Sequences and Series

Matt Rosenzweig

Contents

1 Sequences and Series

1.1 Sequences

Proposition 1. *1. For $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$.*

2. For $p > 0$, $\lim_{n\to\infty} p^{\frac{1}{n}} = 1$.

3. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

4. For $p > 0$ and $\alpha \in \mathbb{R}$, $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

5. For $|x| < 1$, $\lim_{n\to\infty} x^n = 0$.

Proof. In all the following proofs, we assume that $\epsilon > 0$ is given. For (1), by the Archimedean property, we can take $n \geq \epsilon^{-\frac{1}{p}}$ so that

$$\frac{1}{n^p} \leq (\epsilon^{-\frac{1}{p}})^{-p} = \epsilon$$

For (2), fix $p > 0$. If $p > 1$, set $x_n := p^{\frac{1}{n}} - 1$ and observe that $x_n > 0$. By the binomial theorem,

$$1 + nx_n \leq (1 + x_n)^n = p \Longrightarrow x_n \leq \frac{p-1}{n}$$

Hence, $\limsup_{n\to\infty} x_n \leq \lim_{n\to\infty} \frac{p-1}{n} = 0$. If $p = 1$, then the result is trivial. If $0 < p < 1$, then we return to the first case by considering p^{-1}.

For (3), set $x_n := n^{\frac{1}{n}} - 1$. Observe that $x_n \geq 0$ and by the binomial theorem,

$$n = (1 + x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2 \Longrightarrow x_n \leq \sqrt{\frac{2}{n-1}}$$

By (1), $\lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$.

For (4), fix $\alpha \in \mathbb{R}$ and $p > 0$, and choose a positive integer $k > \alpha$. For $n > 2k$, we have by the binomial theorem that

$$(1+p)^n > \binom{n}{k}p^k = \frac{n!}{(n-k)!k!}p^k = \frac{n(n-1)\cdots(n-k+1)}{k!}p^k > \left(\frac{n}{2}\right)^k \frac{p^k}{k!} \Longrightarrow \frac{1}{(1+p)^n} < \frac{2^k k!}{n^k p^k}$$

Hence,

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{n^{k-\alpha}p^k}$$

Since $k-\alpha > 0$, $n^{k-\alpha} \to 0$ by (1).

For (5), take $\alpha = 0$ and $p = |x|^{-1} - 1 > 0$ in (4). □

1.2 Series

The following proposition attributed to Cauchy shows that the convergence of a series with monotonically decreasing terms is determined by the growth of a 'small' subset of its terms.

Proposition 2. *(Cauchy's Convergence Test) Let $(a_n)_{n=1}^\infty$ be a decreasing sequence of real numbers bounded from below by 0. Then the series $\sum_{n=1}^\infty a_n$ converges if and only if the series*

$$\sum_{k=0}^\infty 2^k a_{2^k}$$

converges.

Proof. Suppose $\sum_{n=1}^\infty a_n$ converges, then

$$\sum_{n=1}^\infty a_n \geq 2a_2 + \sum_{k=2}^\infty \left(2^k - 2^{k-1}\right) a_{2^k} = 2a_2 + \sum_{k=2}^\infty (2^{k-1}) a_{2^k},$$

which implies the convergence of $\sum_{k=2}^\infty 2^{k-1}a_{2^k}$ and therefore $\sum_{k=0}^\infty 2^k a_{2^k}$ by the comparison test. If $\sum_{k=0}^\infty 2^k a_{2^k}$ converges, then

$$\sum_{k=0}^\infty 2^k a_{2^k} \geq a_1 + \sum_{k=0}^\infty \sum_{j=2^k+1}^{2^{k+1}} a_j = \sum_{j=1}^\infty a_j,$$

which implies that $\sum_{n=1}^\infty a_n$ converges by the comparison test. □

Cauchy's convergence test allows us to give a short proof of the convergence (and divergence) conditions for the harmonic p-series $\sum_{n=1}^\infty \frac{1}{n^p}$.

