NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II

Lecture Notes Part VI

Chapter 3. Conformal Mappings, Linear Fractional Transformations (continued)

3.5. Geometric properties of Möbius transformations/LFT

Möbius Transformations have several more very nice geometric properties.

Theorem 3.5.1. (Decomposition of Möbius transformations into a composition of basic transformations)
A general Möbius transformation can be decomposed into a series of basic transformations consisting of translations, inversions, rotations and dilations. (More precisely, a Möbius transformation can be written as the composition of at most a translation, an inversion, a rotation, a dilation followed by a translation.)

Proof: Let $f(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$, be a Möbius transformation.
If $c = 0$, then $f(z) = \dfrac{a}{d}z + \dfrac{b}{d}$ which is a obtained from a rotation followed by a dilation followed by a translation.
If $c \neq 0$, we have $\quad f(z) = \dfrac{az+b}{cz+d} = \dfrac{bc-ad}{c^2(z+d/c)} + \dfrac{a}{c}$.
If we let

$$\begin{aligned} z_1 &= f_1(z) = z + d/c, \\ z_2 &= f_2(z_1) = 1/z_1, \\ z_3 &= f_3(z_2) = \frac{bc-ad}{c^2} z_2, \\ w &= f_4(z_3) = z_3 + \frac{a}{c} \end{aligned}$$

Then $f(z) = (f_4 \circ f_3 \circ f_2 \circ f_1)(z)$ where f_1 is a translation by d/c, f_2 is an inversion, f_3 is a rotation followed by a dilation and f_4 is another translation.

Lemma 3.5.2. The general equation of a circle or straight line in the complex plane can be written in the form

$$Az\bar{z} + \overline{B}z + B\bar{z} + C = 0 \tag{*}$$

where $A, C \in \mathbb{R}$ and $|B|^2 - AC > 0$. If $A \neq 0$, then $(*)$ represents a circle. If $A = 0$, then $(*)$ represents a straight line.

Proof. A circle can be written as $|z - z_0| = r$, where z_0 is the center and $r > 0$ is the radius. Then we can rewrite

$$\begin{aligned} |z - z_0| = r &\iff |z - z_0|^2 = r^2 \\ &\iff (z - z_0)(\bar{z} - \overline{z_0}) = r^2 \\ &\iff z\bar{z} - \overline{z_0}z - z_0\bar{z} + (|z_0|^2 - r^2) = 0. \end{aligned} \tag{1}$$

Thus, $(*)$ holds with $A = 1$, $B = -z_0$, $C = |z_0|^2 - r^2$ with $|B|^2 - AC = |z_0|^2 - (|z_0|^2 - r^2) = r^2 > 0$. Also, a straight line in the complex plane can be represented by the equation $|z - z_1| = |z - z_2|$ for some complex constants $z_1 \neq z_2$. Then

$$\begin{aligned} |z - z_1| = |z - z_2| &\iff |z - z_1|^2 = |z - z_2|^2 \\ &\iff (z - z_1)(\bar{z} - \overline{z_1}) = (z - z_2)(\bar{z} - \overline{z_2}) \\ &\iff \overline{(z_2 - z_1)}z + (z_2 - z_1)\bar{z} + (|z_1|^2 - |z_2|^2) = 0. \end{aligned} \tag{2}$$

Thus (*) holds with $A = 0$, $B = z_2 - z_1$, $C = |z_1|^2 - |z_2|^2$ with $|B|^2 - AC = |z_2 - z_1|^2 > 0$.
Conversely, suppose (*) holds. If $A = 0$, then one easily sees that (*) becomes a linear equation in x and y with real coefficients, where $z = x + iy$, and thus (*) represents a straight line. If $A \neq 0$, then by dividing both sides of (*) by A and reversing the calcuations in (1), one easily sees that (*) represents a circle with center at $z_0 = -\frac{\overline{B}}{A}$ and radius $r = \sqrt{\frac{|B|^2 - AC}{A^2}} > 0$.

Lemma 3.5.3. The inversion map $w = \frac{1}{z}$ maps a (circle or extended straight line) bijectively to a (circle or extended straight line).

Proof. By Lemma 3.5.2, we can represent a circle or an extended straight line by the equation

$$Az\bar{z} + \overline{B}z + B\bar{z} + C = 0 \tag{*}$$

where $A, C \in \mathbb{R}$, $|B|^2 - AC > 0$ (with the understanding that ∞ is included when (*) represents a straight line, i.e., when $A = 0$). Now the transformation $w = f(z) = 1/z$ maps equation of the circle or straight line

$$Az\bar{z} + \overline{B}z + B\bar{z} + C = 0$$

in the z-plane to the equation

$$\begin{aligned} & A\frac{1}{w\overline{w}} + \overline{B}\frac{1}{w} + B\frac{1}{\overline{w}} + C = 0 \\ \Longleftrightarrow \qquad & Cw\overline{w} + Bw + \overline{B}\overline{w} + A = 0, \qquad (**) \end{aligned}$$

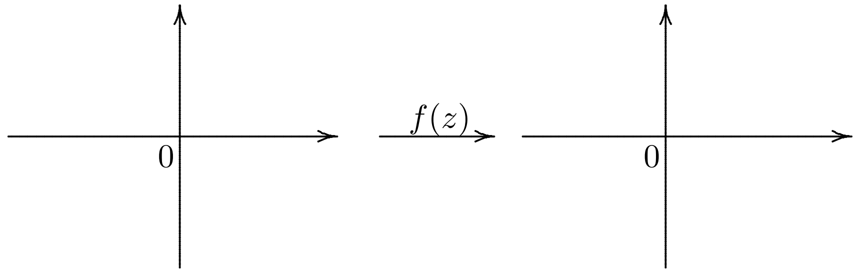
which is again the equation of a circle or extended straight line in the w-plane (again with the understanding that ∞ is included in the case of a straight line, i.e., when $C = 0$). Note that if $A = 0$, then (*) represents a extended straight line (with ∞ included). Note that $f(\infty) = 0$ which also lies in (**). Similarly, if $C = 0$, then the locus of (*) contains the origin 0, where we have $f(0) = \infty$. Note that in this case, (**) represents an (extended) straight line where ∞ is also included. Thus, $w = 1/z$ maps a (circle or extended line) to a (circle or extended line). Since LFT maps $\hat{\mathbb{C}}$ bijectively to $\hat{\mathbb{C}}$, one can deduce that the inversion map $w = 1/z$ also maps a (circle or extended line) bijectively to a (circle or extended line) (Exercise).

Remark 3.5.4. We note that straight lines passing through the origin have $A = 0, C = 0$ and so are mapped to straight lines through the origin again, straight lines that do not pass through the origin have $A = 0, C \neq 0$ so are mapped to circles passing through the origin, circles passing through the origin are mapped to straight lines that do not pass through the origin and finally, circles which do not pass through the origin are mapped to circles which do not pass through the origin.

Example. Find the image of (i)the real and imaginary axes; (ii) the line $y = 1$; (iii) the line $x = 1$; and (iv) the line $x + y = 1$ under the inversion map.

