NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2011-2012

Ph.D. QUALIFYING EXAMINATION

Paper 2

ANALYSIS

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination contains a total of **TEN (10)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. The maximum score for this examination is 100 points.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [10 points] For each of the following statements, prove it if it is true and provide a counterexample if it is false.

(a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function so that the directional derivative of f in any direction exists at $(0,0)$. Then f is differentiable at $(0,0)$.

(b) Let U be a connected open set in $\mathbb{C}$ and let $f : U \longrightarrow \mathbb{C}$ be an analytic function. If there exists $z_0 \in U$ such that $\operatorname{Re} f(z_0) \geq \operatorname{Re} f(z)$ for any $z \in U$, then f is constant on U.

Question 2 [10 points] Let X be a compact metric space. Show that there is a sequence of open sets $(U_n)_{n=1}^{\infty}$ in X such that for any $x_0 \in X$ and any closed set F in X not containing x_0, there exists n so that $x_0 \in U_n$ and $\overline{U_n} \cap F = \emptyset$.

Question 3 [10 points] Let f be a complex function that is analytic on an open set containing the closed ball $\{z \in \mathbb{C} : |z| \leq 1\}$. Assume that $f(0) \neq 0$ and that $f(z) \neq 0$ for any z with $|z| = 1$. Suppose that $(a_k)_{k=1}^{n}$ are the distinct zeros of f in $\{z \in \mathbb{C} : |z| < 1\}$, with respective multiplicities $(m_k)_{k=1}^{n}$. Show that

$$\sum_{k=1}^{n} \frac{m_k}{a_k^2} = \int_C \frac{f'(z)}{z f(z)}\, dz - \frac{f'(0)}{f(0)},$$

where C is the circle $\{z : |z| = 1\}$, traversed once in the counterclockwise direction.

Question 4 [10 points] Let $f : [0,1] \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. Denote Lebesgue measure by λ. Show that the series

$$s_n = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \lambda\left(\left\{x : \frac{k}{2^n} < f(x) \leq \frac{k+1}{2^n}\right\}\right)$$

converges absolutely for each $n \in \mathbb{N}$, and that $\lim_{n\to\infty} s_n = \int_0^1 f\, d\lambda$.

Question 5 [10 points] For any $n \in \mathbb{N}$, the n-th Rademacher function $r_n : [0,1] \longrightarrow \mathbb{R}$ is defined by

$$r_n(t) = \begin{cases} (-1)^{k+1} & \text{if } t \in [\frac{k-1}{2^n}, \frac{k}{2^n}),\ 1 \leq k \leq 2^n, \\ 0 & \text{if } t = 1. \end{cases}$$

Show that $\lim_{n\to\infty} \int_0^1 f r_n d\lambda = 0$ for any $f \in L^1[0,1]$. Here λ denotes Lebesgue measure.

Question 4 [10 points] Let U be an open subset of $\mathbb{C}$ and let (f_n) be a sequence of analytic functions on U. Suppose that (f_n) converges uniformly on compact subsets of U to a function f. Let $w \in U$ be an isolated zero of f. Show that there exist $n_0 \in \mathbb{N}$ and a sequence (z_n) in U converging to w such that $f_n(z_n) = 0$ for all $n \geq n_0$.

Question 5 [10 points] Denote the Lebesgue measure and the Lebesgue outer measure on $\mathbb{R}$ by λ and λ^* respectively. Let D be a subset of $\mathbb{R}$ with $\lambda^*(D) < \infty$. Suppose that for any interval $I \subseteq \mathbb{R}$,

$$\lambda^*(D \cap I) \leq \frac{1}{2}\lambda(I).$$

Show that D is Lebesgue measurable and has Lebesgue measure 0.

Question 6 [10 points] Let (Ω, Σ, μ) be a measure space and let $1 \leq p_1 < p < p_2 < \infty$. If $f \in L^{p_1}(\Omega, \Sigma, \mu) \cap L^{p_2}(\Omega, \Sigma, \mu)$, show that $f \in L^p(\Omega, \Sigma, \mu)$ and that

$$\int |f|^p \, d\mu \leq \Big(\int |f|^{p_1} \, d\mu\Big)^{\alpha} \cdot \Big(\int |f|^{p_2} \, d\mu\Big)^{1-\alpha},$$

where $p = \alpha p_1 + (1-\alpha)p_2$.

Question 7 [10 points] Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative Lebesgue integrable function such that $\int f \, d\lambda > 0$. Show that $\liminf_{k \to \infty} \int |\cos(kx)| f(x) \, d\lambda(x) > 0$, where λ denotes Lebesgue measure.
[Hint: First consider characteristic functions of intervals.]

Question 8 [10 points] Denote Lebesgue measure by λ. Let $(f_k)_{k=1}^{\infty}$ be a sequence of nonnegative Lebesgue integrable functions on $[0,1]$. Assume that $\int_{[0,1]} f_k \, d\lambda = 1$ and that there is a constant $M < \infty$ so that $f_k(t) \leq M$ for all k and all $t \in [0,1]$. If $(a_k)_{k=1}^{\infty}$ is a nonnegative real sequence such that $\sum_{k=1}^{\infty} a_k = \infty$, show that there is a Lebesgue measurable subset E of $[0,1]$ with $\lambda(E) > 0$ such that $\sum_{k=1}^{\infty} a_k f_k(t) = \infty$ for all $t \in E$.
[Hint: Egoroff's Theorem.]

