NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II

Lecture Notes Part VII

Chapter 4. Harmonic functions

4.1. Harmonic Functions and analytic functions

Theorem 4.1.1. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D. Then u and v satisfy the **Laplace equation**:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ v_{xx} + v_{yy} &= 0 \quad \text{on } D. \end{aligned}$$

Example 4.1.2. Consider $f(z) = e^z$. Then $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ and it is easy to verify that

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ v_{xx} + v_{yy} &= 0 \quad \text{on } \mathbb{C}. \end{aligned}$$

Proof of Theorem 4.1.1. $f(z)$ is analytic $\implies u_x = v_y,\ u_y = -v_x$. Furthermore, u and v have continuous partial derivatives of all orders since f is infinitely differentiable. Differentiating, we get

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy},$$

and similarly,

$$v_{yy} = u_{xy} = u_{yx} = -v_{xx}$$

Definition 4.1.3. Let D be a domain in $\mathbb{R}^2$. A function $u : D \longrightarrow \mathbb{R}$ is said to be **harmonic** in $D \subset \mathbb{R}^2$ if
(i) u has continuous first and second partial derivatives, and
(ii) u satisfies the **Laplace equation** in D:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Corollary 4.1.4. *If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harmonic in D.*

Proof. Condition (i) of Definition 4.1.3 follows from the fact (MA3111) that the partial derivatives (of any order) of the real and imaginary parts of an analytic function are always continuous (cf. Chapter 1, (1.4.3)). Condition (ii) follows from Theorem 4.1.1.

Example 4.1.5. The function $\operatorname{Log} z$ is analytic on the cut plane $\mathbb{C} \setminus (-\infty, 0]$. Its real and imaginary parts $u(x, y) = 1/2 \log(x^2 + y^2)$ and $v(x, y) = Arg\ z = \arctan(y/x)$ (with values between $-\pi$ and π) are harmonic on $\mathbb{C} \setminus (-\infty, 0]$.

In general, harmonic functions arise naturally in many physical situations. Consider a thin homogeneous metal plate in $\mathbb{R}^2$ with fixed temperatures at its boundary (At different boundary points, the temperatures may be different). Then it is known (for physical reasons) that the steady state temperature $T(x, y)$ at points (x, y) on the plate is a harmonic function (in particular, $T_{xx} + T_{yy} = 0$). See [Churchill, p. 361(7th ed) or p.299(6th ed)].

4.2. Harmonic conjugates

Definition 4.2.1. Let u, v be two harmonic functions in a domain D. We say that v is a **harmonic conjuate** of u if

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{on } D.$$

Example 4.2.2. Consider the function $u(x, y) = x^2 - y^2$ which is harmonic in $\mathbb{R}^2$ (equivalently, $\mathbb{C}$). It is not difficult to verify that $v(x, y) = 2xy$ is a harmonic conjugate of u.

Theorem 4.2.3. Let u be a harmonic function in a domain D, and let v is a harmonic conjugate of u. Then $f(z) = u + iv$ is an analytic function in D.

Proof. The given conditions mean that the function $f = u + iv$ satisfies the sufficient conditions for differentiability (cf. (1.2.3)). Thus f is differentiable everywhere in D. Therefore, f is analytic in D.

Example 4.2.4. From the above example, $u(x, y) + iv(x, y) = x^2 - y^2 + i(2xy)$ is an entire function.

Proposition 4.2.5. If v and v' are both harmonic conjugates of u on a domain D, then $v' = v + c$ for some real constant c.

Proof. By Theorem 4.2.3, the functions $f = u + iv$ and $g = u + iv'$ are analytic functions on D, since v and v' are harmonic conjugates of u. Then $g - f$ is an analytic function with Re $(g - f) = 0$, hence, $g - f \equiv c$ is a constant function on D (by the Open Mapping Theorem). Thus,

$$v' - v = (u + iv') - (u + iv) = g - f \equiv c$$

is a constant function (note that c is a purely imaginary constant).

4.2.6. Construction of harmonic conjugates

Starting with a harmonic function u on $\mathbb{C}$, we would like to construct a harmonic conjugate v. This is desirable, since by Theorem 4.2.3, we will get an analytic function $f = u + iv$. We can do this by integrating the conditions satisfied by the harmonic conjugate.

Example 4.2.7. We give the simple example $u(x, y) = x^2 - y^2$.

4.3. Further properties of harmonic functions

Theorem 4.3.1. Let u be a harmonic function on a **simply connected** domain D. Then u is the real part of an analytic function on D (or equivalently, u has a harmonic conjugate on D).

Proof. First we show that $g := u_x - iu_y$ is analytic on D. (This is suggested by the C-R equations). Since u has continuous partial derivatives up to 2nd order, g has continuous first order partial derivatives. Moreover, since u is harmonic,

$$\begin{aligned} (u_x)_x - (-u_y)_y &= u_{xx} + u_{yy} = 0, \quad \text{and} \\ (u_x)_y + (-u_y)_x &= u_{xy} - u_{yx} = 0; \\ \text{i.e.,} \quad (u_x)_x = (-u_y)_y \quad &\text{and} \quad (u_x)_y = -(-u_y)_x. \end{aligned}$$

so the real and imaginary parts of g satisfy the C-R equations. Thus from (1.2.3), g is differentiable and thus analytic in D.
On the other hand, by a corollary to the Cauchy-Goursat Theorem for simply connected domains (see (1.4.2), an analytic function on a simply connected domain always has an anti-derivative. Thus, the analytic function g has an anti-derivative f in the simply connected domain D. Write $f = U + iV$, where U, V are the real and imaginary parts of f respectively. Then

$$\begin{aligned} g = f' \implies u_x - iu_y &= U_x + iV_x = U_x - iU_y \\ u_x = U_x \quad &\text{and} \quad u_y = U_y, \end{aligned}$$

which implies $u = U + C$ where C is a real constant. Thus u is the real part of the analytic function $f(z) + C$.

Warning: In the above theorem, the condition that D is simply connected is crucial. There are examples of harmonic functions on non-simply connected domains D which cannot be realised as the real part of an analytic function on D. For example, the harmonic function

$$u(x, y) = \ln \sqrt{x^2 + y^2}$$

is not the real part of any analytic function on the domain $D := \mathbb{C} \setminus \{0\}$. See Tutorial 10 Q5.

