of all contravariant functors from $\mathcal{C}$ to $\mathcal{D}$.

1.3.10 Diagrams

A useful application of the preceding discussion occurs when functors from $\mathcal{C}$ to $\mathcal{D}$ are interpreted as defining *diagrams* in $\mathcal{D}$.

A diagram in a category $\mathcal{D}$ is the same thing as the *graph* of a functor from some category $\mathcal{C}$ to $\mathcal{D}$. That is, it consists of a collection of objects and morphisms of $\mathcal{D}$ which are the images of the objects and morphisms of $\mathcal{C}$ and which satisfy the appropriate commutativity conditions. Although in principle any category $\mathcal{C}$ defines diagrams, the notion is most often used when $\mathcal{C}$ has few objects and morphisms.

For example, let $[n]$ be the totally ordered set $\{0, \ldots, n\}$, which we view as a category with exactly one morphism from i to j whenever $i \leq j$. We can display $[n]$ as a diagram

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \quad \cdots \quad \longrightarrow n-1 \longrightarrow n,$$

in which the 'generating' morphisms are shown; the remaining morphisms in $[n]$ are identities or products of those indicated. (Note that this category is labelled according to the number of generating morphisms, rather than the number of objects. This is due to its topological applications. For example, see Exercise 1.3.17 below.)

A functor from $[n]$ to $\mathcal{D}$, that is, an object F of $[[n], \mathcal{D}]$, is then a sequence of morphisms in $\mathcal{D}$

$$F : D_0 \xrightarrow{\alpha_1} D_1 \xrightarrow{\alpha_2} D_2 \quad \cdots \quad D_{n-1} \xrightarrow{\alpha_n} D_n$$

and a morphism η from F' (with the obvious labelling) to F is a set of morphisms in $\mathcal{D}$

$$\eta_i : D'_i \longrightarrow D_i, \quad i = 0, \ldots, n,$$

If this is the case, F is said to be *left adjoint* to G and G is said to be *right adjoint* to F. (The terms 'left' and 'right' here are convincing only for covariant functors.)

Thus, to show that F and G are adjoint, it is necessary and sufficient to give a family of bijections

$$\eta_{C,D} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D))$$

which are natural in C and D.

The following example typifies the set-up.

1.3.8 Example: free modules

Let X be a set and let $\mathrm{Fr}_R(X)$ be the free right R-module on X. Then $\mathrm{Fr}_R : \mathcal{S}et \to \mathcal{M}od_R$ is a covariant functor (1.2.3). In the other direction, let $\Upsilon : \mathcal{M}od_R \to \mathcal{S}et$ be the forgetful functor, which assigns to a module M its underlying set $\Upsilon(M)$ of elements, as in (1.2.2)(iv). It is easy to see that Fr_R is left adjoint to Υ.

1.3.9 Functor categories

Given a small category $\mathcal{C}$ and an arbitrary category $\mathcal{D}$, the covariant functors from $\mathcal{C}$ to $\mathcal{D}$ can themselves be regarded as the objects of a category $[\mathcal{C}, \mathcal{D}]$. Such categories $[\mathcal{C}, \mathcal{D}]$ and their subcategories are called *functor* categories.

The objects of $[\mathcal{C}, \mathcal{D}]$ are the covariant functors

$$F : \mathcal{C} \longrightarrow \mathcal{D},$$

and a morphism between functors F and G is a natural transformation

$$\eta : F \longrightarrow G.$$

We write $\mathrm{Nat}(F, G)$ for the set of all such natural transformations. This is indeed a set, since $\eta = \{\eta_C \mid C \in \mathcal{C}\}$ is specified by a collection of data which is indexed by the set $\mathrm{Ob}\,\mathcal{C}$, and each η_C must belong to the set $\mathrm{Mor}(FC, GC)$. (The target category $\mathcal{D}$ need not be small.)

To see that we actually have a (right) category, we note that each functor F has an *identity* natural transformation id_F given by

$$(id_F)_C = id_{FC} \text{ for each } C \in \mathcal{C},$$

and that for $\eta \in \mathrm{Nat}(F, G)$ and $\zeta \in \mathrm{Nat}(G, H)$, their product

$$\zeta\eta : F \longrightarrow H$$

the first variable and covariant in the second. It is cochiral in both variables. A version which exhibits mixed chiralities is given in Exercise 2.1.1.

Natural transformations between multifunctors, of the same variance and chirality, are defined in the obvious way.

It is also possible to describe a multifunctor 'term-by-term'. Let F be a multifunctor, and choose any index $h = 1, \ldots, k$ and any set of objects $\{C_i \in \mathcal{C}_i \mid i \neq h\}$. Then F restricts to a functor

$$F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k) : \mathcal{C}_h \longrightarrow \mathcal{D},$$

which is covariant or contravariant according as δ_h is Id or Op, and cochiral or contrachiral according as ϵ_h is Id or Mir.

Furthermore, any collection of morphisms

$$\{\beta_i : \delta_i(B_i) \longrightarrow \delta_i(C_i) \in \delta_i(\mathcal{C}_i) \mid i \neq h\}$$

gives rise to a natural transformation

$$F(B_1, \ldots, B_{h-1}, -, B_{h+1}, \ldots, B_k) \longrightarrow F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k).$$

Conversely, given a fixed index h, a multifunctor can be defined by a collection of functors of the form $F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k)$, one for each set of objects from the categories $\mathcal{C}_i$, $i \neq h$, provided that the appropriate natural transformations exist.

1.3.7 *Adjoint functors*

We now introduce a concept which is very important in both the theory and the applications of categories, particularly in connection with universal constructions, which we consider in detail in the next section.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be (covariant) functors. We can form two bifunctors

$$\mathrm{Mor}_{\mathcal{C}}(-, G(-)) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{S}et$$

and

$$\mathrm{Mor}_{\mathcal{D}}(F(-), -) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{S}et,$$

both of which are contravariant in the first variable and covariant in the second.

The functors F and G are said to form an *adjoint pair*, or to be *adjoint*, if there is a natural isomorphism

$$\eta : \mathrm{Mor}_{\mathcal{D}}(F(-), -) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{C}}(-, G(-)).$$

a formal symbol corresponding to a morphism $\beta : B \to D$ in $\mathcal{C}$, and that the composition of a pair of composable morphisms $\beta^{\odot}$ and $\delta^{\odot}$ in $\mathcal{C}^{\odot}$ is $\beta^{\odot}\delta^{\odot} = (\delta\beta)^{\odot}$.

Thus, if we have a pair of composable morphisms $(\alpha, \beta^{\odot})$, $(\gamma, \delta^{\odot})$ in $\mathcal{C} \times \mathcal{C}^{\odot}$, their product is

$$(\gamma, \delta^{\odot})(\alpha, \beta^{\odot}) = (\gamma\alpha, \beta^{\odot}\delta^{\odot}) = (\gamma\alpha, (\delta\beta)^{\odot}).$$

Methods of handling mixed chiralities become important later (3.1.9), when we consider tensor products. Thus we need a definition which accommodates both mixed variances and mixed chiralities.

Our uniform treatment of the various possible mixtures of variances and chiralities is based on two observations. Firstly, if F is a contravariant functor on $\mathcal{C}$, then $F \circ \mathrm{Op}$ is a covariant functor on $\mathcal{C}^{\mathrm{op}}$; and secondly, if $F : \mathcal{C} \to \mathcal{D}$ is a contrachiral functor, then $F \circ \mathrm{Mir} : \mathcal{C}^{\odot} \to \mathcal{D}$ is a cochiral functor. (Here we rely on the obvious identifications $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ and $(\mathcal{C}^{\odot})^{\odot} = \mathcal{C}$ for any category $\mathcal{C}$.)

Let $\mathcal{D}, \mathcal{C}_1, \ldots, \mathcal{C}_k$ be a finite set of (right) categories. Suppose that

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \longrightarrow \mathcal{D}$$

is given by a function

$$F : \mathrm{Ob}(\mathcal{C}_1) \times \cdots \times \mathrm{Ob}(\mathcal{C}_k) \longrightarrow \mathrm{Ob}(\mathcal{D})$$

and, for each pair C_i', C_i of objects of each $\mathcal{C}_i$, a function

$$F : \mathrm{Mor}_{\mathcal{C}_1} \times \cdots \times \mathrm{Mor}_{\mathcal{C}_k} \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(C_1', \ldots, C_k'), F(C_1, \ldots, C_k)).$$

Then we say that F is a *multifunctor* or a *functor in k variables* on $\mathcal{C}_1, \ldots, \mathcal{C}_k$ provided that for some (necessarily unique) choice of functors δ_i, ϵ_i on $\mathcal{C}_i$, with each δ_i as either Id or Op and each ϵ_i as either Id or Mir,

$$F \circ (\delta_1\epsilon_1 \times \cdots \times \delta_k\epsilon_k) : \epsilon_1\delta_1(\mathcal{C}_1) \times \cdots \times \epsilon_k\delta_k(\mathcal{C}_k) \longrightarrow \mathcal{D}$$

is a covariant and cochiral functor.

If δ_i is Id, then F is said to be *covariant in the i th variable*, while if δ_i is Op, F is said to be *contravariant in the i th variable.*

Likewise, if ϵ_i is Id, then F is said to be *cochiral in the i th variable*, while if ϵ_i is Op, F is *contrachiral in the i th variable.*

The terms *bifunctor*, *trifunctor*,... are also used when $k = 2, 3, \ldots$. Note that a multifunctor in one variable is simply a functor of the given variance and chirality.

Thus $\mathrm{Mor}_{\mathcal{C}}(-,-)$ is a bifunctor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{S}ET$ which is contravariant in

and

$$S(\alpha) = A, \text{ the matrix of } \alpha.$$

For each object $(R^n, F = \{f_j\})$ of $\mathcal{B}_{ASES_R}$, there is an R-module isomorphism η_F from R^n to R^m which associates to each member x of R^n its coordinate vector ${}_F(x)$: thus

$$\eta_F(x) = {}_F(x).$$

(Properly speaking, η should have as suffix the pair of objects between which we map, but the notation is then unwieldy.)

To interpret these isomorphisms as defining a natural isomorphism of functors, we let T be the composite functor $T = \text{Inc} \circ S$, where Inc is the inclusion of $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ in $\mathcal{B}_{ASES_R}$. For any morphism $\alpha : (R^n, F = \{f_j\}) \to (R^l, G = \{g_j\})$ in $\mathcal{B}_{ASES_R}$, there is a commutative diagram:

$$\begin{array}{ccc} (R^n, \{f_j\}) & \xrightarrow{\eta_F} & (R^m, \{e_j\}) \\ \alpha \downarrow & & \downarrow T(\alpha) \\ (R^\ell, \{g_j\}) & \xrightarrow{\eta_G} & (R^k, \{e_j\}) \end{array}$$

which shows $\eta : \text{Id} \to T$ to be a natural transformation, as desired. Note that the homomorphism $T(\alpha)$ is simply left multiplication by the matrix A of α.

1.3.6 Multifunctors

It is often necessary to consider functors of several variables. The guiding example is $\text{Mor}_{\mathcal{C}}(-,-)$ for a (right) category $\mathcal{C}$, which attaches a set $\text{Mor}_{\mathcal{C}}(C, D)$ to each pair of objects C, D of $\mathcal{C}$. By the remarks in (1.2.7), $\text{Mor}_{\mathcal{C}}(-, D)$ is a contravariant functor, while $\text{Mor}_{\mathcal{C}}(C, -)$ is covariant. So we wish to recognize $\text{Mor}_{\mathcal{C}}(-,-) : \mathcal{C} \times \mathcal{C} \to \mathcal{S}_{ET}$ as simultaneously functorial in both variables, even though it is not actually a functor on the product category $\mathcal{C} \times \mathcal{C}$. The point is that, because of its mixed variances, the would-be functor $\text{Mor}(-,-)$ does not obey either of the composition laws Fun 2 or Fun 2°. The same difficulty arises when we consider $\text{Op} \times \text{Id} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}^{\text{op}} \times \mathcal{C}$.