Question 9 [10 points] Let (Ω, Σ, μ) be a finite measure space. Suppose that f is a nonnegative Σ-measurable function on Ω. Define $E_n = \{\omega \in \Omega : f(\omega) \geq n\}$ for each $n \in \mathbb{N} \cup \{0\}$. Show that f is integrable if and only if $\sum_{n=0}^{\infty} \mu(E_n) < \infty$.

Question 1 [10 points]

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose that

$$f_1(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$f_2(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

both exist for all $(x,y) \in \mathbb{R}^2$, and that f_1 and f_2 are both continuous on $\mathbb{R}^2$. If

$$\lim_{h\to 0} \frac{f_1(x,y+h) - f_1(x,y)}{h} = g(x,y)$$

exists for all (x,y) and g is continuous on $\mathbb{R}^2$, show that

$$\lim_{h\to 0} \frac{f_2(x+h,y) - f_2(x,y)}{h} = g(x,y)$$

for all (x,y).

(b) Compute the integral

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx.$$

Question 2 [10 points] Let $f_n : [0,1] \to \mathbb{R}$ be a continuous function that is differentiable on $(0,1)$ for $n = 1,2,3,\ldots$. Assume that

$$\sup_n \sup_{x\in[0,1]} |f_n(x)| \quad \text{and} \quad \sup_n \sup_{x\in(0,1)} |f_n'(x)|$$

are both finite. Show that there is a subsequence (f_{n_k}) of (f_n) that converges uniformly on $[0,1]$.

Question 3 [10 points] Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc. Denote by $\mathcal{F}$ the set of all analytic functions $f : D \to D$ such that $f(0) = f'(0) = 0$. Show that

$$M = \sup\{|f''(0)| : f \in \mathcal{F}\} < \infty$$

and find its value. Determine all functions $f \in \mathcal{F}$ such that $|f''(0)| = M$.

Question 6 [10 points] Let $f, g : [0,1] \to \mathbb{R}$ be integrable functions with respect to Lebesgue measure λ. Assume that for any $a, b \in \mathbb{R}$

$$\lambda(\{t : f(t) \leq a\} \cap \{t : g(t) \leq b\}) = \lambda\{t : f(t) \leq a\} \cdot \lambda\{t : g(t) \leq b\}.$$

Show that fg is integrable on $[0,1]$ with respect to Lebesgue measure and that

$$\int_{[0,1]} fg \, d\lambda = \int_{[0,1]} f \, d\lambda \cdot \int_{[0,1]} g \, d\lambda.$$

Question 7 [10 points] Let $(f_k)_{k=1}^{\infty}$ be a sequence in $L^p(\mathbb{R})$, where $1 \leq p < \infty$. Suppose that $f_1 \leq f_2 \leq \cdots$ and $\sup_k \|f_k\|_p < \infty$. Show that $(f_k)_{k=1}^{\infty}$ converges in L^p norm.

Question 8 [10 points] Let f be a Lebesgue integrable function on $[0,1]$ and denote Lebesgue measure by λ. Suppose that $0 < \alpha < 1$. Show that for almost all $t \in [0,1]$, the function $F_t(x) = f(x)|x-t|^{-\alpha}$ is integrable on $[0,1]$. Define $g(t) = \int_0^1 F_t \, d\lambda$ where the integral exists and 0 otherwise. Show that $g \in L^1[0,1]$.

Question 9 [10 points] Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : (a,b) \to \mathbb{R}$ be a continuous function. Define F to be the set of all $x \in (a,b)$ such that $f'(x)$ exists (as a real number). For each $k \in \mathbb{N}$, and any $p, q, q' \in \mathbb{Q}$ with $a < q < q' < b$, define

$$H(k,p,q,q') = \{x \in (q,q') : |f(y) - f(x) - p(y-x)| \leq \frac{|y-x|}{k} \text{ for all } y \in (q,q')\}.$$

Express F in terms of the sets $H(k,p,q,q')$ and deduce that F is a Borel set.

Question 10 [10 points] Suppose that $1 \leq p < \infty$. Show that there is a linear bijection $T : L^p[0,1] \to L^p(\mathbb{R})$ such that $\int_{\mathbb{R}} |Tf|^p \, d\lambda = \int_0^1 |f|^p \, d\lambda$ for all $f \in L^p[0,1]$, where λ is Lebesgue measure.

END OF PAPER

Ph.D. Qualifying Examination 2013 Jan (Analysis)

(1) A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if given any $x \in \mathbb{R}^n$, there exist $\delta, L > 0$ (depending on x) such that

$$|\phi(z) - \phi(y)| \le L|z - y| \quad \text{for all } z, y \in B_\delta(x) = \{t \in \mathbb{R}^n : |x - t| < \delta\}. \qquad (*)$$

If ϕ is locally Lipschitz on $\mathbb{R}^n$, show that for any compact set $K \subset \mathbb{R}^n$, there exists a constant $M > 0$ (depending on K) such that [10]

$$|\phi(x) - \phi(y)| \le M|x - y| \quad \text{for all } x, y \in K.$$

(2) Let $w \in L^1(\mathbb{R}^d)$ be strictly positive and $\{f_n\} : \mathbb{R}^d \to \mathbb{R}$ be (Lebesgue) measurable functions such that [12]

$$\lim_{m,n\to\infty} \int_{|f_n - f_m| > t} w(x)dx = 0 \quad \text{for any } t > 0.$$

Show that $\{f_n\}$ has a subsequence $\{f_{n_j}\}$ that converges a.e. to a measurable function $g(x)$.