Corollary 4.3.2. If $u : D \to \mathbb{C}$ is a harmonic function in an open set $D \subset \mathbb{C}$, then
(i) for every $z_0 \in D$, there exists an open ball $B(z_0, r) \subset D$ and an analytic function f on $B(z_0, r)$ such that $u = \text{Re}\ (f)$ on $B(z_0, r)$;
(ii) u has continuously partial derivatives of any orders in D.

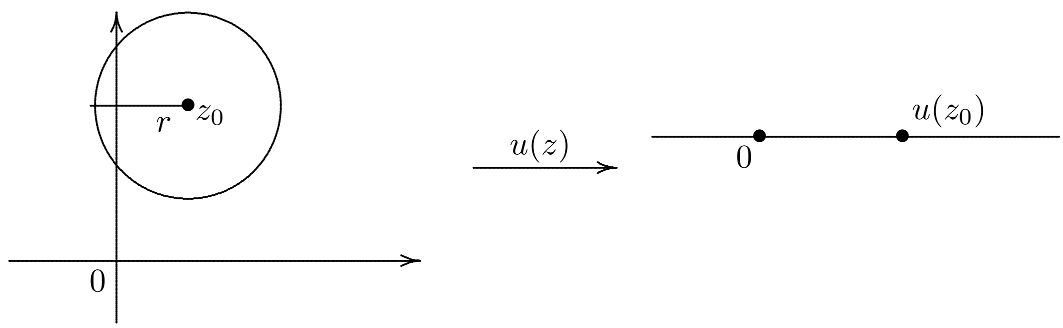
[Roughly speaking, a harmonic function is locally always equal to the real part of some analytic function, and thus it is infinitely differentiable.]

Proof. (i) Let $z_0 \in D$, since D is open, there exists $r > 0$ such that the open ball $B(z_0, r) \subset D$. Since $B(z_0, r)$ is simply connected, it follows from Theorem 4.3.1 that there exists an analytic function f on $B(z_0, r)$ such that u is the real part of f in $B(z_0, r)$. (ii) In particular, since f is analytic at z_0, so its real part u has continuous partial derivatives of all orders at z_0. Since z_0 is aribitrary, it follows that u has continuously partial derivatives of any orders everywhere in D.

Theorem 4.3.3 (Mean value theorem for harmonic functions (MVT)).
If u is harmonic in a domain $D \subset \mathbb{C}$, then for all closed ball $\overline{B(z_0, r)} \subset D$, where $r > 0$,

$$u(z_0) = \frac{1}{2\pi} \int_o^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

[In other words, the value of a harmonic function at the center of a circle is equal to the average of its values along the circle.]



Proof. First note that if $\overline{B(z_0, r)} \subset D$, then $B(z_0, r + \epsilon) \subset D$ for some $\epsilon > 0$ since $\overline{B(z_0, r)}$ is a closed bounded set and D is open. By Corollary 4.3.2, there exists an analytic function f on $B(z_0, r + \epsilon)$ with $u = \text{Re}\ f$ in $B(z_0, r + \epsilon)$. By the Gauss Mean Value Theorem for analytic functions (Theorem 2.2.1),

$$f(z_0) = \frac{1}{2\pi} \int_o^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

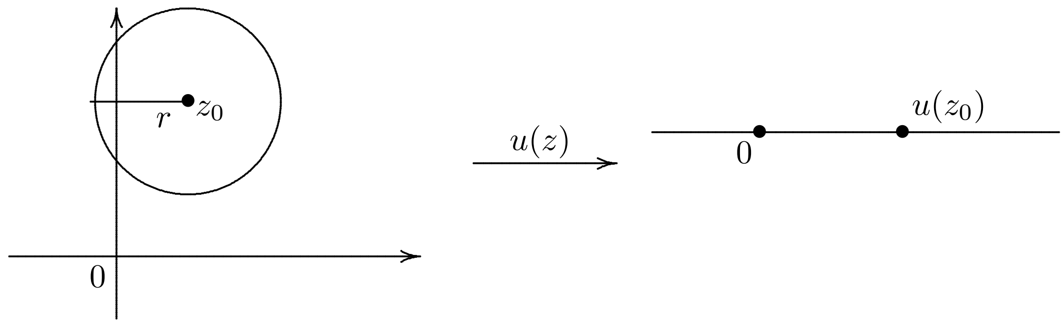
By taking the real part on both sides, we get

$$u(z_0) = \frac{1}{2\pi} \int_o^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

Corollary 4.3.4 (Area MVT for Harmonic functions).
If u is harmonic in a domain $D \subset \mathbb{C}$, then for all closed balls $\overline{B(z_0, r)} \subset D$, where $r > 0$,

$$u(z_0) = \frac{1}{\pi r^2} \int\int_{\overline{B(z_0,r)}} u(z)\, dxdy.$$

[In other words, the value of a harmonic function at the center of a circle is equal to the average of its values over the entire ball bounded by the circle.]



Proof. By Theorem 4.3.3,

$$u(z_0) = \frac{1}{2\pi} \int_o^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta. \qquad (*)$$

for any $0 < \rho \leq r$. Take the integral $\int_o^r \ldots \rho d\rho$ on both sides of (*), we get

$$\begin{aligned} \int_o^r u(z_0)\, \rho d\rho &= \frac{1}{2\pi} \int_o^r \Big(\int_o^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta \Big) \rho d\rho \\ \Longrightarrow \quad \frac{1}{2} r^2 u(z_0) &= \frac{1}{2\pi} \int_o^{2\pi} \int_o^r u(z_0 + \rho e^{i\theta})\, \rho d\rho d\theta \\ &= \frac{1}{2\pi} \int\int_{\overline{B(z_0,r)}} u(z)\, dxdy, \end{aligned}$$

and the result follows.

Theorem 4.3.5 (Maximum-minimum principle for harmonic functions).
Let u be a harmonic function on a domain $D \subset \mathbb{C}$. If u attains a local maximum (or minimum) value at a point in D, then u is a constant function on D.

