NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II

Lecture Notes Part III

Chapter 2. Further properties of analytic/holomorphic functions (continued)

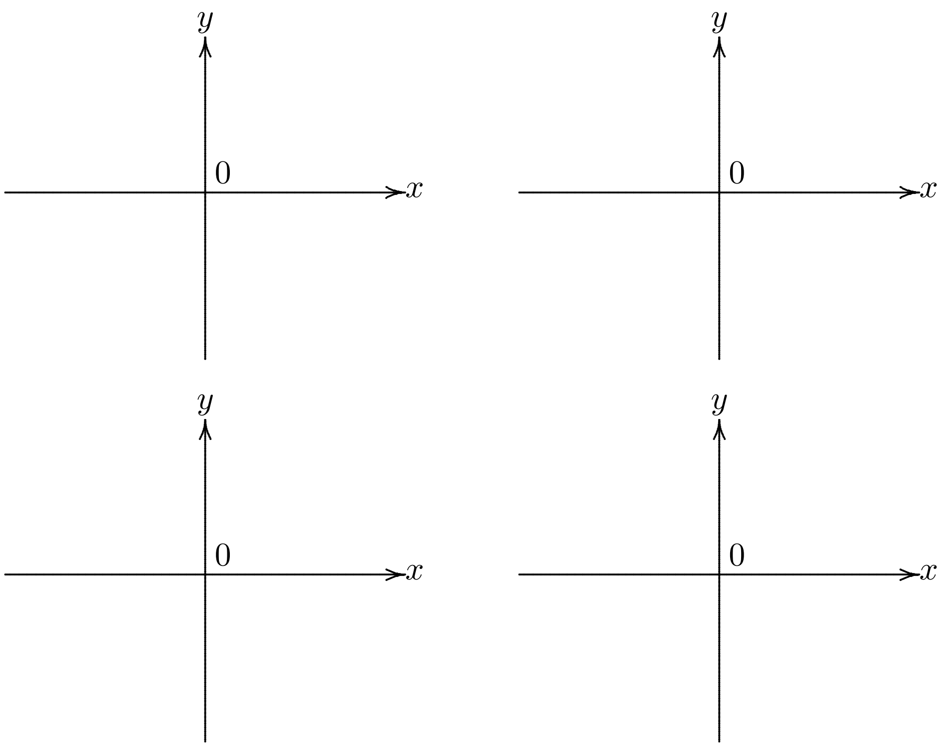
§2.3. Winding numbers, Argument Principle and Rouché's Theorem

Definition. (Winding Numbers) Let γ be an oriented closed contour in $\mathbb{C}$ which does not pass through the origin 0. The **winding number** $n(\gamma; 0)$ of γ about 0 is the number of times the point z winds around the origin 0 as it moves along γ. It can be defined more precisely by

$$n(\gamma; 0) := \frac{1}{2\pi} \Delta_\gamma \arg z, \tag{*}$$

where $\Delta_\gamma \arg z$ denotes the change in the argument of z as z moves along γ.

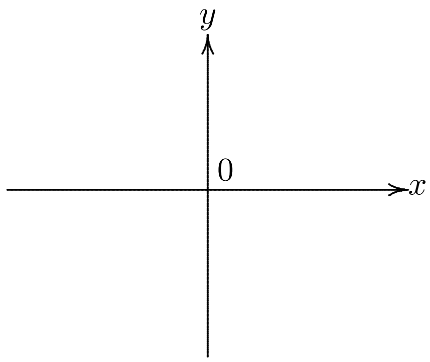
Example.



Proposition 2.3.1.

$$n(\gamma; 0) = \frac{1}{2\pi} \Delta_\gamma \arg z = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}.$$

Proof: Write $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ where each γ_i is differentiable, and each γ_i lies in a region of γ where a branch of $\log z$ is defined. (In other words, each γ_i should miss a cut in the complex plane).



