Continuity

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Contents

1 Continuity

1.1 Rudin Chapter 4 Exercises

1.1.1 Exercise 1

If $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0, \qquad \forall x \in \mathbb{R}$$

then f need not be continuous.

Proof. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & |x| > 0 \\ 0 & x = 0 \end{cases}$$

It is clear that f has a discontinuity at $x = 0$, but is continuous at every $x \neq 0$. The limit statement follows from continuity for all $x \neq 0$. If $x = 0$, then

$$f(x+h) - f(x-h) = f(h) - f(-h) = 1 - 1 = 0, \qquad \forall h \neq 0$$

□

1.1.2 Exercise 2

If $f : (X, d) \to (Y, \rho)$ is a continuous mapping of metric spaces, then

$$f(\overline{E}) \subset \overline{f(E)}, \qquad \forall E \subset X$$

Moreover, the inclusion can be proper.

Proof. Let $y = f(x) \in f(\overline{E})$, where $x \in \overline{E}$. Let $\epsilon > 0$ be given, and consider the open neighborhood $B(y; \epsilon)$. I claim that $B(y; \epsilon) \cap f(E) \neq \emptyset$. Since f is continuous at y, there exists $\delta > 0$ such that

$$d(x, x') < \delta \Longrightarrow \rho(f(x), f(x')) < \epsilon$$

By definition of closure, there exists $x' \in E \cap B(x; \delta)$, so that

$$\rho(f(x), f(x')) = \rho(y, f(x')) < \delta$$

But $f(x') \in f(E)$, which implies that $B(y; \epsilon) \cap f(E) \neq \emptyset$. Since $\epsilon > 0$ was arbitrary, we conclude that $y \in \overline{f(E)}$. □

1.1.3 Exercise 3

Let $f : (X, d) \to \mathbb{R}$ be a continuous real-valued function on a metric space (X, d). Define the zero set of f to be

$$Z(f) := \{p \in X : f(p) = 0\}$$

Then $Z(f)$ is closed.

Proof. Let $(p_n)_{n=1}^{\infty}$ be a sequence in $Z(f)$ such that $p_n \to p \in X$. Since f is continuous, we have that

$$f(p) = \lim_{n\to\infty} f(p_n) = 0 \Longrightarrow p \in Z(f)$$

□

1.1.4 Exercise 4

Let $f, g : (X, d) \to (Y, \rho)$ be continuous mappings, and let E be a dense subset of X. Then $f(E)$ is dense in $f(X)$. Furthermore, if $g(p) = f(p)$ for all $p \in E$, then $g(p) = f(p)$ for all $p \in X$.

Proof. Let $y = f(x) \in f(X)$, and let $\epsilon > 0$ be given. Since f is continuous, there exists $\delta > 0$ such that

$$d(x, x') < \delta \Longrightarrow \rho(y, f(x') < \epsilon$$

Since E is dense in X, there exists $x' \in E \cap B(x; \delta)$, so that $\rho(y, f(x')) < \epsilon$ and therefore $B(y; \epsilon) \cap f(E) \neq \emptyset$.

Suppose $g(p) = f(p)$ for all $p \in E$. Let $\epsilon > 0$ be given. Since $f(E) = g(E)$ are dense in $f(X)$ and $g(X)$, there exists $q \in E$ such that

$$\rho(f(p), f(q)) < \frac{\epsilon}{2}, \qquad \rho(g(p), g(q)) < \frac{\epsilon}{2}$$

Since $f(q) = g(q)$, we conclude from the triangle inequality that

$$\rho(f(p), g(p)) \leq \rho(f(p), f(q)) + \rho(f(q), g(q)) + \rho(g(q), g(p)) = \rho(f(p), f(q)) + \rho(g(q), g(p)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\rho(f(p), g(p)) = 0 \Longleftrightarrow f(p) = g(p)$. □

1.1.5 Exercise 5

If

1.1.6 Exercise 11

Suppose $f : (X, d) \to (Y, \rho)$ is uniformly continuous. Then $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in Y for every Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$x, x' \in X, d(x, x') < \delta \Longrightarrow \rho(f(x'), f(x)) < \epsilon$$

and choose a positive integer N such that

$$n, m \geq N \Longrightarrow d(x_n, x_m) < \delta$$

Then by uniform continuity,

$$n, m \geq N \Longrightarrow \rho(f(x_n), f(x_m) < \epsilon,$$

If f is a uniformly continuous real function defined on a dense subset $E \subset X$. Then f has a unique continuous extension from E to X.