Proposition 3. *(Root Test) Let $(a_n)_{n=1}^\infty$ be a sequence of complex numbers, and set $\alpha := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Then*

1. *if $\alpha < 1$, the series $\sum_{n=1}^\infty a_n$ converges;*
2. *if $\alpha > 1$, the series $\sum_{n=1}^\infty a_n$ diverges;*
3. *if $\alpha = 1$, the root test is inconclusive.*

Proof. Suppose $\alpha < 1$. Since finitely many terms do not affect the convergence of $\sum a_n$, we may assume that $\sup_n |a_n|^{\frac{1}{n}} < 1$. We see that $\sum_{n=1}^\infty a_n$ is absolutely convergent by the comparison test, since

$$\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty (|a_n|^{\frac{1}{n}})^n < \sum_{n=1}^\infty \alpha^n = \frac{\alpha}{1-\alpha}$$

Analogously, if $\alpha > 1$, then there exists a subsequence of indices $(n_k)_{k=1}^\infty$ such that $|a_{n_k}|^{\frac{1}{n}} > 1$, so the necessary condition $\lim_{n\to\infty} a_n$ does not hold.

To see that the case $\alpha = 1$ provides us insufficient information to determine convergence or divergence, consider the following examples. If $a_n = 1$ for all $n \in \mathbb{Z}^{\geq 1}$, then clearly $\sum_{n=1}^{\infty} a_n$ diverges to ∞, but $|a_n|^{\frac{1}{n}} = 1$ for all n. If $a_n = \frac{1}{n^2}$, then

$$\limsup_{n\to\infty} \left|n^{-2}\right|^{\frac{1}{n}} = \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^{-2} = 1,$$

where the penultimate inequality follows from continuity, and $\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$. ☐

Proposition 4. *(Ratio Test) For a sequence $(a_n)_{n=1}^{\infty}$ of complex numbers, the series $\sum_{n=1}^{\infty} a_n$*

1. *converges, if* $\limsup_{n\to 1} \left|\frac{a_{n+1}}{a_n}\right| < 1$*;*
2. *diverges if* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all but finitely many n.*

1.3 Rudin Chapter 3 Exercises

1.3.1 Exercise 1

We first prove the useful reverse triangle inequality.

Lemma 5. *For $a, b \in \mathbb{C}$, $||a| - |b|| \leq |a + b|$.*

Proof. Observe that

$$\begin{aligned} |a+b|^2 = (a+b)(\overline{a}+\overline{b}) = |a|^2 + (a\overline{b} + b\overline{a}) + |b|^2 &= |a|^2 + 2\operatorname{Re}(a\overline{b}) + |b|^2 \\ &\leq |a|^2 - 2|a||b| + |b|^2 \\ &= ||a| - |b||^2 \end{aligned}$$

Taking the square root of both sides completes the proof. ☐

Let $(s_n)_{n=1}^{\infty}$ be a convergent sequence of complex numbers. For $\epsilon > 0$, there exists $N \in \mathbb{Z}^{\geq 1}$ such that $n \geq N$ implies $|s_n - s|$. By the reverse triangle inequality,

$$n \geq N \Longrightarrow ||s_n| - |s|| \leq |s_n - s| < \epsilon$$

The converse is false. Define $s_{2n} = 1$ and $s_{2n+1} = -1$. Then $(s_n)_{n=1}^{\infty}$ oscillates between 1 and -1, but $|s_n| = 1$ for all n.

1.3.2 Exercise 2

$\lim_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$.

Proof. We can write $\sqrt{n+1} = \sqrt{n} + x$, for $x \geq 0$, so that

$$n + 1 = (\sqrt{n} + x)^2 = n + 2\sqrt{n}x + x^2 \Longrightarrow 1 = 2\sqrt{n}x + x^2 \Longrightarrow 0 \leq x \leq \frac{1}{2\sqrt{n}}$$

For any $0 < c < 1$, I claim that $x \geq c\frac{1}{2\sqrt{n}}$ for all but finitely many n. Indeed, otherwise there exists a subsequene $n_k \uparrow \infty$ such that

$$1 = 2\sqrt{n_k}x + x^2 \leq 2\sqrt{n_k}(c\frac{1}{2\sqrt{n_k}}) + c^2\frac{1}{4n_k} = c + \frac{c^2}{4n_k} < 1$$

for all k sufficiently large, which is a contradiction. We see that

$$\frac{1}{2} - \sqrt{n(n+1)} + n = \frac{1}{2} - \sqrt{n}(\sqrt{n} + x) + n = \frac{1}{2} - \sqrt{n}x \geq \frac{1}{2}(1 - c)$$

Letting $c \uparrow 1$ completes the proof. ☐

1.3.3 Exercise 3

Set $s_1 := \sqrt{2}$, and for $n \in \mathbb{Z}^{\geq 1}$, define

$$s_{n+1} := \sqrt{2 + \sqrt{s_n}}$$

Then the sequence $(s_n)_{n=1}^{\infty}$ converges and moreover, $s_n < 2$ for all $n \geq 1$.