Proof. Suppose that u attains a local maximum at $z_0 \in D$ and that $B(z_0, \delta) \subset D$ for some $\delta > 0$. Then by Corollary 4.3.2, there exists an analytic function f on $B(z_0, \delta)$ such that $u = \text{Re } f$ on $B(z_0, \delta)$. Consider the function $g(z) = e^{f(z)}$ on $B(z_0, \delta)$. Then $|g(z)| = e^{u(z)}$ attains a local maximum at z_0. Thus, by the maximum modulus principle for analytic functions, it follows that $g(z)$ is a constant function on $B(z_0, \delta)$. Thus, $g'(z) = e^{f(z)} f'(z) \equiv 0$, which implies $f'(z) \equiv 0$ on $B(z_0, \delta)$. Hence $f(z)$ is a constant function, and thus u is a constant function on $B(z_0, \delta)$. But we still need to show that u is in fact constant on the entire domain D.
Now if z_1 is any other point in D, we can find a simple polygonal line γ joining z_0 to z_1 and a simply connected domain D_1 with $\gamma \subset D_1 \subset D$. Applying Corollary 4.3.2, u is the real part of an analytic function f_1 on D_1, and using the previous argument, f_1 is constant on D_1, so u is constant on D_1 which implies $u(z_0) = u(z_1)$. Since z_1 is arbitrary, u is constant on D.

Corollary 4.3.6. If u is harmonic in a bounded domain D and is continuous on $\overline{D} := D \cup \partial D$ (here ∂D denotes the boundary of D), then u attains its maximum and minimum on ∂D.

Proof. This is certainly true if u is a constant function. So it remains to consider the case when u is not constant. By the Extreme Value Theorem, since u is continuous on the closed bounded set $\overline{D}$, it attains its max. and min. on the set $\overline{D}$. But Theorem 4.3.5 implies that if u is not a constant function, then the maximum and minimum are not achieved at any interior points. Thus, u attains its maximum and minimum at some boundary points.

Corollary 4.3.7. Let u and v be harmonic functions on a bounded domain D and continuous on $\overline{D} = D \cup \partial D$. If u and v are equal on the boundary ∂D, then $u \equiv v$ on $\overline{D}$.

Proof. Consider $u-v$ which is harmonic on D (Exercise) and continuous on $\overline{D}$, with $u-v \equiv 0$ on the boundary ∂D. By Corollary 4.3.6, since the max and min are achieved on the boundary ∂D (and thus both values are 0), this implies $u - v \equiv 0$ on $\overline{D}$. ☐

Remark. The boundedness condition on D is necessary in the above corollary, consider the following example: Let $D := \{z \in \mathbb{C} \,\big|\, |z| > 1\}$, and let

$$u(z) = \ln |z|^2 = \ln(x^2 + y^2), \quad \text{and} \quad v(z) := 0, \quad z \in \overline{D}$$

We can show that u and v are harmonic on D and continuous on $\overline{D}$ and $u(z) = v(z)$ on ∂D but clearly, $u \neq v$.

Theorem 4.3.8. Suppose that an analytic function $w = f(z) = u(x, y) + iv(x, y)$ maps a domain D_1 in the z-plane to a domain $D_2 = f(D_1)$ in the w-plane. If $h(u, v)$ is a harmonic function defined on D_2, then the function

$$H(x, y) = h[u(x, y), v(x, y)]$$

is harmonic in D_1.

[In short, composition of harmonic functions with analytic functions are harmonic.]

Proof. Let $z_0 \in D_1$ and $w_0 = f(z_0) \in D_2$. Since D_2 is open (a domain), there exists $r > 0$ such that $B(w_0, r) \subset D_2$ and let $U = f^{-1}(B(w_0, r))$ which is open in D_1 and contains z_0. Then, by Corollary 4.3.2, there exists an analytic function g on $B(w_0, r)$ such that $h = \text{Re } \ g$. Now $g \circ f$ is analytic on U and the real part of $g \circ f = H(x, y) = h[u(x, y), v(x, y)]$ is harmonic on U. Since z_0 is arbitrary, H for all $z_0 \in D_1$.

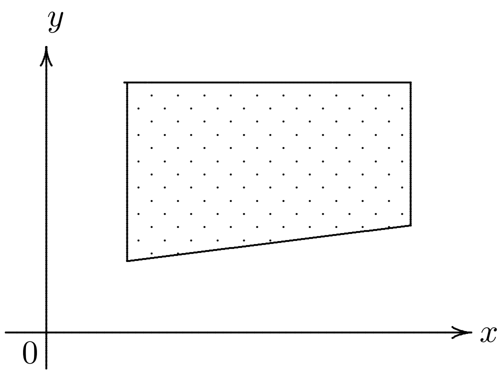
Remark. We can also prove the above by directly using the chain rule for partial derivatives (Exercise), but the calculations are quite involved.

Example. Let $f(z) = z^2$ on the first quadrant, and $h(u, v) = \tan^{-1} v/u$ on the upper half space.

4.4. (The Dirichlet Problem) Suppose that D is a domain and that F is a continuous function defined on the boundary ∂D of D. The Dirichlet problem is the problem of finding a function u which is harmonic in D, and continuous on $\overline{D} := D \cup \partial D$ such that $u(x, y) = U(x, y)$ for $(x, y) \in \partial D$, i.e., we want to solve the following boundary value problem:

$$\begin{cases} u_{xx} + v_{yy} = 0 & \text{on } D \\ u(x,y) = U(x,y) & \text{for } (x,y) \in \partial D. \end{cases} \tag{*}$$

Physically this corresponds to finding the steady state temperature of the homogeneous thin metal plate whose boundary are kept at precribed temperatures.



Remark 4.4.1. (i) Corollary 4.3.7 implies that in the case where D is a bounded domain, then the solution to the above Dirichlet Problem (*), if it exists, is unique.

(ii) Theorem 4.3.8 allows us to transform the Dirichlet Problem of finding a harmonic function on a general domain D satisfying some boundary conditions to a corresponding problem of finding a harmonic function on a "nice" domain, say the unit ball, satisfying an equivalent set of boundary conditions. (This is why the Riemann mapping theorem is so important for applications).