Finally we have

Theorem 3.5.5. A Möbius transformation transforms (circles and extended straight lines) bijectively to (circles and extended straight lines).



Proof. By Theorem 3.5.1, we can write

$$f = f_n \circ f_{n-1} \circ \cdots \circ f_1,$$

where each f_i is either a translation, inversion, rotation or dilation. It is also clear from elementary geometry that translations, dilations and rotations also map a circle bijectively to a circle, and they map an extended straight line bijectively to an extended straight line. Together with Lemma 3.5.3, it follows that each f_i transforms (circles and extended straight lines) bijectively to (circles and extended straight lines), from which one sees that f also has the same property.

Remark:
1. For the purpose of this chapter, when we say straight line, we will mean an extended straight line unless otherwise specified.

2. As we saw above, the inversion map $w = 1/z$ maps 0 to ∞ and so transforms any circle or straight line passing through the origin to an straight line. Similarly, a general Möbius transformation with $c \neq 0$ maps $-d/c$ to ∞ and so transforms any circle or straight line through $-d/c$ to a straight line. If $c = 0$, ∞ is mapped to ∞ and so straight lines are mapped to straight lines, and circles are mapped to circles.

3. Recall from Proposition 3.3.1 that circles and extended lines of $\mathbb{C}$ corresponds to circles on the Riemann sphere S^2 via the stereographic projection map. Thus, Theorem 3.5.5 can be interpreted as follows: A Möbius transformation transforms circles on S^2 bijectively to circles on S^2.

Theorem 3.5.6.
(1) Any Möbius transformation not equal to the identity has either one or two fixed points. (A **fixed point** of a function f is a point z such that $f(z) = z$.)
(2) A Möbius transformation which fixes 3 distinct points is the identity transformation.
(3) The unique Möbius transformation mapping three different points $z_1, z_2,\ z_3$ to ∞, 0 and 1 respectively is given by

$$Tz \equiv T(z) = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}. \qquad (*)$$

(If one of z_1, z_2 or z_3 is ∞, we cancel the two factors which contain that z_i, for example, if $z_2 = \infty$, then the required LFT can be obtained by taking the limit of (*) as $z_2 \to \infty$ (or equivalently, the LFT can be obtained from (*) by removing the two factors involving z_2, i.e., one has

$$T(z) = \lim_{z_2 \to \infty} \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \frac{(z_3 - z_1)}{(z - z_1)} \qquad (**)$$

The case when $z_1 = \infty$ or when $z_3 = \infty$ can be obtained similarly.

(4) There exists a unique Möbius transformation which maps any ordered triple of 3 distinct points in $\hat{\mathbb{C}}$ to any other ordered triple of 3 distinct points.

Proof. (1) Suppose the the Möbius transformation is given by

$$Tz = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

Case 1 ($c \neq 0$):

$$Tz = z \Longrightarrow cz^2 + (d - a)z - b = 0$$

which has one or two solutions. Note that $T\infty = a/c \neq \infty$.
Case 2 ($c = 0$): We have $T(\infty) = \infty$ and for $z \neq \infty$,

$$Tz = z \Longrightarrow \frac{az + b}{d} = z \Longrightarrow (a - d)z + b = 0.$$

If $a \neq d$, there is one other fixed point of f given by $z = -b/(a - d)$.
If $a = d$, then $a = d \neq 0$ (since $ad - bc \neq 0$) and $b \neq 0$ (otherwise, T is the identity mapping). In this case, the above equation has no finite solution, and thus the only fixed point of f is ∞.

(2) This follows from (1) above.

(3) A direct verification shows that the transformation Tz in (*) indeed maps z_1 to ∞, z_2 to 0 and z_3 to 1 (check it!). For uniqueness, if S is another Möbius transformation which satisfies this condition, then $T \circ S^{-1}$ fixes 0, 1 and ∞ and therefore must be the identity mapping by (2). Hence $T \equiv S$.

(3') Say, when $z_2 = \infty$, one can also see directly that (**) is the LFT which maps z_1, ∞, z_3 to ∞, 0, 1 respectively. The other cases when $z_1 = \infty$ and when $z_3 = \infty$ can be treated similarly.

(4) Suppose z_1, z_2, z_3 and w_1, w_2, w_3 are two ordered triples of distinct points. Let T be the Möbius transformation mapping z_1, z_2, z_3 to $\infty, 0, 1$ respectively and S be the Möbius transformation mapping w_1, w_2, w_3 to $\infty, 0, 1$ respectively. Then $S^{-1} \circ T$ is a Möbius transformation mapping z_1, z_2, z_3 to w_1, w_2, w_3 respectively. Proof of uniqueness is similar to (3) above (exercise).

Example. Find the Möbius transformation mapping the points $0, i, -i$ to $1, -1, 0$ respectively.

Solution: Let T be the LFT mapping $0, i, -i$ to $\infty, 0, 1$ respectively, and let S be the LFT mapping $1, -1, 0$ to $\infty, 0, 1$ respectively. Then $w = S^{-1} \circ T$ will be the required LFT sending $0, i, -i$ to $1, -1, 0$ respectively. Thus the required LFT is given by

$$\begin{aligned} w = S^{-1} \circ T(z) &\Longrightarrow Sw = Tz \\ &\Longrightarrow \frac{(w+1)(0-1)}{(w-1)(0+1)} = \frac{(z-i)(-i-0)}{(z-0)(-i-i)} \\ &\Longrightarrow \frac{-w-1}{w-1} = \frac{z-i}{2z} \\ &\Longrightarrow w = \frac{z+i}{-3z+i}, \end{aligned}$$

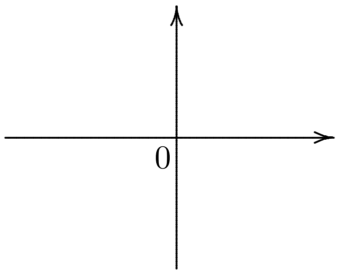
which is the required LFT.

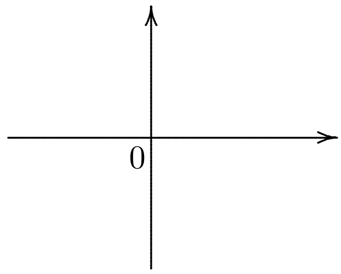
More Examples. (see Tutorial 7)

3.6. The Cross-ratio

Definition 3.6.1. (Cross-ratio) The cross ratio of 4 distinct points z_1, z_2, z_3 and z_4 on the extended complex plane $\hat{\mathbb{C}}$, denoted by $(z_1, z_2; z_3, z_4)$, is the image of z_4 under the Möbius transformation T which maps z_1, z_2 and z_3 to ∞, 0 and 1 respectively. Hence by Theorem 3.5.5,

$$(z_1, z_2; z_3, z_4) = \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)}.$$





