NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

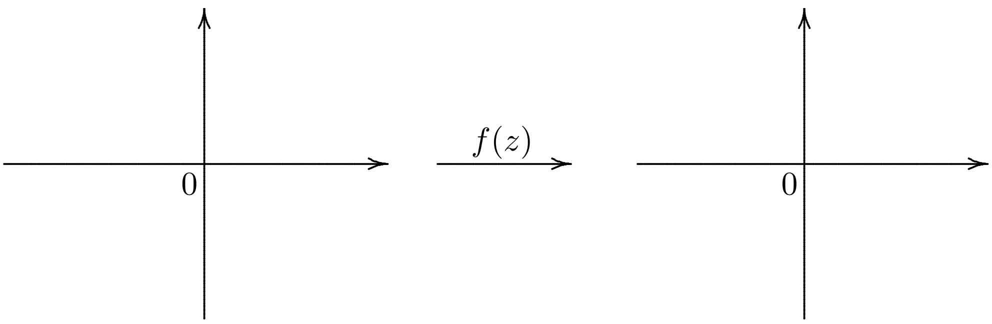
MA4247 Complex Analysis II

Lecture Notes Part IV

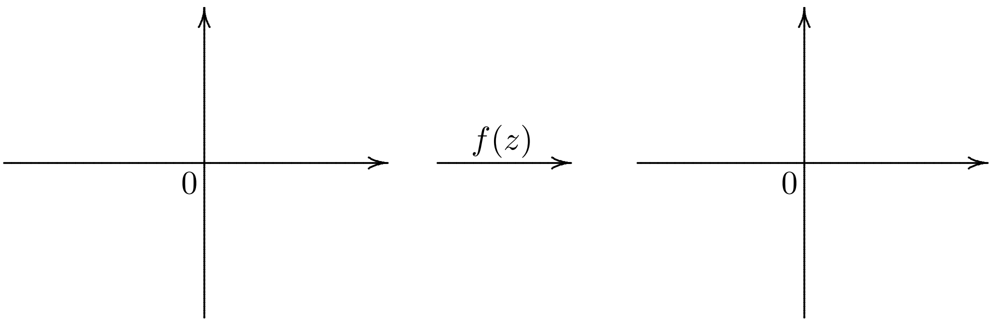
## Chapter 2. Further properties of analytic/holomorphic functions (continued)

### §2.4. The Open mapping theorem

**Theorem 2.4.1. (Open mapping theorem)**
Suppose $f$ is a non-constant analytic function on a domain $D$. If $U$ is an open subset of $D$, then $f(U)$ is open.




*Proof.* We want to show that $f(U)$ is open, i.e., take any point $w_0 \in f(U)$, then we need to show that there exists some $r > 0$ such that the open ball $B(w_0, r) := \{w \in \mathbb{C} \,\big|\, |w - w_0| < r\}$ is contained in $f(U)$. In other words, for any $w \in B(w_0, r)$, we need to show that there exists $z \in U$ such that $f(z) = w$.




Step 1. Since $w_0 \in f(U)$, there exists $z_0 \in U$ such that $f(z_0) = w_0$. Let $g(z) = f(z) - w_0$ so that $g(z_0) = f(z_0) - w_0 = 0$. Since $f(z)$ is a non-constant analytic function, so is $g(z)$. Thus $z_0$ is an isolated zero of $g(z)$ of some order $k$, $k \geq 1$ (see Proposition 2.1.1 and Tutorial 1 Question 4). Then we can choose $\delta > 0$ such that

(i) $B(z_0, \delta) \subset U$ (since $U$ is open);

(ii) $g(z) \neq 0$ for all $z \in \overline{B(z_0, \delta)} - \{z_0\}$ (since zeroes of $g$ are isolated), where $\overline{B(z_0, \delta)} = \{z \in \mathbb{C} : |z - z_0| \leq \delta\}$ ;
(iii) $g'(z) \neq 0$ for all $z \in \overline{B(z_0, \delta)} - \{z_0\}$ (since $g$ is non-constant, $g'$ is analytic and not identically zero).