Proof. By induction, we see that $s_n > \sqrt{2} > 1$ for all n. Hence,

$$s_{n+1}^2 = 2 + \sqrt{s_n} \Longrightarrow s_{n+1}^2 < 2 \Longrightarrow s_{n+1} < \sqrt{2}$$

I claim that $s_n < s_{n+1}$. The base case $n = 1$ follows from the monotonicity of the square root function. Suppose the assertion holds for all $1 \leq j \leq n$. Then

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n$$

□

1.3.4 Exercise 4

Define a real sequence $(s_n)_{n=1}^{\infty}$ by

$$s_n := \begin{cases} 0 & n = 1 \\ \frac{s_{2m-1}}{2} & n = 2^m, m \in \mathbb{Z}^{\geq 1} \\ \frac{1}{2} + s_{2m} & n = 2^m + 1, m \in \mathbb{Z}^{\geq 1} \end{cases}$$

Then $\liminf_{n\to\infty} s_n = 0$ and $\limsup_{n\to\infty} s_n = \frac{1}{2}$.

Proof. □

1.3.5 Exercise 6

If $a_n := \sqrt{n+1} - \sqrt{n}$, then for any $N \in \mathbb{Z}^{\geq 1}$,

$$\sum_{n=1}^{N} a_n = \sqrt{N+1} - 1,$$

so $\sum_{n=1}^{\infty} a_n$ diverges to ∞.

If $a_n := \frac{\sqrt{n+1}-\sqrt{n}}{n}$, then using our above estimate for $\sqrt{n+1} - \sqrt{n}$, we see that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \leq \sum_{n=1}^{\infty} \frac{1}{2n^{\frac{3}{2}}} < \infty$$

Hence, $\sum_{n=1}^{\infty} a_n$ converges by the comparison test.

Suppose $a_n := \frac{1}{1+z^n}$. I claim that the series $\sum_{n=1}^{\infty} a_n$ converges for $|z| > 1$ and diverges for $|z| < 1$. First, suppose $|z| > 1$. For any $N \in \mathbb{Z}^{\geq 1}$,

$$\begin{aligned} \left| \sum_{n=1}^{N} \frac{1}{1+z^n} \right| = \left| \sum_{n=1}^{N} z^{-n} \sum_{k=0}^{\infty} (-z^{-n})^k \right| \leq \sum_{n=1}^{N} |z|^{-n} \sum_{k=0}^{\infty} |z|^{-nk} &= \sum_{n=1}^{N} \frac{1}{|z|^n (1 - |z|^{-n})} \\ &\leq \frac{1}{1 - |z|^{-1}} \sum_{n=1}^{N} \frac{1}{|z|^n} \end{aligned}$$

which converges as $N \to \infty$ by comparison with the geometric series.
Now suppose that $|z| < 1$.

1.3.6 Exercise 7

Suppose $(a_n)_{n=1}^\infty$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^\infty a_n$ converges. Then the series

$$\sum_{n=1}^\infty \frac{\sqrt{a_n}}{n}$$

converges.

Proof. □

1.3.7 Exercise 8

If $\sum_{n=1}^\infty a_n$ converges and $(b_n)_{n=1}^\infty$ is a bounded, monotonically increasing sequence, then $\sum_{n=1}^\infty a_n b_n$ also converges.

Proof. By considering real and imaginary parts separately, it suffices to consider the case where the a_n are real. Without loss of generality, we may assume that $b_n > 0$ for all n. Set $b := \sup_n |b_n|$. Let $\epsilon > 0$ be given, and choose $N_0 \in \mathbb{Z}^{\geq 1}$ such that $N, M \geq N_0$ implies that $\left|\sum_{n=M+1}^N a_n\right| < \epsilon$.

$$\sum_{n=M+1}^N \operatorname{sgn}(a_n)\,|a_n|\, b_n = \sum_{n=M+1}^N a_n b_n \leq b\left(\sum_{n=M+1}^N a_n\right) < b\epsilon$$

If $a_n \geq 0$ then $-ba_n = -b\operatorname{sgn}(a_n)\,|a_n| < a_n b_n$, and if $a_n < 0$, then since $-b \leq -b_n$, $-ba_n = -b\,|a_n| \leq -b_n\,|a_n| = a_n b_n$. Hence,

$$-b\epsilon < -b\left(\sum_{n=M+1}^N a_n\right) \leq \sum_{n=M+1}^N a_n b_n$$

We conclude that $\left|\sum_{n=M+1}^N a_n b_n\right| < \epsilon$, so the series $\sum_{n=1}^\infty a_n b_n$ converges by the Cauchy criterion. □

1.3.8 Exercise 9

The radius of convergence of $\sum_{n=0}^\infty n^3 z^n$ is 1.