(3) Let Ω be an open connected subset of $\mathbb{R}^3$. Suppose $u \in C_0^2(\Omega)$ and $f \in C_0(\Omega)$ are such that

$$\Delta u - 2u = f \quad \text{on } \Omega \text{ where } \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Show that [5]

$$\int |f|^2 dx = \int |\Delta u|^2 + 4|\nabla u|^2 + 4|u|^2 dx \quad \text{where } \nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}).$$

(4) Compute (a) $\int_\gamma e^z/z^3 dz$, where $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(t) = e^{i6\pi t}$ and

(b) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta}$, $a > 0$. [12]

(5) Let $f : \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \to \mathbb{R}$ be such that f is continuously differentiable with $\nabla f(0,0) = (1,2)$ and $f(0,0) = 0$. Show that there exist $\varepsilon > 0$ and $\gamma \in C^1(-\varepsilon, \varepsilon)$ such that $\gamma(0) = 0$ and $f(x, \gamma(x)) = 0$ for $x \in (-\varepsilon, \varepsilon)$. Compute $\gamma'(0)$ if possible. [5]

(6) Explain in details with the help of the fact that $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $|x| < 1$ why [10]

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\log 2.$$

(7) Let $w \in L^1[0,\pi]$ be nonnegative. Suppose $\{f_n\}$ is a sequence of (Lebesgue measurable) functions that converges a.e. to a function f. Suppose $\int_0^\pi |f_n|^2 w dx \to \int_0^\pi |f|^2 w dx < \infty$. Show that $\int_0^\pi |f_n - f|^2 w dx \to 0$. [10].

(8) Prove or disprove not more than six (6) of the following statements. [36]

(a) If ϕ is a function of bounded variation on $[a,b]$ for all $[a,b] \subset \mathbb{R}$ and g is a nondecreasing function on [0,1], then $\phi(g) \in BV[0,1]$.

(b) Let C be the Cantor set. Then χ_C is Riemann integrable on $[0,1]$.

(c) If $\{f_j\}, f : (0,1) \to \mathbb{C}$ are integrable such that

$$\lim_{j\to\infty} \int_K f_j(x)dx = \int_K f(x)dx \quad \text{for all compact subset } K \text{ of } (0,1),$$

then

$$\lim_{j\to\infty} \int_{(0,1)} f_j(x)dx = \int_{(0,1)} f(x)dx.$$

(d) If $\{a_{i,j}\}_{i,j=1}^{\infty}$ is a collection of nonnegative real numbers, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = \lim_{N\to\infty} \sum_{\{i,j: i+j\le N\}} a_{i,j}.$$

(e) Let $\{f_n\} : \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ be analytic. If there exists f on $\mathbb{D}$ such that $f_n \to f$ uniformly on any compact subset of $\mathbb{D}$, then f is also analytic on $\mathbb{D}$.

(f) Let $u : G \to \mathbb{R}$ be a harmonic function where G is an open connected set in $\mathbb{C}$. Then there exists $v : G \to \mathbb{R}$ such that the function $f(z) = u(z) + iv(z)$ is analytic on G.

(g) If a function is piecewise differentiable on $[a,b]$, then it is of bounded variation on $[a,b]$.

(Note that a function f is said to be piecewise differentiable on $[a,b]$ if there exist a partition $a_0 = a < a_1 < \cdots < a_k = b$ of $[a,b]$ and $g_i : [a_{i-1}, a_i] \to \mathbb{R}$ such that its derivative g_i' is continuous on $[a_{i-1}, a_i]$ and $f = g_i$ on (a_{i-1}, a_i) for $i = 1, \cdots, k$.)

(h) Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable and for all $x \in \mathbb{R}^n$, define

$$h(x) = \limsup_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|^{1/2} dy.$$

Then $|f(x)| \le h(x)^2$ a.e..

— **END OF PAPER** —

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2013-2014

Ph.D. QUALIFYING EXAMINATION

Paper 2

ANALYSIS

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination contains a total of **TEN (10)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The maximum score for this examination is 100 points.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

(9) Prove or disprove **eight** of the following statements. [32 marks]

(a) If $f : [0,1] \to \mathbb{R}$ is a measurable function, then given any $\varepsilon > 0$, there exists a compact set $K \subset [0,1]$ with $|[0,1] \setminus K| < \varepsilon$ such that f is continuous on K.

(b) If $\{f_n\}$ is a nondecreasing sequence of Riemann integrable functions on $[0,1]$ that converges to 0 on $[0,1]$, then $\lim_{k\to\infty} \int f_k = 0$.

(c) If f is integrable on $[0,\pi]$, then $\lim_{n\to\infty} \int_0^\pi f(x) \cos nx dx = 0$.

(d) If f is a real function on $\mathbb{R}$ such that it is of bounded variation on $[a,b]$ for all $-\infty < a < b < \infty$, then f is continuous everywhere except countably many points.

(e) Let $\{f_n\}$ be a sequence of harmonic functions on the open unit disk in $\mathbb{R}^2$. If $f_n \to f$ uniformly on the open unit disk, then f is also harmonic on the open unit disk.