Remarks 3.6.2.
1. The order of the points z_1, z_2, z_3 and z_4 is important. Also, some books give a different definition for the cross-ratio based on a different permutation of these 4 points. For example, Ahlfors has a different definition.
2. The definition of the cross-ratio still makes sense if one of the z_i's is ∞, in this case, we take the limit of the expression as this particular z_i approaches ∞ (cf. Theorem 3.5.6 part(3)). This limit exists and the result is equivalent to removing the two factors in the expression containing the z_i, for example,

$$(z_1, z_2; z_3, \infty) = \lim_{z_4 \to \infty} \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)} = \frac{z_3 - z_1}{z_3 - z_2},$$

$$(\infty, z_2; z_3, z_4) = \lim_{z_1 \to \infty} \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)} = \frac{z_4 - z_2}{z_3 - z_2},$$

3. One always has $(\infty, 0; 1, z) = z$.(This follows directly from the definition).

Examples. Find the following cross ratios: (i)$(1, i, -i, 0)$;
(ii) $(\infty, 1 + i, 2, 5)$; (iii) $(0, \infty, 1, z)$

Proposition 3.6.3. Permuting the 4 points have the following effect on the cross-ratios:

$$(z_1, z_2; z_3, z_4) = (z_3, z_4; z_1, z_2),$$
$$(z_1, z_2; z_3, z_4) = \frac{1}{(z_2, z_1; z_3, z_4)} = \frac{1}{(z_1, z_2; z_4, z_3)},$$
$$(z_1, z_2; z_3, z_4) = (z_2, z_1; z_4, z_3) = (z_4, z_3; z_2, z_1).$$

Proof. Exercise.

The importance of the cross-ratio is that it is a geometric invariant under Möbius transformations, namely, we have:

Proposition 3.6.4. The cross-ratio of 4 points is invariant under any Möbius transformation S, that is,

$$(Sz_1, Sz_2; Sz_3, Sz_4) = (z_1, z_2; z_3, z_4).$$

Proof. Let T be the LFT which maps z_1, z_2, z_3 to $\infty, 0$ and 1 respectively. Then $T \circ S^{-1}$ maps Sz_1, Sz_2 and Sz_3 to $\infty, 0$ and 1 respectively, so by the definition of the cross-ratio,

$$(Sz_1, Sz_2; Sz_3, Sz_4) = T \circ S^{-1}(Sz_4) = Tz_4 = (z_1, z_2; z_3, z_4).$$

Remark. The invariance of the cross-ratio can be used to find the LFT which maps any three points z_1, z_2, z_3 to any other 3 points w_1, w_2, w_3 respectively, namely, the transformation $w = f(z)$ is given by the equation

$$\boxed{(z_1, z_2; z_3, z) = (w_1, w_2; w_3, w).}$$

equivalently,

$$\frac{(w - w_2)(w_3 - w_1)}{(w - w_1)(w_3 - w_2)} = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}.$$

Proof. Let $w = f(z)$ be the required LFT. Then for any z, from the invariance of the cross-ratio under f, we have

$$(z_1, z_2; z_3, z) = (f(z_1), f(z_2); f(z_3), f(z)) = (w_1, w_2; w_3, w).$$

Example Find the LFT which maps $1, 2, 7$ to $1, 2, 3$ respectively.

Solution. Suppose $w = f(z)$ is the required LFT. Then by the invariance of cross ratio under f, we have $(1, 2; 3, w) = (1, 2; 7, z)$, or

$$\frac{(w - 2)(3 - 1)}{(w - 1)(3 - 2)} = \frac{(z - 2)(7 - 1)}{(z - 1)(7 - 2)}$$

which gives $w = \frac{7z-4}{2z+1}$.

Example. Find an LFT that maps the real line to the unit circle $|z| = 1$.

Example. Show that an LFT f which is not the identity map fixes the two distinct points $a, b \in \mathbb{C}$ if and only if f is of the form

$$\frac{f(z) - a}{f(z) - b} = K\frac{z - a}{z - b} \qquad (*)$$

for a constant $K \in \mathbb{C} - \{0, 1\}$.

Solution: If f is of the form above, then it is clear that $f(a) = a$ and $f(b) = b$. Conversely, let $w = f(z)$ be the required LFT. Take a point $c \in \mathbb{C}$ such that $f(c) \in \mathbb{C}$ and $f(c) \neq c$ (why must such c exist?), so that f maps b, a, c to $b, a, f(c)$ respectively. Then by the invariance of cross ratio under f, we have

$$\begin{aligned}
(b, a; f(c); w) &= (b, a; c; z) \\
\implies \frac{(w - a)(f(c) - b)}{(w - b)(f(c) - a)} &= \frac{(z - a)(c - b)}{(z - b)(c - a)} \\
\implies \frac{f(z) - a}{f(z) - b} &= K\frac{z - a}{z - b}, \\
\text{where } K &= \frac{(f(c) - a)(c - b)}{(f(c) - b)(c - a)} = (b, a; c, f(c)),
\end{aligned}$$

from which one easily sees that $K \neq 0, 1$.

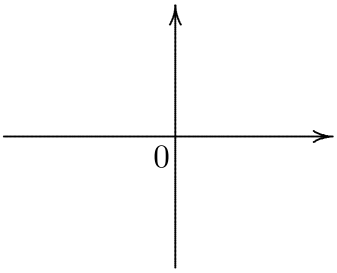
Remark. In the above example, Let f^n be the n-th composite of f, i.e., $f^n = f \circ f \circ \cdots \circ f$ n-times. Then one easily sees from (*) that f^n satisfies

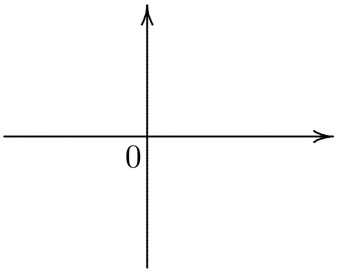
$$\frac{f^n(z) - a}{f^n(z) - b} = K^n\frac{z - a}{z - b}. \qquad \text{(Exercise.)}$$

Question-Exercise: Take a point $z \neq a, b$. What happens to $f^n(z)$ as $n \to \infty$? Distinguish the three cases when $|K| < 1$, $|K| > 1$ and $|K| = 1$.

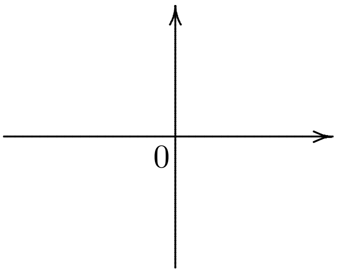
3.7. Orientation principle

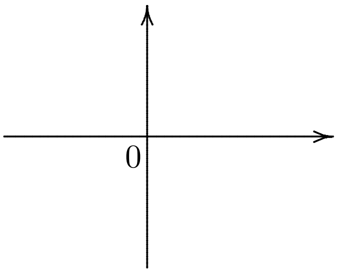
Let γ be an oriented (directed) circle or extended straight line in $\mathbb{C}$. The orientation of γ (as well as γ itself) is determined by three distinct successive points z_1, z_2, z_3 on γ.





When γ is an oriented straight line with three distinct points z_1, z_2, z_3 in succession, it is easy to see that γ divides $\mathbb{C} \setminus \gamma$ into two half planes, one consisting of points on the left side of γ, and the other consisting of points on the right side of γ. Similarly an oriented circle divides $\mathbb{C} \setminus \gamma$ into two domains, the interior and the exterior of the circle. If one travels along the circle following the orientation of γ, then the interior points will all appear to be on one (left or right) side of γ. If the circle γ is positively (resp. negatively) oriented, then the interior will be on the left (resp. right) side of γ.