Proof. Since the finite limit of products of sequences is the product of the limits, we have that

$$\limsup_{n\to\infty}(n^3)^{\frac{1}{n}} = \limsup_{n\to\infty}(n^{\frac{1}{n}})^3 = \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^3 = 1$$

□

The radius of convergence of $\sum_{n=0}^\infty \frac{2^n}{n!} z^n$ is infinite.

Proof. Applying the binomial formula to 2^n, for n large, we obtain the upper bound

$$\begin{aligned}\frac{2^n}{n!} = \frac{1}{n!}\sum_{k=0}^n \frac{n!}{k!(n-k)!} &= \sum_{k=0}^m \frac{1}{k!(n-k)!} + \sum_{k=m+1}^n \frac{1}{k!(n-k)!}\\ &\leq \frac{1}{n-m}\sum_{k=0}^m \frac{1}{k!} + \sum_{k=m+1}^n \frac{1}{k!}\\ &\leq \frac{e}{n-m} + \sum_{k=m+1}^n \frac{1}{k!}\end{aligned}$$

Since $e = \sum_{k=1}^\infty \frac{1}{k!}$, we can choose $N \in \mathbb{Z}^{\geq 1}$ such that $m, n \geq N$ implies that $\sum_{k=m+1}^n \frac{1}{k!} < \frac{\epsilon}{2}$. Choose $N' > N$ such that $n \geq N'$ implies that $\frac{e}{n-N} < \frac{\epsilon}{2}$. Then

$$n \geq N' \implies \frac{2^n}{n!} \leq \frac{e}{n-N} + \sum_{k=N+1}^n \frac{1}{k!} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

We conclude that $\limsup_{n\to\infty} \frac{2^n}{n!} = 0$, and therefore the radius of convergence is infinite. □

The radius of convergence of $\sum_{n=0}^{\infty} \frac{2^n}{n^2} z^n$ is $\frac{1}{2}$.

Proof. Since $\lim_{n\to\infty} a_n = a \neq 0$ implies that $\lim_{n\to\infty} a_n^{-1} = a^{-1}$, we have that

$$\limsup_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2}{(n^{\frac{1}{n}})^2} = \frac{2}{(\lim_{n\to\infty} n^{\frac{1}{n}})^2} = 2$$

□

The radius of convergence $\sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n$ is 3.

Proof. By the same arguments used above, we have that

$$\limsup_{n\to\infty} \left(\frac{n^3}{3^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(n^{\frac{1}{n}})^3}{3} = \frac{(\lim_{n\to\infty} n^{\frac{1}{n}})^3}{3} = \frac{1}{3}$$

□

1.3.9 Exercise 10

Let $(a_n)_{n=1}^{\infty}$ be a sequence of integers, infinitely many of which are nonzero. Then the radius of convergence of $\sum_{n=1}^{\infty} a_n z^n$ is at most 1.

Proof. Denote the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ by $R \in [0, \infty]$. If $\sum_{n=1}^{\infty} a_n z^n$ converges for some $z \in \mathbb{C}$, then

$$\lim_{n\to\infty} |a_n z^n| = 0$$

If $|z| > 1$, then for all n sufficiently large, $|a_n| < 1$. Since $a_n \in \mathbb{Z}$, $a_n = 0$, for all but finitely many indices n, which contradicts our hypothesis. □

1.3.10 Exercise 11

Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers. Suppose $\sum_{n=1}^{\infty} a_n$ diverges. Then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges.