(f) Let U be a bounded open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$. If there exists a sequence of continuously differentiable functions $\{f_n\}$ that converges uniformly to f on U, then f is differentiable on U.

(g) If $f = u + iv$ (u and v are both real-valued functions) is an entire function such that $v(z) < 1$ for all z, then u must be a constant function.

(h) If f is an analytic function on an open connected set $\mathcal{D}$ (in the complex plane), then it is either a constant function or it will map open subsets of $\mathcal{D}$ to open sets.

(i) Let $\sum_{k=1}^{\infty} a_k$ be a convergent series. Then $\sum_{k=1}^{\infty} a_k \sin(k\pi x)$ converges if x is irrational.

(j) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function such that the Jacobian metric ∇f has nonzero determinant at the origin 0. If $f(0) = (1,0)$, then there exists $\varepsilon > 0$ such that for all $y \in \mathbb{R}^2$ with $|y - (1,0)| < \varepsilon$, the equation $f(x) = y$ has at least one solution.

— **END OF PAPER** —

Ph.D. Qualifying Examination 2011 January (Analysis)

(1) If $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal family of functions in a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, show the Bessel's inequality: [8 marks]

$$\sum_{k=1}^{\infty} |\langle x, \phi_k \rangle|^2 \leq \langle x, x \rangle \quad \text{for all } x \in H.$$

(2) Let $\{x_n\}$ be a bounded sequence of real numbers and let S be the collection of limit points of convergent subsequences of $\{x_n\}$. Show that S is closed. [10 marks]

(3) Explain why there is no differentiable function f on $\mathbb{R}$ such that $f' = \chi_{\mathbb{Q}}$ on $\mathbb{R}$. [8 marks]

(4) Let $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Consider a subset of $C[a, b]$ (the collection of continuous functions on $[a, b]$)

$$S = \{\int_a^x f(t, g(t))dt, x \in [a, b] : g \in C[a, b] \text{ such that } g(t) \in [c, d] \text{ for all } t \in [a, b].\}$$

Show that S is precompact in $C[a, b]$ (under the metric $d(\phi_1, \phi_2) = \sup_{x\in[a,b]} |\phi_1(x) - \phi_2(x)|$). [5 marks]

(5) Compute (and justify) **one** of the following: [10 marks]

(i) $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(6) If f is a nonnegative measurable function on $\mathbb{R}^n$ and $\int f(x)dx < \infty$, show that

$$\lim_{\alpha\to\infty} \int_{\{x: f(x) > \alpha\}} f(x)dx = 0.$$

[7 marks]

(7) Let $1 < p < \infty$ and $\{f_k\}$ be a bounded sequence of functions in $L^p(\mathbb{R}^n)$ (i.e., there exists $C > 0$ such that $\|f_k\|_p \leq C$). If $f_k \to f$ a.e., show that

$$\int f_k g dx \to \int f g dx$$

for all $g \in L^q(\mathbb{R}^n)$ where $1/q = (p-1)/p$. [10 marks]

(8) Let $1 \leq p < \infty$ and $1/q = (p-1)/p$. Let f be a measurable function on $[0, 1]$ such that

$$|\int_0^1 f g dx| \leq \|g\|_q \quad \text{for all step functions } g \text{ on } [0, 1] .$$

Show that $\|f\|_p \leq 1$. [10 marks]

Question 5 [10 marks]

Suppose $f(u)$ is a continuous function on $[-1,1]$. Show that

$$\int\int_{x^2+y^2+z^2=1} f(z)ds = 2\pi \int_{-1}^{1} f(z)dz.$$

Question 6 [10 marks]

Suppose that $\sum_{n=1}^{\infty} u_n$ converges. Show that

$$\lim_{n\to\infty} \frac{u_1 + 2u_2 + 3u_3 + \cdots + nu_n}{n} = 0.$$

Question 7 [10 marks]

Let $\phi(t)$ be a positive continuous function on $[0,\infty)$ and $f(t,x)$ be a continuous function of two variables such that $|f(t,x)| \leq \phi(t)|x|$. Suppose $\int_0^\infty \phi(t) < \infty$. Show that if the function y satisfies the inequality

$$|y(t)| \leq \int_0^t |f(s,y(s))|ds,$$

for all $t \in [0,\infty)$, then $y(t) \equiv 0$.

Question 8 [10 marks]

Consider the functions

$$f(x) = (\int_0^x e^{-u^2} du)^2$$

and

$$g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1} dt.$$

Show that $f(x) + g(x) = \frac{\pi}{4}$ for all $x \geq 0$ and hence $\lim_{x\to\infty} \int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

END OF PAPER

Answer **all** *the questions in this paper*

Question 1 [20 marks]

Suppose f is a non-negative function on R^n such that $\int_{R^n} f = 1$. If $0 < p < 1$ is real number, show that

$$\int_E f^p \leq |E|^{1-p}$$

for every measurable set E. Similarly show that if E is a measurable set with $0 < |E| < \infty$, then

$$\int_E \log f \leq -|E| \log |E|.$$

Question 2 [15 marks]

Let Δ be the unit disk of the complex plane. Let f be analytic and bounded by M on Δ. If $a_1, a_2, \cdots, a_n$ are among the zeros of f, and define

$$B(z) = \Pi_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}.$$

(1) Show that B is analytic on Δ and $|B(z)| = 1$ for $|z| = 1$;
(2) Show that $|f(z)| \leq M|B(z)|$ for each $z \in \Delta$.