(4.4.2) The Dirichlet problem for the ball

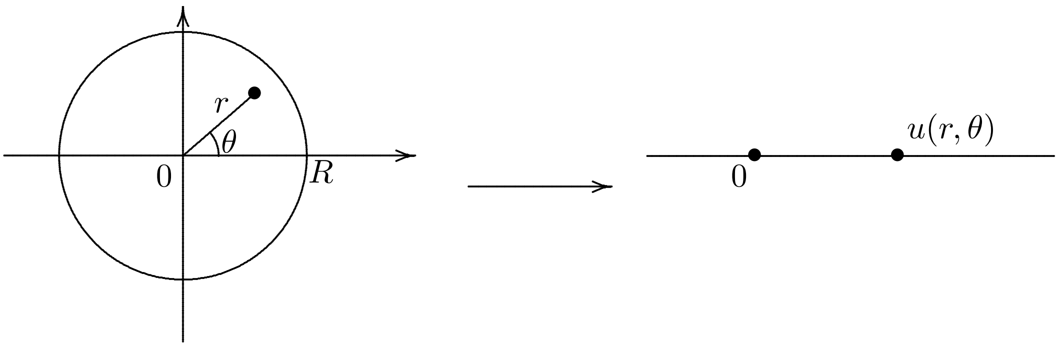
We will consider only the case when D is the open ball centered at the origin 0 and of radius R, i.e., $D = B(0, R)$ so that ∂D is the circle $|z| = R$.

In the following Proposition, we will solve the Dirichlet Problem (*) assuming that the solution u is the real part of an analytic function f, which is analytic on the closed ball $\overline{B(0, R)}$.

Proposition 4.4.3. Suppose $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic on the closed ball $\overline{B(0, R)}$, where $z = re^{i\theta}$, and $u(R, \phi) = U(\phi)$ for $0 \leq \phi \leq 2\pi$. Then for $z = re^{i\theta} \in B(0, R)$, we have, in terms of polar coordinates,

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2} U(\phi) d\phi \qquad (1)$$

for all$0 \leq r < R$ and $0 \leq \theta \leq 2\pi$.



Proof. Fix a point $z = re^{i\theta}$ inside the circle $C : |z| = R$. Then by the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds. \qquad (2)$$

Note that the point z^* which is symmetric to z with respect to C is given by

$$z^* = \frac{R^2}{r} e^{i\phi} = \frac{R^2}{\bar{z}}. \qquad (3)$$

(Exercise), and z^* is necessarily outside C. Thus, by the Cauchy-Goursat Theorem,

$$0 = \int_C \frac{f(s)}{s - z^*} ds. \qquad (5)$$

Subtracting (5) from (2), we get

$$f(z) = \frac{1}{2\pi i} \int_C \Big(\frac{1}{s - z} - \frac{1}{s - z^*} \Big) f(s)\, ds. \qquad (6)$$

Write $s = Re^{i\phi}$, $0 \le \phi \le 2\pi$, so that $ds = iRe^{i\phi} d\phi = isd\phi$. Then (6) gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_0^{2\pi} \left(\frac{1}{s-z} - \frac{1}{s-z^*} \right) f(s)\, is\, d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{s}{s-z} - \frac{s}{s-z^*} \right) f(s)\, d\phi. \end{aligned} \tag{7}$$

Note that $s\bar{s} = R^2$. Thus,

$$\begin{aligned} \frac{s}{s-z} - \frac{s}{s-z^*} &= \frac{s}{s-z} - \frac{s}{s - \frac{s\bar{s}}{\bar{z}}} \\ &= \frac{s}{s-z} - \frac{\bar{z}}{\bar{z}-\bar{s}} \\ &= \frac{s}{s-z} + \frac{\bar{z}}{\bar{s}-\bar{z}} \\ &= \frac{s\bar{s} - z\bar{z}}{(s-z)(\bar{s}-\bar{z})} \\ &= \frac{R^2 - r^2}{|s-z|^2}. \end{aligned} \tag{8}$$

By the Cosine Rule, we have

$$|s-z|^2 = R^2 - 2Rr\cos(\phi - \theta) + r^2. \tag{9}$$

Combining (7), (8) and (9), we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi-\theta) + r^2} f(Re^{i\phi})\, d\phi. \tag{10}$$

Equating the real parts on both sides, we get

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi-\theta) + r^2} u(R,\phi)\, d\phi. \tag{11}$$

Note that $u(R,\phi) = U(\phi)$. Thus, we have

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi-\theta) + r^2} U(\phi)\, d\phi. \quad \square$$

Remark 4.4.4. (i) The real valued function

$$P(R, r, \phi - \theta) := \frac{R^2 - r^2}{R^2 - 2Rr\cos(\phi - \theta) + r^2} \tag{12}$$

is called the **Poisson kernel**. The formula in (1) is called the **Poisson integral formula**.
(ii) From (8) and with the same notation as there, one may also write

$$P(R, r, \phi - \theta) = \frac{R^2 - r^2}{|s - z|^2} = \text{Re}\left(\frac{s + z}{s - z}\right).$$

The last expression implies that for fixed s, $P(R, r, \phi - \theta)$ is a harmonic function with respect to the variable z.

In fact, the Poisson integral formula solves the Dirichlet Problem (*) when $D = B(0, R)$, even when $U(\phi)$ is only piecewise continuous on the boundary circle $|z| = R$. More specifically, we have

Theorem 4.4.5 (Poisson Integral Formula)
Let $U(\phi)$ be a piecewise continuous function on the interval $0 \le \phi \le 2\pi$. Then the function

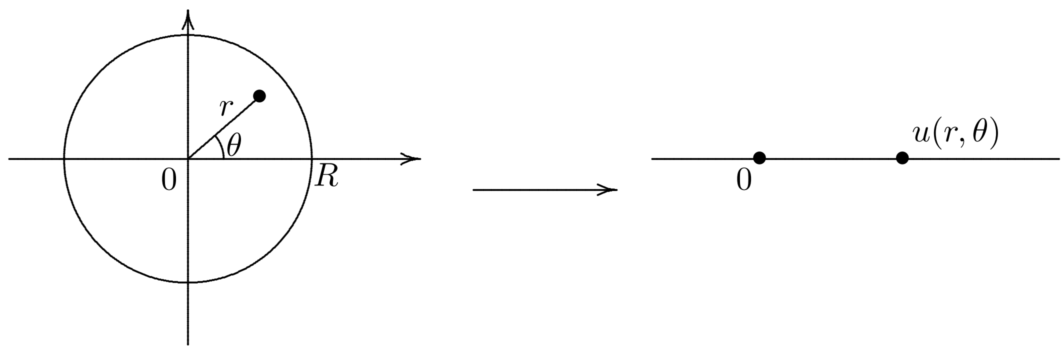
$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} P(R, r, \phi - \theta)U(\phi)\, d\phi \tag{13}$$

is a harmonic function on $B(0, R)$ and

$$\lim_{\substack{(r,\theta)\to(R,\theta_o)\\ (r,\theta)\in B(0,R)}} u(r, \theta) = U(\theta_o)$$

at any point θ_o where U is continuous at θ_o.