Proposition 3.7.1. Let γ be an oriented circle or extended straight line, and let U and V be the two domains of $\mathbb{C} \setminus \gamma$. Let z_1, z_2, z_3 be three distinct points of γ in succession. Then

$$\text{Im } (z_1, z_2; z_3, z) = 0 \Longleftrightarrow z \in \gamma \setminus \{z_1, z_2, z_3\};$$

and

$$\text{Im } (z_1, z_2; z_3, z) > 0 \quad \forall\, z \in U \quad \& \quad \text{Im } (z_1, z_2; z_3, z) < 0 \quad \forall\, z \in V,$$

or vice versa.

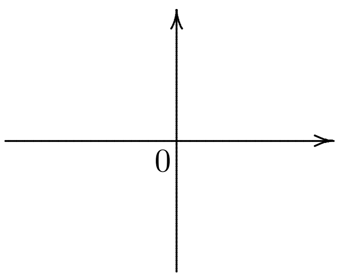
Proof. Let T be the LFT which maps z_1, z_2, z_3 to $\infty, 0, 1$ respectively, so that $Tz = (z_1, z_2; z_3, z)$. By Theorem 3.5.5, T maps γ bijectively to the extended straight line L passing through $\infty, 0, 1$ (i.e. the real axis), and similarly, T^{-1} maps L bijectively to γ. Hence,

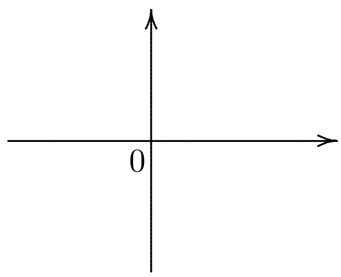
$$z \in \gamma \setminus \{z_1, z_2, z_3\} \Longleftrightarrow Tz \in L \setminus \{\infty, 0, 1\} \Longleftrightarrow \text{Im } (z_1, z_2; z_3, z) = 0.$$

Let U (or V) be one of the two domains of $\mathbb{C}\backslash\gamma$. Then Im $(z_1, z_2; z_3, z)$ is a continuous function on U (in the variable z). Note that the 3 points $\infty, 0, 1$ in succession determine an orientation of L, and T preserves the orientations of γ and L. Note that T maps $\hat{\mathbb{C}}$ bijectively to $\hat{\mathbb{C}}$. Together with Theorem 3.5.5, it follows that T maps $\hat{\mathbb{C}} \setminus \gamma$ bijectively to $\mathbb{C} \setminus L$. Hence

$$\text{Im } (z_1, z_2; z_3, z) \neq 0 \quad \forall z \in \mathbb{C} \setminus \gamma. \tag{*}$$

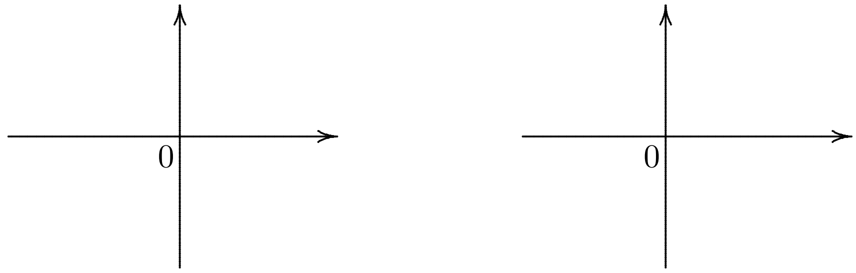
Take a component, say U, of $\mathbb{C} \setminus \gamma$, and take a point $z_0 \in U$, and suppose, say Im $(z_1, z_2; z_3, z_0) > 0$. Then we are going to show that Im $(z_1, z_2; z_3, z) > 0$ for all other $z \in U$. To see this, we let ℓ be a polygonal line segment in U joining z_0 to z. Suppose Im $(z_1, z_2; z_3, z) < 0$. Then one can use the Intermediate Value Theorem to conclude that there is a point z' on ℓ such that Im $(z_1, z_2; z_3, z') = 0$ (Exercise), contradicting (*). Thus, Im $(z_1, z_2; z_3, z)$ has the same sign for all points in the same domain of $\mathbb{C} \setminus \gamma$. On the other hand, the sign cannot be the same for both U and V, since if Im $(z_1, z_2; z_3, z) > 0$ (resp. < 0) for all points of U and V, then T maps the entire complex plane to the upper half plane Im $w \geq 0$ (resp. lower half plane Im $w \leq 0$), contradicting the fact that T maps $\hat{\mathbb{C}}$ bijectively to $\hat{\mathbb{C}}$. This finishes the proof of the proposition.





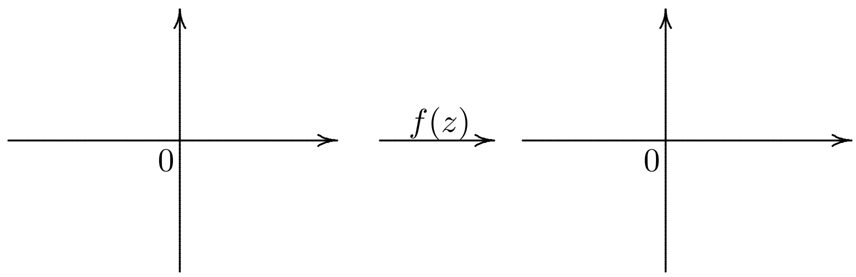
Remark. Using the angle preserving property of conformal mappings, one easily sees that the LFT T which maps z_1, z_2, z_3 to $\infty, 0, 1$ respectively will map points on the left side of γ to points in the upper half plane, and points on the right side of γ to the lower half plane. The above observation and Proposition 3.7.1 lead immediately to the following

Corollary 3.7.2. Let γ be an oriented circle or extended straight line in the complex plane, and let z_1, z_2, z_3 be 3 distinct points of γ in succession. Then $\hat{\mathbb{C}}\backslash\gamma$ consists of two domains. The domain on the left side of γ consists of points z such that Im $(z_1, z_2; z_3, z) > 0$, and the domain on the right side of γ consists of points z such that Im $(z_1, z_2; z_3, z) < 0$.



Note that when γ is a circle, we may also regard ∞ as a point on the left (resp. right) side of γ if Im $(z_1, z_2; z_3, \infty) > 0$ (resp. < 0). Finally we have

Theorem 3.7.3. (Orientation principle) Let γ be an oriented circle or extended straight line, and let z_1, z_2, z_3 be three distinct points on γ in succession, and let $f(\gamma)$ be endowed with the orientation determined by $f(z_1), f(z_2), f(z_3)$ in succession. Then an LFT f will map the component of $\hat{\mathbb{C}} \setminus \gamma$ on the left side (resp. right side) of γ bijectively to the component of $\hat{\mathbb{C}} \setminus f(\gamma)$ on the left side (respectively, right side) of $f(\gamma)$. (Here the component of $\hat{\mathbb{C}} \setminus \gamma$ on the left side of γ means the domain of $\mathbb{C} \setminus \gamma$ on the left side of γ together with ∞ if ∞ also lies on the left side of γ, etc.)



