Baby Rudin Chapter 5

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1 Differentiability

1.1 Definition

We say that $f : [a, b] \to \mathbb{R}$ is **differentiable** at $x \in [a, b]$ if

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

exists. We denote this limit by $f'(x)$.

Differentiability is, in fact, a stronger property than continuity as the following proposition shows.

Proposition 1. *If $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$, then f is continuous at x.*

Proof. Let $\epsilon > 0$ be given. If f is differentiable at x, there exists $\delta > 0$ such that

$$t, x \in [a, b], |t - x| < \delta \Longrightarrow \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

By the triangle inequality,

$$|f(t) - f(x)| \leq |f(t) - f(x) - f'(x)(t - x)| + |f'(x)(t - x)| < (\epsilon + |f'(x)|) |t - x| ,$$

which tends to 0 as $t \to x$. $\square$

Proposition 2. *Suppose $f, g : [a, b] \to \mathbb{R}$ are differentiable at $x \in [a, b]$. Then $f + g, fg$ are differentiable, and if $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable. Moreover,*

1. $(f + g)'(x) = f'(x) + g'(x)$;
2. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;
3. $\left(\frac{f}{g}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

Proof. To see (1), we have by the additivity of limits that

$$f'(x) + g'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} + \lim_{t \to x} \frac{g(t) - g(x)}{t - x} = \lim_{t \to x} \frac{(f(t) + g(t)) - (g(t) + g(x))}{t - x} = (f + g)'(x)$$

To see (2), recall that differentiability implies continuity and the limit of products is the product of limits, so that

$$\begin{aligned}(fg)'(x) = \lim_{t \to x} \frac{f(t)g(t) - f(x)g(x)}{t - x} &= \lim_{t \to x} \frac{f(t)g(t) - f(x)g(t) + f(x)g(t) - f(x)g(x)}{t - x} \\ &= \lim_{t \to x} g(t) \lim_{t \to x} \frac{f(t) - f(x)}{t - x} + f(x) \lim_{t \to x} \frac{g(t) - g(x)}{t - x} \\ &= g(x)f'(x) + g'(x)f(x)\end{aligned}$$

To see (3), we first show that if $h = g^{-1}$, then $h'(x) = -\frac{g'(x)}{g^2(x)}$. Observe that

$$h'(x) = \lim_{t \to x} \frac{1}{g(t)} - \frac{1}{g(x)} t - x = \lim_{t \to x} \frac{g(x) - g(t)}{g(t)g(x)(t - x)} = \frac{1}{g^2(x)} \lim_{t \to x} \frac{g(x) - g(t)}{t - x} = \frac{-g'(x)}{g^2(x)}$$

Applying (2) with f and h, we obtain

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g^2(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$\square$

We can use the preceding proposition to show that every polynomial is differentiable, and every rational function (i.e. quotient of polynomials) is differentiable, except where the denominator is zero. Indeed, by induction we can show that, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. It is evident that this holds for $n = 1$. Suppose the result is true for all $j < n$. Then $f(x) = x^n = g(x)h(x)$, where $g(x) = x$ and $h(x) = x^{n-1}$, so that

$$f'(x) = (gh)'(x) = g'(x)h(x) + g(x)h'(x) = x^{n-1} + (n-1)xx^{n-2} = nx^{n-1}$$

By additivity (really, linearity) of the derivative, every polynomial is differentiable and every rational function is differentiable by (3), except at the points where the denominator is zero.

We now prove a useful result, known as the chain rule, for computing the derivative of the composition of differentiable functions.

Proposition 3. *(Chain Rule) Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, $f'(x)$ exists for some $x \in [a,b]$, and $g : I \to \mathbb{R}$, where I is an interval containing $f([a,b])$, is differentiable at $f(x)$. If*

$$h : [a,b] \to \mathbb{R}, \qquad h(t) := g(f(t))$$

then h is differentiable at x and $h'(x) = g'(f(x))f'(x)$.

Proof. Since f is continuous on $[a,b]$, and g is differentiable at $y := f(x)$, we have that

$$\lim_{s \to y} \frac{g(s) - g(y)}{s - y} = \lim_{t \to x} \frac{g(f(t)) - g(y)}{f(t) - f(x)}$$

Recalling that the limit of products is the product of limits, we obtain

$$\begin{aligned}
\lim_{t \to x} \frac{g(f(t)) - g(y)}{t - x} = \lim_{t \to x} \frac{g(f(t)) - g(y)}{f(t) - f(x)} \frac{f(t) - f(x)}{t - x} &= \left(\lim_{s \to y} \frac{g(s) - g(y)}{s - y} \right) \left(\lim_{t \to x} \frac{f(t) - f(x)}{t - x} \right) \\
&= g'(f(x))f'(x)
\end{aligned}$$

□

1.2 Mean Value Theorem

Let $f : X \to \mathbb{R}$ be a real-valued function on a metric space (X, d). We say that f has a **local maximum** at $x \in X$ if there exists $\delta > 0$ such that $f(y) \leq f(x)$ for all $d(x, y) < \delta$. We say that f has a **local minimum** at x if $f(y) \geq f(x)$ for all $d(x, y) < \delta$.