Clearly,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = \frac{1}{2\pi i}\left(\int_{\gamma_1} \frac{dz}{z} + \cdots + \int_{\gamma_n} \frac{dz}{z}\right)$$

For each γ_i, choose a branch of the log function where the argument is chosen so that θ at the terminal point of γ_i agrees with θ at the initial point of γ_{i+1}. For each i, γ_i is a curve with initial point at some point z_i and terminal point at some point z_{i+1}. Then

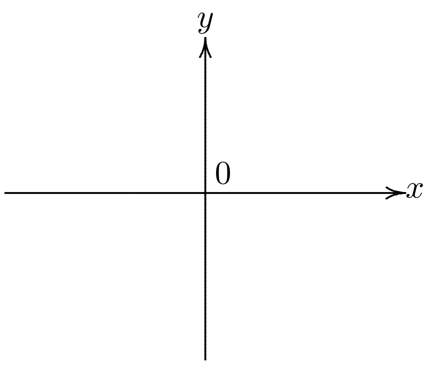
$$\begin{aligned}\int_{\gamma_i} \frac{dz}{z} &= \log z\Big|_{z_i}^{z_{i+1}} \\ &= (\ln|z_{i+1}| - \ln|z_i|) + i(\arg z_{i+1} - \arg z_i).\end{aligned}$$

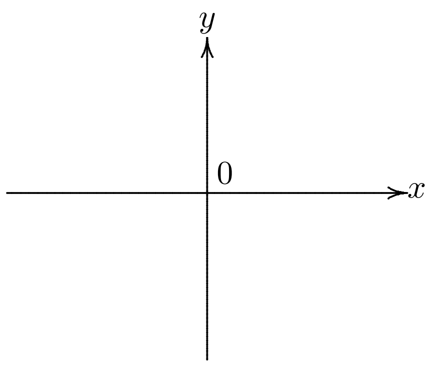
Note that $\arg z_{i+1} - \arg z_i = \Delta_{\gamma_i} \arg z$. Hence

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} &= \sum_{i=1}^{n} \big[(\ln|z_{i+1}| - \ln|z_i|) + i(\arg z_{i+1} - \arg z_i)\big] \\ &= \frac{1}{2\pi i}\big[(\ln|z_{n+1}| - \ln|z_1|) + i\sum_{i=1}^{n} \Delta_{\gamma_i} \arg z\big] \\ &= 0 + \frac{1}{2\pi}\Delta_\gamma \arg z,\end{aligned}$$

where the last line follows from the fact that $z_{n+1} = z_1$ (since γ is closed), and the identity $\Delta_\gamma \arg z = \Delta_{\gamma_1} \arg z + \cdots + \Delta_{\gamma_n} \arg z$. $\square$

Remark. $n(\gamma; 0)$ is always an integer and is positive if γ winds around 0 in the anti-clockwise direction and negative if γ winds around 0 in the clockwise direction.

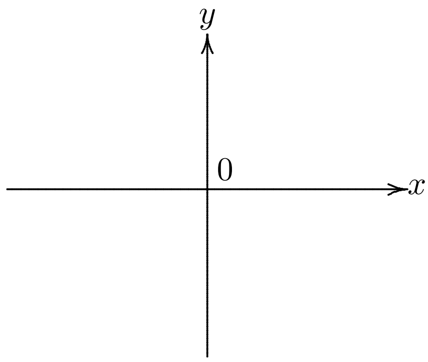


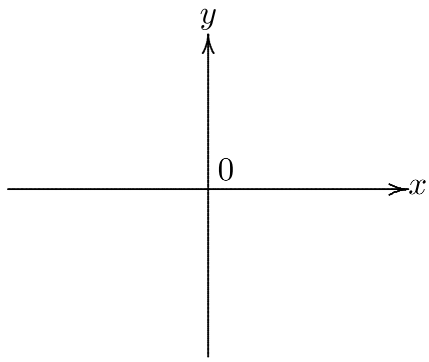


Similarly, one can define the winding number of a closed curve γ about any point $z_0 \in \mathbb{C}$ which does not lie on γ by

