NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

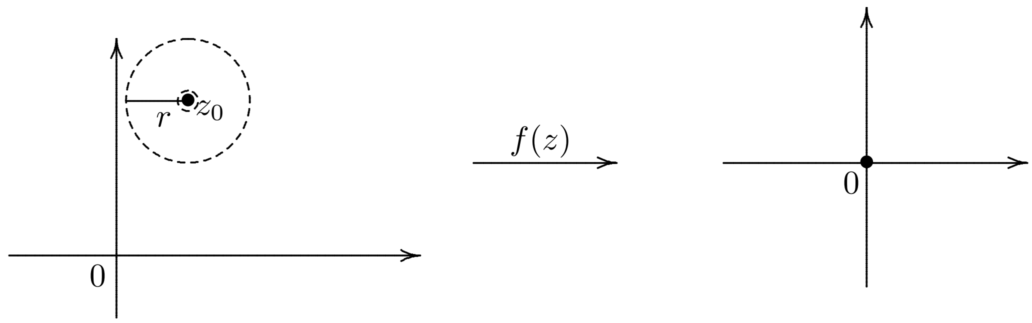
MA4247 Complex Analysis II

Lecture Notes Part II

## Chapter 2. Further properties of analytic functions

### 2.1. Local/Global behavior of analytic functions; The Identity Theorem

**Defnition.** A point $z \in \mathbb{C}$ is said to be a **zero** of $f$ if $f(z_0) = 0$.
A point $z_0$ is said to be an **isolated zero** of $f$ if $f(z_0) = 0$, and there exists some $r > 0$ such that $f(z) \neq 0$ for all $z$ satisfying $0 < |z - z_0| < r$.




**Proposition 2.1.1.** Suppose that $f$ is analytic at $z_0$ and $f(z_0) = 0$, then either
(i) $z_0$ is an isolated zero of $f$, or
(ii) $f$ is identically zero in some open ball centered at $z_0$ (i.e., $\exists\, \delta > 0$ such that $f(z) = 0$ for all $z \in B(z_0, \delta)$).

[Roughly speaking, **zeros of non-constant analytic functions are always isolated**.]

*Proof.* Since $f$ is analytic at $z_0$, there exists an open set, which can be taken to be an open ball $B(z_0, r)$ with $r > 0$, such that $f$ is differentiable (and hence analytic) everywhere in $B(z_0, r)$. By Taylor's theorem, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ for all } |z - z_0| < r, \text{ where } a_k = \frac{f^{(k)}(z_0)}{k!}. \quad (1)$$

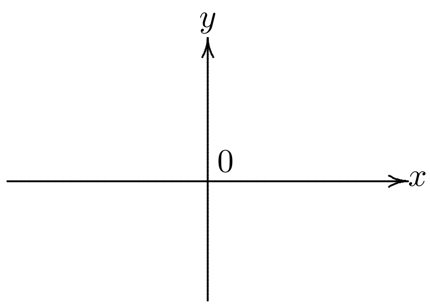
If all the Taylor coefficients $a_k$'s are zero, then by (1), $f(z) = 0$ for all $|z - z_0| < r$, and thus (ii) holds. Otherwise, let $m$ be the smallest positive integer such that $a_m \neq 0$. Then by (1), we have

$$\begin{aligned} f(z) &= a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + \cdots \\ &= (z - z_0)^m \big[ a_m + a_{m+1}(z - z_0) + a_{m+1}(z - z_0)^2 + \cdots \big] \\ &= (z - z_0)^m \phi(z) \quad \text{for all } |z - z_0| < r, \end{aligned} \quad (2)$$

where $\phi(z) := a_m + a_{m+1}(z - z_0) + a_{m+1}(z - z_0)^2 + \cdots$. The power series $\phi(z)$ has the same radius of convergence as the Taylor series of $f(z)$ about $z_0$, which is $\geq r$, hence it represents an analytic (and hence continuous) function on $B(z_0, r)$, with $\phi(z_0) = a_m \neq 0$. By continuity, there exists $\delta > 0$ such that $\phi(z) \neq 0$ for all $|z - z_0| < \delta$. See tutorial 1 for details of argument.

Hence, there exists $\delta > 0$ such that $\phi(z) \neq 0$ for all $z \in B(z_0, \delta)$. Shrinking $\delta$ if necessary, we may assume that $\delta < r$ (so that both (2) and (3) hold on $B(z_0, \delta)$). Together with the fact that $(z-z_0)^m \neq 0$ if $z \neq z_0$, it follows from (2) that $f(z) \neq 0$ for all $z \in B'(z_0, \delta) = B(z_0, \delta) \setminus \{z_0\}$, and thus (i) holds. Thus, either (i) or (ii) has to hold, and this finishes the proof of the proposition.

**Exercise.** Find an infinitely differentiable non-constant function from $\mathbb{R}$ to $\mathbb{R}$ which has a non-isolated zero.