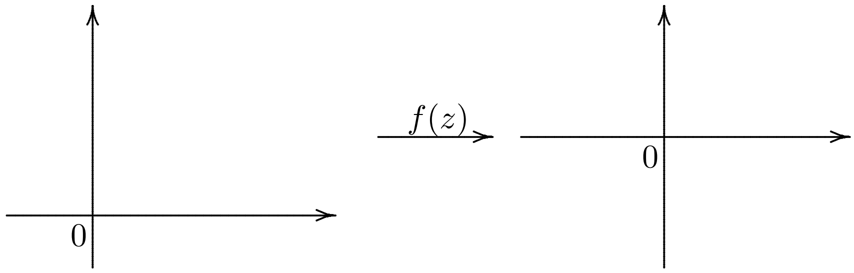
Proof. (Reading Exercise) See Churchill p. 419-422 or Ahlfors, p. 169-170 for details.



Chapter 5. Analytic Continuation

5.1. Analytic continuation

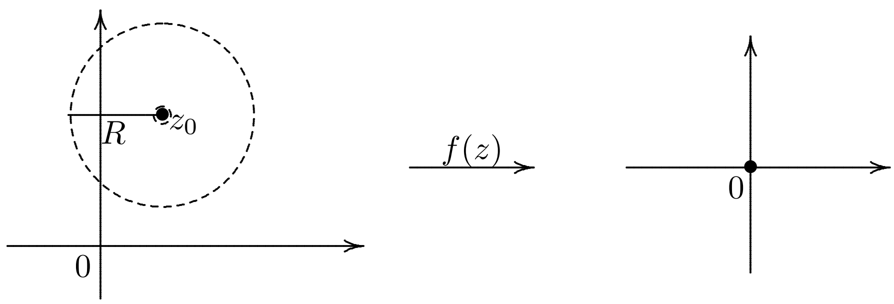
Given an analytic function f on a domain D, we are interested in the following question: Whether there exists an analytic function g defined on a bigger domain $D' \supset D$ such that the restriction of g to D is f?



Example 5.1.1. (Removing a removable singular point)
Suppose a function $f(z)$ defined on the punctured ball $0 < |z - z_0| < R$ has a removable singular point at a point z_0. By Laurent's theorem,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (0 < |z - z_0| < R) \qquad (*) \\ &= a_0 + a_1(z - z_0) + \cdots \end{aligned}$$

f(recalling that all $b_n = 0$). Note that the RHS of (*), denoted by $g(z)$, is a convergent power series, and thus $g(z)$ is an analytic function on $|z - z_0| < R$. Then one extend $f(z)$ across the point z_0 by letting $f(z_0) = a_0$, then (*) will hold everywhere on $|z - z_0| < R$. Alternatively, we say that $g(z)$ is the *analytic continuation* of f (defined orignally on the punctured ball $0 < |z - z_0| < R$) to the bigger domain $|z - z_0| < R$.

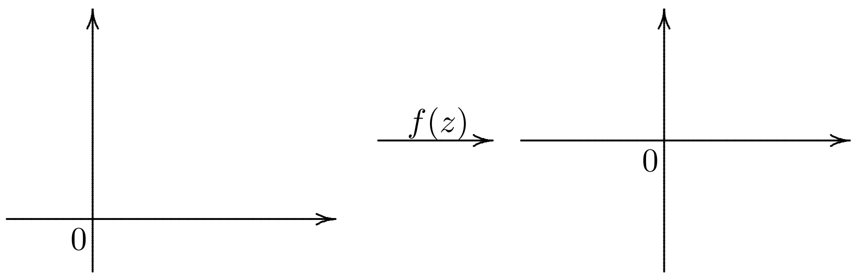


Example 5.1.2. The function $f(z) = \dfrac{\sin z}{z}$, $z \in \mathbb{C} \setminus \{0\}$, has a removable singular point at $z = 0$. For $0 < |z| < \infty$,

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots . \qquad (*)$$

Then the entire function $g(z) = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} + \cdots$ is the analytic continuation of $f(z)$ to $\mathbb{C}$.

Definition 5.1.3. Suppose that f is analytic in a domain D_1 and that g is analytic in a domain D_2. Then we say that g is a **(direct) analytic continuation** of f to D_2 if $D_1 \cap D_2$ is non-empty and $f(z) = g(z)$ for all $z \in D_1 \cap D_2$.



Remark 5.1.4. (i) In the above definition, D_1 need not be a subset of D_2.
(ii) If f is analytic in a domain D_1 and g is a direct analytic continuation of f to the domain D_2, then the function

$$F(z) := \begin{cases} f(z) & \text{for } z \in D_1, \\ g(z) & \text{for } z \in D_2, \end{cases}$$

is a (single-valued) analytic function on $D_1 \cup D_2$.

Theorem 5.1.5 (Uniqueness of analytic continuation). If f is analytic in a domain D_1 and D_2 is a domain such that $D_1 \cap D_2$ is non-empty, then the direct analytic continuation of f to D_2, if it exists, is unique.