Proof. Using the invariance of the cross-ratio under f, one has

$$(z_1, z_2; z_3, z) = (f(z_1), f(z_2); f(z_3), f(z)).$$

Thus, by Corollary 3.7.2,

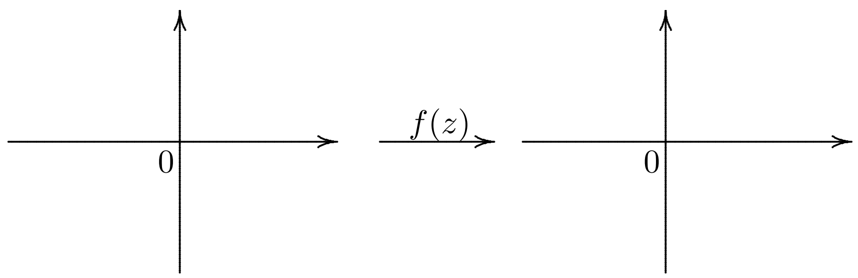
$$\begin{aligned} & z \text{ lies on the left side of } \gamma \\ \iff & \text{Im } (z_1, z_2; z_3, z) > 0 \\ \iff & \text{Im } (f(z_1), f(z_2); f(z_3), f(z)) > 0 \\ \iff & f(z) \text{ lies on the left side of } f(\gamma). \end{aligned}$$

The points on the right side of γ can be treated similarly. The property that f maps each component of $\hat{\mathbb{C}} \setminus \gamma$ bijectively to the corresponding component of $\hat{\mathbb{C}} \setminus f(\gamma)$ can be deduced easily from the fact that f is a bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, using an argument similar to the one used in the proof of Proposition 3.7.1(Exercise). $\square$

Example. Find the image of the interior of the circle $C : |z-2| = 2$ under the LFT

$$w = f(z) = \frac{z}{2z-8}.$$

Solution. Let $z_1 = 0$, $z_2 = 2+2i$ and $z_3 = 4$. These are points on the circle C and the interior of C lies on the right as one goes along C in the direction $z_1 \to z_2 \to z_3$. The images under the LFT are $f(z_1) = 0$, $f(z_2) = -i/2$ and $f(z_3) = \infty$. So the image of C is the extended imaginary axis and the right side as we go along $f(z_1) \to f(z_2) \to f(z_3)$ is the left half plane, so f maps the interior of C to the left half plane $\text{Re } w < 0$.



3.8. Some Special LFT's

Theorem 3.8.1. The set of analytic automorphisms of the unit ball $|z| < 1$ consists of mappings of the form

$$f(z) = e^{i\theta}\phi_\alpha(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z},$$

where $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, and $\theta \in \mathbb{R}$.

Proof. See tutorial.

Proposition 3.8.2. The LFT

$$w = f(z) = \frac{z-\alpha}{z-\overline{\alpha}},$$

where Im $\alpha > 0$, is an analytic isomorphism from the **upper half plane (UHP or just H)** Im $z > 0$ to the unit ball $|w| < 1$, mapping α to 0.

Proof. Direct verification. Note that if z is real, then $|f(z)| = 1$ so that f maps the extended real line to the unit circle $|z| = 1$.

Theorem 3.8.3. The set of analytic isomorphisms from the UHP Im $z > 0$ to the unit ball $|w| < 1$ consists of mappings of the form

$$f(z) := e^{i\theta}\frac{z-\alpha}{z-\overline{\alpha}}, \text{ where Im } \alpha > 0 \text{ and } \theta \in \mathbb{R}. \qquad (*)$$

Proof. ($\Longleftarrow$) Given such an LFT f as in (*), for any $a \in \mathbb{R}$, $|f(a)| = 1$ so f maps the real line to the unit circle, it also maps $\alpha \in UHP$ to $0 \in B(0,1)$. Using the orientation principle, one easily sees that f is an analytic isomorphism as stated.
($\Longrightarrow$) Suppose f is an analytic isomorphism from the UHP to $B(0,1)$, then $f(\alpha) = 0$ for some α in the UHP. Now if h is an LFT given in the previous proposition, it also has the same property, so $f \circ h^{-1}$ is an analytic automorphism of the unit ball $B(0,1)$ mapping 0 to 0, so

$$f \circ h^{-1}(w) = e^{i\theta}w \Longrightarrow f(z) = e^{i\theta}h(z) = e^{i\theta}\frac{z-\alpha}{z-\overline{\alpha}}, \ (\text{Im } \alpha > 0, \ \theta \in \mathbb{R}).$$

Theorem 3.8.4. The set of analytic automorphisms of the UHP Im $z > 0$ consists of mappings of the form

$$f(z) = \frac{az+b}{cz+d}, \quad \text{with } a, b, c, d \in \mathbb{R},\ ad - bc > 0.$$

Proof. Let f be an analytic automorphism of the UHP. Take an analytic isomorphism g from the UHP to the unit ball $|w| < 1$ as given in Proposition 3.8.2, so that in particular, g is an LFT. Then $h = g \circ f \circ g^{-1}$ is an analytic automorphism of the unit ball, so h is an LFT by Theorem 3.8.3. Hence $f = g^{-1} \circ h \circ g$ is also an LFT since it is the composition of LFT's. It remains to show that the LFT can be chosen so that all the coefficients a, b, c, d are real and $ad - bc > 0$. This can be done directly by computing $f = g^{-1} \circ h \circ g$. However, we can show this in another way. Since f maps the real line to the real line, the same holds for f^{-1} and the points $\infty, 0, 1$ are the images of some z_1, z_2, z_3 on the extended real line under f. Hence

$$(\infty, 0, 1, w) = (z_1, z_2, z_3, z) \Longrightarrow w = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$$

$$\Longrightarrow f(z) = \frac{az+b}{cz+d},\ a, b, c, d \in \mathbb{R}$$

Now

$$f(i) = \frac{ai+b}{ci+d} = \frac{(bd+ac) + (ad-bc)i}{c^2+d^2}$$

which is in the UHP if and only if $ad - bc > 0$.

