NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

Ph.D. QUALIFYING EXAMINATION

ANALYSIS (SAMPLE PAPER 1)

Time allowed : 3 hours

Note that the passing mark is usually around 60.

(1) Prove or disprove each of the following statements. [40 marks]

(a) If f is of bounded variation on [0,1], then it is continuous on [0,1].

(b) If $f : [0,1] \to [0,1]$ is a continuous function, then there exists $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

(c) Let $\{f_n\}$ be a sequence of uniformly continuous functions on an interval I. If $\{f_n\}$ converges uniformly to a function f on I, then f is also uniformly continuous on I.

(d) If $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists, then $\lim_{n\to\infty} |a_n|^{1/n}$ exists and the two limits are equal.

(e) If $\sum_{n=1}^{\infty} a_n x^n$ converges for all $x \in [0,1]$, then

$$\lim_{x\to 1^-} \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n.$$

(f) If $E \subset \mathbb{R}$ and

$$\mu(E) = \inf\{\sum_{I_i \in S} |I_i| : S = \{I_i\}_{i=1}^n \text{ such that } E \subset \cup_{i=1} I_i \text{ for some } n \in \mathbb{N}\},$$

then μ coincides with the outer measure of E.

(g) If E is a Borel set and f is a measurable function, then $f^{-1}(E)$ is also measurable.

(h) If f is differentiable on a connected set $E \subset \mathbb{R}^n$, then for any $x, y \in E$, there exists $z \in E$ such that $f(x) - f(y) = \nabla f(z)[x - y]$.

(2) If f is a finite real value measurable function on a measurable set $E \subset \mathbb{R}$, show that the set $\{(x, f(x)) : x \in E\}$ is measurable. [10 marks]

(3) Let $g : [0,1] \times [0,1] \to [0,1]$ be a continuous function and let $\{f_n\}$ be a sequence of functions such that [15 marks]

$$f_n(x) = \begin{cases} 0, & 0 \le x \le 1/n, \\ \int_0^{x-\frac{1}{n}} g(t, f_n(t))dt, & 1/n \le x \le 1. \end{cases}$$

With the help of the Arzela-Ascoli theorem or otherwise, show that there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = \int_0^x g(t, f(t))dt$$

for all $x \in [0,1]$. (Hint: first show that $|f_n(x_1) - f_n(x_2)| \le |x_1 - x_2|$.)

(4) Find the number of zeroes, counting multiplicities, of the polynomial [5 marks]

$$f(z) = 2z^5 - 6z^2 - z + 1 = 0$$

in the annulus $1 \le |z| \le 2$.

(5) Find an analytic isomorphism from the open region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ to the upper half plane $\text{Im}z > 0$. (You may leave your result as a composition of functions). [5 marks]

(6) Use Green theorem or otherwise to prove the Cauchy theorem. [10 marks]

(7) Evaluate the improper integral [5 marks]

$$\int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)}.$$

(8) State and prove the divergence theorem on any rectangle in $\mathbb{R}^2$. [10 marks]