Proof. Let g_1 and g_2 be direct analytic continuations of f to D_2. Then $g_1 = g_2 = f$ on $D_1 \cap D_2 \neq \emptyset$, which necessarily has an accumulation point in D_2. Hence $g_1 = g_2$ on D_2 by the identity theorem for analytic functions. $\square$

Example 5.1.6.
(i) The power series

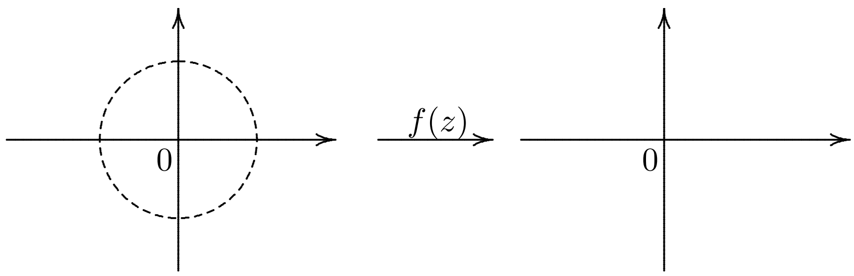
$$f(z) = 1 + z + z^2 + \cdots$$

has radius of convergence $R = 1$, and thus $f(z)$ is an analytic function on the unit ball $D_1 = \{z \in \mathbb{C} \,|\, |z| < 1\}$. On the other hand, one easily sees that

$$f(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}, \quad |z| < 1.$$

But the function $g(z) = \dfrac{1}{1-z}$ is analytic on $D_2 = \mathbb{C} \setminus \{1\}$. Thus, $g(z) = \frac{1}{1-z}$ is an analytic continuation $f(z)$ to the much bigger domain D_2. Note that in this case,

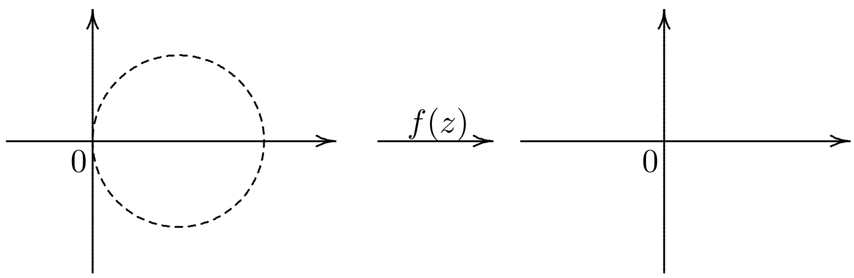
$$f(z) = g(z) \quad \text{on } D_1 \cap D_2 = D_1.$$



(ii) The function $f(z)$ defined by the series

$$f(z) := \sum_{j=1}^{\infty} \frac{(-1)^{j-1}(z-1)^j}{j}$$

converges in the open ball $D_1 : |z-1| < 1$. We can verify that this is in fact the Taylor series for $\operatorname{Log} z$ at $z = 1$, so that $\operatorname{Log} z$ an analytic continuation of $f(z)$ to $D_2 = \mathbb{C} \setminus (-\infty, 0]$. Note that by Taylor's Theorem, $\operatorname{Log} z = f(z)$ on $D_1 \cap D_2 = D_1$.



Theorem 5.1.7. Suppose a power series $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ has radius of convergence R and thus $f(z)$ is analytic on $B(z_0, R)$. Let $g(z)$ be an analytic continuation of $f(z)$ (to any bigger domain $D \supsetneq B(z_0, R)$). Then $g(z)$ has a singular point on the circle $|z| = R$.

Proof. (by contradiction.) Suppose $g(z)$ has no signular points on the cirlce $|z| = R$. Then $g(z)$ is analytic everywhere inside and on the circle $|z - z_0| = R$.
Claim: Then $g(z)$ is analytic on an open ball $B(z_0, R')$ for some $R' > R$. Otherwise by constructing a decreasing sequence of open balls $B(z), R_n)$, with $R_n \searrow R$ as $n \to \infty$, we can then find a sequence of distinct singular points z_j of f such that each $|z_j - z_0| > R$, but

$$|z_j - z_0| \to R, \quad \text{as } j \to \infty. \tag{1}$$

In particular, the $\{z_j\}_{1 \le j < \infty}$ lies in the closed and bounded subset $|z| \le r$ for some finite $r > R$. Hence by the Bolzano-Weierstrass Theorem, $\{z_j\}_{1 \le j < \infty}$ has a convergent subsequence, say $\lim_{\ell \to \infty} z_{j_\ell} = z_*$. From (1), we must have

$$\lim_{\ell \to \infty} |z_{j_\ell} - z_0| = R \implies |z_* - z_0| = R.$$

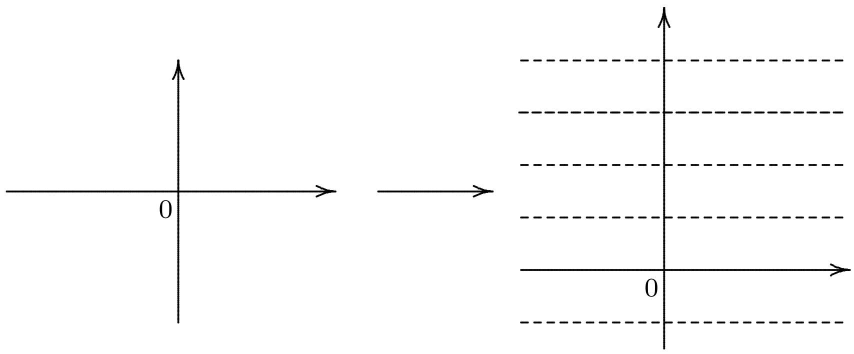
In other words, z_* lies on the circle $|z - z_0| = R$. On the other hand, it is a simple exercise that a limit point of a sequence of distinct singular points of a function f must be a singular point of f (Exercise). Thus z_* is a singular point of f on $|z - z_0| = R$. This contradicts the assumption that $g(z)$ is analytic on the circle $|z - z_0| = R$, and thus the claim is proved.
Since $g(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ on $B(z_0, R)$, it follows from the uniqueness of power series representation that $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ is the Taylor series of g at z_0. On the other hand, by Taylor's theorem, since $g(z)$ analytic on $B(z_0, R')$, it follows that $g(z)$ is equal to its Taylor series on $B(z_0, R')$, i.e.,

$$\sum_{k=0}^{\infty} a_k(z - z_0)^k = g(z) \quad \text{(converges) on } B(z_0, R').$$