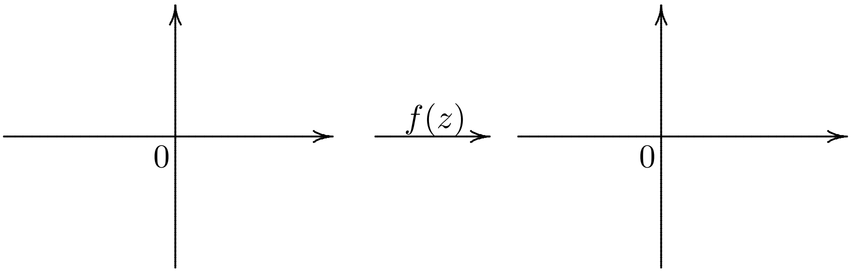
Corollary 3.8.5. The set of analytic isomorphisms from the UHP to the lower half plane Im $z < 0$ consists of LFTs of the form

$$f(z) = \frac{az+b}{cz+d}, \text{ with } a, b, c, d \in \mathbb{R}, ad - bc < 0.$$

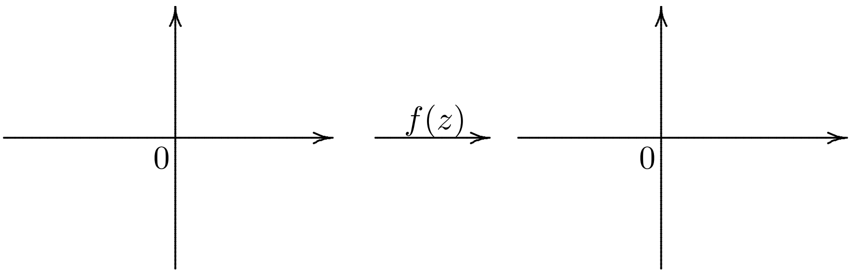
(Exercise. Hint: Use Theorem 3.8.4.)

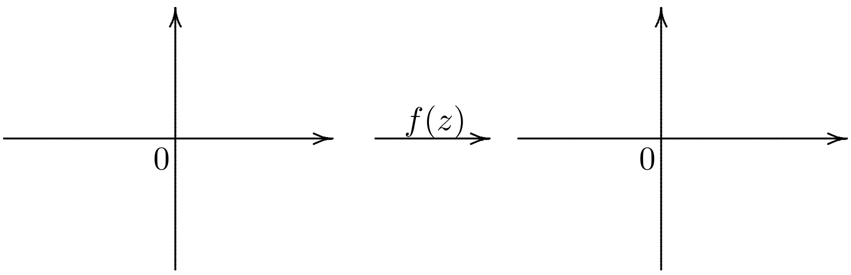
3.9. Symmetry principle

The points z and $\bar{z}$ are symmetric with respect to the real axis, in the sense that they are mirror images of each other with respect to the mirror given by the real axis. Also, if we have an LFT which preserves the real axis, it can be represented by an LFT f with real coefficients so that the images $f(z)$ and $f(\bar{z})$ are still symmetric with respect to the real axis since $f(\bar{z}) = \overline{f(z)}$. We are going to generalize this to circles and other extended straight lines.



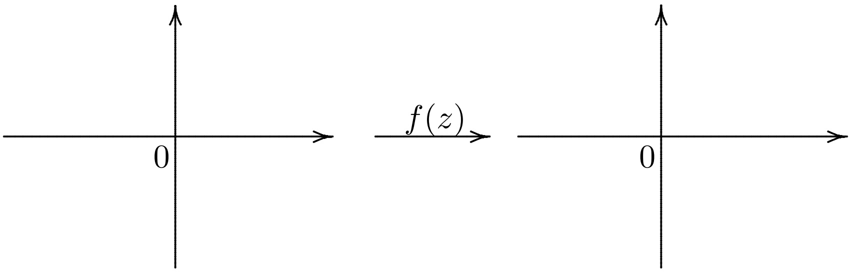
Definition 3.9.1. Two points z and z^* in $\hat{\mathbb{C}}$ are **symmetric** with respect to a circle or extended straight line C through z_1, z_2, z_3 if and only if $(z_1, z_2; z_3, z^*) = \overline{(z_1, z_2; z_3, z)}$.





Remark 3.9.2. The above definition depends only on C, and it does not depend on the choice of the points z_1, z_2, z_3. More precisely, if z_1', z_2', z_3' are another 3 distinct points on C. Then

$$(z_1, z_2; z_3, z^*) = \overline{(z_1, z_2; z_3, z)} \Longleftrightarrow (z_1', z_2'; z_3', z^*) = \overline{(z_1', z_2'; z_3', z)}.$$



Proof. Let T be the LFT sending z_1, z_2, z_3 to $\infty, 0, 1$ respectively, so that $Tz = (z_1, z_2; z_3, z)$. Let S be the LFT sending z_1', z_2', z_3' to $\infty, 0, 1$ respectively, so that $S(z) = (z_1', z_2'; z_3', z)$. Then $S \circ T^{-1}$ is an LFT sending the extended real axis to the extended real axis itself. Thus, $S \circ T^{-1}$ can be given as an LFT with real coefficients. In particular, we have

$$S \circ T^{-1}(\bar{z}) = \overline{S \circ T^{-1}(z)} \quad \text{for all } z. \tag{*}$$

Now,

$$\begin{aligned}
& (z_1, z_2; z_3, z^*) = \overline{(z_1, z_2; z_3, z)} \\
\Longleftrightarrow \quad & T(z^*) = \overline{Tz} \\
\Longleftrightarrow \quad & S \circ T^{-1}(Tz^*) = S \circ T^{-1}(\overline{Tz}) \\
\Longleftrightarrow \quad & S(z^*) = \overline{S \circ T^{-1}(Tz)} \quad \text{(by (*))} \\
\Longleftrightarrow \quad & S(z^*) = \overline{Sz} \\
\Longleftrightarrow \quad & (z_1', z_2'; z_3', z^*) = \overline{(z_1', z_2'; z_3', z)}.
\end{aligned}$$

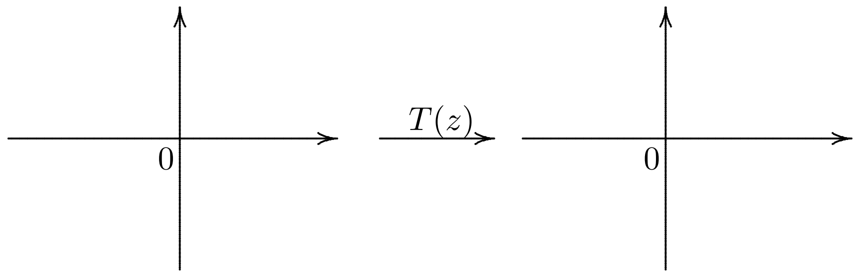
Remark.
(i) The points on C, and only those are symmetric to themselves. (Recall that the cross-ratio of 4 points is real if and only if they all lie on the same circle or extended straight line).
(ii) Let T be the LFT such that $Tz = (z_1, z_2, z_3, z)$. The mapping which sends z to z^* is a one-to-one correspondence and is called a **reflection** with respect to C, and is given by $z^* = T^{-1}(\overline{Tz})$ (Exercise).
(iii) The composition of two reflections (about different circles or extended straight lines) is an LFT (Exercise. You may find it helpful to write the LFTs in matrix form).

