NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 9

1. Find a formula for all analytic isomorphisms of

   (i) the first quadrant of $\mathbb{C}$ to itself;

   (ii) the right half plane to itself;

   (iii) the open ball $|z| < 2$ onto the unit ball $|z| < 1$.

   Remark: The expressions are not unique.

   [Hint: Use results from Tutorial 8 and the lecture notes:

   (a) The set of analytic automorphisms of the unit ball $|z| < 1$ consists of mappings of the form
   $$f(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z},$$
   where $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, and $\theta \in \mathbb{R}$.

   (b) The set of analytic automorphisms of the UHP $\text{Im } z > 0$ consists of mappings of the form
   $$f(z) = \frac{az+b}{cz+d}, \quad \text{with } a,b,c,d \in \mathbb{R},\ ad-bc > 0.]$$

2. (a) Let C denote the circle passing through the three points $1, i, 1+i$. Find the point z if z and $1-i$ are symmetric with respect to C.

   (b)Find a conformal isomorphism mapping the upper half plane onto $B(0,1)$ and sending i to 0 and ∞ to -1.

   [Hint: Use the Symmetry Principle.]

3. (a) Suppose that C_1 and C_2 are two distinct concentric circles with centre a. Show that the only pair of points z and z^* in $\hat{\mathbb{C}}$ which are symmetric with respect to both C_1 and C_2 are a and ∞ (you may use the geometric interpretation of symmetry).

   (b) Find two points z_1 and z_2 which are symmetric with respect to both the imaginary axis as well as the circle $|z + \frac{5}{2}| = 2$. Hence or otherwise, find a linear fractional transformation which maps the imaginary axis and the circle $|z + \frac{5}{2}| = 2$ to concentric circles centred at the origin.

   [Hint: For part (b), you may need to use the result in part (a).]

4. Find a conformal isomorphism mapping the infinite vertical strip

   $0 < \text{Re } z < 2$ to the unit ball $|z| < 1$.

   [Hint: At an intermediate stage, we may need to map a strip to the unit ball centered at the origin.

5. Find an analytic isomorphism from the region $0 < \arg z < \frac{\pi}{3}$ to the open ball $|z - 1| < 2$.