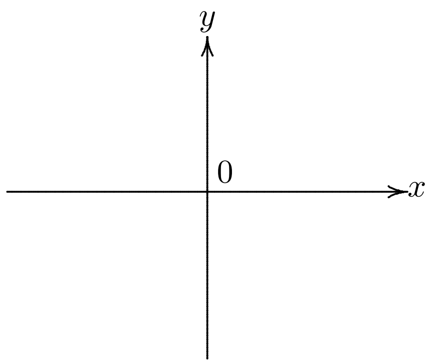
$$n(\gamma; z_0) = \frac{1}{2\pi}\Delta_\gamma \arg(z - z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}.$$

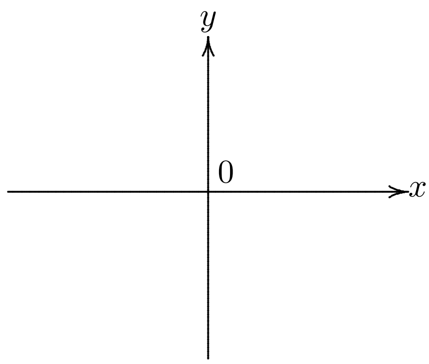
Here, similar to (*), $\Delta_\gamma \arg(z - z_0)$ denotes the change in the argument of $z - z_0$ as z moves along γ. Note that $n(\gamma; z_0)$ is equal to $n(\gamma', 0)$ where γ' is the curve obtained by translating γ by $-z_0$.





Remark. Let γ be a positively oriented simple closed contour. Then $n(\gamma; z_0) = 1$ if z_0 lies in the interior of γ, and $n(\gamma; z_0) = 0$ if z_0 lies outside γ.





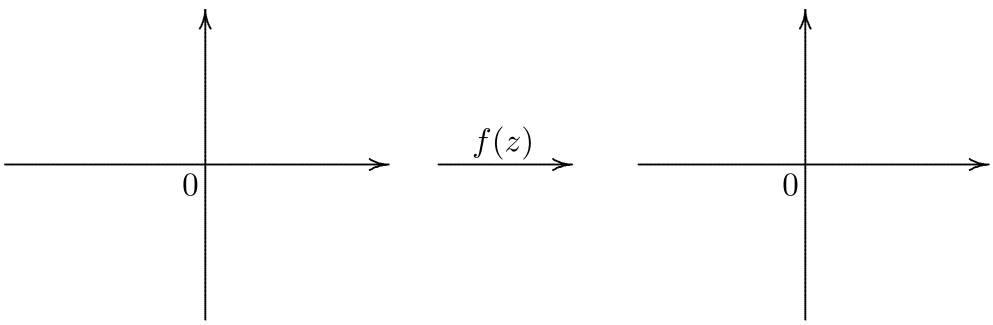
Definition 2.3.2. (meromorphic functions) A function is said to be **meromorphic** in a domain D if it is analytic throughout D except possibly for poles. (If there are removable singularities, we usually extend the function so that it is analytic over these points).

Note: The poles are isolated singularities (by definition).

Example. The function $f(z) = \tan z$ is meromorphic in $\mathbb{C}$. All rational functions $p(z)/q(z)$ where $p(z)$ and $q(z)$ are polynomials, $q(z) \not\equiv 0$, are meromorphic in $\mathbb{C}$. In particular, all polynomials and entire functions are meromorphic in $\mathbb{C}$.