Proof. Uniqueness follows from Exercise 4. For existence, first fix $x \in X$. By the density of E, there exists a sequence $(x_n)_{n=1}^{\infty}$ in E such that $x_n \to x$. Then (x_n) is Cauchy and hence $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ by our previous result. By completeness, $f(x_n)$ converges to some limit y. Define

$$f(x) := y = \lim_{n\to\infty} f(x_n)$$

We need to show that this definition is well-defined, i.e. does not depend on our choice of sequence (x_n). If $(x'_n)_{n=1}^{\infty}$ is another sequence in E such that $x'_n \to x$. Then by uniform continuity, for any $\epsilon > 0$,

$$|f(x'_n) - f(x_n)| < \epsilon,$$

for all n sufficiently large, which implies that $f(x'_n) \to f(x)$. To see that the defined function $f : X \to \mathbb{R}$ is continuous, first choose $\delta > 0$ such that $p, q \in E$ with $d(p,q) < \delta$ implies that $|f(p) - f(q)| < \frac{\epsilon}{3}$. Choose $p, q \in E$ such that $\max\{|f(x) - f(p)|, |f(y) - f(q)|\} < \frac{\epsilon}{3}$. Then

$$\begin{aligned} x, y \in X, d(x,y) < \delta \Longrightarrow |f(x) - f(y)| &\leq |f(x) - f(p)| + |f(p) - f(q)| + |f(q) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

□

1.1.7 Exercise 12

Let $f : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (Z, d_Z)$ be uniformly continuous functions between metric spaces. Then $h := g \circ f : (X, d_X) \to (Z, d_Z)$ is uniformly continuous.

Proof. Let $\epsilon > 0$ be given. Choose $\delta_1 > 0$ such that

$$\forall y, y' \in Y,\ d_Y(y, y') < \delta_1 \Longrightarrow d_Z(f(y), f(y')) < \epsilon$$

and choose $\delta_2 > 0$ such that

$$\forall x, x' \in X,\ d_X(x, x') < \delta_2 \Longrightarrow d_Y(f(x), f(x')) < \delta_1 \Longrightarrow d_Z(g(f(x)), g(f(x'))) < \epsilon$$

We conclude that h is uniformly continuous. □

1.1.8 Exercise 14

Let $I := [0,1]$ be the closed unit interval, and suppose that $f : I \to I$ is continuous. Then f has at least one fixed point in I.

Proof. Consider the continuous function $g : I \to \mathbb{R}$ defined by $g(x) := f(x) - x$. If f has no fixed points, then $g(x) \neq 0$ for all $x \in I$. By the intermediate value theorem, we must have $g(x) > 0$ for all $x \in I$ or $g(x) < 0$ for all $x \in I$. By hypothesis that $f(I) \subset I$, $g(0) > 0$ and $g(1) = f(1) - 1 < 0$, which results in a contradiction. □

1.1.9 Exercise 15

Every continuous open mapping $f : \mathbb{R} \to \mathbb{R}$ is monotonic.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. □

1.1.10 Exercise 20

Let E be a nonempty subset of a metric space (X, d). Define a function $\rho_E : X \to [0, \infty)$ by

$$\rho_E(x) := \inf_{z \in E} d(x, z)$$

$\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

Proof. Observe that

$$\rho_E(x) = 0 \Longleftrightarrow \forall \epsilon > 0, \exists z \in E \text{ s.t.} d(x,z) < \epsilon \Longleftrightarrow \forall \epsilon > 0, B(x;\epsilon) \cap E \neq \emptyset \Longleftrightarrow x \in \overline{E}$$

□

ρ_E is a uniformly continuous function on X.