Consider the function $|g(z)|$ restricted to the circle $\gamma : |z - z_0| = \delta$. By the Extreme Value Theorem, $|g(z)|$ attains its minimum value on the circle $\gamma$. From (ii), the minimum value of $|g(z)|$ on the circle $\gamma$ is $\neq 0$, and we denote it by $r$ (so that $|g(z)| \geq r$ on $\gamma$). Now consider the open ball $B(w_0, r)$ and let $w_1 \in B(w_0, r)$ so that $|w_1 - w_0| < r$. Let $h(z) = w_0 - w_1$ (constant function). Then on the circle $\gamma : |z - z_0| = \delta$,

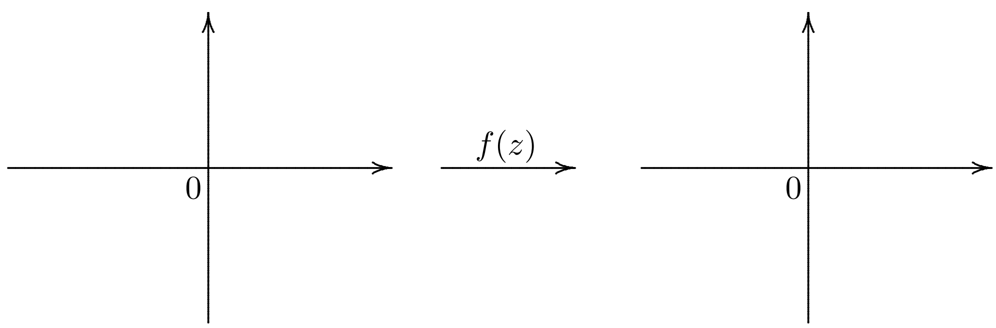
$$|g(z)| \geq r > |h(z)|$$

since $|h(z)| = |w_0 - w_1| < r$, so by Rouché's theorem, $g(z)$ and $g(z) + h(z)$ has the same number of zeroes inside the circle $\gamma$. Since $g(z) = f(z) - w_0$ has $k$ zeroes inside $\gamma$ (counting multiplicity), so does $g(z) + h(z) = f(z) - w_1$. Now since by (iii), $f'(z) = g'(z) \neq 0$ in the punctured ball $B(z_0, \delta) - \{z_0\}$, all the zeroes of $f(z) - w_1$ are distinct and of order 1, so there are $k$ distinct solutions of the equation $f(z) = w_1$ inside the open ball $B(z_0, \delta)$. Hence,

$$B(w_0, r) \subset f(B(z_0, \delta)) \subset f(U). \quad \square$$

Note that we have in fact proven a stronger result, namely the following theorem from which the open mapping theorem is a corollary:

**Theorem 2.4.2.** Suppose that $f$ is a non-constant function on a domain $D$ and $z_0 \in D$ with $f(z_0) = w_0 \in f(D)$. If $z_0$ is a zero of order $k$ of $f(z) - w_0$, then we can find open balls $B(z_0, \delta)$ and $B(w_0, r)$ with $\delta$, $r > 0$ such that every $w \in B(w_0, r) \setminus \{w_0\}$ has exactly $k$ distinct preimages in $B(z_0, \delta)$.

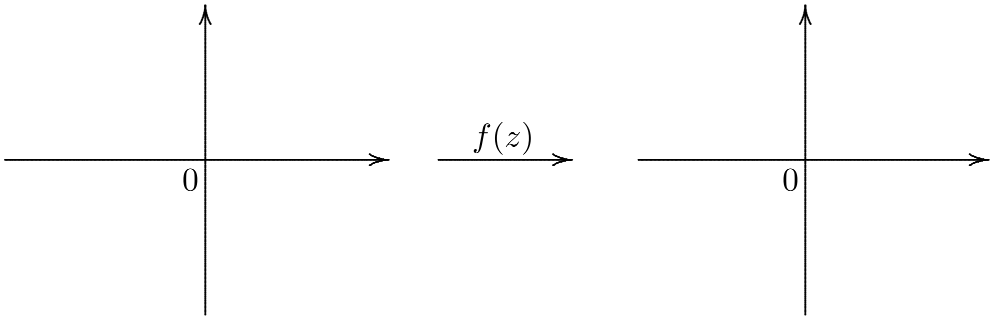


**Example. 2.4.3.** Consider the function $f(z) = z^3 + z^4 - 2z^5$.

**Remark 2.4.4.** The open mapping theorem implies that if $f$ is analytic and non-constant on a domain $D$, then $f(D)$ is open.