Thus, the radius of convergence of $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is at least $R' > R$, contradicting that its radius of convergence is R. $\square$

Example 5.1.8. (Successive analytic continuation may lead to a multi-valued function)
Let H_1, H_2, H_3 and H_3 be the right, upper, left and lower half planes respectively. Define the following branches of $\log z$ on each of the planes as follows:

$$\begin{aligned}
&\text{On } H_1: \quad \log_1 z = \log|z| + i\arg z, \qquad -\pi/2 < \arg z < \pi/2\\
&\text{On } H_2: \quad \log_2 z = \log|z| + i\arg z, \qquad 0 < \arg z < \pi\\
&\text{On } H_3: \quad \log_3 z = \log|z| + i\arg z, \qquad \pi/2 < \arg z < 3\pi/2\\
&\text{On } H_4: \quad \log_4 z = \log|z| + i\arg z, \qquad \pi < \arg z < 2\pi
\end{aligned}$$



Note that

$$\log_2 z = \log_1 z \quad \text{on } H_1 \cap H_2 = \{z : 0 < \arg z < \pi/2\}.$$

Thus, $\log_2 z$ is a direct analytic continuation of $\log_1 z$.
Similarly, $\log_3 z$ is a direct analytic continuation of $\log_2 z$, and
$\log_4 z$ is a direct analytic continuation of $\log_3 z$.
But we note that $\log_4 z$ is NOT a direct analytic continuation of $\log_1 z$.
This shows that direct analytic continuation is not transitive.

5.2. The Gamma function $\Gamma(z)$ (Optional reading)

The **gamma function** is a very interesting function which occurs in analytic number theory as well as in other mathematical contexts such as statistics and probability. It may be regarded as a 'continuous' version of the factorial function ($(n-1)!$).

The gamma function is defined initially in the right half-plane $Re(z) > 0$, by the improper integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \text{for } Re(z) > 0, \tag{1}$$

where the variable t is real and taken along the positive axis, and the principal branch of t^{z-1} is taken, i.e., $t^{z-1} = e^{(z-1)\operatorname{Log} t} = e^{(z-1)\ln t}$. First we have

Proposition 5.2.1.
(i) When Re $(z) > 0$, the improper integral in (1) converges, i.e., $\Gamma(z)$ is finite when $Re(z) > 0$.
(ii) $\Gamma(z)$ is an analytic function in the right half plane Re $(z) > 0$.

Proof. To see (i), we just need to check that the integral in (1) converges absolutely, i.e., $\int_0^\infty |e^{-t}t^{z-1}|\, dt < \infty$. Note that for $t > 0$,

$$|t^{z-1}| = |e^{(z-1)\ln t}| = e^{(\text{Re } z-1)\ln t} = t^{\text{Re } z-1}.$$

Note that one can use L'Hopital's rule to check that

$$\lim_{t\to+\infty} \frac{t^{\text{Re } z-1}}{e^{\frac{t}{2}}} = 0.$$

From this, one can deduce that there exists a constant $C > 0$ such that

$$t^{\text{Re } z-1} \le Ce^{\frac{t}{2}} \quad \text{for all } t \ge 1.$$

$$\Longrightarrow \int_1^\infty |e^{-t}t^{z-1}|\, dt \le C\int_1^\infty e^{-t}e^{t/2}\, dt = C\int_1^\infty e^{-t/2}\, dt = 2C\sqrt{e}. \tag{2}$$

Note that if $Re(z) > 0$, then it is easy to check that the integrand

$$|e^{-t}t^{z-1}| = e^{-t}t^{\text{Re } z-1} \le 1 \quad \text{for } 0 \le t \le 1$$

$$\Longrightarrow \int_0^1 |e^{-t}t^{z-1}|\, dt \le 1. \tag{3}$$

Together with (2), it follows that $\int_0^\infty |e^{-t}t^{z-1}|\, dt < \infty$.

Hence $\int_0^\infty e^{-t}t^{z-1}\,dt$ converges (i.e., it has a finite value), which gives (i). We will leave (ii) as an exercise to show that

$$\Gamma'(z) = \int_0^\infty e^{-t}(\ln t)t^{z-1}\,dt \quad \text{for Re } z > 0.$$

Remark 5.2.2. When Re $z \le 0$, the integral in (1) does not converge. For example, when $z = 0$,

$$\int_0^\infty e^{-t}t^{0-1}\,dt = \int_0^\infty \frac{1}{te^t}\,dt \ge \int_0^1 \frac{1}{te}\,dt = +\infty.$$

Proposition 5.2.3. (i) When Re $(z) > 0$, one has $\Gamma(z+1) = z\Gamma(z)$.
(ii) $\Gamma(1) = 1$.
(iii) When n is a positive integer, one has $\Gamma(n) = (n-1)!$.

Proof. (i) Using integration by parts, we have, for Re $(z) > 0$,

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty e^{-t}t^z\,dt \\
&= \int_0^\infty t^z\,d(-e^{-t}) \\
&= \lim_{R\to+\infty} -\frac{t^z}{e^t}\Big|_0^R - \int_0^\infty -e^{-t}zt^{z-1}\,dt \\
&= 0 - 0 + z\Gamma(z) \quad \text{(Exercise)},
\end{aligned}$$

which gives (i). To prove (ii), we just observe that

$$\Gamma(1) = \int_0^\infty e^{-t}t^0\,dt = \lim_{R\to+\infty} -e^{-t}\Big|_0^R = 1.$$

Finally, by repeated use of (i), we get

$$\begin{aligned}
\Gamma(n) = (n-1)\Gamma(n-1) &= (n-1)(n-2)\Gamma(n-2) \\
&= \cdots \\
&= (n-1)(n-2)\cdot\cdots\cdot 2\cdot 1\cdot\Gamma(1) \\
&= (n-1)(n-2)\cdot\cdots\cdot 2\cdot 1\cdot 1 = (n-1)!,
\end{aligned}$$

since $\Gamma(1) = 1$ by (ii), which gives (iii). $\square$

Finall we have

Theorem 5.2.4. $\Gamma(z)$ has an analytic continuation to $\mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}$.