Proof. To show that ρ_E is uniformly continuous, it suffices to show that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

Let $\epsilon > 0$ be small. By the definition of ρ_E and the triangle inequality, we obtain the inequalities

$$\rho_E(x) \leq d(x, z) \leq d(x, y) + d(z, y), \ \rho_E(y) \leq d(y, z) \leq d(x, y) + d(x, z) \qquad \forall z \in E, x, y \in X$$

Taking the infimum over all $z \in E$. We see that $\rho_E(x) \leq d(x, y) + \rho_E(y)$ and $\rho_E(y) \leq d(x, y) + \rho_E(x)$, which shows that

$$-d(x, y) \leq \rho_E(x) - \rho_E(y) \leq d(x, y) \iff |\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

1.1.11 Exercise 21

Let K, F be disjoint subsets of a metric space (X, d), with K compact and F closed. There exists $\delta > 0$ such that

$$p \in K, q \in F \Longrightarrow d(p, q) > \delta$$

If neither K nor F are compact, then the conclusion may fail.

Proof. Since K, F are disjoint and closed, it follows from Exercise 20 that $\rho_F(p) > 0$ for all $p \in K$ and ρ_F is continuous on K. Since K is compact, by Weierstrass' extreme value theorem, ρ_F attains its minimum at some $p_0 \in K$. Hence,

$$0 < \rho_F(p_0) \leq \rho_F(p) = \inf_{q \in F} d(p, q) \leq d(p, q), \qquad \forall p \in K, q \in F$$

Choosing $\delta = \rho_F(p_0)$ completes the proof.

To see that the conlusion may fail if neither K nor F are compact, let our metric space be the rationals equipped with the Euclidean metric. Then $K := (\sqrt{2}, \infty)$ and $F := (-\infty, \sqrt{2})$ are closed in $\mathbb{Q}$, but neither is compact. However, $\inf_{p \in K, q \in F} d(p, q) = 0$ since points in K and F become arbitrarily close as they approach $\sqrt{2}$ from the right and left, respectively. □

1.1.12 Exercise 22

Let A and B be disjoint nonempty closed sets in a metric space (X, d), and define a function $f : X \to \mathbb{R}$ by

$$f(p) := \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}, \qquad p \in X$$

Then f is a continuous function with $f(X) \subset [0, 1]$, $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$.

Proof. Recall from Exercise 20 that $\rho_A(p) = 0$ and $\rho_B(p) = 0$ if and only if $p \in \overline{A} = A$ and $p \in \overline{B} = B$, respectively. Since A and B are disjoint, we see that

$$\rho_A(p) = 0 \Longrightarrow \rho_B(p) > 0, \qquad \rho_B(p) = 0 \Longrightarrow \rho_A(p) > 0$$

Thus,

$$f(p) = 0 \iff \rho_A(p) = 0 \iff p \in A, \qquad f(p) = 1 \iff \rho_B(p) = 0 \iff p \in B$$

For any $p \in X \setminus A$,

$$0 \leq \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \leq \frac{\rho_A(p)}{\rho_A(p)} = 1$$

which shows that $f(X) \subset [0, 1]$. f is continuous since the it is the composition of the continuous functions ρ_A and ρ_B. □

Note that this exercise establishes a converse to Exercise 3: namely, every closed subset of a metric space is the zero set of some continuous function.

Since the image of f is contained in $[0, 1]$, we have that

$$V := f^{-1}((-\infty, \frac{1}{2})) = f^{-1}([0, \frac{1}{2}), \qquad W := f^{-1}((\frac{1}{2}, +\infty)) = f^{-1}((\frac{1}{2}, 1])$$

where V and W are open, since f is continuous. Hence, A and B are separated by disjoint open sets. □

1.1.13 Exercise 23

For Exercises 23 and 24, see my post 'Continuity, Convexity, and Jensen's Inequality.'

1.1.14 Exercise 24