Proof. If $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges, then $\frac{a_n}{1+a_n} \to 0$, so for $\epsilon > 0$ small,

$$\frac{a_n}{1+a_n} < \epsilon \Longrightarrow a_n < \epsilon + \epsilon a_n \Longrightarrow a_n < \frac{\epsilon}{1-\epsilon}$$

If (a_n) is unbounded, then $\frac{a_n}{1+a_n}$ does not tend to 0 as $n \to \infty$, hence $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ does not converge. Suppose (a_n) is bounded by some positive constant $M > 1$. Then

$$\sum_{n=1}^{N} \frac{a_n}{1+a_n} > \frac{a_1(1+a_1) + \cdots + a_N(1+a_N)}{M+1} > \frac{1}{M+1} \sum_{n=1}^{N} a_n$$

Hence, $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges by the comparison test. □

If $s_n := a_1 + \cdots + a_n$, then

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}, \qquad \forall k \in \mathbb{Z}^{\geq 1}$$

and therefore $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges.

Proof. For any $k \in \mathbb{Z}^{\geq 1}$, $s_{N+1} < \cdots < s_{N+k}$. Hence,

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$$

This lower bound shows that the series $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges by the Cauchy criterion. □

The sequence $(\frac{a_n}{s_n^2})_{n=1}^\infty$ satisfies

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and therefore the series $\sum_{n=1}^\infty \frac{a_n}{s_n^2}$ converges.

Proof. Since $s_{n-1} < s_n$, we have that

$$\frac{a_n}{s_n^2} < \frac{a_n}{s_n s_{n-1}} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

Since $s_n \uparrow \infty$ and the RHS in the above inequality is telescoping, we see that

$$\sum_{n=1}^N \frac{a_n}{s_n^2} \leq \frac{a_1}{s_1^2} + \sum_{n=2}^N \left[\frac{1}{s_{n-1}} - \frac{1}{s_n}\right] = \frac{a_1}{s_1^2} + \left(\frac{1}{s_1} - \frac{1}{s_N}\right) \rightarrow \frac{a_1}{s_1^2} + \frac{1}{s_1}$$

as $N \rightarrow \infty$. □

The series $\sum_{n=1}^\infty \frac{a_n}{1+n^2 a_n}$ converges. The convergence or divergence of $\sum_{n=1}^\infty \frac{a_n}{1+na_n}$ depends on the sequence $(a_n)_{n=1}^\infty$.

Proof. The second claim follows from noting that $\frac{a_n}{1+n^2a_n} \leq \frac{1}{n^2}$ and that $\sum_{n=1}^\infty \frac{1}{n^2} < \infty$. Hence, $\sum_{n=1}^\infty \frac{a_n}{1+n^2 a_n}$ converges by the comparison test. For the first claim, first consider $a_n := \frac{1}{n}$. Then

$$\sum_{n=1}^\infty \frac{a_n}{1+na_n} = \sum_{n=1}^\infty \frac{1}{2n},$$

which diverges since the harmonic series diverges. Now define a_n by

$$a_n := \begin{cases} \frac{1}{n} & n = m^2, m \in \mathbb{Z} \\ 2^{-n} & \text{otherwise} \end{cases}$$

I claim that $\sum_{n=1}^\infty a_n$ diverges. Let $\alpha > 0$. Since $\sum_{n=1}^\infty \frac{1}{n}$ diverges, we can choose an integer N sufficiently large so that $M \geq N$ implies that $\sum_{n=1}^M \frac{1}{n} > \alpha$. Then

$$\sum_{n=1}^{N^2} a_n > \sum_{m=1}^N \frac{1}{m} > \alpha$$

But $\sum_{n=1}^\infty \frac{a_n}{1+na_n}$ converges since

$$\sum_{n=1}^N \frac{a_n}{1+na_n} = \sum_{\substack{1 \leq n \leq N \\ n=m^2, m \in \mathbb{Z}}} \frac{\frac{1}{n}}{2} + \sum_{\substack{1 \leq n \leq N \\ n \neq m^2, m \in \mathbb{Z}}} \frac{2^{-n}}{1+n2^{-n}} \leq \sum_{m=1}^\infty \frac{1}{2m^2} + \sum_{n=1}^\infty 2^{-n} < \infty$$

Many thanks to the math.stackexchange community for helping me out with this last part. □

1.3.11 Exercise 12

Suppose $(a_n)_{n=1}^\infty$ is a sequence of positive reals such that the series $\sum_{n=1}^\infty a_n$ converges. Define tails

$$r_n := \sum_{m=n}^\infty a_m, \qquad \forall n \in \mathbb{Z}^{\geq 1}$$

If $m < n$, then

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

and therefore $\sum_{n=1}^\infty \frac{a_n}{r_n}$ converges.