Proposition 4. *Suppose $f : [a,b] \to \mathbb{R}$ has a local maximum (minimum) at $x \in (a,b)$ and $f'(x)$ exists. Then $f'(x) = 0$.*

Proof. Choose $\delta > 0$ such that $x, y \in (a,b)$ with $|x - y| < \delta$ implies that $f(y) \leq f(x)$. If $y \in (x, x + \delta)$, then

$$\frac{f(y) - f(x)}{y - x} \geq 0 \Longrightarrow f'(x) = \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x} \geq 0$$

If $y \in (x - \delta, x)$, then

$$\frac{f(y) - f(x)}{y - x} \leq 0 \Longrightarrow f'(x) = \lim_{y \to x^-} \frac{f(y) - f(x)}{y - x} \leq 0,$$

which implies that $f'(x) = 0$. The argument for the case where f has a local minimum at x is completely analogous. □

1.3 Intermediate Value Property

1.4 L'hospital's Rule

Theorem 5. *Suppose $f, g : (a,b) \to \mathbb{R}$ are differentiable, and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$. Suppose*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \in \mathbb{R}^{\pm\infty}$$

If
$$f(x) \to 0, g(x) \to 0, x \to a\ (x \to b),$$
or if $g(x) \to +\infty$ *(*$g(x) \to -\infty$*) as* $x \to a$ *(*$x \to b$*), then*
$$\frac{f(x)}{g(x)} \to A, x \to a\ x \to b$$

Proof. □

1.5 Chapter 5 Exercises

1.5.1 Exercise 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function satisfying
$$|f(x) - f(y)| \leq (x - y)^2, \qquad \forall x, y \in \mathbb{R}$$
Then f is constant.

Proof. Let $x, y \in \mathbb{R}$ with $x \neq y$. Then
$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y| \Longrightarrow \lim_{y \to x} \left|\frac{f(x) - f(y)}{x - y}\right| = 0,$$
so $f'(x) = 0$. Hence, f' is indentically zero on $\mathbb{R}$, which implies that f is constant by the mean value theorem. □

1.5.2 Exercise 2

Let $f : (a, b) \to \mathbb{R}$ be differentiable and satisfy $f'(x) > 0$ for all $x \in (a, b)$. Then f is strictly increasing in (a, b). If g denotes the inverse function of f, then g is differentiable and
$$g'(f(x)) = \frac{1}{f'(x)}, \qquad \forall x \in (a, b)$$

Proof. Let $a < x < y < b$. Then by the mean value theorem, there exists $c \in (x, y)$ such that
$$\frac{f(y) - f(x)}{y - x} = f'(c) > 0 \Longleftrightarrow f(y) - f(x) = f'(c)(y - x) > 0 \Longrightarrow f(y) > f(x)$$
Thus, f is injective on (a, b), so we can define an inverse function $g : (f(a), f(b)) := \{y \in \mathbb{R} : y = f(x), x \in (a, b)\} \to \mathbb{R}$, where $f(a) := \inf_{x \in (a,b)} f(x)$ and $f(b) := \sup_{x \in (a,b)} f(x)$. To see that g is differentiable on $(f(a), f(b))$, fix $y_0 = f(x_0) \in (f(a), f(b))$. Since f is continuous,
$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to \infty} \left(\frac{f(x) - f(x_0)}{x - x_0}\right)^{-1}$$
Since $z \mapsto z^{-1}$ is continuous, we see that the last expression is equal to $\frac{1}{f'(x_0)}$. □

1.5.3 Exercise 3

Suppose $g : \mathbb{R} \to \mathbb{R}$ is differentiable with $|g'| \leq M$, for some positive constant M. Fix $\epsilon > 0$, and define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := x + \epsilon g(x)$. Then f is injective, for ϵ sufficiently small.

Proof. Let $x < y \in \mathbb{R}$, and observe that
$$\frac{f(y) - f(x)}{y - x} = \frac{(y - x) + \epsilon[g(y) - g(x)]}{y - x} = 1 + \epsilon \frac{g(y) - g(x)}{y - x}$$
By the mean value theorem,
$$-M \leq \frac{g(y) - g(x)}{y - x} \leq M$$
So if we choose $\epsilon < \frac{1}{2M}$, we conclude that
$$\frac{1}{2} = 1 - \frac{1}{2} < 1 - \epsilon M \leq \frac{f(y) - f(x)}{y - x},$$
which implies that $f(y) \neq f(x)$, hence f is injective. □

1.5.4 Exercise 4

If $C_0, \cdots, C_n$ are real constants with

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

then the polynomial defined by

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n$$

has at least one real root in $[0, 1]$.

