NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 11

1. (i) Let D be a bounded domain whose boundary is denoted by ∂D. Suppose u is a continuous function on $D \cup \partial D$ such that u is harmonic on D. If $u(z) = 0$ for all $z \in \partial D$, show that $u(z) = 0$ for all $z \in D$.

   (ii) Consider the function $u(x, y) = \ln \sqrt{x^2 + y^2}$ on the closed ball $x^2 + y^2 \leq 1$. Explain why u does not contradict (i).

2. Use the Poisson integral formula to find the function $u(r, \theta)$ in polar coordinates such that u is a harmonic function in the unit ball $r < 1$, and

$$\lim_{(r,\theta)\to(1,\phi),\ |r|<1} u(r, \theta) = \begin{cases} 1 & \text{if } 0 < \phi < \frac{\pi}{2}, \\ 0 & \text{if } \frac{\pi}{2} < \phi < 2\pi. \end{cases}$$

   [You may use freely the formula:

$$\int \frac{dx}{b + c\cos x} = \frac{2}{a\sqrt{b^2 - c^2}} \tan^{-1}\left(\sqrt{\frac{b-c}{b+c}} \tan \frac{ax}{2}\right) + C, \quad \text{if } b^2 > c^2.$$

   [Answer:

$$\frac{1}{\pi} \tan^{-1}\left(\frac{1+r}{1-r} \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right) + \frac{1}{\pi} \tan^{-1}\left(\frac{1+r}{1-r} \tan \frac{\theta}{2}\right).]$$

3. Show that the function

$$f_2(z) = \frac{1}{z^2 + 1}, \quad (z \neq \pm i),$$

   is the analytic continuation of the function

$$f_1(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad (|z| < 1)$$

   to the domain $\mathbb{C} \setminus \{\pm i\}$.

4. Consider the analytic function $f(z) = \sum_{n=0}^{\infty} 2^n z^n, \quad |z| < \frac{1}{2}$. Let g be the analytic continuation of f to the domain $\mathbb{C} \setminus \{\frac{1}{2}\}$. Find $g(i)$.

5. Consider the analytic function on the right half plane given by

$$f(z) = \int_0^{\infty} te^{-zt} dt, \quad \text{Re } z > 0,$$

   Find the analytic continuation of f to the domain $\mathbb{C} \setminus \{0\}$.