A related problem arises when we wish to take the variables from categories of differing chiralities, since the law of composition in the product of categories of mixed chiralities can be rather convoluted. For instance, let $\mathcal{C}$ be a right category, so that its mirror $\mathcal{C}^{\odot}$ (1.1.5) is a left category. Our rule then is to view the product category $\mathcal{C} \times \mathcal{C}^{\odot}$ as a right category (1.1.11).

To see how this works, recall that a morphism $\beta^{\odot} : B^{\odot} \to D^{\odot}$ in $\mathcal{C}^{\odot}$ is

If R^ℓ is another standard free right R-module with basis $\{g_j\} = \{g_1, \ldots, g_k\}$ and $\alpha : R^n \to R^\ell$ is a homomorphism of right R-modules, we associate to α a $k \times m$ matrix $A = (a_{ij})$, which is determined by the formula

$$\alpha(f_j) = \sum_{i=1}^{k} g_i a_{ij}, \quad j = 1, \ldots, m.$$

The coordinate vector ${}_G(\alpha x)$ of αx with respect to $G = \{g_j\}$ is then given by the relation

$${}_G(\alpha x) = A_F(x).$$

The association of a matrix to a homomorphism is easily seen to be multiplicative; that is, if $\beta : R^p \to R^n$ is an R-module homomorphism that is composable with α, and if B is the matrix of β with respect to a basis $\{h_j\}$ of R^p and the given basis $\{f_j\}$ of R^n, then the matrix of $\alpha\beta$ with repect to the pair $\{h_j\}$, $\{g_j\}$ is AB.

The association is also additive, for if α and α' are homomorphisms from R^n to R^ℓ with matrices A and A' respectively (relative to the pair of bases $\{f_j\}$, $\{g_j\}$), then the matrix of $\alpha + \alpha'$ will be $A + A'$ (relative to the same pair of bases).

In particular, fix an n-element basis of R^n and use it at 'both ends'. Then the matrix A of an endomorphism α will be $n \times n$, and the map $\alpha \mapsto A$ is a ring isomorphism from $\mathrm{End}(R^n)$ to the matrix ring $M_n(R)$.

1.3.5 Example: matrices and bases of free modules

Now we interpret the above discussion in terms of categories and functors. We define $\mathcal{B}_{ASES_R}$ to be the category whose objects are free right R-modules of the form R^n equipped with a preferred basis $\{f_j\}$, that is, a pair $(R^n, \{f_j\})$. The morphisms in $\mathcal{B}_{ASES_R}$ are just the R-module homomorphisms from R^n to R^ℓ (here the choice of basis is ignored). The full subcategory $\mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$ of $\mathcal{B}_{ASES_R}$ has as objects the pairs $(R^n, \{e_j\})$ where $\{e_j\}$ is the standard basis of R^n. (We use the notation $\{e_j\}$ ambiguously for a standard basis of any size.)

The fact that the operation of taking a matrix of a homomorphism is multiplicative now reveals that we obtain a functor

$$S : \mathcal{B}_{ASES_R} \longrightarrow \mathcal{S}_{TAN}\mathcal{B}_{ASES_R}$$

by setting

$$S(R^n, \{f_j\}) = (R^m, \{e_j\}), \quad m \text{ being the number of members of } F = \{f_j\},$$

volume, we review these facts in greater generality than we need for our immediate purposes.

Let R be a ring and let M be a right R-module. A *basis* of M is an ordered subset $\{f_\lambda\}_{\lambda\in\Lambda}$ (where Λ is an ordered indexing set), which satisfies the following property.

Bas. Given any element m of M, there are *unique* elements $r_\lambda(m)$, $\lambda \in \Lambda$, of R such that

$$m = \sum_{\lambda\in\Lambda} f_\lambda r_\lambda(m).$$

The set Λ may be infinite, but at most a finite number of coefficients $r_\lambda(m)$ can be nonzero for any particular m.

An arbitrary module M need not have a basis. If M does have a basis, then M is a *free right* R-module.

Note that a basis must be ordered. An 'unordered basis' is called a *free generating set*. Thus, the free module $\mathrm{Fr}_R(X)$ on an unordered set X (1.2.3) has X as a free generating set. If we impose an ordering on X, then we obtain a basis of $\mathrm{Fr}_R(X)$.

When the index set Λ is finite, we usually take it to be $\{1,\dots,m\}$ for some integer m and write the basis as

$$\{f_j\} = \{f_1,\dots,f_m\}.$$

We then say that the free module has *finite rank*.

Now let R^n be the standard free right R-module of finite rank n, which we view as the 'space of column vectors' of length n, with entries in R. The evident choice of 'unit' vectors $\{e_1,\dots,e_n\}$ gives the *standard basis* $\{e_j\}$ of R^n. Next, take any finite basis

$$\{f_j\} = \{f_1,\dots,f_m\}$$

of R^n. If it must be the case that $m = n$, the ring R is said to have *invariant basis number*. However, we wish to work with an arbitrary ring, so we must allow the possibility that $m \neq n$. Some examples and results on invariant basis number and the lack of it are to be found in §2.3 of [BK: IRM].

An element x of R^n has a *coordinate vector* ${}_F(x)$ with respect to the basis $F = \{f_j\}$, which is given by the formula

$${}_F(x) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \quad \text{where } x = f_1y_1 + \cdots + f_my_m,\ y_1,\dots,y_m \in R.$$

rise to a commuting square

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{-\oplus 1} & \mathrm{GL}_{n+1}(R) \\ \Big\downarrow{\scriptstyle \mathrm{GL}_n(f)} & & \Big\downarrow{\scriptstyle \mathrm{GL}_{n+1}(f)} \\ \mathrm{GL}_n(S) & \xrightarrow{-\oplus 1} & \mathrm{GL}_{n+1}(S) \end{array}$$

(viii) We continue with the above notation, but now restrict our attention to the category of commutative rings. We can then define the determinant map $\det : M_n(R) \to R$ by the usual formula involving sums and differences of products of entries. (See [Cohn 1982], Chapter 7 for example.) Because the determinant preserves products, that is, $\det(AB) = \det(A)\det(B)$ for $n \times n$ matrices A and B, it restricts to a homomorphism of the groups of units $\det : \mathrm{GL}_n(R) \to U(R)$. Again, this homomorphism is natural, meaning that there is a commuting square

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{\det} & U(R) \\ \Big\downarrow{\scriptstyle \mathrm{GL}_n(f)} & & \Big\downarrow{\scriptstyle U(f)} \\ \mathrm{GL}_n(S) & \xrightarrow{\det} & U(S) \end{array}$$

whenever f is a ring homomorphism.

1.3.3 Natural isomorphisms

If for each object C of $\mathcal{C}$ the morphism η_C is an isomorphism in $\mathcal{D}$, we say that the functors F and G are *naturally isomorphic* through the *natural isomorphism* η. The notation is $F \simeq G$. The terms *naturally equivalent* and *natural equivalence* are also used, especially in older texts.

Among the above examples, that in (vi) is a natural isomorphism because the matrix transpose operation is evidently invertible. In (ii) and (viii), we clearly have natural isomorphisms when $n = 1$.

1.3.4 Matrices and bases of free modules: a summary

Our next example gives a categorical interpretation of some results on the matrix representation of homomorphism between free modules. To get started, we need some facts about free modules and their bases which are given in detail in [BK: IRM], particularly §2.2. For future ease of reference in this

(vi) For another example, where neither functor is the identity functor, we use the operation of transposition on matrices. Given a matrix $A = (a_{ij})$ over a ring R, the *transpose* of A is defined to be the matrix $A^t = (a^\circ_{ji})$, which has entries in the opposite ring R°. Applying this definition to matrices over R° rather than R itself, we obtain a map

$$(-)^t : M_n(R^\circ) \longrightarrow (M_n(R))^\circ$$

which is readily seen to be a ring isomorphism. (Further details can be found in Exercises 1.2.13 and 1.2.14 of [BK: IRM].)

Given a ring homomorphism $f : R \to S$ there is a commutative square

$$\begin{array}{ccc} M_n(R^\circ) & \xrightarrow{(-)^t} & (M_n(R))^\circ \\ \downarrow & & \downarrow \\ M_n(S^\circ) & \xrightarrow{(-)^t} & (M_n(S))^\circ \end{array}$$

so that $(-)^t$ is a natural transformation between the functors $M_n((-)^\circ)$ and $(M_n(-))^\circ$.

(vii) For a ring R, the general linear group $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices over R, that is, the group of units of the matrix ring $M_n(R)$.

Evidently, $\mathrm{GL}_n(R)$ can be embedded in $\mathrm{GL}_{n+1}(R)$ by the insertion of an extra (last) row and column with diagonal entry 1 and other entries 0 to each matrix A in $GL_n(R)$:

$$A \longmapsto \left(\begin{array}{c|c} & 0 \\ A & \vdots \\ & 0 \\ \hline 0 \cdots 0 & 1 \end{array}\right).$$

We abbreviate the right-hand matrix to $A \oplus 1$. Since

$$AB \oplus 1 = (A \oplus 1)(B \oplus 1),$$

we obtain a group homomorphism $- \oplus 1 : \mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$. Moreover, $- \oplus 1$ is natural because any ring homomorphism $f : R \to S$ gives

(ii) Let R be a ring and let $M_n(R)$ be the ring of $n \times n$ matrices over R. The inclusion map

$$\iota_R : R \longrightarrow M_n(R)$$

sends an element r of R to the scalar matrix $\mathrm{diag}(r, \ldots, r)$. (The structure of matrix rings is discussed at length in section 2.2 of [BK: IRM].)

If $f : R \to S$ is a ring homomorphism, there is an obvious commuting diagram

$$\begin{array}{ccc} R & \xrightarrow{\iota_R} & M_n(R) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle M_n(f)} \\ S & \xrightarrow{\iota_S} & M_n(S) \end{array}$$

of ring homomorphisms. Thus ι is a natural transformation from the identity functor on the category $\mathcal{R}_{ING}$ to the functor $M_n(-)$ on $\mathcal{R}_{ING}$. Typically, one says just that the embedding ι is natural.

(iii) A ring R contains a smallest subring, namely its *prime ring*, which is the subring $\langle 1_R \rangle$ comprising the multiples $n.1_R$, $n \in \mathbb{Z}$, of the multiplicative identity 1_R of R. The assignment $P : R \mapsto \langle 1_R \rangle$ is clearly a functor from the category $\mathcal{R}_{ING}$ of rings to itself. Then the embedding of $\langle 1_R \rangle$ in R is a natural transformation from P to the identity functor because a ring homomorphism $f : R \to S$ gives a commuting square

$$\begin{array}{ccc} \langle 1_R \rangle & \hookrightarrow & R \\ {\scriptstyle Pf}\downarrow & & \downarrow{\scriptstyle f} \\ \langle 1_S \rangle & \hookrightarrow & S \end{array}$$

(iv) Given a nonunital ring R, the *enveloping ring* or *unitalization* of R is

$$\overline{R} = \{(r, a) \mid r \in R, a \in \mathbb{Z}\},$$

which is a ring under the addition and multiplication given by

$$(r, a) + (s, b) = (r + s, a + b) \quad \text{and} \quad (r, a) \cdot (s, b) = (rs + br + as, ab).$$

The identity of $\overline{R}$ is $(0, 1)$. Then, in the category $\mathcal{R}_{NG}$ of nonunital rings, the embedding of R in its enveloping ring is natural.