3.9.3. The geometric interpretation of symmetry:

Case 1. C is an extended straight line.
In this case, we can take $z_1 = \infty$. Then

$$\begin{aligned}
& (z_1, z_2; z_3, z^*) = \overline{(z_1, z_2; z_3, z)} \qquad (1) \\
\Longrightarrow \quad & \frac{z^* - z_2}{z_3 - z_2} = \overline{\left(\frac{z - z_2}{z_3 - z_2}\right)} \\
\Longrightarrow \quad & |z^* - z_2| = |z - z_2|.
\end{aligned}$$

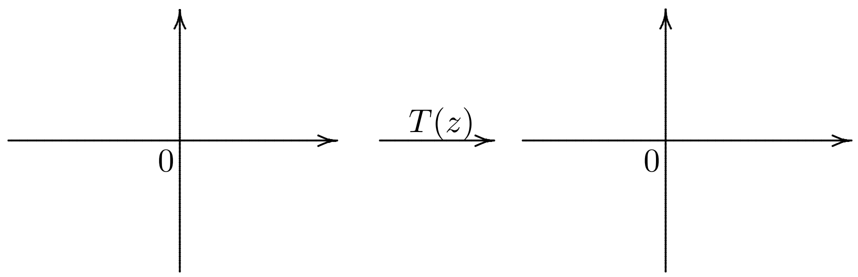
Since z_2 can be any arbitrary point on the line C, this means that z and z^* are equidistant from all points on C. Also, by Remark (i) above, z and z^* are distinct if $z \notin C$. In other words, C is the perpendicular bisector of the line segment joining z to z^* and $z^* = z$ if and only if z lies on C.



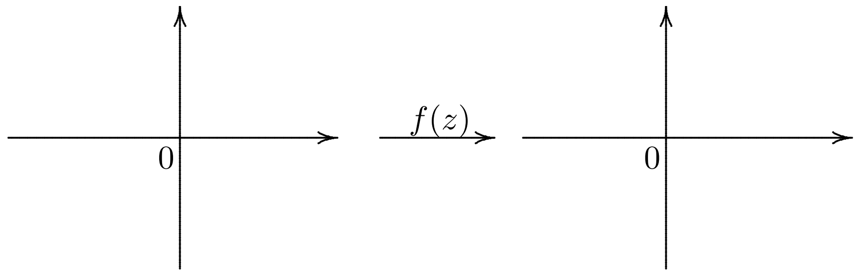
Case 2. C is a circle with centre at a and radius R.
Systematic use of the invariance of the cross-ratio under LFTs gives us

$$\begin{aligned}
(z_1, z_2; z_3, z^*) &= \overline{(z_1, z_2; z_3, z)} \\
&= \overline{(z_1 - a, z_2 - a; z_3 - a, z - a)} \\
&= (\overline{z_1 - a}, \overline{z_2 - a}; \overline{z_3 - a}, \bar{z} - \overline{a}) \\
&= (\frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}; \frac{R^2}{z_3 - a}, \bar{z} - \overline{a}) \\
&\qquad (\text{since } (z_1 - a)\overline{(z_1 - a)} = |z_1 - a|^2 = R^2, \text{ etc}) \\
&= (\frac{1}{z_1 - a}, \frac{1}{z_2 - a}; \frac{1}{z_3 - a}, \frac{\bar{z} - \overline{a}}{R^2}) \\
&= (z_1 - a, z_2 - a; z_3 - a, \frac{R^2}{\bar{z} - \overline{a}}) \\
&= (z_1, z_2; z_3, \frac{R^2}{\bar{z} - \overline{a}} + a) \\
\implies z^* &= \frac{R^2}{\bar{z} - \overline{a}} + a \\
\implies (z^* - a)(\bar{z} - \overline{a}) &= R^2 \\
\implies |z^* - a||\bar{z} - \overline{a}| &= R^2, \quad \text{and} \qquad (*) \\
\arg(z^* - a) &= \arg(z - a)
\end{aligned}$$

which implies that z and z^* lie on the same half line emanating from the center a. Also, (*) implies that z and z^* lie on the two different sides (interior and exterior) of the circle C. Note that the symmetric point of a is ∞.



Theorem 3.9.4. (The symmetry principle) If an LFT f maps a circle or extended straight line C_1 to another circle or extended straight line C_2, then f maps any pair of symmetric points with respect to C_1 to a pair of symmetric points with respect to C_2.
(Briefly speaking, LFTs preserve symmetry).



Proof. Take 3 distinct points z_1, z_2, z_3 on C_1. Then $f(z_1), f(z_2), f(z_3)$ are 3 distinct points on C_2. Suppose z and z^* is a pair of symmetric points with respect to C_1. Then we have

$$(z_1, z_2; z_3, z^*) = \overline{(z_1, z_2; z_3, z)}.$$

Together with the invariance of the cross ratios under LFTs, we have

$$\begin{aligned}(f(z_1), f(z_2); f(z_3), f(z^*)) &= (z_1, z_2; z_3, z^*)\\ &= \overline{(z_1, z_2; z_3, z)}\\ &= \overline{(f(z_1), f(z_2); f(z_3), f(z))}.\end{aligned}$$

Therefore, $f(z)$ and $f(z^*)$ are symmetric with respect to C_2. $\square$

Example/Exercise (difficult). Suppose that C_1 and C_2 are two circles which do not intersect. Show that there are exactly two points z, z^* in $\hat{\mathbb{C}}$ which are symmetric with respect to both C_1 and C_2.

3.10. Uniformisation of simple regions

The Riemann mapping theorem says that any domain D which is not the entire complex plane $\mathbb{C}$ is analytically isomorphic to the unit ball $|z| < 1$. This is important both from a theoretical and practical point of view. Unfortunately, the proof is highly non-constructive so that it is difficult in general to write down such analytic isomorphism explicitly. In some special cases, this is possible. In this section, we will only consider the simplest possible cases, where the domain D is
(1) One of the two components of the complement of a circle or line. (the transformation can be chosen to be an LFT).
(2) A wedge shaped region which is the intersection of two disks.
(3) Infinite or semi-infinite strips

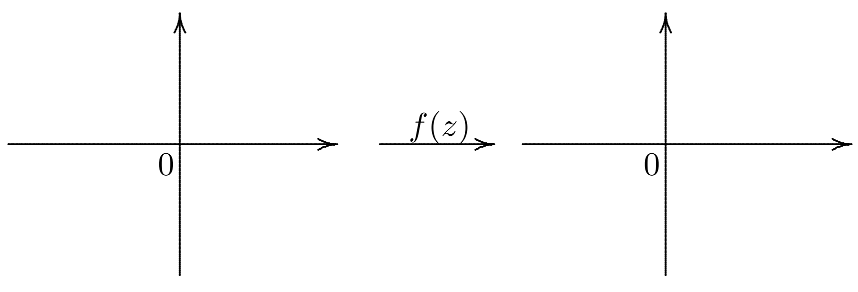
Remark. It is useful to recall that
(i) the power map $f(z) = z^\alpha$ will change the angle of a wedge at the origin by a factor of α.
(ii) the exponential map $f(z) = e^z$ maps the infinite strip $0 < \text{Im } z < \pi$ to the upper half plane $\text{Im } w > 0$.
(iii) Together the LFTs which also include inversion, translations, rotations and dilations, we can construct explicitly conformal isomorphisms from quite a lot of simple regions to the unit ball.

Example 3.10.1.

(1) Find a conformal isomorphism between the half plane $y < x$, where $z = x + iy$, and the unit ball $|w| < 1$.