Proof. First, note that $r_{n+1} < r_n$ and since $\sum_{n=1}^{\infty} a_n$ converges, $r_n \downarrow 0$. Hence,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} = \frac{r_m - r_{n+1}}{r_m} = 1 - \frac{r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m}$$

where $m < n$. Let $\epsilon > 0$ and $m \in \mathbb{Z}^{\geq 1}$ be given. Since $r_n \downarrow 0$, we can choose $n > m$ sufficiently large so that $1 - \frac{r_n}{r_m} > \epsilon$, which implies that the series $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ does not satisfy the Cauchy criterion. □

For $n \in \mathbb{Z}^{\geq 1}$,

$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right),$$

so that the series $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$ converges.

Proof. We can write $a_n = r_n - r_{n+1}$, so that

$$\begin{aligned}\frac{a_n}{\sqrt{r_n}} = \frac{r_n - r_{n+1}}{\sqrt{r_n}} = \frac{\left(\sqrt{r_n} + \sqrt{r_{n+1}}\right)\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)}{\sqrt{r_n}} &= \left(1 + \frac{\sqrt{r_{n+1}}}{\sqrt{r_n}}\right)\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right) \\ &< 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right),\end{aligned}$$

since $\frac{r_{n+1}}{r_n} < 1$. For any $N \in \mathbb{Z}^{\geq 1}$,

$$\sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < 2\sum_{n=1}^{N}(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2(\sqrt{r_1} - \sqrt{r_{N+1}}) \to 0$$

as $N \to \infty$. Hence, $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$ converges. □

1.3.12 Exercise 13

If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converge absolutely, then the Cauchy product $\sum_{k=0}^{\infty} c_k$ converges absolutely and

$$\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right)$$

Proof. Let $\epsilon \in (0,1)$ be given. Set $A = \sum_{k=0}^{\infty} a_k$ and $B = \sum_{k=0}^{\infty} b_k$. Choose $M > 0$ such that $\sum_{k=0}^{\infty} |a_k|, \sum_{k=0}^{\infty} |b_k| \leq M$, choose $m > 0$ such that $\max\{\sup_k |a_k|, \sup_k |b_k|\} \leq m$, and choose $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 \Rightarrow \max\left\{\sum_{k=N+1}^{\infty} |a_k|, \sum_{k=N+1}^{\infty} |b_k|\right\} < \epsilon$$

Then for $N \geq N_0$, we have

$$\begin{aligned}\left|AB - \sum_{k=0}^{N} c_k\right| &= \left|\left(A - \sum_{k=0}^{N} a_k\right)\left(B - \sum_{k=0}^{N} b_k\right) + \left(A - \sum_{k=0}^{N} a_k\right)\sum_{k=0}^{N} b_k + \left(B - \sum_{k=0}^{N} b_k\right)\sum_{k=0}^{N} a_k + \sum_{N<i+j\leq 2N} a_i b_j\right| \\ &< M\epsilon + 2M\epsilon + \left(\sum_{i=N}^{\infty} |a_i|\right)\left(\sum_{j=N}^{\infty} |b_j|\right) \\ &< 4M\epsilon\end{aligned}$$

To see absolute convergence, observe that for $N \in \mathbb{N}$,

$$\sum_{k=0}^{N} |c_k| \leq \sum_{k=0}^{N}\sum_{i=0}^{k} |a_i|\,|b_{k-i}| \leq \left(\sum_{i=1}^{N} |a_i|\right)\left(\sum_{j=1}^{N} |b_j|\right) \leq \left(\sum_{i=1}^{\infty} |a_i|\right)\left(\sum_{j=1}^{\infty} |b_j|\right) < \infty$$

Since the partial sums $\sum_{k=0}^{N} |c_k|$ are monotonically nondecreasing, they converge by the monotone convergence theorem. □

It follows by induction that the Cauchy product of a finitely many absolutely convergent series $\sum_{k=0}^{\infty} a_{k,j}$, $1 \leq j \leq r$, is absolutely convergent with limit

$$\sum_{n=0}^{\infty} \sum_{k_1+\cdots+k_r=n} a_{k_1,1} a_{k_2,2} \cdots a_{k_r,r}$$

1.3.13 Exercise 14

Let $(s_n)_{n=0}^{\infty}$ be a sequence of complex numbers. For $n \in \mathbb{Z}^{\geq 0}$, we define the n^{th} arithmetic mean σ_n by

$$\sigma_n := \frac{s_0 + s_1 + \cdots + s_n}{n+1}$$

If $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} \sigma_n = s$.