Theorem 2.3.3. (The Argument Principle.) Let f be meromorphic in the domain D interior to a positively oriented simple closed contour γ and suppose that f is analytic and non-zero on γ. Let Z be the number of zeroes and P be the number of poles of f in D (both counted with multiplicities), then

$$Z - P = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = n(f(\gamma); 0).$$



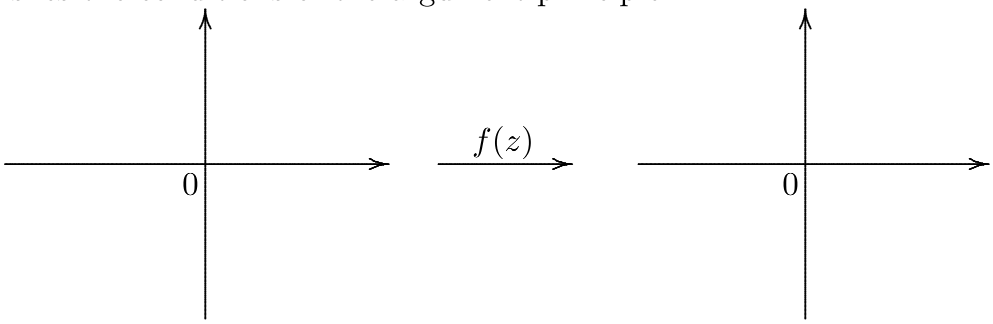
Remark 2.3.4. (i) Here Z is counted with multiplicities means that if $f(z)$ has a zero of order m at z_0 inside γ, then z_0 will be counted (as a zero of f) m times. Similarly, P is counted with multiplicity means that if $f(z)$ has a pole of order m at z_0 inside γ, then z_0 will be counted (as a pole of f) m times.
(ii) Here $n(f(\gamma); 0)$ is the winding number of $f(\gamma)$ about 0. Note that we may write

$$n(f(\gamma); 0)) = \frac{1}{2\pi} \Delta_\gamma \arg f(z),$$

where $\Delta_\gamma \arg f(z)$ is the total change in the argument of $f(z)$ as z moves along γ.

Example. Consider the function $f(z) = \frac{z}{(z-1)(z-3)}$. If γ_1 is the positively oriented circle of radius 2 about the origin and γ_2 is the positively oriented circle of radius 4 about the origin, what is $n(f(\gamma_1); 0)$ and $n(f(\gamma_2); 0)$?

Example. What is $Z - P$ if γ and $f(\gamma)$ are as in the picture, and f satisfies the conditions of the argument principle?



Proof of Theorem 2.3.3. We first show that

$$Z - P = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz.$$

(1). We show that the quantity $Z-P$ is well-defined, that is, the number of zeroes and poles in D is finite. Recall the Jordan curve theorem in MA3111 that a simple closed contour separates its complement in the complex plane into two domains, one bounded and the other unbounded. In particular, $D \cup \gamma$ is a closed and bounded set in $\mathbb{C}$.
(1a) First we claim that the number of poles of f in D must be finite. Suppose not. Then by the Heine-Borel Theorem, the infinite set of poles must have an accumulation point z_0 in the closed bounded set $D \cup \gamma$. In particular, there exists a sequence of distinct poles $\{z_n\}$ of f such that

$$\lim_{n\to\infty} z_n = z_0. \qquad (*)$$

If $f(z)$ is analytic at z_0, then there exists an open ball $B(z_0, r)$ with $r > 0$ such that $f(z)$ is differentiable everywhere (and hence analytic everywhere) in $B(z_0, r)$, which contradicts with (*). Therefore, z_0 is a singular point of f. Also, z_0 cannot lie on γ, since f is assumed to be analytic everywhere on γ. Hence, z_0 is a singular point of f in D. Since f is meromorphic in D, it follows that z_0 is a pole of f. Since a pole is an isolated singular point, it follows that there exists an open ball $B(z_0, \delta)$ with $\delta > 0$ such that f is analytic on $B(z_0, \delta) \setminus \{z_0\}$, which contradicts with (*) again. Therefore, the number of poles of f in D is finite.

(1b) Similarly, using the identity theorem for analytic functions, one can show that the number of zeros of f in D is finite (Exercise).