**Exercise:** Show that $f(D)$ is connected, so that $f(D)$ is also a domain.

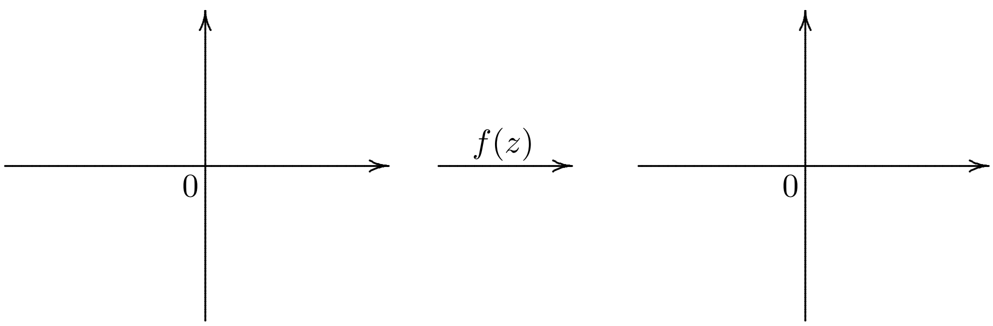
**Hint.** The proof that $f(D)$ is connected is similar to Theorem 2.1.2. A sketch of the proof is outlined below, and it will be left to the student as an exercise to fill in the details. To show that $f(D)$ is connected, it means that for any two distinct points $w_1$, $w_2 \in f(D)$, we have to find a polygonal line $L \subset f(D)$ joining $w_1$ to $w_2$.




Note that $w_1, w_2 \in f(D)$ implies that there exist $z_1, z_2 \in D$ such that $f(z_1) = w_1$ and $f(z_2) = w_2$. Since $D$ is connected, there exists a polygonal line $\ell = \ell_1 + \cdots + \ell_k \subset D$ joining $z_1$ to $z_2$ (unfortunately $f(\ell)$ needs not be a polygonal line in $f(D)$). Similar to Theorem 2.1.2, we shall first show that each point of $f(\ell_1)$ is connected to $w_1$ by a polygonal line in $f(D)$, and then the argument is repeated to show that each point of $f(\ell)$ (and in particular, the point $w_2$) is connected to $w_1$ by a polygonal line in $f(D)$. Thus it remains to show that each point of $f(\ell_1)$ is connected to $w_1$ be a polygonal line in $f(D)$. Let $\gamma : [0,1] \to D$ be a parametrization of the line segment $\ell_1$, so that $\gamma(a) = w_1$. Consider the interval

$$I = \{\, s \in [0,1] : f(\gamma(t)) \text{ is connected to } w_1 \text{ by some polygonal line in } f(D) \text{ for all } 0 \leq t \leq s\}.$$

Let $s_o := \sup I$. As in Theorem 2.1.2, we need to show that
(i) $s_o > 0$ (basically because $f(D)$ is open);
(ii) for any $0 \leq s \leq s_o$, $f(\gamma(s))$ is connected to $w_1$ by some polygonal line in $f(D)$; and
(iii) $s_o = 1$ (again basically because $f(D)$ is open).



The proof of the above statements is similar to that of Theorem 2.1.2, and it will be left as an exercise. An alternative proof is to use the fact that a connected open set cannot be decomposed into two non-trivial open subsets.

**Corollary 2.4.5.** If $f$ is analytic and non-constant on a domain $D$ and $z_0 \in D$ such that $f'(z_0) = 0$, then $f$ is never one-to-one on any open set $U$ containing $z_0$.