Question 3 [15 marks]

Show that the polynomial $P(z) = 3z^{15} + 4z^8 + 6z^5 + 19z^4 + 3z + 1$ has (i) 4 zeros for $|z| < 1$ and (ii) 11 zeros for $1 < |z| < 2$.

Question 4 [10 marks]

Let $\{f_k\}$ be a sequence of non-negative measurable functions on a measurable set E with $|E| < \infty$. Is it true that f_k converges to 0 on E in measure as $k \to \infty$ if and only if

$$\lim_{k \to \infty} \int_E \frac{f_k}{1 + f_k} = 0?$$

If it is true, prove it. If it is not true, provide a counterexample.

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

Qualification Examination

Analysis

January, 2010 — Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper comprises **THREE (3)** printed pages.
2. This paper consists of **EIGHT (8)** questions. Answer **ALL** of them. Marks for each question are not necessarily the same; marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

(a) If f is absolutely continuous on $[a,b]$, then it is a difference of two continuous nondecreasing functions on $[a,b]$.

(b) Let $D_1 = [\frac{1}{3}, \frac{2}{3}]$, $D_2 = [\frac{1}{3^2}, \frac{2}{3^2}] \cup [\frac{7}{3^2}, \frac{8}{3^2}]$,

$$D_3 = [\frac{1}{3^3}, \frac{2}{3^3}] \cup [\frac{7}{3^3}, \frac{8}{3^3}] \cup [\frac{19}{3^3}, \frac{20}{3^3}] \cup [\frac{25}{3^3}, \frac{26}{3^3}].$$

Define $D_4, \cdots$ similarly. Then $[0,1] \setminus \cup_k D_k$ is countable.

(c) Let $\{f_n\} : \mathcal{D} = \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ be a sequence of analytic functions. Suppose there exists f on $\mathcal{D}$ such that $f_n \to f$ uniformly on any compact subset of $\mathcal{D}$. Then there exists $N > 0$ such that f_n has the same number of zeroes (counting multiplicities) for $n \geq N$.

(d) Given any compact set K and open set Ω, if $K \subset \Omega$, then there exists infinitely differentiable function f such that

$$\chi_K \leq f \leq \chi_\Omega.$$

(e) Let f be an analytic function on a neighborhood of $\mathcal{D}$ (see part (c)). If $|f(z)| < 1$ for $|z| = 1$, then there is a unique $z_0 \in \mathcal{D}$ such that $f(z_0) = z_0$.

(f) If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then both $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ diverges. Note that $a_k^+ = \max\{a_k, 0\}$ and $a_k^- = a_k^+ - a_k$.

(g) If f is a Borel measurable function on $\mathbb{R}^n$, then $f = \sum_{k=1}^{\infty} a_k \chi_{A_k}$, for some $\{a_k\} \subset \mathbb{R}$ and Borel sets $\{A_k\}$ in $\mathbb{R}^n$.

(h) $\int_{\mathbb{R}^2} det Df dx = 0$, if $f \in C_0^2(\mathbb{R}^2 \to \mathbb{R}^2)$, i.e., $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that it is twice continuously differentiable and with compact support.

(7) Let Ω be an open set in $\mathbb{R}^n$ and $f \in L^p(\Omega)$, $1 \leq p < \infty$. Define [14]

$$\|f\|_{p,\sharp,\Omega} = \inf\{\|f - a\|_{L^p(\Omega)} : a \in \mathbb{R}\}.$$

Show that there exists $a \in \mathbb{R}$ such that $\|f\|_{p,\sharp,\Omega} = \|f - a\|_{L^p(\Omega)}$.

Suppose $\{\Omega_i\}$ is a sequence of bounded domains such that $\Omega_i \subset \Omega_{i+1}$ for all i and $f \in L^p(\Omega_i)$ for all i such that $\sup_i \|f\|_{p,\sharp,\Omega_i} < \infty$, show that the sequence $\{\|f\|_{p,\sharp,\Omega_i}\}$ is a convergent sequence and

$$\|f\|_{p,\sharp,\cup\Omega_i} = \lim_{i\to\infty} \|f\|_{p,\sharp,\Omega_i}.$$

(Note that $\|f\|_{p,\sharp,\Omega}$ will be defined even if $f \notin L^p(\Omega)$ provided f is measurable.)

Ph.D. Qualifying Examination 2013 Aug (Analysis)

(1) If $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $1 \leq p < q \leq \infty$, show that for any $p < r < q$, there exists $0 < \lambda < 1$ such that [6]

$$\|f\|_{L^r(\mathbb{R}^n)} \leq \|f\|^{\lambda}_{L^p(\mathbb{R}^n)} \|f\|^{1-\lambda}_{L^q(\mathbb{R}^n)} \quad \text{for all } f \in L^r(\mathbb{R}^n).$$

(2) (i) State Schwarz's lemma.

(ii) Find an 1-1 conformal mapping from $\mathcal{D}$ onto the right half plane of $\mathbb{C}$ ($\mathcal{D}$ is the open unit Disc centered at the origin).