Proof. Choose $\epsilon > 0$, and let $N \in \mathbb{Z}^{\geq 1}$ be sufficiently large such that $n \geq N$ implies that $|s_n - s| < \frac{\epsilon}{2}$. Then

$$|\sigma_n - s| = \left|\frac{s_0 + s_1 + \cdots + s_n}{n+1} - s\right| = \left|\frac{s_0 + s_1 + \cdots + s_n}{n+1} - (n+1)\frac{s}{n+1}\right| \\ \leq \frac{|s_0 - s| + \cdots + |s_n - s|}{n}$$

Choose $N' \geq N$ such that $\frac{1}{N'+1}\sum_{j=0}^{N} |s_j - s| < \frac{\epsilon}{2}$. Then

$$n \geq N' \Longrightarrow |\sigma_n - s| \leq \sum_{j=0}^{N} \frac{|s_j - s|}{n+1} + \sum_{j=N+1}^{n} \frac{|s_j - s|}{n+1} \leq \sum_{j=0}^{N} \frac{|s_j - s|}{N+1'} + \frac{\epsilon}{2} < \epsilon$$

□

Note that a sequence (s_n) need not converge in order for its arithmetic means (σ_n) to converge. Let $s_n := (-1)^n$. Then

$$|\sigma_n| = \begin{cases} 0 & n \equiv 1 \pmod 2 \\ \frac{1}{n+1} & n \equiv 0 \pmod 2 \end{cases}$$

from which it is immediate that $\sigma_n \to 0$.

We can even construct a sequence (s_n) satisfying $s_n > 0$ for all n, $\limsup_{n\to\infty} s_n = \infty$, yet $\lim_{n\to\infty} \sigma_n = 0$. Define

$$s_n := \begin{cases} k & n = 2^k, \text{ for some } k \in \mathbb{Z}^{\geq 0} \\ \dfrac{1}{2^{2k}} & n \in (2^k, 2^{k+1}) \end{cases}$$

For a given $n \in \mathbb{Z}^{\geq 0}$, let $k = k(n)$ be the maximal integer such that $2^k \leq n$, so that

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \leq \frac{\sum_{j=0}^{k} j + 2^j \cdot 2^{-2j}}{n+1} = \frac{k(k+1) + 4(1 - 2^{-(k+1)})}{2(n+1)} \leq \frac{k(k+1) + 4}{2(2^k + 1)} \to 0$$

as $n \to \infty$.

For $n \geq 1$, set $a_n := s_n - s_{n-1}$. Then

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} k a_k$$

Proof. Observe that

$$s_n - \sigma_n = s_0 + \sum_{k=1}^{n}[s_k - s_{k-1}] - \frac{1}{n+1}\sum_{k=0}^{n} s_k = s_0 + \sum_{k=1}^{n} a_k - \frac{s_0}{n+1} - \frac{1}{n+1}\sum_{k=1}^{n}\left(s_0 + \sum_{j=1}^{k}[s_j - s_{j-1}]\right) \\ = \sum_{k=1}^{n} a_k - \frac{1}{n+1}\sum_{k=1}^{n}\sum_{j=1}^{k} a_j \\ = \sum_{k=1}^{n} a_k - \frac{1}{n+1}\sum_{k=1}^{n}(n-k)a_k \\ = \frac{1}{n+1}\sum_{k=1}^{n} k a_k$$

□

Now assume that $\lim_{n\to\infty} na_n = 0$ and $\lim_{n\to\infty} \sigma_n = \sigma$ exists. Then $\lim_{n\to\infty} s_n = \sigma$.