*Proof.* Let $g(z) = f(z) - f(z_0)$, so that $g(z_0) = 0$. If $f'(z_0) = 0$, then $g(z_0) = g'(z_0) = 0$, and thus $g(z)$ has a zero of order $k \geq 2$ at $z_0$. By the proof of the Open Mapping Theorem (or Theorem 2.4.2), there exists open balls $B(z_0, \delta)$ and $B(f(z_0), r)$ with $\delta$, $r > 0$ (and we can choose $\delta$ so that $B(z_0, \delta) \subset U$ such that every $w \in B(f(z_0), r) \setminus \{f(z_0)\}$ has at least $k$ inverse images of $f$ in $B(z_0, \delta)$, so $f$ cannot be one-to-one.

**Corollary 2.4.6.** If $f$ is analytic and one-to-one on a domain $D$, then $f'(z) \neq 0$ for all $z \in D$.

*Proof.* Corollary 2.4.6 follows immediately from Corollary 2.4.5.

**Remark:** The converse to Cor 2.4.6 is not true, give an example.

**Example 2.4.7.** Suppose that $f$ is analytic on a domain $D$ and $f(z) \in \mathbb{R}$ for all $z \in D$. Then $f$ is constant.

*Proof.* $f(D)$ is not an open set, hence $f$ must be constant.

**Example 2.4.8.** Suppose $f(z)$ is analytic and non-constant on $B(0,1)$ and $f(B(0,1)) \subset \overline{B(0,1)}$. Then $f(B(0,1)) \subset B(0,1)$. (Note: This also follows from the MMP).

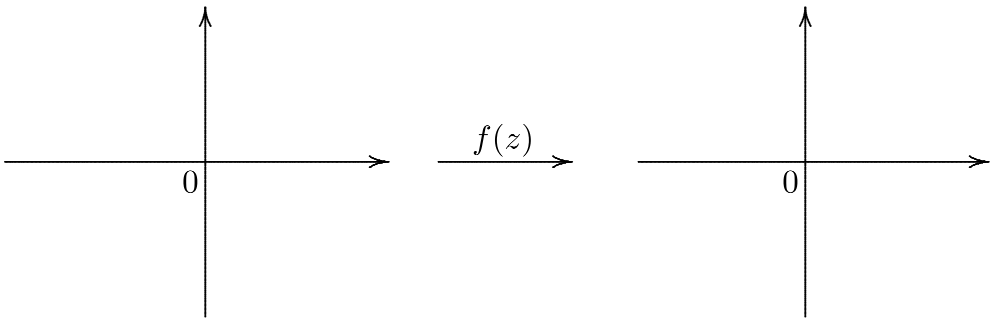
**Theorem 2.4.9. (Inverse function theorem)**
If $f$ is one-to-one and analytic on a domain $D$, then each of the following holds:
(i) The inverse function $f^{-1}$ is continuous on $f(D)$.
(ii) $f^{-1}$ is analytic in $f(D)$.
(iii)If $w \in f(D)$, then

$$(f^{-1})'(w) = \frac{1}{f'(z)} \qquad \text{where } z = f^{-1}(w).$$

(Note that $f'(z) \neq 0$ from Corollary 2.4.6 since $f$ is one-to-one. Also, (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i))

*Proof.* (i) First we note that since $f$ is one-to-one on $D$, its inverse $f^{-1}$ is a well-defined map from $f(D)$ to $D$. Fix $w_0 \in f(D)$ and let $z_0 \in D$ such that $w_0 = f(z_0)$, i.e., let $z_0 = f^{-1}(w_0)$. To prove (i), we need to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|w - w_0| < \delta \Longrightarrow |f^{-1}(w) - z_0| < \epsilon.$$




Since $D$ is open, shrinking $\epsilon$ if necessary, we may assume that $B(z_0, \epsilon) \subset D$. By the open mapping theorem,

$$U := f(B(z_0, \epsilon))$$

is open in $f(D)$ and since $w_0 \in U$, there exists a $\delta > 0$ such that $B(w_0, \delta) \subset U$. Hence

$$w \in B(w_0, \delta) \Longrightarrow w \in U \Longrightarrow f^{-1}(w) \in B(z_0, \epsilon)$$

or

$$|w - w_0| < \delta \Longrightarrow |f^{-1}(w) - z_0| < \epsilon.$$

Hence, $f^{-1}$ is continuous at $w_0$ and since $w_0$ is an arbitrary point of $f(D)$, $f^{-1}$ is continuous on $f(D)$.
(ii) and (iii). Suppose $w_0 \in f(D)$. Then for each $w \in f(D)$ such that $w \neq w_0$, we have

$$\begin{aligned} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} &= \frac{z - z_0}{f(z) - f(z_0)} \qquad \text{where } w = f^{-1}(z) \\ &= \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}} \qquad \text{where } z \neq z_0 \text{ since } w \neq w_0. \end{aligned} \tag{*}$$