(iii) Let $f : \mathcal{D} \to \mathbb{C}$ be an analytic function such that $\mathbf{Re}f(z) \geq 0$ for all $z \in \mathcal{D}$ and $f(0) = 1$. Show that

(a) $\mathbf{Re}f(z) > 0$ for all $z \in \mathcal{D}$;

(b) $|f(z)| \leq \dfrac{1+|z|}{1-|z|}$ for all $z \in \mathcal{D}$;

(c) $|f(z)| \geq \dfrac{1-|z|}{1+|z|}$ for all $z \in \mathcal{D}$. [15]

(3) Let f be a continuous function on (a, b) such that $D_+f(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \geq 0$ exists for all $x \in (a, b)$. Show that $f(x_1) \geq f(x_0)$ for all $x_1 \geq x_0, x_0, x_1 \in (a, b)$. [6]

(4) Let $u : [0, \infty) \to \mathbb{R}$ be a monotone function such that $\int_0^\infty |u(r)|r^2 dr < \infty$. If $f : \mathbb{R}^3 \to \mathbb{R}$ is such that $f(x) = u(|x|)$ for all x. Show that [11]

(i) f is measurable on $\mathbb{R}^3$;

(ii) $\int_{\mathbb{R}^3} f dx = 4\pi \int_0^\infty u(r)r^2 dr$. (Note that f may not be continuous and thus you cannot use Polar coordinates directly to get the answer.)

(5) Let $1 < p < \infty$. Suppose $\{a_i\}$ is a sequence of nonnegative real numbers and $\{B_{r_i}(x_i)\}$ is a sequence of open balls in $\mathbb{R}^n$. [12]

Let $g \in L^q(\mathbb{R}^n)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and define

$$g^*(y) = \sup\{\frac{1}{|B|}\int_B |g|dx : B \text{ is any open ball containing } y\}.$$

For each i, let $B_i = B_{r_i}(x_i)$ and $3B_i = B_{3r_i}(x_i)$. Show that there exists $C_0 > 0$ such that

$$\int_{\mathbb{R}^n} \sum_i a_i \chi_{3B_i}(x)|g(x)|dx \leq \int_{\mathbb{R}^n} C_0 \sum_i a_i \chi_{B_i}(x) g^*(x) dx.$$

Hence show that there exists $C > 0$ (independent of a_i's) such that

$$\|\sum_i a_i \chi_{3B_i}\|_{L^p} \leq C\|\sum_i a_i \chi_{B_i}\|_{L^p}.$$

(6) Prove or disprove **SIX** (6) of the following eight (8) statements. [36]

(7) Let Ω be an open connected subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be analytic. Is it true that f is conformal (on Ω)? Explain what "conformal" means and justify your answer. [5]

(8) For each $z \in \mathbb{C}$, evaluate [10]

$$\int_0^1 \int_0^{2\pi} \frac{1}{re^{i\theta} + z} d\theta dr.$$

(9) Prove or disprove **Six** (6) of the following statements. [30]

(a) If f is an entire function on $\mathbb{C}$, then the function $g(z) = \overline{f(\bar{z})}$ is also entire.

(b) Let Ω be an open connected set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2 : [0, 1] \to \Omega$ are piecewise differentiable such that for all $t \in [0, 1]$,

$$|\gamma_1(t) - \gamma_2(t)| \leq \min\{t, 1 - t\}d(t)/2 \text{ where } d(t) = \inf\{|\gamma_1(t) - z| : z \notin \Omega\},$$

then $\int_0^1 f(\gamma_1(t))\gamma_1'(t)dt = \int_0^1 f(\gamma_2(t))\gamma_2'(t)dt$.

(c) If $\sum_{n=1}^{\infty} a_n(-1)^n$ converges, then $\sum_{n=1}^{\infty} a_n x^n$ converges to a C^∞ function on $(-1, 1)$.

(d) There exists a harmonic function f on $\{z = (x, y) : 0 < x^2 + y^2 < 1\}$ such that

(i) $\lim_{z \to z_0, |z|<1} f(z) = 1$ for all $|z_0| = 1$;

(ii) $\lim_{z \to 0} f(z) = -1$.

(e) If $f : \mathbb{R}^2 \to \mathbb{R}$ is Borel measurable, then for each $x \in \mathbb{R}$, $f_x(y) = f(x, y)$ is Borel measurable on $\mathbb{R}$.

(f) The series $\sum_{k=1}^{\infty} \cos kx/k$ converges conditionally for almost all $x \in \mathbb{R}$.

(g) Let Ω be a connected open set in $\mathbb{C}$. If $f : \Omega \to \mathbb{C}$ is continuous and $\gamma : [0, 1] \to \Omega$ is a rectifiable curve, then the line integral $\int_\gamma f dz$ is defined.

(h) Let $\{a_n\}$ be a bounded sequence of positive real numbers such that $\lim_{n\to\infty} a_{n+1}/a_n = 1$, then $\{a_n\}$ is a convergent sequence.

