NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 4

1. Let $h(z) = \frac{1}{2}(z + \frac{1}{z})$. Prove that if w is any complex number **not** in the closed interval $[-1, 1]$, then there is exactly one z in the unit open ball $B(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ such that $h(z) = w$.

   Hint: Fix a w not in $[-1, 1]$, and let

   $$f(z) = h(z) - w.$$

   Show that as z traces around the unit circle $C : |z| = 1$ once in the positive direction $f(C)$ does not pass through the origin in the w plane and has winding number 0 about the origin. Then apply the argument principle.

   Alternatively, you can solve this by elementary methods.

2. Use Rouché's theorem to show that the polynomial $z^5 + 3z^2 + 1$ has exactly two zeroes in the disk $|z| < 1$ counting multiplicity.

3. Prove that the equation $z^3 + 9z + 27 = 0$ has no roots in the disk $|z| < 2$.

4. Find the number of roots of the equation $6z^4 + z^3 - 2z^2 + z - 1 = 0$ in the disk $|z| < 1$.

5. Give an example to show that the conclusion of Rouché's theorem may be false if the strict inequality $|g(z)| < |f(z)|$ is replaced by $|g(z)| \le |f(z)|$ on C.

6. Prove that all the roots of the equation $z^6 - 5z^2 + 10 = 0$ lie in the annulus $1 < |z| < 2$.

7. Let a, $b \in \mathbb{C}$, and $n \in \mathbb{Z}^+$. Show that $az^n + be^z$ has n zeroes counting multiplicity in the interior of the unit circle $|z| = 1$ if $|a| > |b|e$.

8. Prove that the equation $z = 2 - e^{-z}$ has exactly one root in the right half plane. Why must this root be real?

9. Determine the number of zeroes of the polynomial $3z^7 + 5z - 1$ counting multiplicity which lie in the annulus $1 < |z| < 2$.