(2). Next, we note that the singularities of $\dfrac{f'(z)}{f(z)}$ can only occur at the zeroes and poles of f (Why?), which can only occur in D and not on γ since f is analytic and non-zero on γ. Let $\{\alpha_i\}_{1\le i\le n}$ be the zeroes of f inside γ, and let $\{\beta_j\}_{1\le j\le \ell}$ be the poles of f inside γ. Suppose $f(z)$ has a zero of order m_i at each α_i, and $f(z)$ has a pole of order p_j at each j.

(3). We now calculate the residue of $\dfrac{f'(z)}{f(z)}$ at its singularities.

(3a) First consider the case where $f(z)$ has a zero of order m_i at α_i. Then there exists some open ball $B(\alpha_i, r)$ with $r > 0$ such that

$$f(z) = (z - \alpha_i)^{m_i}\phi(z) \quad \text{on } B(\alpha_i, r)$$

for some $\phi(z)$ analytic at α_i and such that $\phi(\alpha_i) \neq 0$. Since $\phi(z)$ is continuous at α_i and $\phi(\alpha_i) \neq 0$, it follows that by shrinking r if necessary, we may assume that $\phi(z) \neq 0$ for all $z \in B(\alpha_i, r)$. Then

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{m_i(z-\alpha_i)^{m_i-1}\phi(z) + (z-\alpha_i)^{m_i}\phi'(z)}{(z-\alpha_i)^{m_i}\phi(z)} \\ &= \frac{m_i}{z-\alpha_i} + \frac{\phi'(z)}{\phi(z)} \quad \text{on } B(\alpha_i, r) \setminus \{\alpha_i\}, \qquad (1)\end{aligned}$$

where $\phi'(z)/\phi(z)$ is analytic on $B(\alpha_i, r)$ (why?). Apply Taylor's theorem to $\phi'(z)/\phi(z)$, then we may rewrite (1) as

$$\frac{f'(z)}{f(z)} = \frac{m_i}{z-\alpha_i} + \sum_{k=0}^{\infty} a_k(z-\alpha_i)^k \quad \text{for } 0 < |z-\alpha_i| < r, \qquad (2)$$

where $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ is the Taylor series of $\phi'(z)/\phi(z)$ at α_i. Note that equation (2) gives the Laurent series of $f'(z)/f(z)$ at α_i, and thus, $f'(z)/f(z)$ has a simple pole at α_i with residue m_i.

(3b) Next consider the case where $f(z)$ has a pole of order p_j at β_j. Then there exists some open ball $B(\beta_j, r)$ with $r > 0$ such that

$$f(z) = \frac{\varphi(z)}{(z-\beta_j)^{p_j}} \quad \text{on } B(\beta_j, r) \setminus \{\beta_j\}$$

where $\varphi(z)$ is analytic at β_j and $\varphi(\beta_j) \neq 0$. Shrinking r if necessary, we may assume that $\varphi(z) \neq 0$ for all $z \in B(\beta_j, r)$.

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{(z-\beta_j)^{p_j}}{\varphi(z)} \cdot \frac{(z-\beta_j)^{p_j}\varphi'(z) - \varphi(z)p_j(z-\beta_j)^{p_j-1}}{(z-\beta_j)^{2p_j}} \\ &= \frac{-p_j}{z-\beta_j} + \frac{\varphi'(z)}{\varphi(z)} \quad \text{on } B(\beta_j, r) \setminus \{\beta_j\}.\end{aligned}$$

Then we may argue as in (3a) to show that $f'(z)/f(z)$ has a simple pole at β_j with residue $-p_j$.
(4). By Cauchy's residue theorem, the integral of f'/f around γ is equal to $2\pi i$ times the sum of the residues of f'/f inside γ. Together with (3a) and (3b), we thus have

