NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 1

1. Let $f(z)$ be an analytic function in a domain D.

   (i) Show that if $\overline{f(z)}$ is also analytic in D, then $f(z)$ is a constant function on D.

   (ii) Using part (i), show that if $|f(z)|$ is constant in D, then $f(z)$ is also a constant function on D.

   [Hint: For (ii), write $|f(z)| \equiv c$ on D. Distinguish the two cases when $c = 0$ and when $c > 0$. Note also that $f(z)\overline{f(z)} = |f(z)|^2$.] [You may use freely the following theorem:

   If $f(z)$ is an analytic function in a domain D such that $f'(z) \equiv 0$ in D, then $f(z)$ is constant in D.]

2. Let f be analytic in the closed disk $|z| \le r$, where $r > 0$. Suppose $|f(z)| \le M$ for all $|z| \le r$, where $M > 0$ (sometimes, we simply say f is bounded by M on the disk $|z| \le r$). Show that for any integer $n \ge 1$,

$$|f^{(n)}(z)| \le \frac{n!M}{(r - |z|)^n} \quad \text{for all } |z| < r.$$

   [Hint: Cauchy integral formula for derivatives]

3. If $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial such that $|p(z)| \le 1$ for all z satisfying $|z| \le 1$, show that $|a_k| \le 1$ for each $k = 0, 1, \cdots, n$.

   [Hint: Use the result in Question 2.]

4. Let $f(z)$ be an analytic function which has a zero of order m at z_0. (Recall that an analytic function $f(z)$ is said to have **a zero of order** m at z_0 if

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0, \quad \text{but } f^{(m)}(z_0) \neq 0.)$$

   (i) Show that there exists $r > 0$ and an analytic function $\phi(z)$ on the open ball $B(z_0, r) = \{z \ : \ |z - z_0| < r\}$ such that $\phi(z) \neq 0$ for all $z \in B(z_0, r)$ (first show that $\phi(z_0) \neq 0$ and then reduce r if necessary), and

$$f(z) = (z - z_0)^m \phi(z) \quad \text{for all } |z - z_0| < r.$$

   (ii) Show that $\dfrac{f'(z)}{f(z)}$ has a simple pole at z_0, and show that the residue of $\dfrac{f'(z)}{f(z)}$ at z_0 is equal to m.

5. Let $f(z)$ and $g(z)$ be two entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there is a constant c such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$. **Justify your arguments carefully.** [Remark: $g(z)$ may be equal to zero at some points. So $f(z)/g(z)$ is apriori not an entire function.]

6. (i) Find the Laurent series of the function $\dfrac{5z-7}{(z-4)(z+9)}$ for the annular domain $4 < |z| < 9$.

   (ii) Using part (i) or otherwise, find the Laurent series of the function

$$\frac{(z-2)^2(5z-17)}{z^2+z-42}$$

   for the annular domain $4 < |z-2| < 9$.