Proof. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1},$$

so that $f(0) = f(1) = 0$. By the mean value theorem, there exists $t \in (0, 1)$ such that

$$0 = \frac{f(1) - f(0)}{1 - 0} = f'(t) = C_0 + C_1 t + \cdots + C_{n-1}t^{n-1} + C_n t^n$$

□

1.5.5 Exercise 5

Suppose $f : [0, \infty) \to \mathbb{R}$ is differentiable and $f'(x) \to 0$ as $x \to \infty$. If $g(x) := f(x+1) - f(x)$, then $g(x) \to 0$ as $x \to \infty$.

Proof. Choose $a \geq 0$ such that $x \geq a$ implies that $f'(x) < \epsilon$, for $\epsilon > 0$ given. Then for all $x \geq a$, by the mean value theorem, there exists $t \in (x, x+1)$ such that

$$g(x) = \frac{f(x+1) - f(x)}{(x+1) - x} = f'(t) \Longrightarrow |g(x)| = |f'(t)| < \epsilon,$$

since $t \geq a$. □

1.5.6 Exercise 6

Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$, and f' is monotonically increasing. If $g : (0, \infty) \to \mathbb{R}$ is defined by $g(x) := \frac{f(x)}{x}$, then g is monotonically increasing.

Proof. To show that g is monotonically increasing, it suffices to show that $g' \geq 0$ on $(0, \infty)$. By the product rule,

$$g'(x) = \frac{f'(x)}{x} - \frac{f(x)}{x^2}$$

□

1.5.7 Exercise 7

1.5.8 Exercise 8

1.5.9 Exercise 11

If f is defined in an open interval about x and $f''(x)$ exists, then

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Proof. The conditions of L'hospital's rule are satisfied, so since f' is continuous at x by the existence of $f''(x)$, we have that

$$\begin{aligned}\lim_{h \to 0} \frac{f'(x+h) + f'(x-h) - 2f'(x)}{2h} = \frac{1}{2}\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} + \frac{1}{2}\lim_{h \to 0} \frac{f'(x-h) - f'(x)}{h} &= \frac{f''(x)}{2} + \frac{f''(x)}{2} \\ &= f''(x)\end{aligned}$$

□

The existence of the limit on the LHS does not necessarily that $f''(x)$ exists. Consider the function

1.5.10 Exercise 14

Let $f : (a,b) \to \mathbb{R}$ be differentiable. Then f is convex if and only if f' is monotonically increasing.

Proof. Suppose f is convex. We proved in Chapter 4 that the function $R(x,y) = \frac{f(x)-f(y)}{x-y}$ is monotonically increasing in each argument. Suppose $a < x < y < b$. Then

$$f'(x) = \lim_{t \to x^-} \frac{f(t)-f(x)}{t-x} \leq \frac{f(x)-f(y)}{x-y} \leq \lim_{s \to y^+} \frac{f(s)-f(y)}{s-y} = f'(y)$$

Now suppose that f' is monotonically increasing on (a,b). □

Let $f : (a,b) \to \mathbb{R}$ be twice differentiable. Then f is convex if and only if $f' \geq 0$.

Proof. By the preceding result it suffices to show that f' is monotonically increasing if and only if $f'' \geq 0$. But this result is immediate from the mean value theorem. □

1.5.11 Exercise 22

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(t) \neq 1$ for every $t \in \mathbb{R}$. Then f has at most one fixed point.

Proof. Suppose $x < y \in \mathbb{R}$ satisfy $f(x) = x, f(y) = y$. If $x \neq y$, then by the mean value theorem,

$$1 = \frac{f(x)-f(y)}{x-y} = f'(t)$$

for some $t \in (x,y)$, which is a contradiction. □

The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) := t + (1+e^t)^{-1}, \qquad t \in \mathbb{R}$$

has no fixed points, even though $f'(t) \in (0,1)$ for all $t \in \mathbb{R}$.