Solution: Consider the extended straight line $L : y = x$. Take 3 points, say, $\infty, 0, 1 + i$ on the line $y = x$, which determine an orientation of L. Then the half plane $y < x$ is the component of $\mathbb{C}\backslash L$ on the right side of L. Take 3 points of the circle $C : |w| = 1$, which determine an orientation of C such that the right side of C corresponds to the interior of C. Such points can be given by $0, -i, i$. From the Orientation Principle, an required conformal isomorphism is given by the LFT

$$(0, -i; i, w) = (\infty, 0, 1 + i, z).$$



(2) Find an analytic isomorphism between the first quadrant of $\mathbb{C}$ and the unit ball $|z| < 1$.

Solution. First open the quadrant to a half plane, using

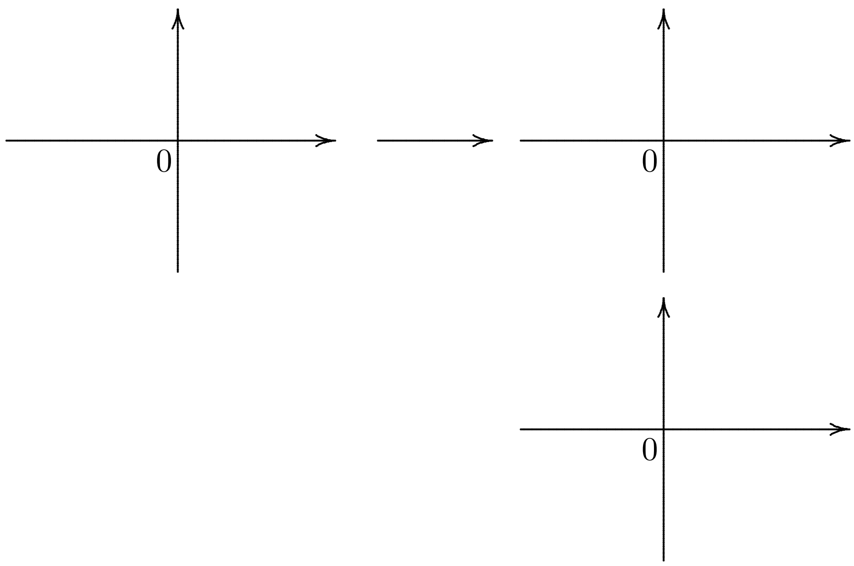
$$z_1 = f_1(z) = z^2$$

which maps the first quadrant isomorphically to the UHP. Next, map the UHP to the unit ball $|w| < 1$ with the LFT

$$w = f_2(z_1) = \frac{z_1 - i}{z_1 + i}.$$

So an required transformation can be chosen to be

$$w = f_2(f_1(z)) = f_2(z^2)$$
$$\Longrightarrow \; w = \frac{z^2 - i}{z^2 + i}.$$



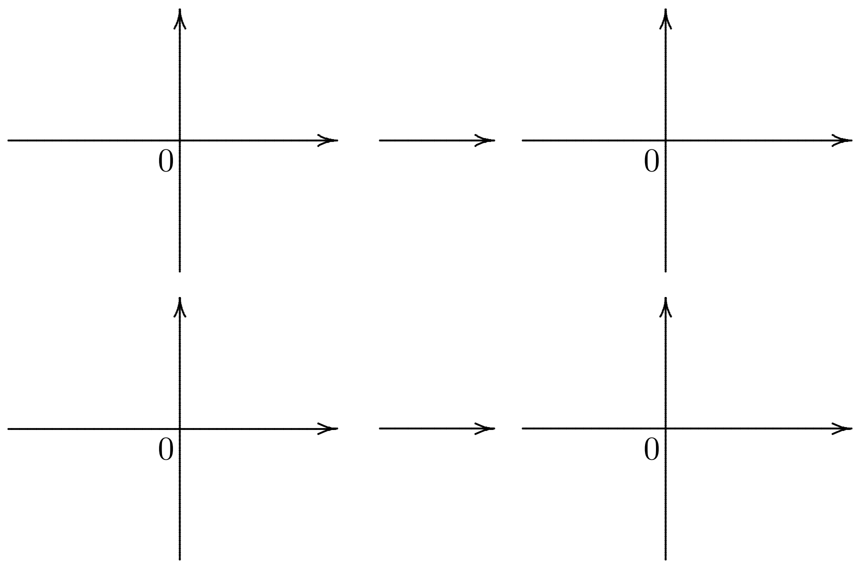
(3) Find an analytic isomorphism from the intersection of the unit balls $|z| < 1$ and $|z-1| < 1$ to the inside of the unit circle.

Solution. The intersection is a wedge with angle $2\pi/3$ at the two corners $a = e^{i\pi/3}$ and $b = e^{-i\pi/3}$. We first transform this region to a standard wedge by an LFT mapping a to ∞, b to 0, and 1 to 1 so that $|z| = 1$ is mapped to the real line. The transformation is given by

$$z_1 = f(z) = \frac{z - e^{-i\pi/3}}{z - e^{i\pi/3}} \frac{1 - e^{i\pi/3}}{1 - e^{-i\pi/3}} = e^{-2\pi i/3} \frac{z - e^{-i\pi/3}}{z - e^{i\pi/3}}.$$

Note that the circle $|z-1| = 1$ is mapped to the straight line passing through the origin at angle $2\pi/3$ with the positive real axis and that the intersection is mapped to the wedge between these two lines lying in the first and part of the second quadrant. Next open this up to a half plane, by the map $z_2 = f_2(z_1) = z_1^{3/2}$. Finally we map the half plane to the inside of the unit disk given by $w = (z_2 - i)/(z_2 + i)$. Thus one such analytic isomorphism is given by

$$\begin{aligned} w =& f_3 \circ f_2 \circ f_1(z) \\ &= f_3(f_2(e^{-2\pi i/3} \frac{z - e^{-i\pi/3}}{z - e^{i\pi/3}})) \\ &= f_3\big(\big(e^{-2\pi i/3} \frac{z - e^{-i\pi/3}}{z - e^{i\pi/3}}\big)^{3/2}\big) \\ &= \frac{\big(e^{-2\pi i/3} \frac{z-e^{-i\pi/3}}{z-e^{i\pi/3}}\big)^{3/2} - i}{\big(e^{-2\pi i/3} \frac{z-e^{-i\pi/3}}{z-e^{i\pi/3}}\big)^{3/2} + i}. \end{aligned}$$



(4) Consider the infinite horizontal strip between the lines $y = 0$ and $y = 1$. Find an analytic isomorphism from the above horizontal strip to the half plane Im $w > 1$.

Solution: First we use a scaling factor $f_1(z) = \pi z$ to map this strip to the strip between the horizontal lines $y = 0$ and $y = \pi$, Then we use the exponential function to map this strip to the upper half plane Im $z > 0$ given by $f_2(z_1) = e^{z_1}$. Finally we map the upper half plane to the half plane Im $z > 1$ by the translation $f_3(z_2) = z_2 + i$. Hence an required analytic isomorphism is given by

$$w = f_3 \circ f_2 \circ f_1(z) = f_3(f_2(\pi z)) = f_z(e^{\pi z}) = e^{\pi z} + i.$$