(v) In the category of commutative domains, the embedding of a commutative domain in its field of fractions is also natural – see [BK: IRM] (1.1.12).

gram to commute:

$$\begin{array}{ccc} FC & \xrightarrow{\eta_C} & GC \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC' & \xrightarrow{\eta_{C'}} & GC' \end{array}$$

(There is little to be gained by attempting to compare functors of different variances, as can be seen from the example of 'unnatural' behaviour that is given in Exercise 1.3.18.)

1.3.2 Examples

(i) Basic and important examples of natural transformations arise from the morphism functors (1.2.7) associated with a (right) category C.

Let $\lambda : L \to M$ and $\xi : X \to Y$ be morphisms in C and recall that

$$\xi_* = \xi_*^L : \mathrm{Mor}(L, X) \longrightarrow \mathrm{Mor}(L, Y)$$

is given by

$$\xi_*(\alpha) = \xi\alpha : L \longrightarrow Y \text{ for } \alpha : L \longrightarrow X \text{ in } \mathrm{Mor}(L, X),$$

and

$$\lambda^* = \lambda_X^* : \mathrm{Mor}(M, X) \longrightarrow \mathrm{Mor}(L, X)$$

is given by

$$\lambda^*\beta = \beta\lambda \text{ for } \beta \text{ in } \mathrm{Mor}(M, X).$$

It is easy to verify that there is a commutative square

$$\begin{array}{ccc} \mathrm{Mor}(M, X) & \xrightarrow{\xi_*^M} & \mathrm{Mor}(M, Y) \\ \downarrow{\scriptstyle \lambda_X^*} & & \downarrow{\scriptstyle \lambda_Y^*} \\ \mathrm{Mor}(L, X) & \xrightarrow{\xi_*^L} & \mathrm{Mor}(L, Y) \end{array}$$

which can be interpreted in two ways, one as showing that ξ_* is a natural transformation from $\mathrm{Mor}(-, X)$ to $\mathrm{Mor}(-, Y)$ and the other, that λ^* is a natural transformation from $\mathrm{Mor}(M, -)$ to $\mathrm{Mor}(L, -)$.

1.3 NATURAL TRANSFORMATIONS

Functors themselves can be related to one another by natural transformations. This ability to compare functors enables us to define the concepts of multifunctors and of adjoint functors, vital for our discussion of universal properties in the next section. It also leads to the notion of a functor category, consisting of all the functors from one category to another. When the domain category has an especially simple form, this gives the important special case of a diagram in a category.

We also consider equivalences between categories; these are analogous to isomorphisms between modules or groups in the sense that equivalent categories can be regarded as identical for most purposes. The definition of isomorphism of categories, given in (1.2.8) above, is usually too restrictive for applications.

1.3.1 Natural transformations

Natural transformations are the morphisms between functors themselves.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be covariant functors. A *natural transformation*

$$\eta : F \longrightarrow G$$

consists of a family of morphisms

$$\eta_C : FC \longrightarrow GC,$$

one for each object C of $\mathcal{C}$, such that for each morphism

$$\alpha : C' \longrightarrow C,$$

the diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

commutes. Somewhat informally, a typical member of this family is sometimes said to be a natural transformation.

If, instead, F and G are both contravariant, we require the following dia-

is also a group viewed as a category in this way, a functor F from $\mathcal{B}H$ to $\mathcal{B}G$ is the same as a group homomorphism γ from H to G.

The right-fibre category $G\backslash\gamma$ consists of pairs (H, g) where H is the single object of $\mathcal{B}H$ and g is an arbitrary element of G; we can therefore identify $G\backslash\gamma$ with the set $\{g\}$ of elements of G. A morphism from g to g' is an element h of H with $\gamma(h) \circ g = g'$.

Thus the set of morphisms in $G\backslash\gamma$ from id_G to itself is the kernel of γ, while the set of morphisms from id_G to g' is $\{h \mid \gamma(h) = g'\}$, the inverse image of g'.

In particular, $G\backslash \mathrm{Id}_{\mathrm{Cat}(G)}$ is the category $\mathcal{E}G$ of Exercise 1.1.3.

Exercises

1.2.1 Let X and Y be sets regarded as discrete categories, as in Exercise 1.1.1.

Show that a functor from X to Y is simply a map $f : X \to Y$.

Verify that for any $y \in Y$, the fibre $y\backslash f$ is $f^{-1}(y)$, the inverse image of y.

1.2.2 We may regard a group G as a category, $\mathcal{B}G$ or $\mathcal{E}G$, in two ways as in Exercise 1.1.3. In (1.2.10), we noted that a functor from $\mathrm{Cat}(H)$ to $\mathrm{Cat}(G)$ is the same thing as a group homomorphism from H to G.

Are there any functors from $\mathcal{B}G$ to $\mathcal{E}G$ or vice versa?

Describe the functors from $\mathcal{E}H$ to $\mathcal{E}G$. In particular, show that each element of the group G acts, by multiplication, as a functor on $\mathcal{E}G$ that is transitive on the objects.

Remark. So, if we identify the objects of $\mathcal{E}G$ with one another via this multiplication, we obtain $\mathcal{B}G$. Given a method of passing from groups to topological spaces, this procedure may be used to form the classifying space BG of the group G [Segal 1968].

1.2.3 Recall (or look up in a standard text on group theory) that the commutator subgroup $[G, G]$ of a group G is the normal subgroup of G generated by all commutators $ghg^{-1}h^{-1}$, and that the *commutator quotient group* $G_{ab} = G/[G, G]$ is an abelian group. Sometimes the group G_{ab} is said to be G *made abelian.*

Define $\mathrm{Ab} : \mathcal{G}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{B}}$ by $\mathrm{Ab}(G) = G_{ab}$, where $\mathcal{G}_{\mathcal{P}}$ is the category of groups and $\mathcal{A}_{\mathcal{B}}$ is the category of abelian groups. Verify that, with the obvious action on homomorphisms, Ab is a functor, sometimes called *abelianization.*

the zero element of a module, we must define a right-fibre category $C\backslash F$ (of F at C) for each object C in $\mathcal{C}$.

An object of the *right-fibre category* consists of a pair (D, α), where D is an object of $\mathcal{D}$ and $\alpha : C \to F(D)$ is a morphism of $\mathcal{C}$; such a pair is said to be *under* C *with respect to* F, or F-*under* C.

A morphism in $C\backslash F$ from (D, α) to (D', α') is given by a morphism $\beta : D \to D'$ in $\mathcal{D}$ such that $\alpha' = F(\beta) \circ \alpha$. Diagrammatically, this may be visualized as:

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & F(D) \\ & \searrow^{\alpha'} & \downarrow{\scriptstyle F(\beta)} \\ & & F(D') \end{array} \qquad \begin{array}{c} D \\ \downarrow{\scriptstyle \beta} \\ D' \end{array}$$

The definition of the dual notion of the *left-fibre category* F/C of pairs (D, α) F-*over* C, should be apparent.

The justification for the name right-fibre category (replacing comma category, surely one of the least descriptive names ever in mathematics) comes from Theorem B of [Quillen 1973] in the homotopy theory of categories, which, however, lies beyond the scope of this book. We use right-fibre categories in section 1.4, where we consider universal constructions.

Here are some simple examples which shed light on the definition, as does Exercise 1.2.1.

1.2.10 Examples

1. Let $\mathcal{C}$ be the category having a unique object C and a unique morphism, id_C. Then, for the unique functor $F : \mathcal{D} \to \mathcal{C}$, the category $C\backslash F$ is isomorphic to the category $\mathcal{D}$ through the functors

$$(D, id_C) \leftrightarrow D, \quad g \leftrightarrow g.$$

2. For the identity functor $\mathrm{Id}_{\mathcal{C}}$ of an arbitrary category $\mathcal{C}$ with C an object of $\mathcal{C}$, the right-fibre category $C\backslash \mathrm{Id}_{\mathcal{C}}$ is just the category of objects of $\mathcal{C}$ *under* C, which is sometimes called the *slice* category $C\backslash\mathcal{C}$. Similarly, there is the category $\mathcal{C}/C = \mathrm{Id}_{\mathcal{C}}/C$ of objects *over* C. These categories are widely used by topologists.
3. Given a group G, form the category $\mathcal{B}G$ with one object G, the morphisms in $\mathcal{B}G$ being simply the elements of the group, as in Exercise 1.1.3. If H

which shows that $\mathrm{Mor}(-, X)$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{S}_{ET}$.

If $\mu : M \to N$ is another morphism, then, for any $\gamma : N \to X$,

$$\mathrm{Mor}(\mu\lambda, X)(\gamma) = \gamma\mu\lambda = \mathrm{Mor}(\lambda, X)\,\mathrm{Mor}(\mu, X)\gamma.$$

When the fixed object L or X need not be mentioned, it is usual to make the abbreviations $\mathrm{Mor}(L, \xi) = \xi_*$ and $\mathrm{Mor}(\lambda, X) = \lambda^*$. The formulas for products then become

$$(\xi\vartheta)_* = \xi_*\vartheta_* \text{ for } \xi : X \longrightarrow Y \text{ and } \vartheta : Y \longrightarrow Z$$

and

$$(\mu\lambda)^* = \lambda^*\mu^* \text{ for } \lambda : L \longrightarrow M \text{ and } \mu : M \longrightarrow N.$$

Note that these formulas do depend on the fact that both the categories $\mathcal{C}$ and $\mathcal{S}_{ET}$ are assumed to be right categories, that is, the functors $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$ are cochiral. If instead we take $\mathcal{C}$ to be a left category, then the product formulas read

$$(\vartheta\xi)_* = \xi_*\vartheta_* \text{ for } \xi : X \longrightarrow Y \text{ and } \vartheta : Y \longrightarrow Z$$

and

$$(\lambda\mu)^* = \lambda^*\mu^* \text{ for } \lambda : L \longrightarrow M \text{ and } \mu : M \longrightarrow N,$$

despite the fact that the variances of the functors $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$ are unchanged. This is an instance of the variability of notation that we discussed in the preceding section.

1.2.8 The category $\mathcal{C}_{AT}$

Given a class of small categories which contains all the small categories that we wish to discuss, we can form the category $\mathcal{C}_{AT}$ of all such categories. The objects of this category are themselves categories $\mathcal{C}, \mathcal{D}, \ldots$, and the set of morphisms from $\mathcal{C}$ to $\mathcal{D}$ is the set $[\mathcal{C}, \mathcal{D}]$ of all covariant functors from $\mathcal{C}$ to $\mathcal{D}$.

Note that an isomorphism in the category $\mathcal{C}_{AT}$ is an invertible functor. Accordingly, we call any invertible functor an *isomorphism.*

1.2.9 Fibre categories

For a functor $F : \mathcal{D} \to \mathcal{C}$, there is a counterpart of the fundamental concept of the kernel of a module homomorphism, namely, that of a right-fibre category. However, as a general category has no distinguished object corresponding to

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Reinterpreting the definition of the product in $\mathcal{C}^{\odot}$, we find that

$$\mathrm{Mir}(\alpha\beta) = \mathrm{Mir}(\beta)\,\mathrm{Mir}(\alpha).$$

As we noted above, the opposite functor can be used to convert a contravariant functor into a covariant functor. Similarly, $\mathrm{Mir} : \mathcal{D} \to \mathcal{D}^{\odot}$ converts a contrachiral functor $F : \mathcal{C} \to \mathcal{D}$ to a cochiral functor $\mathrm{Mir} \circ F : \mathcal{C} \to \mathcal{D}^{\odot}$, and likewise a cochiral functor can be changed into a contrachiral functor.