Proof. Since $e^t > 0$ for all t, we have that

$$f(t) - t = (1+e^t)^{-1} > 0, \qquad \forall t \in \mathbb{R}$$

For the second claim, observe that by the chain rule, $f'(t) = 1 - e^t(1+e^t)^{-1}$, which evidently lies in $(0,1)$ for all real t. □

If $f : \mathbb{R} \to \mathbb{R}$ is a function for which there is a constant $0 \leq A < 1$ such that $|f'(t)| \leq A$ for all $t \in \mathbb{R}$, then f has a unique fixed point x such that $x = \lim_{n\to\infty} x_n$, where

$$x_1 \in \mathbb{R}, \qquad x_{n+1} = f(x_n) \; \forall n \in \mathbb{Z}^{\geq 1}$$

Proof. For all $x < y \in \mathbb{R}$, we have by the mean value theorem that

$$\exists t \in (x,y) \text{ s.t. } \left|\frac{f(y)-f(x)}{y-x}\right| = |f'(t)| \leq A \Longrightarrow |f(y)-f(x)| \leq A\,|y-x|\,,$$

so that f is Lipschitz on $\mathbb{R}$ with Lipschitz constant at most A. Choose some $x_1 \in \mathbb{R}$, and define a sequence $(x_n)_{n=1}^\infty$ inductively as above. I claim that (x_n) is Cauchy. To show this, it suffices to show that the series $\sum_{n=1}^\infty |x_{n+1}-x_n|$ converges. Observe that, for $n \geq 2$,

$$|x_{n+1}-x_n| = |f(x_n) - f(x_{n-1})| \leq A\,|x_n - x_{n-1}|\,,$$

so by induction, $|x_{n+1}-x_n| \leq A^{n-1}\,|x_2 - x_1|$. Since $0 \leq A < 1$, the convergence of $\sum_{n=1}^\infty |x_{n+1}-x_n|$ follows by comparison with the geometric series $\sum_{n=0}^\infty A^n$. Denote the limit of (x_n) by x. By continuity,

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x),$$

which shows that f has a fixed point. By our first result, we see that this fixed point is unique. □

1.5.12 Exercise 23

The function f defined by $f(x) := \frac{x^3+1}{3}$ has three fixed points α, β, γ where

$$\alpha \in (-2, -1), \qquad \beta \in (0, 1), \qquad \gamma \in (1, 2)$$

Proof. Define a function g by $g(x) := x^3 - 3x + 1$. Locating the fixed points of f is equivalent to the locating the zeroes of f. Since g has degree three, g has at most three distinct real roots. Observe that

$$g(-2) = -8 + 6 + 1 = -1, \qquad g(-1) = -1 + 3 + 1 > 0$$

so by the intermediate value theorem (IVT), there exists $\alpha \in (-2, -1)$ such that $g(\alpha) = 0$. $g(0) = 1$ and $g(1) = -1$, so by the IVT, there exists $\beta \in (0, 1)$ such that $g(\beta) = 0$. Lastly, $g(2) = 3$, so by the IVT, there exists $\gamma \in (1, 2)$ such that $g(\gamma) = 0$. $\square$

In fact, we can give an algorithm for locating β. Let $x_1 \in \mathbb{R}$ be chosen arbitrarily. Define a sequence $(x_n)_{n=1}^{\infty}$ inductively by $x_{n+1} := f(x_n)$, for $n \geq 1$.

I claim that if $x_1 < \alpha$, then $x_n \to -\infty$ as $n \to \infty$. Indeed,

I claim if $\alpha < x_1 < \gamma$, then $x_n \to \beta$ as $n \to \infty$.

I claim that if $\gamma < x_1$, then $x_n \to +\infty$ as $n \to \infty$.

1.5.13 Exercise 25

For a solution to Exercise 25, see the appendix of my paper on mathematical methods for wind power estimation.

1.5.14 Exercise 26

Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there exists $A \in \mathbb{R}$ such that

$$|f'(x)| \leq A\,|f(x)|, \qquad \forall x \in [a, b]$$

Then $f(x) = 0$ for all $x \in [a, b]$.

Proof. If $A = 0$, then it is an immediate consequence of the mean value theorem that $f = 0$ on $[a, b]$, so assume that $A > 0$. Let $x_0 \in [a, b]$. Since f is *a fortiori* continuous and $[a, x_0]$ is compact, $M_0 := \sup_{x \in [a, x_0]} |f(x)|$ is finite. Our hypothesis that $|f'(x)| \leq A\,|f(x)|$ implies that $|f'(x)| \leq AM_0$. By the mean value theorem,

$$|f(x)| = |f(x) - f(a)| \leq AM_0(x - a) \leq AM_0(x_0 - a)$$

So if we choose $x_0 = \max\left\{a + \frac{1}{2A}, b\right\}$, then

$$|f(x)| \leq AM_0 \frac{1}{2A} = \frac{M_0}{2}, \qquad \forall x \in [a, x_0]$$

which implies that $M_0 = 0$, otherwise we obtain a contradiction from the definition of supremum.

We repeat the argument with a replaced by x_0 and x_0 replaced by x_1, and so on. The argument ends after finitely many steps since, at the j^{th} step, $x_j - x_{j-1} > \frac{1}{2A}$, and by the archimedean property of the reals, there exists a positive integer N such that $\frac{N}{2A} > b - a$. $\square$