NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 6

1. Does there exist a non-constant entire function $f(z)$ such that its image $f(\mathbb{C}) \subset \mathbb{C} \setminus \{z = x + iy \in \mathbb{C} : x \geq 0 \ \&\ y \geq 0\}$?

   [Hint: Riemann Mapping Theorem.]

2. Give an example of an entire function which is conformal on the entire complex plane $\mathbb{C}$ but not one-to-one on $\mathbb{C}$.

3. Show that if $w = f(z)$ is analytic at a point z_o and $f'(z)$ has a zero of order n at z_o, then f maps two oriented smooth curves in the z-plane intersecting at angle θ at z_o to two curves in the w-plane intersecting at the angle $(n+1)\theta$.

   [Hint: Let γ_1 and γ_2 be the two oriented smooth curves intersecting at z_o. You may assume that θ is given by

   $$\theta = \lim_{z \to z_o, z \in \gamma_1} \arg(z - z_o) - \lim_{z \to z_o, z \in \gamma_2} \arg(z - z_o)$$

   with the understanding that one-sided limits are taken, i.e., z always lie on one of the two branches of γ_1 (or γ_2) separated by z_o. Similarly, you may assume that $f(\gamma_1)$ and $f(\gamma_2)$ intersect at $f(z_o)$ at the angle given by

   $$\lim_{z \to z_o z \in \gamma_1} \arg(f(z) - f(z_o)) - \lim_{z \to z_o, z \in \gamma_2} \arg(f(z) - f(z_o)).$$

4. Consider the principal logarithmic function $w = \text{Log } z$ defined on $\mathbb{C} \setminus (-\infty, 0]$. Write $w = u + iv$. Describe and sketch the level curves of u and v associated to the function.

5. Describe the image of each of the following domains under the mapping $w = e^z$:

   (i) the strip $0 < \text{Im } z < \pi/2$;

   (ii) the half strip $\text{Re } z < 0$, $0 < \text{Im } z < \pi$;

   (iii) the half planes $\text{Re } z > 0$ and $\text{Re } z < 0$.

6. Find the image of the circle $|z| = 1$ under the maps

   (i) $w = \dfrac{1}{z-1}$;

   (ii) $w = \dfrac{1}{z-2}$.