Proof. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{Z}^{\geq 1}$ sufficiently large so that $n \geq N$ implies that $|na_n| < \epsilon$ and $|\sigma_n - \sigma| < \epsilon$. Choose $N' \geq N$ such that $n \geq N'$ implies that

$$\frac{1}{n+1}\sum_{k=1}^{N} k\,|a_k| < \epsilon$$

Then

$$\begin{aligned} n \geq N' \Longrightarrow |s_n - \sigma| \leq |s_n - \sigma_n| + |\sigma_n - \sigma| < \epsilon + \frac{1}{n+1}\sum_{k=1}^{n} ka_k &< 2\epsilon + \frac{1}{n+1}\sum_{k=N+1}^{n} k\,|a_k| \\ &< 2\epsilon + \frac{1}{n+1}\sum_{k=N+1}^{n} \epsilon \\ &\leq 3\epsilon \end{aligned}$$

□

It turns out that we can relax our hypothesis above that $\lim_{n\to\infty} na_n = 0$ to just $(na_n)_{n=1}^{\infty}$ is a bounded sequence.

Proof. Choose $M > 0$ such that $|na_n| < M$ for all n. Let $n \in \mathbb{Z}^{\geq 1}$ be large. For $m < n$, write

$$\begin{aligned} s_n - \sigma_n = \frac{n-m}{n-m}s_n - \sigma_n &= -\sigma_n + \frac{1}{n-m}\sum_{i=m+1}^{n} s_i + \frac{1}{n-m}\sum_{i=m+1}^{n}[s_n - s_i] \\ &= \frac{-(n-m)\sum_{i=0}^{n} s_i}{(n+1)(n-m)} + \frac{(n+1)\sum_{i=m+1}^{n} s_i}{(n-m)(n+1)} + \frac{1}{n-m}\sum_{i=m+1}^{n}[s_n - s_i] \\ &= \frac{\sum_{i=m+1}^{n}(m+1)s_i - (n-m)\sum_{i=0}^{m} s_i}{(n-m)(n+1)} + \frac{1}{n-m}\sum_{i=m+1}^{n}[s_n - s_i] \\ &= \frac{m+1}{n-m}\sigma_n - \frac{m+1}{n-m}\sigma_m + \frac{1}{n-m}\sum_{i=m+1}^{n}[s_n - s_i] \end{aligned}$$

For $i \geq m+1$,

$$|s_n - s_i| = \left|\sum_{k=i+1}^{n} a_k\right| \leq \sum_{k=i+1}^{n}|a_k| \leq \sum_{k=i+1}^{n}\frac{M}{k} \leq \sum_{k=i+1}^{n}\frac{M}{i+1} = \frac{M(n-i)}{i+1} \leq \frac{M(n-m-1)}{m+2}$$

Choose $\epsilon > 0$ small. Then $\frac{n-\epsilon}{1+\epsilon}$ lies in the interval $[m, m+1)$, for some nonnegative integer m, so that

$$m(1+\epsilon) \leq n - \epsilon < (m+1)(1+\epsilon) \Longleftrightarrow (m+1)\epsilon \leq n - m < 1 + (m+1)\epsilon,$$

which implies that $\frac{m+1}{n-m} \leq \frac{1}{\epsilon}$ and therefore $|s_n - s_i| < M\epsilon$, for $m+1 \leq i \leq n$. Since $m \to \infty$ as $n \to \infty$ and by our hypothesis that $\sigma_n \to \sigma$, we see that

$$\begin{aligned} \limsup_{n\to\infty}|s_n - \sigma| &\leq \limsup_{n\to\infty}\left[|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{i=m+1}^{n} M\epsilon\right] \\ &= M\epsilon + \limsup_{n\to\infty}|\sigma_n - \sigma_m| \\ &= M\epsilon \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $s_n \to \sigma$. □

1.3.14 Exercise 15

For solutions to exercises 16, 17, and 18, see my note on square root algorithms.

1.3.15 Exercise 16

1.3.16 Exercise 17

1.3.17 Exercise 18

1.3.18 Exercise 19

1.3.19 Exercise 20

If $(p_n)_{n=1}^{\infty}$ is a Cauchy sequence in a metric space (X, d), and some subsequence $(p_{n_k})_{k=1}^{\infty}$ converges to a point $p \in X$. Then $p_n \to p$.

Proof. Choose $N \in \mathbb{Z}^{\geq 1}$ such that $n, m \geq N$ implies that $d(p_n, p_m) < \frac{\epsilon}{2}$. Choose $k_0 \in \mathbb{Z}^{\geq 1}$ such that $k \geq k_0$ implies that $n_k \geq N$ and $d(p_{n_k}, p) < \frac{\epsilon}{2}$. By the triangle inequality,

$$d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

1.3.20 Exercise 21

For solutions to exercises 23, 24, and 25, see my blog post entitled 'How to Complete Your Metric Space.'