By (i), $f^{-1}$ is continuous. Hence, as $w \longrightarrow w_0$, we have $f^{-1}(w) \longrightarrow f^{-1}(w_0)$ or equivalently, $z \longrightarrow z_0$, which, in turn, implies that

$$\frac{1}{\frac{f(z)-f(z_0)}{z-z_0}} \to \frac{1}{f'(z_0)}.$$

Together with (*), we thus have

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{1}{f'(z_0)}.$$

Since $w_0$ is an arbitrary point of $f(D)$, the result follows. $\square$

**Example.** Let $w = f(z) = e^z$ on the domain $D = \{z \ : \ -\pi < \operatorname{Im} z < \pi\}$. Then $f$ is one-to-one on $D$ and $f(D)$ is the cut-plane $\mathbb{C} \setminus (-\infty, 0]$, and $f^{-1}(w) = \log w$, the principal log function. Furthermore, by (iii), $f'(w) = 1/f'(z) = 1/w$.

**Definition 2.4.10.** A one-to-one analytic function from a domain $U$ onto the domain $V = f(U)$ is called an **analytic isomorphism** from $U$ to $V$.
(If $U = V$, $f$ is called an **analytic automorphism** of $U$.) $U$ and $V$ are said to be **analytically isomorphic** if there exists an analytic isomorphism $f : U \longrightarrow V$.

**Example.**(i) $f(z) = e^z$ on the domain $D = \{z \ : \ -\pi < \text{Im } z < \pi\}$.
(ii) $f(z) = \frac{z-i}{z+i}$ on the upper half plane $H := \{z \ : \ \text{Im } z > 0\}$.

**Proposition 2.4.11.** Let $S$ denote the set of domains in $\mathbb{C}$. (For example, the entire complex plane $\mathbb{C}$ is an element of $S$, the upper half plane $H$ is an element of $S$, all balls $B(z_0, r)$ where $r > 0$ are elements of $S$). Define the following binary relation $\sim$ on $S$:

For any $U$, $V \in S$, we say that $U \sim V$ if $U$ is analytically isomorphic to $V$.

Then $\sim$ is an equivalence relation on $S$, i.e.,
(i) (reflexivity) $U \sim U$ for any $U \in S$;
(ii) (symmetry) If $U \sim V$, then $V \sim U$;
(iii) (transitivity) If $U \sim V$ and $V \sim W$, then $U \sim W$.

*Proof.* (i) For any domain $U \in S$, clearly the identity map $f(z) = z$ on $U$ is an one-to-one analytic function from $U$ onto $U$, i.e., it is an analytic isomorphism from $U$ to $U$; in other words, $U \sim U$, which gives (i). To prove (ii), suppose $U \sim V$. Then there exists an analytic isomorphism $f$ from $U$ to $V$, i.e., $f$ is a one-to-one analytic function from $U$ onto $V$ Then the inverse function implies that if $f^{-1} : V \longrightarrow U$ is also an analytic isomorphism (Exercise), so that $V \sim U$. Similarly, to prove (iii), it is an easy exercise (for you) to show that if $f : U \longrightarrow V$ is an analytic isomorphism and $g : V \longrightarrow W$ is also an analytic isomorphism, then using the Chain Rule, one can show that $g \circ f : U \longrightarrow W$ is also an analytic isomorphism. $\square$