Thus, a judicious use of the functors Op and Mir allows one to work only with covariant and cochiral functors.

1.2.7 The morphism functors

Let $\mathcal{C}$ be a (right) category. The morphism sets $\mathrm{Mor}(L, X) = \mathrm{Mor}_{\mathcal{C}}(L, X)$ can be viewed as defining two functors, $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$, from $\mathcal{C}$ to the category $\mathcal{S}_{ET}$ of sets, by holding one term of $\mathrm{Mor}(L, X)$ constant, and allowing the other to vary, as the notation is meant to suggest. The functors arising from Mor in one way or another are called *morphism functors.*

Choose an object L of $\mathcal{C}$. The functor

$$\mathrm{Mor}(L, -) : \mathcal{C} \longrightarrow \mathcal{S}_{ET}$$

sends an object X to the set $\mathrm{Mor}(L, X)$, and for a morphism $\xi : X \to Y$ in $\mathcal{C}$, the map

$$\mathrm{Mor}(L, \xi) : \mathrm{Mor}(L, X) \longrightarrow \mathrm{Mor}(L, Y)$$

is given by

$$\mathrm{Mor}(L, \xi)\alpha = \xi\alpha : L \longrightarrow Y \text{ for } \alpha : L \longrightarrow X \text{ in } \mathrm{Mor}(L, X),$$

so that $\mathrm{Mor}(L, -)$ is a covariant functor from $\mathcal{C}$ to $\mathcal{S}_{ET}$.

(We write $\mathrm{Mor}(L, X)$ and $\mathrm{Mor}(L, \xi)$ in preference to $\mathrm{Mor}(L, -)(X)$ and $\mathrm{Mor}(L, -)(\xi)$.)

If $\vartheta : Y \to Z$ is another morphism, then

$$\mathrm{Mor}(L, \vartheta\xi)\alpha = \vartheta\xi\alpha = \mathrm{Mor}(L, \vartheta)\,\mathrm{Mor}(L, \xi)\alpha.$$

On the other hand, suppose that we fix the object X. Then $\mathrm{Mor}(-, X)(L) = \mathrm{Mor}(L, X)$ for any object L of $\mathcal{C}$, and for a morphism $\lambda : L \to M$, we define

$$\mathrm{Mor}(\lambda, X) : \mathrm{Mor}(M, X) \longrightarrow \mathrm{Mor}(L, X)$$

by

$$\mathrm{Mor}(\lambda, X)\beta = \beta\lambda \text{ for } \beta \in \mathrm{Mor}(M, X),$$

Chirality of $\mathcal{C}$	Chirality of $\mathcal{D}$	F covariant	F contravariant
left	left	$F(\alpha\beta) = F(\alpha)F(\beta)$	$F(\alpha\beta) = F(\beta)F(\alpha)$
left	right	$F(\alpha\beta) = F(\beta)F(\alpha)$	$F(\alpha\beta) = F(\alpha)F(\beta)$
right	left	$F(\beta\alpha) = F(\alpha)F(\beta)$	$F(\beta\alpha) = F(\beta)F(\alpha)$
right	right	$F(\beta\alpha) = F(\beta)F(\alpha)$	$F(\beta\alpha) = F(\alpha)F(\beta)$

We have listed these variations in detail since they are ignored in all texts on category theory that we have come across. Presumably, this is because it is the usual practice in category theory to deal only with what we have called here right categories.

When concentrating on right categories only, it is tempting to characterize covariant and contravariant in terms of the appearance of axiom Fun 2: *in this case*, covariant functors are those which 'preserve' products, while contravariant functors 'reverse' products. However, as we have seen, this characterization does not adequately describe the intrinsic distinction between covariant and contravariant functors. It is an artefact of the notation and so needs to be treated with caution. The essential difference is that the *direction* of a morphism is preserved in one case but reversed in the other.

In an ideal world, there would be a systematic notation devised to handle functors of differing variances between categories of differing chiralities. In reality we are obliged to use well-established notations for functors that take no account of chirality and variance.

The above table can be simplified by introducing the notion of the *chirality* of a functor. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *cochiral* if the categories $\mathcal{C}$ and $\mathcal{D}$ have the same chirality, and *contrachiral* otherwise. Then, for any composable morphisms α, β in $\mathcal{C}$, we have

$$F(\alpha\beta) = F(\alpha)F(\beta)$$

whenever F has the same variance as chirality, that is, both are co- or both are contra-, and

$$F(\alpha\beta) = F(\beta)F(\alpha)$$

whenever the variance of F differs from its chirality.

One example of a product reversing functor is the opposite functor Op (1.2.2)(iii) which is contravariant and cochiral.

Another such functor is the *mirror* functor. For any category $\mathcal{C}$, we define $\mathrm{Mir} : \mathcal{C} \to \mathcal{C}^{\odot}$ from $\mathcal{C}$ to its mirror category $\mathcal{C}^{\odot}$ (1.1.5) by putting $\mathrm{Mir}(C) = C^{\odot}$ and $\mathrm{Mir}(\alpha) = \alpha^{\odot}$. This functor is evidently covariant and contrachiral.

is given by

$$GF(C) = G(F(C))$$

and

$$GF(\alpha) = G(F(\alpha)).$$

The product GF is sometimes written $G \circ F$ if we wish to emphasise that we are performing a composition of functors.

It is easy to verify that GF will be covariant if F and G are either both covariant or both contravariant, and contravariant otherwise.

Notice that the opposite functor $\mathrm{Op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ provides a method of interchanging contravariant and covariant functors, since a contravariant functor $G : \mathcal{D} \to \mathcal{C}$ defines a covariant functor $\mathrm{Op} \circ G : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$, and vice versa.

1.2.5 Restriction

Given a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$, the *restriction* of F to $\mathcal{C}'$ is the composite

$$F \circ \mathrm{Inc} : \mathcal{C}' \longrightarrow \mathcal{D}.$$

If $\mathcal{D}'$ is a subcategory of $\mathcal{D}$ such that FC and $F\alpha$ are in $\mathcal{D}'$ for every object and morphism of $\mathcal{C}'$, we can also regard F as a functor from $\mathcal{C}'$ to $\mathcal{D}'$, which we again call the restriction of F.

In applications, $\mathcal{D}'$ will often be a full subcategory of $\mathcal{D}$; in this case, F restricts to a functor from $\mathcal{C}'$ to $\mathcal{D}'$ if and only if $F(\mathrm{Ob}\,\mathcal{C}')$ is contained in $\mathrm{Ob}\,\mathcal{D}'$.

To save notation, a restriction of a functor is usually denoted by the same symbol as the original functor.

1.2.6 Appearances and chirality

So far, we have discussed functors between right categories only. When we have to consider left categories as well, we find that the appearance of axiom Fun 2 depends on the chiralities of the categories involved, as well the variance of the functor itself.

To help the reader through this tangle, we list the various possibilities; here, F is a functor from $\mathcal{C}$ to $\mathcal{D}$, and, as hitherto, $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C', C)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(C, C'')$.

1.2.3 Free modules

Another important functor is

$$\mathrm{Fr}_R : \mathcal{S}ET \longrightarrow \mathcal{M}od_R,$$

which associates to a set X the free right R-module $\mathrm{Fr}_R(X)$ on X, where R is a fixed ring. Since this functor will be encountered frequently in the sequel, we give a sketch of its construction. A fuller account is given in [BK: IRM] (2.1.16).

An element m of $\mathrm{Fr}_R(X)$ is a formal sum

$$m = \sum_{x \in X} x r_x(m) \text{ with } r_x(m) \in R,$$

in which only a finite set of coefficients $r_x(m)$ may be nonzero. (In other words, m is a function from X to R whose value at x we prefer to write as $r_x(m)$ rather than $m(x)$, and which takes the value 0 except possibly at a finite number of members of X.) Two such sums m, n are the same precisely when $r_x(m) = r_x(n)$ for all x in X, and addition and scalar multiplication are defined by the rules

$$m + n = \sum_{x \in X} x(r_x(m) + r_x(n)) \quad \text{and} \quad mr = \sum_{x \in X} x(r_x(m)r).$$

It is easy to verify that $\mathrm{Fr}_R(X)$ is a right R-module. We view X as a subset of $\mathrm{Fr}_R(X)$ by identifying an element x of X with the module element that has $r_x = 1$ and $r_y = 0$ for $y \neq x$.

Now, given a mapping $\alpha : X \to Y$ of sets, there is one and only one way to extend α to a homomorphism

$$\mathrm{Fr}_R(\alpha) : \mathrm{Fr}_R(X) \longrightarrow \mathrm{Fr}_R(Y),$$

since we must have

$$\mathrm{Fr}_R(\alpha)\left(\sum_{x \in X} x r_x(m)\right) = \sum_{x \in X} \alpha(x) r_x(m).$$

It is straightforward to check that Fr_R is a functor as claimed.

1.2.4 The product

Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors. Then the *composite* or *product* functor

$$GF : \mathcal{C} \longrightarrow \mathcal{E}$$

A *contravariant functor* reverses the direction of morphisms. Thus for each morphism

$$\alpha : C' \longrightarrow C \text{ in } \mathcal{C}$$

there is a morphism

$$F\alpha : FC \longrightarrow FC' \text{ in } \mathcal{D}.$$

The second axiom now reads

Fun 2^o If $\mathcal{C}$ and $\mathcal{D}$ are both right categories and

$$\alpha : C' \longrightarrow C \text{ and } \beta : C \longrightarrow C''$$

are morphisms in $\mathcal{C}$, then

$$F(\beta\alpha) = F(\alpha)F(\beta) : FC'' \longrightarrow FC'.$$

The unqualified term *functor* will mean a covariant functor, unless the context demands otherwise.

1.2.2 Some examples

Here are some basic examples.

(i) The *identity* functor $\mathrm{Id}_{\mathcal{C}}$ from a category $\mathcal{C}$ to itself; this 'does nothing' to any object or morphism.

(ii) If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, there is the *inclusion* functor

$$\mathrm{Inc} : \mathcal{C}' \longrightarrow \mathcal{C};$$

again this leaves objects and morphisms unchanged.

(iii) On any category, there is a standard contravariant functor, the *opposite functor*

$$\mathrm{Op} : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}},$$

which has $\mathrm{Op}(C) = C^{\mathrm{op}}$ and $\mathrm{Op}(\alpha) = \alpha^{\mathrm{op}}$ – see (1.1.6).

(iv) The *forgetful* or *underlying* functor

$$\Upsilon : \mathcal{C} \longrightarrow \mathcal{D};$$

this arises when each object and morphism of the category $\mathcal{C}$ can be regarded as belonging to another category $\mathcal{D}$ by forgetting some of its structure. Thus there are forgetful functors from $\mathcal{M}_{OD_R}$ to $\mathcal{A}_B$, from $\mathcal{A}_B$ to $\mathcal{G}_P$ and from $\mathcal{G}_P$ to $\mathcal{S}_{ET}$.

for right categories and dismiss the left-handed version as obvious. However, the fact that categories may have different chiralities does now have an effect on our notation. The reason is that we have to take into account the possibility that a functor may well relate categories of different chiralities. The outcome is that some entrenched conventions must be treated with caution, since a covariant functor may act in a way that is superficially similar to a contravariant functor.

