NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 5

1. Let $f_n(z)$, $n = 1, 2, \cdots$, be a sequence of functions analytic in the open ball $D: \ |z| < R$ with $R > 0$. Suppose the sequence of functions $\{f_n(z)\}$ converges uniformly to an analytic function $f(z)$ on D. Prove that if $f(z) \neq 0$ for all z on the circle $|z| = \delta$, where $0 < \delta < R$, then there exists a positive integer N such that for all $n > N$, $f_n(z)$ and $f(z)$ have the same number of zeroes (counting multiplicity) in the ball $|z| < \delta$.

   [Hint: Rouché's Theorem]

2. Use the open mapping theorem to give a quick proof of the following familiar facts: If f is analytic in a domain D, then f is identically constant in D if any of the following conditions holds:

   (a) Re $f(z)$ is constant in D.

   (b) Im $f(z)$ is constant in D.

   (c) $|f(z)|$ is constant in D.

3. Fix any complex constant α such that $|\alpha| < 1$. Consider the function

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

   (i) Show that ϕ_α maps the unit circle C: $|z| = 1$ to itself.

   (ii) Show also that ϕ_α is an analytic function from the open ball $B(0,1)$ into itself.

   [Hint: Use (i).]

   (iii) Show that ϕ_α is an analytic automorphism of $B(0,1)$.

   (iv) Show that the inverse of ϕ_α on $B(0,1)$ is $\phi_{-\alpha}$, i.e., show that

$$\phi_\alpha \circ \phi_{-\alpha}(z) = z = \phi_{-\alpha} \circ \phi_\alpha(z), \quad \text{for all } z \in B(0,1).$$

   (v) Show that $\phi'_\alpha(0) = 1 - |\alpha|^2$ and $\phi'_\alpha(\alpha) = (1 - |\alpha|^2)^{-1}$. [Remark: The $\phi'_\alpha s$ are very useful analytic automorphisms on $B(0,1)$, which we will see in the next question and in a number of occasions later. An important property of ϕ_α is that $\phi_\alpha(\alpha) = 0$ (check it).]

4. (Another generalization of Schwarz's lemma).

   Let $f(z)$ be an analytic function on the open ball $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ such that $|f(z)| < 1$ for all $|z| < 1$. Show that

$$|f'(0)| \leq 1 - |f(0)|^2.$$

   [Hint: Let $\alpha = f(0)$. Consider the composite function $g = \phi_\alpha \circ f$, where ϕ_α is as in Question 3. Then apply Schwarz's lemma to the function. You may also need to use the results in Question 3.]