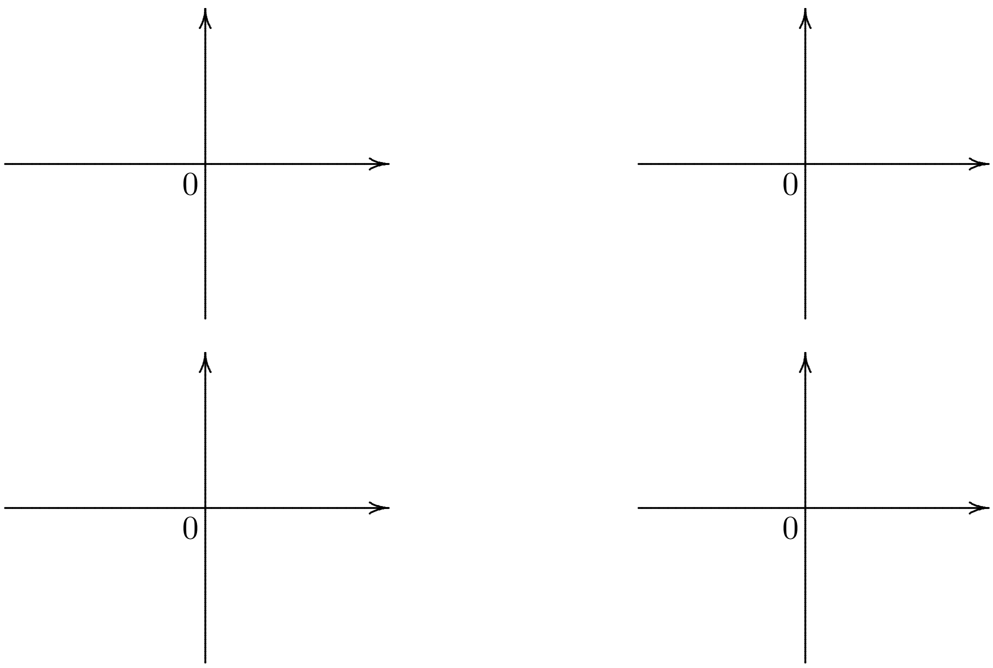
**Remark.** From general theory of equivalence relations, one deduces from Proposition 2.4.11 that the equivalence relation $\sim$ partitions $S$ into equivalence classes of domains such that the domains in each equivalence class are analytically isomorphic to each other.

Thus, it is natural to attempt to solve the classification problem, i.e., one wants to know how many equivalence classes of domains are there, and what are those domains in each equivalence class. In this generality, this is a very difficult problem and is still "unsolved", you may want to read up about the Koebe conjecture. To concentrate on the complex analytic problem, we will try to classify domains with "trivial" topology. We recall the following definition:

**Definition 2.4.12.** A domain $D \subset \mathbb{C}$ is **simply connected** if every simple closed contour in $D$ encloses only points in $D$.

In particular, a domain in $\gamma$ is simply connected if it has no "punctures" or "holes".

**Questions.**(i) How many equivalence classes of analytically isomorphic simply connected domains are there in $\mathbb{C}$? (ii) Which simply connected domains in $\mathbb{C}$ are analytically isomorphic? (Equivalently, what are the simply connected domains in each equivalence class of simply connected domains?) For example, is the interior of the unit square analytically isomorphic to the interior of the unit disk?




**Proposition 2.4.12.** The unit ball $B(0,1) := \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ is not analytically isomorphic to $\mathbb{C}$.

*Proof.* Suppose not, so suppose there exists an analytic isomorphism $f : \mathbb{C} \to B(0,1)$. Apply Liouville's theorem to get a contradiction (Exercise: fill in the details). $\square$

**Theorem 2.4.13. (Riemann mapping theorem)**
Any simply connected domain $\Omega$ which is not the whole complex plane $\mathbb{C}$ is analytically isomorphic to the unit ball $B(0,1)$. Furthermore, if $z_0 \in \Omega$, there is a unique analytic isomorphism $f : \Omega \to B(0,1)$ satisfying

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0. \qquad (*)$$

**Remarks:**
1. The theorem was formulated by Riemann but the first successful proof was given by Koebe. This is a remarkable and very important theorem as it allows us to have existence and uniqueness theorems which allow us to define important analytic functions without actually using analytic expressions.
2. The conditions $f(z_0) = 0$ and $f'(z_0) > 0$ are often called normalising conditions to pin down the uniqueness of $f$. Note that $f'(z_0) > 0$ means it is a positive real number.