Nevertheless, we continue to work with right categories as far as possible, leaving statements and proofs in the left-handed case to the reader, unless there are particular points which we need to emphasise.

1.2.1 Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ associates with each object C of $\mathcal{C}$ an object FC in $\mathcal{D}$, and with each morphism

$$\alpha : C' \longrightarrow C \text{ in } \mathcal{C}$$

it associates a morphism

$$F\alpha : FC' \longrightarrow FC \text{ in } \mathcal{D}.$$

Thus F is a collection of mappings of two types (all of which are, by custom, denoted by the same letter F): there is one map of the first type

$$F : \mathrm{Ob}\,\mathcal{C} \longrightarrow \mathrm{Ob}\,\mathcal{D},$$

and for each pair C' and C of objects of $\mathcal{C}$ there is a map of the second type

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC).$$

These mappings are subject to the axioms

Fun 1 $F(id_C) = id_{FC}$

and

Fun 2 if $\mathcal{C}$ and $\mathcal{D}$ are both right categories and

$$\alpha : C' \longrightarrow C \text{ and } \beta : C \longrightarrow C''$$

are morphisms in $\mathcal{C}$, then

$$F(\beta\alpha) = F(\beta)F(\alpha) : FC' \longrightarrow FC''.$$

Remark. The existence of an anti-isomorphism between a group and its opposite is sometimes used implicitly to identify the two groups. An example is given by the symmetric group S_n, the group of all permutations of the set $\{1, \ldots, n\}$. If we consider the permutations to act as right operators, we obtain one version of the symmetric group, while if we view the permutations as left operators, we obtain the opposite group. Usually, a group theorist will take the right-handed version to be S_n, so that the left-handed version is then properly S_n^{op}.

A practical situation which requires both the left and the right handed versions of S_n arises in Chapter 12 of [Loday 1992].

1.1.6 **Morphisms on morphisms**

Describe the category $\mathcal{M}or^2\mathcal{C} = \mathcal{M}or\,(\mathcal{M}or\,\mathcal{C})$. Compare the categories $\mathcal{A}ut\,(\mathcal{M}or\,\mathcal{C})$ and $\mathcal{M}or\,(\mathcal{A}ut\mathcal{C})$, etc.

1.1.7 Compare $(\mathcal{C}^{\mathrm{op}})^{\odot}$ and $(\mathcal{C}^{\odot})^{\mathrm{op}}$.

1.1.8 Show that if each category $\mathcal{C}_\lambda$ is a module category, then the product $\prod_\Lambda \mathcal{C}_\lambda$ is also a module category.

1.1.9 **Groupoids**

A *groupoid* is a small category G in which every morphism is invertible.

Let $\{v_\lambda \mid \lambda \in \Lambda\}$ be the set of objects of G, where Λ is some set. Show that $G_\lambda = \mathrm{End}(v_\lambda, v_\lambda)$ is a group for each λ; G_λ is called the *vertex group* at λ.

Show also that if $\mathrm{Mor}(v_\lambda, v_\mu) \neq \emptyset$, then the groups G_λ and G_μ are isomorphic.

(Any set $\{G_\lambda\}$ of groups can be viewed as defining a groupoid in which $\mathrm{Mor}(v_\lambda, v_\mu) = \emptyset$ if $\lambda \neq \mu$.)

1.2 FUNCTORS

A major impact of category theory on mathematics is its emphasis on describing the morphisms between mathematical objects, rather than considering the objects in isolation. This philosophy can be applied to the study of categories themselves, leading to the definition of a functor, that is, a morphism between categories.

The language of functors plays an indispensable role in the remainder of this text, and in K-theory in general.

Up to this point, our distinction between right and left categories has not had much influence on the text, since we have been able to develop the theory

$\mathcal{B}G$: this has one object, G itself, and $\mathrm{Mor}(G,G) = \{g \mid g \in G\}$. Composition is multiplication.

$\mathcal{E}G$: here, $\mathrm{Ob}\,\mathcal{E}G = \{g \mid g \in G\}$ is the set of elements of G, and $\mathrm{Mor}(g,h) = \{hg^{-1}\}$ has one element.

What are the subcategories of $\mathcal{B}G$ and of $\mathcal{E}G$?

Investigate the various morphism categories associated with $\mathcal{B}G$ and $\mathcal{E}G$.

For example, observe that the class $\mathrm{Mor}\,\mathcal{C}$ of all morphisms of a category $\mathcal{C}$ forms a group precisely when $\mathcal{C}$ may be identified with the category $\mathcal{B}(\mathrm{Mor}\ \mathcal{C})$.

1.1.4 **A monoid as a category**

A *monoid* (or *semigroup with identity*) is a set M with an associative multiplication for which there is an identity element 1 (see [BK: IRM] (1.1.1)). Repeat, as far as possible, the previous exercise, substituting a monoid M in place of a group. Since an element of M need not be invertible, we must take the morphisms in $\mathcal{E}M$ to be

$$\mathrm{Mor}(m,n) = \{x \in M \mid xm = n\},$$

so that $id_m = 1$ always.

Show that $m \cong 1$ in $\mathcal{E}M$ if and only if m has a (twosided) inverse in M.

1.1.5 **Opposites of groups**

Groups G and H are said to be *anti-isomorphic* if there is a bijective map $\phi : G \to H$ with $\phi(gh) = \phi(h)\phi(g)$ for all g, h in G.

Show that $\iota : g \to g^{-1}$ is an anti-isomorphism of G, and that ι is an isomorphism of G if and only if G is abelian.

Define the *opposite* of a group to be a formal set

$$G^{\mathrm{op}} = \{g^{\mathrm{op}} \mid g \in G\}$$

in bijective correspondence with G, with multiplication given by

$$g^{\mathrm{op}} h^{\mathrm{op}} = (hg)^{\mathrm{op}}.$$

Verify that G^{op} is a group and that

$$\tau : G \longrightarrow G^{\mathrm{op}}, \quad g \mapsto g^{\mathrm{op}},$$

is an anti-isomorphism.

Show also that any anti-isomorphism $\phi : G \to H$ factors as a product $\phi = \overline{\phi}\tau$ where $\overline{\phi}$ is a group isomorphism.

Deduce that $\mathrm{Mor}((\mathcal{B}G)^{\odot})$ and $\mathrm{Mor}((\mathcal{B}G)^{\mathrm{op}})$ are both isomorphic to G^{op}.

commutes; that is, $\chi''\gamma = \delta\chi$.

Clearly, $\mathcal{M}or_R\, \mathcal{C}$ is a right category, which can be regarded as a subcategory of $\mathcal{C} \times \mathcal{C}$ by ignoring the morphism χ. If $\mathcal{C}$ is a module category, say a subcategory of $\mathcal{M}od_R$, then $\mathcal{M}or_R\, \mathcal{C}$ is also a module category, being a subcategory of $\mathcal{M}od_{R\times R}$.

It is also clear that $\mathcal{M}or_R\, \mathcal{C}$ is small if $\mathcal{C}$ is small.

Further morphism categories arise as full subcategories of $\mathcal{M}or_R\, \mathcal{C}$ which are defined by placing restrictions on the morphism appearing in (C, D, χ). Here are a few examples.

$\mathcal{E}nd\, \mathcal{C}$, the *endomorphism* category. The objects are morphisms (C, C, χ), that is, $\chi : C \to C$ must be an *endomorphism* of C.

$\mathcal{I}so\, \mathcal{C}$, the *isomorphism* category, which has objects (C, D, χ) where χ is an *isomorphism* – in an arbitrary category, this means that there is a morphism $\chi^{-1} : D \to C$ with $\chi^{-1}\chi = id_C$ and $\chi\chi^{-1} = id_D$, or, in other words, χ is *invertible*. We may then write $C \cong D$ to indicate that C and D are isomorphic.

$\mathcal{A}ut\, \mathcal{C}$, the *automorphism* category. Here, objects have the form (C, C, χ) with χ invertible.

Exercises

1.1.1 **Discrete categories**

A category $\mathcal{C}$ is *discrete* if

$$\mathrm{Mor}(C, D) = \begin{cases} \{id_C\} & \text{if } C = D, \\ \emptyset & \text{if } C \neq D. \end{cases}$$

Show that

(a) any set X can be regarded as a discrete category,

(b) if Y is any subclass of the objects of a given category $\mathcal{D}$, then Y defines a discrete subcategory of $\mathcal{D}$.

1.1.2 Let λ and μ be elements of a partially ordered set Λ. Show that the following are equivalent.

(a) $\lambda \cong \mu$ when we regard Λ as a category.

(b) Both $\lambda \leq \mu$ and $\mu \leq \lambda$.

1.1.3 **A group as a category**

Given a group G, we can define two categories, called $\mathcal{B}G$ and $\mathcal{E}G$, as follows.

in [BK: IRM] (2.6.6)ff.) Thus the categories $\mathcal{M}\mathit{od}_{R\times S}$ and $\mathcal{M}\mathit{od}_R \times \mathcal{M}\mathit{od}_S$ are in effect the same, but we cannot substantiate this claim formally until we have the language of equivalences at our disposal – see (1.3.16) and (1.3.17).

1.1.13 *Infinite products*

The generalization of the definition of a direct product to an arbitrary set of categories $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$, where Λ is any ordered index set, is evident. An object of the product category $\prod_\Lambda \mathcal{C}_\lambda$ is a sequence (C_λ), where each C_λ is an object of the corresponding category $\mathcal{C}_\lambda$, and a morphism $(\gamma_\lambda) : (C_\lambda) \to (C'_\lambda)$ is given by a sequence of morphisms $\gamma_\lambda : C_\lambda \to C'_\lambda$, $\lambda \in \Lambda$.

If each $\mathcal{C}_\lambda$ is a right or left category, then the product of morphisms in $\prod_\Lambda \mathcal{C}_\lambda$ is defined so that $\prod_\Lambda \mathcal{C}_\lambda$ is also right or left accordingly; again, if the chiralities are mixed, we make the product a right category.

In the case that the index set is finite, we usually take it to be $\{1, \ldots, n\}$ for some natural number n, and write the product as $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$. When the component categories $\mathcal{C}_\lambda$ are all the same category, $\mathcal{C}$ say, we use the natural notations $\mathcal{C}^n$ and $\mathcal{C}^\Lambda$.

Products of more than two categories arise naturally when we consider direct products of rings. If $R = R_1 \times \cdots \times R_n$ is a finite direct product of rings and $\mathcal{C}_i$ is a category of R_i-modules for each i, an obvious generalization of (1.1.12) shows that $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ is a category of right R-modules.

1.1.14 *Morphism categories*

We can form new categories by regarding some of the morphisms of a given category as themselves objects of a category. Such categories are called *morphism categories*. We give the constructions only for right categories, the left-handed versions being obvious.

The first of this type is the category $\mathcal{M}\mathit{or}\,\mathcal{C}$, the *morphism category* of the category $\mathcal{C}$.

The objects of $\mathcal{M}\mathit{or}\,\mathcal{C}$ are triples (C, D, χ) where χ is a morphism from C to D in $\mathcal{C}$, and a morphism from (C, D, χ) to (C'', D'', χ'') is a morphism $(\gamma, \delta) : (C, D) \to (C'', D'')$ in the product category $\mathcal{C}\times\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & C'' \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi''} \\ D & \xrightarrow{\delta} & D'' \end{array}$$

in $\mathcal{C} \times \mathcal{D}$ is a pair of morphisms

$$\gamma : C \longrightarrow C'' \text{ and } \delta : D \longrightarrow D'',$$

belonging to $\mathcal{C}$ and $\mathcal{D}$ respectively.

