NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 8

1. Using the cross-ratio notation, write an equation defining a Möbius transformation that maps the half plane below the line $y = 2x - 3$ onto the interior of the circle $|w - 4| = 2$. Repeat for the exterior of the circle.

   [Hint: Recall the Orientation Principle.]

   Remark: The transformation is not unique.

2. Find a Möbius transformation that maps

   (i) the region $\{z : |z - 1| > 1\}$ onto the open ball $|z - 1| < 1$.

   (ii) the region $\{z : |z - 1| > |z - i|\}$ onto the open ball $|z - 1| < 1$.

3. Using the Orientation Principle, show that the map $f(z) = \frac{(1+z)}{(1-z)}$ is an analytic isomorphism of the upper half unit ball $\{z \in \mathbb{C} : |z| < 1, \text{ Im } z > 0\}$ onto the first quadrant of $\mathbb{C}$.

   [Hint: What are the images of $-1, 0, 1$ under f? What are the images of $-1, 1, i$ under f? Explain carefully why f is both injective and surjective with respect to the given domains.]

4. Let $w = f(z)$ be the Möbius transformation that maps the points $0, \lambda, \infty$ to $-i, 1, i$ respectively, where λ is real. For what values of λ is the upper half plane mapped onto the unit ball $B(0, 1)$? Justify your answer.

5. Prove that the set of analytic automorphisms of the unit ball $B(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ consists of mappings of the form

   $$f(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z}$$

   where α and θ are constants such that $|\alpha| < 1$ and $\theta \in \mathbb{R}$.

   [Hint: First show that if f maps 0 to 0, then $f(z) = e^{i\theta} z$. For the general case, compose f by a suitable ϕ_a in Tutorial 5 Question 3. (You may use the results in Tutorial 5 Question 3 that each

   $$\phi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z},$$

   with $|\alpha| < 1$, is an automorphism of $B(0, 1)$ such that $\phi_\alpha(\alpha) = 0$. Also, recall that the inverse of ϕ_α is $\phi_{-\alpha}$.]