NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 7

1. Consider the linear fractional transformations
$$f(z) = \frac{2z+1}{3z+2} \quad \text{and} \quad g(z) = \frac{iz+2}{z+3}.$$
Find $f \circ g$, $g \circ f$ and also g^{-1} in the form of an LFT.

   Answer:
$$f\circ g(z) = \frac{(1+2i)z+7}{(2+3i)z+12}; \quad g\circ f(z) = \frac{(6+2i)z+4+i}{11z+7}; \quad g^{-1}(z) = \frac{-3z+2}{z-i}$$

2. Write the LFT $f(z) = \frac{3z-4-i}{iz-1}$ as a composition of basic transformations (i.e., the inversion, rotations, dilations, and translations).

3. Find a linear map (i.e. a map of the form $f(z) = az + b$) that maps the circle $|z| = 1$ onto the circle $|w - 5| = 3$ and taking the point $z = i$ to $w = 2$.

   [Hint: What are the basic transformations needed?]

4. Write the linear fractional transformation which sends the points

   (i) $2, 3i, 4$ to $\infty, 0, 1$ respectively;

   (ii) $0, i, \infty$ to $\infty, 0, 1$ respectively. Answer: (i) $Tz = \frac{(4-3i)z-8+6i}{2z+6i}$. (ii) $Tz = \frac{z-i}{z}$.

5. Find the Möbius transformation which sends the points

   (i) $-2, 2, i$ to $-1, 1, i$ respectively;

   (ii) $i, -i, 0$ to $1, -1, i$ respectively;

   (iii) $\infty, i, 0$ to $0, i, \infty$ respectively.

   (iv) $0, 1, \infty$ to $-1, -i, 1$ respectively.

   Express your answers in (ii), (iii) and (iv) in the form $w = f(z) = \frac{az+b}{cz+d}$.

6. Find the fixed points in $\hat{\mathbb{C}}$ of the mappings:

   (a) $w = \frac{z-1}{z+1}$, (b) $w = \frac{z}{z+1}$, (c) $w = z + 1$.

   [Recall that fixed points of f are points z such that $f(z) = z$.]

7. (a) Find the linear fractional transformation which has 0 and ∞ as fixed points and which maps $1 + i$ onto $2 + 3i$.

   (b) Suppose the transformation $w = \dfrac{az+b}{cz+d}$, where $ad - bc \neq 0$, is the same as its inverse. Show that either (i) $d = -a$; or (ii) $a = d \neq 0$ and $b = c = 0$.