Composition is given by the rule

$$(\gamma, \delta)(\gamma', \delta') = (\gamma\gamma', \delta\delta'),$$

where $(\gamma', \delta') : (C', D') \to (C, D)$.

It is easily verified that $\mathcal{C} \times \mathcal{D}$ is a right category, the identity map on a pair (C, D) being the morphism (id_C, id_D). It is clear that the product category is small if both $\mathcal{C}$ and $\mathcal{D}$ are small.

If instead $\mathcal{C}$ and $\mathcal{D}$ are both left categories, we make the obvious change in the definition of composition so that $\mathcal{C} \times \mathcal{D}$ is also a left category. Should we need to consider products of categories of mixed chiralities, we fix the notation (by using mirrors) so that the product is always a right category. The reader is invited to write out the various forms for composition. An illustration is encountered in (1.3.6).

We next show that if $\mathcal{C}$ and $\mathcal{D}$ are both module categories, so is their product. Given rings R and S, their *direct product* is

$$R \times S = \{(r, s) \mid r \in R, s \in S\},$$

with addition and multiplication defined componentwise:

$$(r, s) + (r', s') = (r + r', s + s'), \quad (r, s) \cdot (r, s) = (rr', ss').$$

We state the result only for right categories; the reader may supply the variations needed for left categories and for mixed chiralities. (Hint: 'it's all done by mirrors'.)

1.1.12 Lemma

Let $\mathcal{C}$ be a subcategory of $\mathcal{M}_{OD_R}$ and let $\mathcal{D}$ be a subcategory of $\mathcal{M}_{OD_S}$. Then $\mathcal{C} \times \mathcal{D}$ is a subcategory of $\mathcal{M}_{OD_{R \times S}}$.

Proof

An object (C, D) in the product category can be made into an $R \times S$-module by the rule $(c, d) \cdot (r, s) = (cr, ds)$, and a morphism in $\mathcal{C} \times \mathcal{D}$ will then be an $R \times S$-module homomorphism. □

Remark. It is not difficult to show that an $R \times S$-module M must be (isomorphic to) a direct sum $L \oplus N$, in which L is an R-module and N is an S-module. Furthermore, a homomorphism of $R \times S$-modules $L \oplus N \to L' \oplus N'$ must arise from a pair of homomorphisms $L \to L'$ and $N \to N'$. (Details are spelt out

In K-theory there are technical reasons for working with small categories, or at least, categories which are equivalent to small categories in a sense to be made precise in (1.3.15). The purpose of the following remarks is to show that the categories of greatest concern to us can indeed be taken to be small.

To illustrate this point, consider $\mathcal{M}_R$, the category of finitely generated R-modules. From one point of view, the objects in $\mathcal{M}_R$ must form a class, rather than a set, for the trivial reason that any set can be regarded artificially as the single element of a zero module. However, we can impose a very natural restriction on the objects which are permitted in $\mathcal{M}_R$ to ensure that it is small, as follows.

An object M in $\mathcal{M}_R$ is the homomorphic image of some free module R^n of finite rank ([BK: IRM] Lemma 2.5.7), and so $M \cong R^n/\operatorname{Ker}\alpha$ for a suitable surjective homomorphism α, by the Induced Mapping Theorem ([BK: IRM] (1.2.11)). But, by construction ([BK: IRM] (1.2.10)), the quotient module $R^n/\operatorname{Ker}\alpha$ is a set of subsets of R^n. We can therefore regard each object of $\mathcal{M}_R$ as being a set of subsets of the countably infinite cartesian product $\prod_\omega R$. Since a fundamental axiom of set theory assures us that the class of subsets of a set is again a set, we see that $\mathcal{M}_R$ is small.

An alternative way of interpreting this argument is that it shows that $\mathcal{M}_R$ has a set of isomorphism classes of objects, and therefore that $\mathcal{M}_R$ is equivalent to a small category. In applications, it makes no difference whether we take $\mathcal{M}_R$ in the 'large', or replace it by some convenient, equivalent, small category, which we give the same name.

The category $\mathcal{M}_R$ is typical of the type of category that we deal with. The objects (and morphisms) of these categories are constructed from a given set of objects by operations which formally involve taking sets of subsets of cartesian products of the form $\prod_{\lambda\in\Lambda} X_\lambda$, where each X_λ is a set and Λ is some index set. Such a method of construction must generate a set of objects.

More detailed accounts of the bearing of set theory on category theory can be found in [Cameron 1999], [Herrlich & Strecker 1979], [Mac Lane 1971], §1.6, and [Schubert 1972], §3.

We now show how to construct some new categories from given categories.

1.1.11 Product categories

Given a pair of (right) categories $\mathcal{C}$ and $\mathcal{D}$, we form the *product* category $\mathcal{C}\times\mathcal{D}$ as follows. The objects of $\mathcal{C}\times\mathcal{D}$ are pairs (C, D), where C is an object of $\mathcal{C}$ and D is an object of $\mathcal{D}$, and a morphism

$$(\gamma,\delta):(C,D)\longrightarrow(C'',D'')$$

by computing $\mathrm{Mor}(0, S)$ in the two cases: if S is a ring other than the zero ring, then $\mathrm{Mor}(0, S) = \{0\}$ in the first case but $\mathrm{Mor}(0, S) = \emptyset$ in the second.

1.1.9 Full subcategories

A *full* subcategory $\mathcal{D}$ of $\mathcal{C}$ is one in which

$$\mathrm{Mor}_{\mathcal{D}}(C, D) = \mathrm{Mor}_{\mathcal{C}}(C, D)$$

for any two objects of $\mathcal{D}$.

It is obvious that any subclass $\mathcal{D}$ of the class of objects of $\mathcal{C}$ gives rise to a unique full subcategory of $\mathcal{C}$.

This method of constructing subcategories of $\mathcal{M}\mathit{od}_R$, for various rings R, will be very useful to us in the sequel. For this reason, we define a *module category* to be any category which is a full subcategory of $\mathcal{M}\mathit{od}_R$ (or ${}_R\mathcal{M}\mathit{od}$) for some ring R.

Here are some important examples.

$\mathcal{F}\mathit{ree}_R$, the category of all free right R-modules.
$\mathcal{P}\mathit{roj}_R$, the category of all projective right R-modules.
$\mathcal{M}_R$, the category of finitely generated right R-modules.
$\mathcal{F}_R$, the category of free right modules R^k of finite rank.
$\mathcal{P}_R$, the category of finitely generated projective right R-modules.

The corresponding subcategories of ${}_R\mathcal{M}\mathit{od}$ are denoted ${}_R\mathcal{F}\mathit{ree}$, ${}_R\mathcal{P}\mathit{roj}$, ${}_R\mathcal{M}$, ${}_R\mathcal{F}$ and ${}_R\mathcal{P}$ respectively.

1.1.10 Some remarks on set theory and small categories

In our definition of a category, we use the expression *'class'*, rather than the word *'set'* which the reader might have expected. The reason for this is that we wish to make a naïve distinction between sets, on which all mathematical constructions are allowed, and 'non-sets', which are too large to permit some operations. The aim is to avoid Russell's Paradox and similar traps: if X is a set of sets, it is sensible to ask whether or not the set X is itself in X. The paradox arises by taking X to be the set of all sets that do not contain themselves. Our avoiding action is to declare that this particular choice of X is a class but not a set; essentially, X is too large to be considered to be a set. The kind of set theory that we have in mind is expanded in detail in [Herrlich & Strecker 1979], [van Dalen, Doets & de Swart 1979], and [Levy 1979].

If the class $\mathrm{Ob}\mathcal{C}$ of objects of the category $\mathcal{C}$ is in fact a set, then the category is said to be *small*.

On the other hand, a right R-module I is injective if instead we have the following.

Inj. Given any injective R-module homomorphism $\mu : I \to M$, there is a splitting homomorphism $\rho : M \to I$, that is, $\rho\mu = id_I$.

(More details are given in §2.5 of [BK: IRM].)

It is reasonable to assume that if we can generalize surjective and injective homomorphisms of modules to abstract categories, then a surjective map in $\mathcal{C}$ will correspond to an injective map in the opposite category $\mathcal{C}^{\mathrm{op}}$, and so a projective object in $\mathcal{C}$ will correspond to an injective object in the opposite category. (This assumption is justified in the next chapter – see (2.2.2).) Thus, provided that we confine ourselves to statements that involve only objects and morphisms, we need only prove results about projective objects, since the corresponding results for injective objects are then true by duality.

This technique for pairing together definitions, arguments, etc. in a category and its opposite is the *principle of duality*, and the phrase *by duality* is used to indicate this method of argument.

1.1.8 Subcategories

A *subcategory* $\mathcal{D}$ of a (right) category $\mathcal{C}$ is defined by the following data.

Sub 1. A subclass $\mathrm{Ob}\,\mathcal{D}$ of $\mathrm{Ob}\,\mathcal{C}$, which specifies the objects that belong to $\mathcal{D}$.

Sub 2. For each pair of objects C, D of $\mathcal{D}$, a subset $\mathrm{Mor}_{\mathcal{D}}(C, D)$ of $\mathrm{Mor}_{\mathcal{C}}(C, D)$; these are the morphisms from C to D in $\mathcal{D}$.

Sub 3. If C is an object of $\mathcal{D}$, then the identity morphism id_C in $\mathrm{Mor}_{\mathcal{C}}(C, C)$ shall belong to $\mathrm{Mor}_{\mathcal{D}}(C, C)$ (where it is again the identity morphism).

Sub 4. If $\alpha \in \mathrm{Mor}_{\mathcal{D}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{D}}(D, E)$, then

$$\beta\alpha \in \mathrm{Mor}_{\mathcal{D}}(C, E),$$

(where the product is the product in $\mathcal{C}$).

It is clear that a subcategory of a right category is itself a right category with identities and products inherited from $\mathcal{C}$.

The corresponding definitions for a left category are obvious.

Examples of subcategories are $\mathcal{A}_B$ in $\mathcal{G}_P$ and $\mathcal{R}_{ING}$ in $\mathcal{R}_{NG}$. Note that any group homomorphism from G to H is also a morphism in $\mathcal{A}_B$ if the groups G and H happen to be abelian. In contrast, a morphism in $\mathcal{R}_{NG}$ from a ring R to a ring S need not be allowable as a morphism in $\mathcal{R}_{ING}$. This can be seen

We remind the reader that if a category has no obvious chirality, for example, $\mathcal{G}_{\mathcal{P}}$ or $\mathcal{R}_{ING}$, we always take it to be a right category. However, many group theorists prefer to view $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{B}}$ as left categories.

Remark. Some authorities in category theory do not regard a category and its mirror as distinct categories, but simply as alternative notations for composition of morphisms in the one category. From this point of view, a left R-module and the corresponding right R°-module are the same object, in differing notations. However, a ring and its opposite are definitely different rings, as can be seen from the use of opposites in Brauer theory ([Cohn 1979], §10.3). We therefore feel that the distinction between a ring and its opposite should be extended to that between a category and its mirror.

1.1.6 The opposite category

We now give the definition of the *opposite* category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$. This is distinct from the notion of the mirror category that we introduced above.