Proof. Recall from Proposition 5.2.1 that $\Gamma(z)$ is an analytic function on the right half plane $D_0 = \{z \in \mathbb{C} : \text{Re }(z) > 0\}$. Also, from Proposition 5.2.3(i), we know that

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad \text{for } z \in D_0. \tag{4}$$

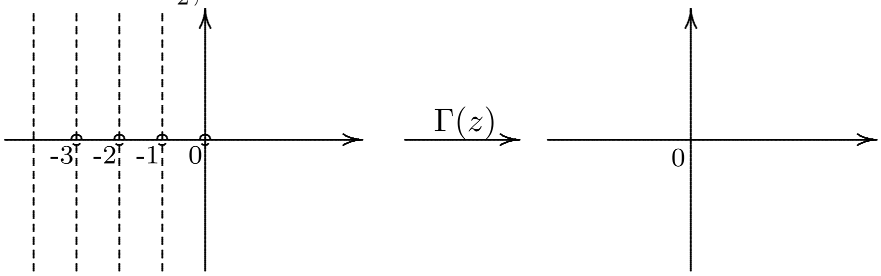
For each positive integer k, we consider the domain

$$D_k := \{z \in \mathbb{C} \,\big|\, \text{Re }(z) > -k, \text{ and } z \neq -k+1, -k+2, \cdots, -1, \cdots, 0\}.$$

It is easy to see that

$$D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq D_3 \subsetneq \cdots, \text{ and}$$
$$\bigcup_{k=1}^{\infty} D_k = \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}.$$

We are going to analytic continue $\Gamma(z)$ first to an analytic function on D_1, and then to D_2, and so on.



First we define the function $\Gamma_1(z)$ on D_1 given by

$$\Gamma_1(z) = \frac{\Gamma(z+1)}{z}, \quad z \in D_1. \tag{5}$$

Observe that if $z \in D_1$, then $z+1 \in D_0$, so that $\Gamma(z+1)$ is defined by the integral in (1). Moreover, by Proposition 5.2.1, the right hand side of (5) is analytic at z as long as $z+1 \in D_0$ and $z \neq 0$, i.e., as long as $z \in D_1$. Hence $\Gamma_1(z)$ is analytic on D_1. For $z \in D_0$, one has

$$\begin{aligned}\Gamma_1(z) &= \frac{\Gamma(z+1)}{z} \quad (\text{by } (5)) \\ &= \Gamma(z) \quad (\text{by } (4)).\end{aligned} \tag{6}$$

Therefore, $\Gamma_1(z)$ is an analytic continuation of $\Gamma(z)$ to the domain D_1. Observe that we have

$$\Gamma_1(z) = \frac{\Gamma_1(z+1)}{z} \quad \text{for all } z \in D_1. \tag{7}$$

To see this, we observe that both sides of (7) are analytic functions on D_1, and it follows from (5) that the equality in (7) holds on D_0, since

$\Gamma_1(z) = \Gamma_1(z)$ and $\Gamma_1(z) = \Gamma(z)$ on D_0. Thus by the identity theorem of analytic functions, the inequality in (7) holds on D_1.
Similarly, we define

$$\Gamma_2(z) = \frac{\Gamma_1(z+1)}{z}, \quad z \in D_2. \tag{8}$$

It is easy to see that the right hand side of (8) is analytic as long as $z+1 \in D_1$, i.e., if $z \in D_2$. Hence $\Gamma_2(z)$ is analytic on D_2. As in (6), one easily checks from (7) and (8) that

$$\Gamma_2(z) = \Gamma_1(z) \quad \text{on } D_1, \tag{9}$$

and thus $\Gamma_2(z)$ is an analytic continuation of $\Gamma_1(z)$ to D_2. Similar to (7), one can show that

$$\Gamma_2(z) = \frac{\Gamma_2(z+1)}{z} \quad \text{for all } z \in D_2. \tag{10}$$

By repeating the above arguments, one can then construct a sequence of functions $\{\Gamma_k(z)\}_{1 \le k < \infty}$ such that for each $k \ge 1$,
(i) $\Gamma_k(z)$ is an analytic function on D_k;
(ii) $\Gamma_{k+1}(z)$ is an analytic continuation of $\Gamma_k(z)$ to D_{k+1};
(iii) $\Gamma_k(z+1) = z\Gamma_k(z), \quad z \in D_k$.
Finally we define a function $\tilde{\Gamma}(z)$ on $\mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}$ by letting

$$\tilde{\Gamma}(z) = \Gamma_k(z), \quad \text{whenever } z \in D_k. \tag{11}$$

By (ii), one easily checks that if $\Gamma_k(z) = \Gamma_{k'}(z)$ if $z \in D_k$ and $z \in D_{k'}$. Thus, $\tilde{\Gamma}(z)$ is a well-defined function. Also, (i) implies readily that $\tilde{\Gamma}(z)$ is analytic on $\mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}$. By (6), we have

$$\tilde{\Gamma}(z) = \Gamma(z) \quad \text{on } D_0.$$

Thus, $\tilde{\Gamma}(z)$ is an analytic continuation of $\Gamma(z)$ to $\mathbb{C} \setminus \{0, -1, -2, \cdots\}$. $\square$

Remark 5.2.5. From (iii) above, one easily deduces that

$$\tilde{\Gamma}(z+1) = z\tilde{\Gamma}(z), \quad z \in \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}. \tag{12}$$

(ii) Following standard practices in literature, we will abuse notation and simply denote the analytic continuation (i.e. $\tilde{\Gamma}(z)$) in Theorem 5.2.4 by $\Gamma(z)$, and the analytic continuation will also be simply called the Gamma function. In other words, the **Gamma function** $\Gamma(z)$ will mean the improper integral in (1) as well as its analytic continuation to $\mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}$. In particular, by (12), we have

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{C} \setminus \{0, -1, -2, -3, \cdots\}. \tag{13}$$

(iii) The Gamma function $\Gamma(z)$ has isolated singular points at $z = 0, -1, -2, -3, \cdots$. Using (13) and that fact that $\Gamma(z)$ is analytic on the right half plane Re $(z) > 0$ with $\Gamma(1) = 1$, one can show that $\Gamma(z)$ has a simple pole at each $z = -n$ and

$$\operatorname*{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}. \quad \text{(Exercise)}$$