3. The proof (of existence) is beyond the scope of the module and uses the theory of normal families but is standard for first year graduate courses. The uniqueness part is relatively simple.
4. (Some terminnology) A analytic function on a domain $\Omega$ is also said to be **univalent** or **schlicht** if it is one-to-one.

To help to prove the uniqueness, we have the following result:

**Proposition 2.4.14. (Consequence of Schwarz's lemma)** Suppose that $f$ is an analytic isomorphism of the unit ball $B(0,1)$ to itself satisfying $f(0)=0$ and $f'(0)>0$. Then $f(z)=z$ for all $z$ in $B(0,1)$.

*Proof.* Since $f(B(0,1)) \subset B(0,1)$ (i.e., $|f(z)|<1$ for all $|z|<1$), it follows from Schwarz's lemma that

$$|f(z)| \leq |z| \quad \text{for all } |z|<1.$$

Similarly, since $f^{-1}(B(0,1)) \subset B(0,1)$, it follows from Schwarz's lemma that

$$|f^{-1}(w)| \leq |w| \quad \text{for all } |w|<1.$$

Equivalently, writing $w=f(z)$, we have

$$|z| \leq |f(z)| \quad \text{for all } |z|<1.$$

Hence we have

$$|f(z)| = |z| \quad \text{for all } |z|<1.$$

Again by Schwarz's lemma, this implies

$$f(z)=Cz, \qquad \text{where } |C|=1.$$

Now

$$f'(0)=C>0 \Longrightarrow C=1 \Longrightarrow f(z)=z. \quad \square$$

**Exercise.** Deduce the uniqueness part of the statement of the Riemann mapping theorem from the above result, i.e., show that if $f$ and $g$ are two analytic isomorphisms from $\Omega$ to $B(0,1)$ satisfying the conditions in (*) of the Riemann Mapping Theorem, then $f(z) \equiv g(z)$ on $B(0,1)$.

[Hint: Consider $g \circ f^{-1}$, and apply Proposition 2.4.14.]

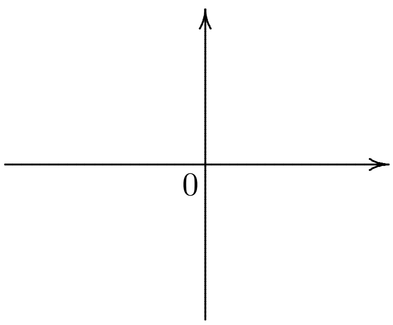
From the Riemann Mapping Theorem and Proposition 2.4.12, one knows that there are only two equivalence classes of analytically isomorphic simply connected domains. One equivalence class consists only of the complex plane $\mathbb{C}$, and the other equivalence class consists of all other domains $\Omega \neq \mathbb{C}$ (and all such $\Omega$ are analytically isomorphic to the unit ball $B(0,1)$).

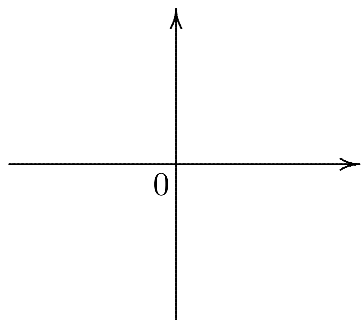
We remark that the classification problem is much more complicated for non-simply connected domains. In fact, one knows that there are uncountably infinite number of equivalence classes of non-simply connected domains. For example, for any $R > 1$, consider the annulus

$$\mathcal{A}(1,R) := \{z \in \mathbb{C}, \big|\, 1 < |z| < R\}.$$

**Proposition 2.4.15.** For $R$, $R' > 1$, $\mathcal{A}(1,R)$ is analytically isomorphic to $\mathcal{A}(1,R')$ iff $R = R'$.

*Proof.* (Skipped) See W. Rudin, *Real and Complex Analysis*, page 312, Theorem 14.22 for a proof. (Project: Write up a proof and submit to the MA4247 IVLE forum page).




**Example.**(Nov 2004 exam, question 1b) Consider the domain $D = \{z = x + iy \in \mathbb{C} : y < x^2 \text{ and } z \neq -i\}$. Does there exist a non-constant entire function $F$ such that $F(\mathbb{C}) \subset D$? Justify your answer.