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)}dz &= 2\pi i\Big[\sum_{i=1}^{n} \operatorname*{Res}_{z=\alpha_i} \frac{f'}{f} + \sum_{j=1}^{\ell} \operatorname*{Res}_{z=\beta_j} \frac{f'}{f}\Big] \\ &= 2\pi i\Big[\sum_{i=1}^{n} m_i + \sum_{j=1}^{\ell}(-p_j)\Big] \\ &= 2\pi i\,(\,Z - P\,) \\ \Longrightarrow Z - P &= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz.\end{aligned}$$

(5) Finally we need to prove

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = n(f(\gamma);0). \qquad (*)$$

For simplicity, we consider only the case when γ is a curve parametrized by a single differentiable function given by $\gamma(t) : [a,b] \to \mathbb{C}$. (Otherwise, we write γ as a union of such curves and apply the argument below to each curve). Then $f(\gamma)$ is parametrized by $f(\gamma(t)) : [a,b] \to \mathbb{C}$. Note that by the Chain Rule,

$$\frac{d}{dt}f(\gamma(t)) = f'(\gamma(t))\gamma'(t).$$

Then, writing $w = f(\gamma(t))$,

$$\begin{aligned}n(f(\gamma);0) &= \frac{1}{2\pi i}\int_{f(\gamma)} \frac{dw}{w} \\ &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}dt.\end{aligned}$$

On the other hand, we clearly have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt.$$

Therefore, the two sides of (*) agree, and this finishes the proof of Theorem 2.3.3. □

Corollary 2.3.5. If f is analytic in the domain interior to a positively oriented simple closed contour γ and suppose that f is analytic and non-zero on γ, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \text{no. of zeroes of } f \text{ inside } \gamma.$$

Example. Consider $f(z) = z^3$, γ the positively oriented unit circle about the origin. Describe $f(\gamma)$. What happens when we perturb f very slightly, say to $f_1(z) = z^3 + 0.01z - 0.001$. What can we say about $f_1(\gamma)$?

Theorem 2.3.6. (Rouché's Theorem)
If $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour γ and if

$$|f(z)| > |g(z)| \text{ at each point } z \text{ of } \gamma, \tag{1}$$

then $f(z)$ and $f(z)+g(z)$ must have the same number of zeroes (counting multiplicities) inside γ.

Example. Consider $f(z) = z^3$, $g(z) = 0.01z - 0.001$, and γ the unit circle about the origin.

Remark. The inequality must be strict, otherwise there is a counterexample which shows that the conclusion does not hold. For example, let γ be the circle $|z| = 1$, and let $f(z) = z$, $g(z) = -z$. Then $|f(z)| = |g(z)| = 1$ at each point of γ.
The number of zeroes of $f(z)$ inside γ is 1.
But $f(z) + g(z) = z - z \equiv 0$. So there are infinitely many zeros of $f(z) + g(z)$ inside γ.

Proof of Theorem 2.3.6. We first note that the strict inequality implies that f and $f+g$ do not have any zeroes on γ (Why?). Let Z_f and Z_{f+g} denote the number of zeroes (counted with multiplicity) of f and $f+g$ inside γ respectively. By the Argument Principle,

$$Z_f = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz \quad \text{and} \quad Z_{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g}\,dz. \tag{*}$$

Claim: We have

$$Z_{f+g} - Z_f = Z_{1+\frac{g}{f}} - P_{1+\frac{g}{f}},$$

where $Z_{1+\frac{g}{f}}$ and $P_{1+\frac{g}{f}}$ denote the number of zeroes and poles of $1 + \frac{g}{f}$ inside γ counting multiplicity respectively.

