NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 2

1. Using the identity theorem for analytic functions, show that there exists at most one entire function f such that $f(x) = x + \sin x$ for all $x \in \mathbb{R}$.

2. Use the identity $\sin 2x = 2 \sin x \cos x$ for all $x \in \mathbb{R}$ to deduce that
$$\sin 2z = 2 \sin z \cos z \quad \text{for all } z \in \mathbb{C}.$$

3. Find all functions f analytic in $B(0, R) := \{z \in \mathbb{C} : |z| < R\}$ that satisfy $f(0) = i$ and $|f(z)| \le 1$ for all z in $B(0, R)$. (Hint: where does the maximum modulus occur?)

4. If f is analytic in the annulus $1 \le |z| \le 2$ and $|f(z)| \le 5$ for $|z| = 1$, and $|f(z)| \le 20$ for $|z| = 2$, prove that $|f(z)| \le 5|z|^2$ for $1 \le |z| \le 2$. (Hint: Consider $f(z)/5z^2$.)

5. Suppose that f is analytic inside and on the simple closed curve C and that $|f(z) - 1| < 1$ for all z on C. Prove that f has no zeroes inside C. (Hint: Suppose $f(z_0) = 0$ for some z_0 inside C and consider the function $g(z) := f(z) - 1$.)

6. Let f and g be analytic in the bounded domain D and continuous up to and including its boundary B. Suppose that g never vanishes. Prove that if the inequality $|f(z)| \le |g(z)|$ holds for all z on B, then it must hold for all z in D.

7. Prove the **minimum modulus principle**: Let $R \subset \mathbb{C}$ be a closed bounded set whose interior is a domain. Suppose f is continuous on R and analytic and not constant in the interior of R. If $f(z) \ne 0$ for any $z \in R$, then $|f(z)|$ attains its minimum value at the boundary of R but not in the interior of R. (Hint: Consider the function $1/f(z)$.)

   Give an example to show why the non-zero condition is necessary.

8. Let the non-constant function f be analytic in the bounded domain D and continuous up to and including its boundary B. Prove that if $|f(z)|$ is constant on B, then f must have at least one zero in D.

9. Suppose that f is analytic inside and on the simple closed curve C and that $|f(z) - 1| < 1$ for all z on C. Prove that f has no zeroes inside C. (Hint: Suppose $f(z_0) = 0$ for some z_0 inside C and consider the function $g(z) := f(z) - 1$.)

10. Let f and g be analytic in the bounded domain D and continuous up to and including its boundary B. Suppose that g never vanishes. Prove that if the inequality $|f(z)| \le |g(z)|$ holds for all z on B, then it must hold for all z in D. (Hint: Consider the function $f(z)/g(z)$.)