Given a right category $\mathcal{C}$, the objects C^{op} of the right category $\mathcal{C}^{\mathrm{op}}$ are in bijective correspondence with the objects C of $\mathcal{C}$, and for each pair of objects C^{op}, D^{op} of $\mathcal{C}^{\mathrm{op}}$, there is a bijection between the morphisms α in $\mathrm{Mor}_{\mathcal{C}}(C, D)$ and the morphisms α^{op} in $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(D^{\mathrm{op}}, C^{\mathrm{op}})$; thus $\alpha : C \to D$ corresponds to $\alpha^{\mathrm{op}} : D^{\mathrm{op}} \to C^{\mathrm{op}}$. If $\beta : D \to E$ is a morphism in $\mathcal{C}$, then composition in $\mathcal{C}^{\mathrm{op}}$ is given by the rule

$$(\alpha^{\mathrm{op}})(\beta^{\mathrm{op}}) = (\beta\alpha)^{\mathrm{op}}.$$

(The definition for left categories is left to the reader, who, we are confident, will get it right ...)

Note that $\mathcal{C}^{\mathrm{op}}$ has the same chirality as $\mathcal{C}$.

1.1.7 The principle of duality

If we can make a definition or state a result by using only the objects and morphisms of a category $\mathcal{C}$, then we obtain a *dual* definition or result in the opposite category $\mathcal{C}^{\mathrm{op}}$ in which objects $C, D, \ldots$ of $\mathcal{C}$ are replaced by the corresponding objects $C^{\mathrm{op}}, D^{\mathrm{op}}, \ldots$ of $\mathcal{C}^{\mathrm{op}}$ and similarly morphisms $\alpha, \beta, \ldots$ are replaced by $\alpha^{\mathrm{op}}, \beta^{\mathrm{op}}, \ldots$. An example is provided by the relationship between the definitions of projective and injective modules.

A projective module may be defined as a right R-module P for which the following holds.

Pro. Given any surjective R-module homomorphism $\pi : M \to P$, there is a splitting homomorphism $\sigma : P \to M$, that is, $\pi\sigma = id_P$.

The product in $\mathcal{C}^{\odot}$ is defined by putting

$$\beta^{\odot}\alpha^{\odot} = (\alpha\beta)^{\odot},$$

where $\beta : D \to E$ is a morphism in $\mathcal{C}$ that is composable with α.

Observe that $\mathcal{C}$ and $\mathcal{C}^{\odot}$ are genuinely distinct categories, rather than the same category with two notational conventions, since the order of a pair of morphisms in their product is specified by the axioms.

It is clear that given a right category $\mathcal{D}$, we can construct a left category $\mathcal{D}^{\odot}$ by a similar method, and that there is a formal identification of $\mathcal{C}^{\odot\odot}$ with $\mathcal{C}$ for any category $\mathcal{C}$.

In [BK: IRM] (1.2.6), we introduced a method of turning a left R-module M into a right R°-module M°, where R° is the opposite ring to R. A typical element of R° has the form r° for r in R, and addition and multiplication in the opposite ring are given by

$$r^{\circ} + s^{\circ} = (r+s)^{\circ} \quad \text{and} \quad r^{\circ}s^{\circ} = (sr)^{\circ}.$$

The elements m° of M° are in bijective correspondence with those of M, and the addition and scalar multiplication for M° are defined by

$$m^{\circ} + n^{\circ} = (m+n)^{\circ} \quad \text{and} \quad m^{\circ}r^{\circ} = (rm)^{\circ}.$$

If $\alpha : M \to N$ is a homomorphism of left R-modules, we define a homomorphism α° of right R°-modules by

$$\alpha^{\circ}m^{\circ} = (m\alpha)^{\circ}.$$

Thus we have an identification

$$({}_R\mathcal{M}\mathit{od})^{\odot} = \mathcal{M}\mathit{od}_{(R^{\circ})},$$

which shows that the use of the mirror category is a generalization of the use of the opposite ring. This identity was used implicitly in Exercise 1.2.14 of [BK: IRM], where we explored the consequences of this technique for switching between left R-modules and right R°-modules. Some authors use this method to avoid dealing both with left and with right modules.

Suppose now that the ring R is commutative. Then $R = R^{\circ}$, so that $({}_R\mathcal{M}\mathit{od})^{\odot} = \mathcal{M}\mathit{od}_R$. Again, this identity is used implicitly to write scalars and operators on the same side in many elementary textbooks that are concerned only with commutative rings.

We can now elucidate the comment in (1.1.2) about the category $\mathcal{A}\mathit{b}$ of abelian groups. For, we regard $\mathcal{A}\mathit{b}$ as the right category $\mathcal{M}\mathit{od}_{\mathbb{Z}}$, which is the mirror of the category ${}_{\mathbb{Z}}\mathcal{M}\mathit{od}$ of left $\mathbb{Z}$-modules.

right. Thus, given a homomorphism of left R-modules $\alpha : {}_RM \to {}_RN$ and an element m in M, α sends m to $m\alpha$ rather than αm. If $\beta : {}_RN \to {}_RP$ is also a homomorphism, then the product $\alpha\beta$ is given by the rule

$$m(\alpha\beta) = (m\alpha)\beta.$$

With this convention, we obtain a modified list of axioms that defines a *left* category, as follows.

Cat 4^ℓ. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$, then there is a morphism

$$\alpha\beta \in \mathrm{Mor}_{\mathcal{C}}(C, E).$$

Cat 5^ℓ. For any morphism $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,

$$(id_C)\alpha = \alpha = \alpha(id_D).$$

Cat 6^ℓ. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$ and $\gamma \in \mathrm{Mor}_{\mathcal{C}}(E, F)$, then

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

The prime example of a left category is ${}_R\mathcal{M}\mathrm{o}_D$, the category of all left R-modules. Also, most group theorists in effect treat $\mathcal{G}_P$ as a left category.

As we noted in (1.1.2), a partially ordered set Λ can be viewed as a category. It is convenient to regard Λ as a left category, since the left-category composition reads

$$\iota^{\lambda\mu}\iota^{\mu\nu} = \iota^{\lambda\nu},$$

where $\iota^{\lambda\mu}$ is the unique morphism from λ to μ (when $\lambda \leq \mu$).

We introduce the term *chirality* to distinguish the two kinds of category, a left or right category having correspondingly left or right chirality.

1.1.5 The mirror

There is a purely formal method that allows us to switch between categories of opposite chiralities. Given a left category $\mathcal{C}$, we manufacture a right category $\mathcal{C}^\odot$, the *mirror* of $\mathcal{C}$, as follows. The objects of $\mathcal{C}^\odot$ are symbols $C^\odot$, corresponding bijectively to the objects C of $\mathcal{C}$, and the morphisms from $C^\odot$ to $D^\odot$ in $\mathcal{C}^\odot$ are symbols $\alpha^\odot$ in bijective correspondence with the morphisms α from C to D in $\mathcal{C}$. Thus for each morphism

$$\alpha : C \longrightarrow D \text{ in } \mathcal{C},$$

there is exactly one morphism

$$\alpha^\odot : C^\odot \longrightarrow D^\odot \text{ in } \mathcal{C}^\odot.$$

1.1.3 The axioms

Now that we have some examples in mind, we list the axioms that the morphisms of an abstract category $\mathcal{C}$ are required to satisfy. These correspond to the properties of the homomorphisms in the category $\mathcal{M}_{OD_R}$, provided that we write homomorphisms on the left and that we use the consequent natural convention for composing homomorphisms. Thus, if α is a right R-module homomorphism from M_R to N_R and m is in M, then αm denotes the image of m in N, and if β is a homomorphism from N_R to P_R, then

$$(\beta\alpha)m = \beta(\alpha m).$$

Cat 3. For each object C of $\mathcal{C}$, there is a distinguished morphism

$$id_C \in \mathrm{Mor}_{\mathcal{C}}(C, C),$$

called the *identity* morphism of C.

Cat 4. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$, then there is a morphism

$$\beta\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, E),$$

called the *product* or *composite* of α and β; α and β are said to be *composable*.

Cat 5. For any morphism $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,

$$(id_D)\alpha = \alpha = \alpha(id_C).$$

Cat 6. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$ and $\gamma \in \mathrm{Mor}_{\mathcal{C}}(E, F)$, then

$$\gamma(\beta\alpha) = (\gamma\beta)\alpha.$$

With regard to Cat 4, note that the product of two morphisms α, β is defined precisely when the codomain of α is the same as the domain of β. Observe also that the manner of writing the product (as $\beta\alpha$ rather than $\alpha\beta$) forms a part of this axiom.

Statements such as Cat 6 are sometimes formulated as 'the product is associative when defined'.

1.1.4 Chirality

It seems to us that a category satisfying the axioms above should, strictly speaking, be called a *right* category, since the form of the axioms mimics the natural form of composition of homomorphisms of *right* R-modules when these homomorphisms are written on the left, as is our convention.

When we consider left modules, we prefer to put homomorphisms on the

the fact that any right $\mathbb{Z}$-module can be regarded equally as a left $\mathbb{Z}$-module, with $am = ma$ for any integer a and element m of M, it is not quite true that $\mathcal{A}_B$ is the same as the category ${}_{\mathbb{Z}}\mathcal{M}_{OD}$ of left $\mathbb{Z}$-modules. The reason for the distinction will be discussed further in (1.1.5).

$\mathcal{T}_{OP}$ This important non-algebraic example of a category has topological spaces as objects and continuous maps as morphisms.

Λ It is sometimes convenient to regard a partially ordered set as a category. A *partially ordered* set is a set Λ together with an order relation $\leq$ on Λ which satisfies the following requirements.

PO 1. **Reflexivity:**
$\lambda \leq \lambda$ for each $\lambda \in \Lambda$.

PO 2. **Transitivity:**
if $\lambda \leq \mu$ and $\mu \leq \nu$ for λ, μ and ν in Λ, then $\lambda \leq \nu$ also.

A partially ordered set is said to be *proper* if the following axiom also holds.

PO 3. If $\lambda \leq \mu$ and $\mu \leq \lambda$ for λ and μ in Λ, then $\lambda = \mu$.

Here are two examples that will be generalized in (5.1).

(a) Let Λ be the set of nonzero ideals of $\mathbb{Z}$, ordered by reverse inclusion:

$$I \leq J \iff J \subseteq I.$$

Then Λ is a proper partially ordered set.

(b) Take Σ to be the set of nonzero elements of $\mathbb{Z}$, ordered by division:

$$a \leq b \iff ax = b \text{ for some } x \in \mathbb{Z}.$$

Then Σ is partially ordered, but not proper.

Given a partially ordered set Λ, we can view Λ as a category whose objects are the elements $\lambda, \mu, \ldots$ of Λ. If $\lambda \leq \mu$, then $\mathrm{Mor}(\lambda, \mu)$ contains a single element $\iota^{\lambda\mu}$, and $\mathrm{Mor}(\lambda, \mu)$ is empty otherwise.

$\mathcal{O}_{RD}$ The class of all ordered sets can be considered to be a category. We say that a set Λ is *totally ordered*, or simply *ordered*, if it is a proper partially ordered set in which any two members are comparable, that is, the following axiom holds.

TO. If λ and μ are in Λ, then either $\lambda \leq \mu$ or $\mu \leq \lambda$.

Then the objects of $\mathcal{O}_{RD}$ are the ordered sets Λ and a morphism $\alpha : \Lambda \to \Sigma$ of ordered sets is a mapping that preserves the order: if $\lambda \leq \mu$ in Λ, then $\alpha\lambda \leq \alpha\mu$ in Σ.

is often used to indicate that α is a morphism from C to D.

These objects and morphisms obey some axioms, but before listing these we note two deeply entrenched conventions and give some examples.

The first convention is to write $C \in \mathcal{C}$ rather than $C \in \mathrm{Ob}\,\mathcal{C}$ when C is an object of $\mathcal{C}$. This reflects the fact that a category is usually considered to be defined in terms of the class $\mathrm{Ob}\,\mathcal{C}$. For this reason, a category is often named after its objects.