Proof of Claim. From the assumption, one easily sees that $1 + \frac{g(z)}{f(z)}$ is analytic and non-zero for all z on γ. By the Argument Principle,

$$\begin{aligned} Z_{1+\frac{g}{f}} - P_{1+\frac{g}{f}} &= \frac{1}{2\pi i}\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f}{f+g}\cdot\frac{fg'-gf'}{f^2}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{fg'-gf'}{f(f+g)}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(f'+g')-(f+g)f'}{f(f+g)}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f'+g'}{f+g}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz \\ &= Z_{f+g} - Z_f \quad \text{(by (*))} \end{aligned}$$

which gives the Claim. Now we continue the proof of Theorem 2.3.6. Write

$$F(z) = 1 + \frac{g(z)}{f(z)}.$$

Applying the Argument Principle to F, we have

$$Z_F - P_F = \frac{1}{2\pi}\Delta_\gamma \arg F(z).$$

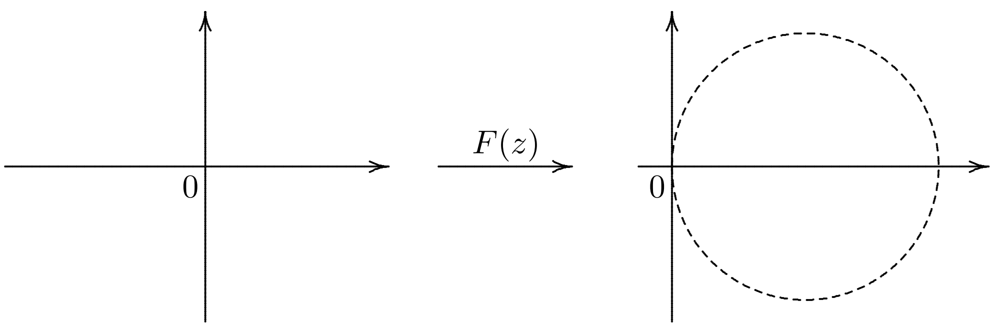
But

$$|F(z)-1| = |\frac{g(z)}{f(z)}| < 1$$

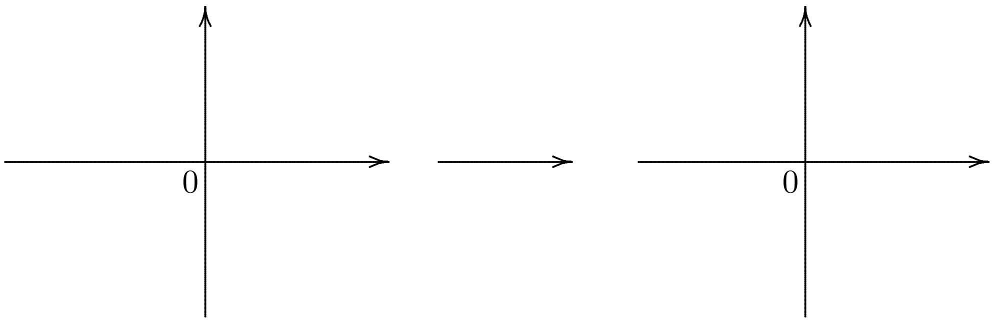
for all z on γ, so that the image $F(\gamma)$ lies completely inside the open disk $\{z : \; |z-1| < 1\}$ and hence does not enclose the origin 0 in the w plane. Hence

$$\Delta_\gamma \arg F(z) = 0.$$

Thus, $Z_F - P_F = 0$. Together with the Claim, it follows that $Z_{f+g} - Z_f = Z_F - P_F = 0$, and the theorem follows. □



An informal (intuitive) proof- The leash argument. By the argument principle it suffices to show that the winding number of $f(\gamma)$ about the origin in the w plane is equal to the winding number of $(f+g)(\gamma)$ about the origin in the w plane. If we think of a human walking a dog with a leash of variable length, and (for concreteness, suppose there is a lamppost at the origin in the w plane), we can inteprete the path $f(\gamma)$ as the path of the human and the path $(f+g)(z)$ as the path of the dog. The length of the leash is then $|g(z)|$ and the property that $|f(z)| > |g(z)|$ implies that for each $z \in \gamma$, the leash never extends from the human to the lamppost. The number of times that the human and the dog winds around the lamppost would therefore be the same.



Example. Determine the number of roots, counting multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \leq |z| \leq 2$.