— **END OF PAPER** —

Ph.D. Qualifying Examination 2012 August (Analysis)

(1) Let $n > 1$. For any $M > 1$, show that there exists $C_M > 1$ such that $\int_{MQ} |x|^{-1}dx \leq C_M \int_Q |x|^{-1}dx$ for any cube $Q \subset \mathbb{R}^n$ where MQ is the concentric cube with M times length as Q and $|x|$ is the Euclidean norm of x. [6]

(2) Let $< X, d >$ be a compact metric space. For any $0 < \alpha < 1$, let $C^\alpha(X)$ be the collection of continuous functions on X such that $\sup_{x \neq y} \frac{|f(x)-f(y)|}{d(x,y)^\alpha} < \infty$. Suppose $\{f_n\}$ is a sequence in $C^\alpha(X)$ such that

$$\sup_n(\sup_{x\neq y} \frac{|f_n(x) - f_n(y)|}{d(x,y)^\alpha} + \sup_{x \in X} |f_n(x)|) < \infty.$$

Show that for each $0 < \beta < \alpha$, $\{f_n\}$ has a convergent subsequence converging uniformly to a function $f \in C^\beta(X)$. [5]

(3) Let $f(x) = \sqrt{x^2 + y^2 + z^2}$. Use the function ∇f on $D = \{(x, y, z) \in \mathbb{R}^3 : 1 < x^2 + y^2 + z^2 < 4\}$ to illustrate divergence theorem. (You will need to compute both integrals.) [7]

(4) Let $f : [0, 1] \to \mathbb{R}$ be measurable and $g \in L^1[0, 1]$ such that $\int_{|f(x)|>t} |g(x)|dx \leq 3/t^2$ for all $t > 0$, show that $\int_0^1 |f(x)|^p|g(x)|dx < \infty$ for $1 < p < 2$. [6]

(5) (i) Let $1 \leq p < \infty$ and w be a nonnegative integrable function on $[0, 1]$. Show that given any interval $I = [a, b] \subset [0, 1]$ and $\varepsilon > 0$, there exists a continuous function ϕ on $[0, 1]$ such that $\phi \geq \chi_I$ and $\int_0^1 |\phi|^p w(x)dx \leq \int_a^b w dx + \varepsilon$. Hence show that $C[0, 1]$ (space of continuous functions on $[0, 1]$) is dense in $L^p_w[0, 1]$ (with norm $(\int_0^1 |f|^p w dx)^{1/p}$). [11]

(ii) Assume that

$$\int_0^1 |f(x) - f_{av}|^p w(x)dx \leq C \int_0^1 |f'(x)|^p w(x)dx \quad \text{where } f_{av} = \int_0^1 f(x)dx \qquad (1)$$

for all $f \in C^1[0, 1]$. Show that inequality (1) holds for all absolutely continuous functions f on $[0, 1]$ such that $f' \in L^p_w([0, 1])$. [10]

(6) Let $f, g \in L^p[a, b]$, $1 < p < \infty$. Show that the function $I(t) = \int_a^b |f(x) + tg(x)|^p dx$ is differentiable at $t = 0$ and compute its derivative. [10]

(8) Prove or disprove **Eight** (8) of the following statements. [32]

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, that is, for all $x \in \mathbb{R}$, there exist $L_x, \delta_x > 0$ such that

$$|f(x) - f(y)| \leq L_x|x - y| \text{ if } |y - x| < \delta_x.$$

Then $f(E)$ is measurable whenever E is measurable.

(b) Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a continuous function. If Ω is simply connected and $\int_\gamma f(z)dz = 0$ for any closed contour γ in Ω, then f is analytic on Ω.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is measurable and nonnegative, then $f = \sum_{k=1}^{\infty} \frac{1}{k}\chi_{A_k}$ where A_k are measurable sets.

(d) Let $\Omega = \{z \in \mathbb{C} : 1 < |z| < 3\}$ and f is a bounded analytic function on Ω. If there exists $z_0 \in \mathbb{C}$, $|z_0| = 2$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$, then $|f(z)| = |f(z_0)|$ for all $z \in \Omega$.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable a.e., then its derivative is measurable.

(f) If $\{a_k\}$ is a monotone sequence of real numbers that converges to 0, then $\sum_{k=1}^{\infty} a_k \cos kx$ converges for almost all $x \in \mathbb{R}$.

(g) If a function is continuous a.e., then it is measurable.

(h) Let $f : [a, b] \to \mathbb{R}$. If there exists $M > 0$ such that for all $a < x_0 < x_1 \cdots < x_n < b$,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M,$$

then f is a function of bounded variation on $[a, b]$.

(i) Let $\{f_n\}$, $\{g_n\}$ be two sequences of measurable functions on $\mathbb{R}$ such that $f_n \to f$ a.e. and $|f_n| \leq g_n$ a.e.. If there exists a measurable function g such that

$$\lim_{n\to\infty} \int g_n dx = \int g dx,$$

then

$$\lim_{n\to\infty} \int f_n dx = \int f dx.$$

(j) Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$. If for each $x, y \in \mathbb{R} \setminus \{0\}$, there exists L such that

$$\lim_{t\to 0} |f(xt, yt) - f(-xt, -yt) - Lt|/t = 0,$$

then f is differentiable at $(0, 0)$.