Second, the term 'morphism' is used mostly when we are considering categories in the abstract; in concrete situations, the morphisms are often the homomorphisms between some familiar objects, such as modules, groups or rings. In such a case, one writes $\mathrm{Hom}(C, D)$ rather than $\mathrm{Mor}_{\mathcal{C}}(C, D)$.

1.1.2 *Some examples*

$\mathcal{S}ET$ The class $\mathrm{Ob}\,\mathcal{S}ET$ of objects is the class of all sets and the set of morphisms from X to Y is the set of all mappings from X to Y. (Some authors denote this category $\mathcal{E}NS$).

We write $\mathrm{Map}(X, Y)$ rather than $\mathrm{Mor}(X, Y)$.

$\mathcal{G}P$ The objects are the groups, with group homomorphisms as morphisms.

$\mathcal{R}ING$ The objects are the rings (which we require to have an identity element), the morphisms from R to S being the ring homomorphisms, which, we insist, send the identity of R to the identity of S.

$\mathcal{R}NG$ The objects are now the nonunital rings, which are rings except that they need not possess an identity element. A morphism from R to S is a homomorphism of nonunital rings, which is the same as a ring homomorphism except that there can be no requirement that the identity of R is sent to the identity of S, even when R and S are actually rings.

$\mathcal{M}od_R$ Given a ring R, we form the category of all right R-modules. Thus $\mathrm{Ob}(\mathcal{M}od_R)$ is the class of all right R-modules and the set of morphisms from M to N is the set of all R-module homomorphisms from M to N, which we write $\mathrm{Hom}(M, N)$, $\mathrm{Hom}_R(M, N)$ or $\mathrm{Hom}(M_R, N_R)$ according to context.

${}_R\mathcal{M}od$ Similarly, we form the category of left R-modules and homomorphisms of left R-modules.

$\mathcal{A}B$ The category of abelian groups – the objects are abelian groups, the morphisms are group homomorphisms.

Since an abelian group has a uniquely defined structure as a right $\mathbb{Z}$-module ([BK: IRM] (1.2.2)), $\mathcal{A}B$ is $\mathcal{M}od_{\mathbb{Z}}$ with another name. Despite

natural way, by 'associativity' – more formal definitions are given below in (1.1.3) and (1.1.4). The consequence is that we obtain two versions of the axiom for the composition of morphisms in an abstract category. Thus we arrive at two kinds of abstract category, a 'right' category and a 'left' category, which are modeled on the corresponding categories of modules. We call this phenomenon *chirality*. It is worth remarking that the distinction between left and right modules was made at a very early stage in the modern development of module theory [Noether & Schmeidler 1920].

The distinction between right and left categories does not seem to have been made explicit before now, and it could be avoided by using a technique given in (1.1.5). However, it seems to us to be more natural to allow both notations for categories, rather than suppressing one notation artificially.

We also discuss some points from set theory which arise from future applications in K-theory, where one needs to be able to work with 'small' categories, that is, categories whose objects can all be taken to be members of some set, which may itself be very big. In general, the objects of a category need not be contained in a set. We are particularly indebted to Wilfrid Hodges for his helpful comments on these questions.

1.1.1 The definition

Informally, a category consists of a collection of mathematical entities, such as the right R-modules over a given ring R, which can be recognised as sharing a common structure, together with a collection of mappings between these entities that respect this structure; for R-modules, we would expect these to be the R-module homomorphisms.

The entities which share the common structure are known as the 'objects' of the category, while the structure-preserving maps are the 'morphisms' of the category. Thus, to define a *category* $\mathcal{C}$ in general, we must specify the following data.

Cat 1. A class $\mathrm{Ob}\,\mathcal{C}$. Members of $\mathrm{Ob}\,\mathcal{C}$ are called the *objects* of $\mathcal{C}$.

Cat 2. For each ordered pair C, D of objects of $\mathcal{C}$, there is a set $\mathrm{Mor}_{\mathcal{C}}(C, D)$; the elements of $\mathrm{Mor}_{\mathcal{C}}(C, D)$ are called the *morphisms* from C to D in $\mathcal{C}$.

It may happen that $\mathrm{Mor}_{\mathcal{C}}(C, D)$ is the empty set.

Given $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, the object C is called the *domain* of α and D the *codomain*. An arrow

$$\alpha : C \longrightarrow D$$

1
CATEGORIES

Category theory provides the language for the discussions in this text and it is also an inescapable foundation for any treatment of K-theory. In this chapter, we set out the fundamental ideas of category theory, and we give a first application of the power of categorical methods. These ideas are introduced in three stages, each of which occupies one section. To start, we define categories themselves. Next, we consider functors, which are the tools for moving between different categories, and then natural transformations, which allow us to compare different functors. In the final, fourth, section of the chapter, we apply these basic notions to investigate universal constructions and universal objects. The idea of 'universality' reveals the common properties of apparently diverse objects, such as free modules and free groups, and it provides the framework for many of the definitions and constructions that we make later in this text.

1.1 FUNDAMENTAL PROPERTIES OF CATEGORIES

We commence our discussion of the theory of categories with the axiomatic definition of a category, together with a selection of techniques for the manufacture of new categories from old.

Because our main interest in this text is the application of category theory to module theory, many of our illustrations are obtained by considering various types of module. We shall also see some categories based on other algebraic entities such as sets and groups.

The principal innovation in our discussion stems from our need to use both right modules and left modules. Our view is that it is best to write operators opposite scalars as far as is practicable, so that a homomorphism between right modules is to be a left operator, while a homomorphism of left modules is to be a right operator. We also wish to compose homomorphisms in the

use the exercises to provide examples and counterexamples that indicate the limitations of our results.

A number of historical remarks are distributed within the text. Readers in search of further details may wish to consult [Bourbaki 1994], [Dieudonné 1989], [Mac Lane 1975] and [Weibel].

We again thank our departments and our families, for their patience and encouragement during the lengthy gestation period of this text and its companion [BK: IRM]. David Tranah, Roger Astley and the TeXline at the Cambridge University Press have provided us with helpful advice.

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some specified exact sequences, which is different to the usage of the term by people who are primarily concerned with category theory. To complicate the issue a bit more, there are two types of exact category in K-theory. The first type has sufficient structure for the purposes of 'lower' K-theory, while the second type has extra structure that is needed to support 'higher' K-theory. We therefore coin the terms 'G-exact category' and 'Q-exact category' for the two types of exact category used in K-theory.

- We give direct proofs of some results that are normally obtained by using the methods of homological algebra, which appears in this text only incidentally. These results mainly concern flat modules (particularly section 3.2), and the local-global properties of modules over orders (section 7.3).
- Recently, Quillen has developed a module theory for nonunital rings. We incorporate some of his results, mostly as exercises.
- The treatments of localization of categories (section 6.3), projective module lifting (section 7.2) and local-global methods (section 7.3) use the minimum of apparatus, so as to make them more accessible to a beginning graduate student.

This text requires the reader to have prior knowledge of some basic results in ring theory, module theory and number theory, since these provide the examples that motivate and illustrate our discussions. These prerequisites are generally met in the first year of graduate studies, although it is unlikely that any one graduate course includes all of them. We have therefore collected them together in a companion volume, [BK: IRM]. Much of the ring theory and module theory can also be found in the expository works [Cohn 1979], [Cohn 1982], [Jacobson 1985] and [Jacobson 1980], with some extra topics that can be met in [Cohn 1985], [McConnell & Robson 1987] or [Rowen 1988]. Sources for the number theory are [Fröhlich & Taylor 1991] and [Marcus 1977]. However, for brevity, we usually give references only to [BK: IRM].

We try to keep to a fairly leisurely pace throughout this text, and, when in doubt, we tend to include an explanation rather than exclude it. Some of the more frequently used prerequisites are recapitulated for convenience.

The symbol □ indicates either the end of a proof, or that a proof is immediate from the preceding discussion. The symbol ○ after a statement indicates that the assertion is quoted from another source, rather than being a consequence of arguments in this text.

There are about two hundred exercises for more ambitious or enthusiastic readers. Some of them (among the exercises!) are fairly routine, others are rather deeper and explore further developments of our results or give brief introductions to topics that are not covered in the body of the text. We also

The second chapter shows how this extra structure can be axiomatized and thus imposed on abstract categories. We first analyse the properties of the groups of homomorphisms in a module category to obtain a list of axioms that define additive and abelian categories; these are abstract categories that share most of the important properties of module categories. The final topic in this chapter stems from the fact that modules can be arranged in short exact sequences, which is basic to the definitions in algebraic K-theory. We are thus led to the idea of an exact category, which is a category together with a nicely behaved collection of short exact sequences. Such categories are the domains on which K-theory operates.

The larger part of this text is more directly concerned with categories of modules, rather than categories in the abstract. In the third chapter, we introduce the tensor product and use it to change the ring of scalars of a module. This leads to the Morita theory (Chapter 4), which tells us when two rings have, in some sense, the same module theory.

In Chapters 5 and 6, we look at several related methods for constructing new rings and categories from old, by taking direct and inverse limits and by performing localizations and completions.

Finally, we exploit these techniques to obtain some local-global results, mainly in the theory of orders. These tackle the problem of gaining information about the projective modules over a ring from knowledge of the modules over its localizations and completions. Roughly speaking, the structure of a projective module is transparent locally, that is, over the completions of a ring. The game is to assemble this information to give the structure over the original ring. The full story requires K-theory for its telling.

Although the novelty of a book such as this lies mainly in its selection and presentation of material, we do make some innovations and give some new arguments, including the following.

- In our treatment of modules, we deal both with categories of left modules and with categories of right modules, and it seems expedient to have separate systems of notation for their homomorphisms, according to the rule that homomorphisms should act 'opposite scalars'. This separation leads us to two types of abstract category, namely a left category and a right category, in which the rules for composition of morphisms follow those of the corresponding module category. These two types do not seem to have been distinguished explicitly before. We introduce the term *chirality* for the 'handedness' of a category, and with each category we associate a formal *mirror* category of opposite chirality.
- In K-theory, the term 'exact category' is used for a category together with

Preface

This book is an introduction to the theory of categories, together with applications of this theory to some constructions with rings and modules. We start with a discussion of categories in general, and then concentrate on the types of category – additive, abelian – which enjoy to an increasing extent the properties of categories of modules. Our applications are the Morita theory, the localization and completion of rings and modules, and finally some 'local-global' methods, in which the properties of a module are compared to those of its localizations and completions. We also develop the tools that we need for these applications, namely, the tensor product and limits, both direct and inverse.

The selection and presentation of our material is motivated by the needs of algebraic K-theory. Indeed, this book started out as some preliminary remarks within a text on that subject. Thus the categorical foundations are those needed to set up algebraic K-theory, and our applications are chosen since they underly some fundamental results in algebraic K-theory. However, the content of this text will be, we hope, of interest to a wider readership than potential K-theorists.

Here is a more detailed survey of the material that we cover. Our first chapter sets out the basics of category theory. There are three fundamental definitions, those of a category, a functor, and a natural transformation, and we show how to use these notions to define universal objects and universal constructions. An understanding of these is important for two reasons. On the one hand, many definitions in module theory (kernels, cokernels, ...) can be reinterpreted in the language of universal objects and so extended to more general situations; on the other, many of the objects in K-theory itself arise as universal objects in one category or another.

A category of modules has a richer structure than an abstract category, since the additive structure on modules extends to their homomorphisms.

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To

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