NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

Ph.D. QUALIFYING EXAMINATION

ANALYSIS (SAMPLE PAPER 2)

Time allowed : 3 hours

Note that the passing mark is usually around 60.

(1) If $\limsup_{n\to\infty} a_n \le l$, show that $\limsup_{n\to\infty} \sum_{i=1}^n a_i/n \le l$. [5 marks]

(2) If f is a nonnegative measurable function on $\mathbb{R}$ and $p > 0$, show that

$$\int f^p dx = \int_0^\infty pt^{p-1} |\{x : f(x) > t\}| dt,$$

where $|\{x : f(x) > t\}|$ is the Lebesgue measure of the set $\{x : f(x) > t\}$. [10 marks]

(3) If f is a nonnegative measurable function on $[0, \pi]$ and $\int_0^\pi f(x)^3 dx < \infty$, show that

$$\lim_{\alpha\to\infty} \int_{\{x:f(x)>\alpha\}} f(x)^2 dx = 0.$$

[5 marks]

(4) Prove or disprove each of the following statements. [40 marks]

(a) If $f : [0, 1] \to \mathbb{R}$ is a measurable function, then given any $\varepsilon > 0$, there exists a compact set $K \subset [0, 1]$ such that f is continuous on K relative to K.

(b) If $\{f_n\}$ is a sequence of measurable functions that converges uniformly to f on $\mathbb{R}$, then $\displaystyle\int f = \lim_{k\to\infty} \int f_k$.

(c) If f is Borel measurable on $\mathbb{R} \times \mathbb{R}$, then for any $x \in \mathbb{R}$, the function $g(y) = f(x, y)$ is also Borel measurable on $\mathbb{R}$.

(d) If $E \subset \mathbb{R}$, then E is measurable if and only if given any $\varepsilon > 0$, there exist a closed set F and an open set G such that $F \subset E \subset G$ and the measure of $G\mathcal{F}$ is less than ε.

(e) If $\{f_k\}$ is a sequence of function in $L^p[0,\infty)$ that converges to a function $f \in L^p[0,\infty)$, then $\{f_k\}$ has a subsequence that converges to f almost everywhere.

(f) If f is Riemann integrable on $[\varepsilon, 1]$ for all $0 < \varepsilon < 1$, then f is Lebesgue integrable on [0,1] if f is nonnegative and the following limit exists $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f dx$.

(g) If f is integrable on [0,1], then $\lim_{n\to\infty} \int_0^1 f(x) \sin n\pi x = 0$.

(h) If f is continuous on $[0,1]$, then it is of bounded variation on $[0,]$.

(5) [15 marks]

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(-1) < 2$ and $f'(1) > 2$, show that there exists $x_0 \in (-1,1)$ such that $f'(x_0) = 2$. (Hint: consider the function $f(x) - 2x$ and recall the proof of Rolle's theorem)

(b) Let $f : (-1,1) \to \mathbb{R}$ be a differentiable function on $(-1,0) \cup (0,1)$ such that $\lim_{x\to 0} f'(x) = l$. If f is continuous on $(-1,1)$, show that f is indeed differentiable at 0 and $f'(0) = l$.

(6) Find an analytic isomorphism from the open region between $x = 1$ and $x = 3$ to the upper half unit disk $\{|z| < 1, \text{Im} z > 0\}$. (You may leave your result as a composition of functions). [8 marks]

(7) Use Cauchy theorem to prove the argument principle. [10 marks]

(8) Evaluate the following by the method of residues: [7 marks]

$$\int_0^{\pi/2} \frac{1}{3 + \sin^2 x} dx.$$