— **END OF PAPER** —

Ph.D. Qualifying Examination 2011 August (Analysis)

(1) Let $\{a_n\}$ be a sequence of real numbers such that $\sum_{k=1}^{\infty} |a_k| < \infty$. If $\{b_k\}$ is a permutation of $\{a_k\}$, show that [10]

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k.$$

(2) Let f be a finite measurable function on E with $|E| < \infty$. Show that given any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and a compact set $F \subset E$ such that $|E \setminus F| < \varepsilon$ and $|f(x)| < N$ for $x \in F$. [8]

(3) If $f : \mathbb{C} \to \mathbb{C}$ is bounded analytic on a deleted neighborhood of a point z_0, show that z_0 is a removable singularity of f. [10]

(4) Let $f_n, f : \mathbb{R} \to \mathbb{R}$ be measurable functions. If $f_n \to f$ in measure and $f_n \geq 0$ for all n, show that [8]

$$\int f dx \leq \liminf_{n\to\infty} \int f_n dx.$$

(5) Let f be an entire function. Suppose for each $z \in \mathbb{C}$, there exists $n \in \mathbb{N}$ such that $f^{(n)}(z) = 0$. Show that f must be a polynomial. [10]

(6) Let $1 \leq p < \infty$. Show that there exists $C > 0$ such that

$$\left(\int |\sum_{i=1}^{\infty} a_i \chi_{2Q_i}|^p dx\right)^{1/p} \leq C\left(\int |\sum_{i=1}^{\infty} |a_i| \chi_{Q_i}|^p dx\right)^{1/p}$$

for any sequence of real numbers a_i, cubes Q_i ($2Q_i$ is the cube with the same center as Q_i but twice its length). [10]

(7) (i) Let $R = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$. Let $u : R \to \mathbb{R}$ be a twice continuously differentiable function. Prove (without using divergence theorem) by using integration by parts or direct integration that [8]

$$\int_R \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dA = \int_{\partial R} \nabla u \cdot \vec{n} ds$$

where $\vec{n}$ is the outward unit normal vector.

(ii) Let $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$ on a domain Ω in $\mathbb{R}^2$. If ϕ is any C^∞ function that vanishes outside a compact subset of Ω, show that [4]

$$\int_\Omega \phi f dA = -\int_\Omega \nabla u \cdot \nabla \phi dA.$$

Question 5 [10 marks]

Let h be a holomorphic function from the unit disk into itself. If $h(0) = h'(0) = \cdots = h^{(k)}(0) = 0$ for some integer $k > 0$, show that $|h(z)| \leq |z|^{k+1}$ for all $|z| \leq 1$. Further show that there exists a z_0 with $|z_0| < 1$ such that $|h(z_0)| = |z_0|^{k+1}$ if and only if $h(z) = e^{i\theta} z^{k+1}$ for some constant $\theta \in [0, 2\pi)$.

Question 6 [10 marks]

Use two methods to show that

$$\lim_{n \to \infty} \int_0^1 \frac{n^{3/2} x}{1 + n^2 x^2} dx = 0.$$

Question 7 [15 marks]

Let $\{x_n\}$ be a strictly monotone increasing sequence with $x_0 \geq 0$. Show that the series

$$\sum_{n=1}^{\infty} (1 - \frac{x_n}{x_{n+1}})$$

converges if $\{x_n\}$ is bounded and diverges if $\{x_n\}$ is unbounded. (**Hint:** set $d_k = x_{k+1} - x_k$ for $k \geq 1$. Express the series in terms of d_n.)

Question 8 [15 marks]

Let $\varphi_1(x, y, z)$ and $\varphi_2(x, y, z)$ be continuously differentiable functions up to order 2 in the domain $\{(x, y, z) : x^2 + y^2 + z^2 < 4\}$. Show that (1) $\nabla\varphi_1 \times \nabla\varphi_2 = Curl(\varphi_1 \nabla\varphi_2)$ where $Curl V = \nabla \times V$ for any vector field V and $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ for any real valued function f; (2) the total flux $\oint_\Sigma A \cdot d\vec{S}$ of the vector $A = \nabla\varphi_1 \times \nabla\varphi_2$ through the surface $\Sigma := \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$ is zero, where $d\vec{S}$ is the oriented surface area element.

END OF PAPER

Answer **all** *the questions in this paper*

Question 1 [10 marks]

(a) Consider the complex valued function $f(x+iy) = (x^2+y^2)+(xy)i$. Is f holomorphic on **C**? If not, find a complex valued function $g(x+iy) = u(x,y)+iv(x,y)$ with $v(x,y) = 3x^2y - y^3$ such that $h := f+g$ is holomorphic on **C**.

(b) Let h be the holomorphic function found in (a). Let Ω be the set $\{h(z) : |z| \leq 1\}$. Find the area of Ω.

Question 2 [10 marks]

Let f be a holomorphic function on **C**. Suppose there exists a natural number n such that the limit, $\lim_{z\to\infty} \frac{f(z)}{z^n} = M$, exists for some constant M. Show that f is a polynomial with degree less than or equal to n.

Question 3 [15 marks]

Let u be a continuous, positive, integrable function on the interval $[0,\infty)$. Suppose there exist two positive constants a and b such that $\frac{du}{dt} \leq u(a+bu)$. Show that $\lim_{t\to\infty} u = 0$.

Question 4 [15 marks]

Let $\{f_n\}$ be a sequence of non-negative measurable functions on a measurable set E. If for any $\epsilon > 0$, $\sum_{n=1}^{\infty} |\{x \in E : f_n(x) > \epsilon\}| < \infty$, show that $\lim_{n\to\infty} f_n(x) = 0$, *a.e.* on the set E. Is the converse also true? Justify your answer.

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

Qualification Examination

Analysis

August, 2010 — Time allowed : 3 hours

<u>**INSTRUCTIONS TO CANDIDATES**</u>

1. This examination paper comprises **THREE (3)** printed pages.

2. This paper consists of **EIGHT (8)** questions. Answer **ALL** of them. Marks for each question are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.