NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 3

1. Prove the following variations of Schwarz's lemma:

   (i) Suppose that f is analytic in the open ball $|z| < R$ for some $R > 0$ and that $|f(z)| \leq 1$ for $|z| < R$, and $f(0) = 0$. Show that $|f(z)| \leq \frac{|z|}{R}$ for $|z| < R$.

   (ii) Suppose that f is analytic in the open ball $|z| < 1$, and that $|f(z)| \leq M$ for $|z| < 1$, where $M > 0$, and $f(0) = 0$. Show that $|f(z)| \leq M|z|$ for $|z| < 1$.

   [Hint: Use the Schwarz's lemma.]

2. Find all the analytic functions $f(z)$ on the open ball $|z| < 1$ such that $|f(z)| < 1$ for all $|z| < 1$, $f(0) = 0$ and $f(\frac{1}{2}) = -\frac{i}{2}$.

3. Which of the following functions are meromorphic in the whole plane?

   (a) $iz + z^5$ (b) $\operatorname{Log} z$ (c) $\dfrac{\cos z}{z^3 + 1}$ (d) $e^{1/z}$

   (e) $\tan z$ (f) $\dfrac{2i}{(z-2)^3} + e^z$.

   Give brief explanations for your answers.

4. Consider the polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Explain why for each sufficiently large value of R,

$$\int_{|z|=R} \frac{P'(z)}{P(z)} dz = 2n\pi i.$$

   Here the circle $|z| = R$ is positively oriented.

   [Hint: Recall the Fundamental Theorem of Algebra.]

5. Evaluate

$$\frac{1}{2\pi i} \int_{|z|=3} \frac{f'(z)}{f(z)} dz,$$

   where $f(z) = \dfrac{z^2(z-i)^3 e^z}{3(z+2)^4(3z-18)^5}$.

6. Let $f(z)$ be analytic on the closed disk $|z| \leq \rho$, and suppose that $f(z) \neq w_0$ for all z on the positively oriented circle $|z| = \rho$. Explain why the value of the integral

$$\frac{1}{2\pi i} \int_{|z|=\rho} \frac{f'(z)}{f(z) - w_0} dz$$

   is bigger than or equal to the number of distinct solutions of the equation $f(z) = w_0$ inside the disk.

7. Suppose that a function f is analytic inside and on a positively oriented simple closed contour γ and that $f(z) \neq 0$ for all $z \in \gamma$. Show that if f has n zeroes z_k, $k = 1, \cdots, n$, inside γ, and each z_k is a zero of f of order m_k, then

$$\int_\gamma \frac{zf'(z)}{f(z)}\, dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$