NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 10

1. Find a conformal isomorphism mapping the semi-infinite strip $x > 1$, $-1 < y < 1$ to the unit ball $|w| < 1$.

   [Hint: Recall that the function $f(z) = \sin z$ is a conformal isomorphism from the semi-infinite strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$ to the upper half plane Im $z > 0$.] Remark: The transformation is not unique.

2. Show that the function $u(x, y) = 2xy + e^x \cos y$ is a harmonic function on $\mathbb{R}^2$ and find a harmonic conjugate to $u(x, y)$. [Answer: $v(x, y) = y^2 + e^x \sin y - x^2 + C$.]

3. Suppose that $f(z) = u + iv$ is analytic on a domain D. Show that $u + v$, $u^2 - v^2$ and uv are harmonic on D. What about $u^2 + v^2$? [You may use freely the fact that the real and imaginary parts of an analytic function are harmonic.]

4. Show that ϕ_x and ϕ_y are harmonic on a domain D if ϕ is harmonic on D. [Hint: Locally, write ϕ as the real part of an analytic function.]

5. Consider the function

$$u(x, y) = \frac{1}{2} \ln(x^2 + y^2), \quad z = x + iy \in \mathbb{C} \setminus \{0\}.$$

   (i) Show that u is harmonic on $\mathbb{C} \setminus \{0\}$.

   (ii) Show that u has no harmonic conjugate on $\mathbb{C} \setminus \{0\}$.

   [Hint: First show that if v is a harmonic conjugate of u on $\mathbb{C} \setminus \{0\}$, then $u + iv = \text{Log } z + iC$ on $\mathbb{C} \setminus (-\infty, 0]$ for some real constant C.]