**Solution.** Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

As a consequence of the above, we have the following theorem:

**Theorem 2.1.2. (Locally zero implies globally zero)**
Suppose that $f$ is analytic in a domain $D$ and $f \equiv 0$ on a non-empty open subset of $D$. Then $f \equiv 0$ on $D$.

*Proof.* Suppose $f$ is analytic on $D$ and $f \equiv 0$ on a non-empty open subset $U$ of $D$. Fix a point $z_0 \in U$ so that $f(z_0) = 0$. Take any other point $w \in D$, we want to show that $f(w) = 0$. Since $D$ is connected, there exists a polygonal line $L = L_1 + \cdots + L_k$ in $D$ joining $z_0$ to $w$, i.e., the initial point of $L_1$ is $z_0$ and the terminal point of $L_k$ is $w$.

First we show that $f \equiv 0$ along $L_1$. Let the terminal point of $L_1$ be $z_1$ (without loss of generality, we may assume $z_1 \neq z_0$) so that $L_1$ may be parametrized by

$$L_1: \ \gamma(t) = z_0 + t(z_1 - z_0), \quad 0 \le t \le 1.$$

Consider the set

$$I := \{s \in [0,1] \,\big|\, f(\gamma(t)) = 0 \text{ for all } 0 \le t \le s\}. \tag{*}$$

Since $f(z_0) = 0$, it follows that $0 \in I$, and thus $I \neq \emptyset$. Let $s_o := \sup I$.

Claim 1: $s_o > 0$.
*Proof of Claim 1.* Since $U$ is open, there exists an open ball $B(z_0, r) \subset U$ with $r > 0$. Then $f \equiv 0$ on $B(z_0, r)$. Then for all $0 \le t \le \dfrac{r}{2\,|z_1 - z_0|}$, we must have $f(\gamma(t)) = 0$, since for such $t$,

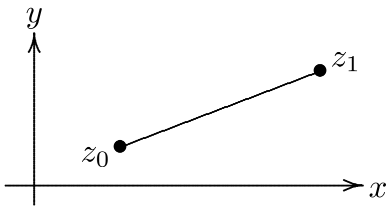
$$\begin{aligned} |\gamma(t) - z_0| &= |z_0 + t(z_1 - z_0) - z_0| \\ &= |t| \cdot |z_1 - z_0| \\ &\le \frac{r}{2\,|z_1 - z_0|} \cdot |z_1 - z_0| = \frac{r}{2} < r. \end{aligned} \tag{1}$$

Therefore, $s_o \ge \frac{r}{2\,|z_1 - z_0|} > 0$.

Claim 2: $f(\gamma(t)) = 0$ for all $0 \le t \le s_o$.
*Proof of Claim 2.* For any $t < s_o = \sup I$, there exists $s > t$ such that $s \in I$. Then $f(\gamma(t)) = 0$ since $t < s$ (see (*)). Thus, $f(\gamma(t)) = 0$ for all $0 < t \le s_o$. It remains to show that $f(\gamma(s_o)) = 0$. Since $f$ is continuous at $\gamma(s_o)$ and $\gamma(t) \to \gamma(s_o)$ as $t \to s_o^-$, it follows that

$$f(\gamma(s_o)) = \lim_{t \to s_o^-} f(\gamma(t)) = \lim_{t \to s_o^-} 0 = 0.$$

(Note $t \to s_o^-$ implies that $t < s_o$ and thus $f(\gamma(t)) = 0$.) This finishes the proof of Claim 2.




Claim 3: $s_o = 1$.
*Proof of Claim 3.* We prove the claim by contradiction. Suppose $s_o < 1$, so that by Claim 1, $0 < s_o < 1$. By Claim 2, $f(\gamma(t)) = 0$ for all $0 \leq t \leq s_o$. Also, for any $r > 0$, $B(\gamma(s_o), r)$ contains some point $\gamma(t)$ with $0 \leq t < s_o$ (and thus $f(\gamma(t)) = 0$) (Exercise, find an explicit $t$ using a calculation similar to (1)). Thus, $\gamma(s_o)$ is a zero but not an isolated zero of $f$. Hence by Proposition 2.1.1, there exists $\delta > 0$ such that $f \equiv 0$ on $B(\gamma(s_o), \delta)$. Shrinking $\delta$ if necessary, we may assume that $B(\gamma(s_o), \delta) \subset D$. Then for all $s_o \leq t \leq s_o + \dfrac{\delta}{2|z_1 - z_0|}$, one can check that $\gamma(t) \subset B(\gamma(s_o), \delta)$ and thus $f(\gamma(t)) = 0$ (Exercise). Together with Claim 2, it follows that $f(\gamma(t)) = 0$ for all $0 \leq t \leq s_o + \dfrac{\delta}{2|z_1 - z_0|}$, and thus $\sup I \geq s_o + \dfrac{\delta}{2|z_1 - z_0|}$, contradicting that $\sup I = s_o$. Therefore, we must have $\sup I = 1$, and this finishes the proof of Claim 3.

*Completion of proof of Theorem 2.1.2.* From Claim 2 and Claim 3, it follows that $f \equiv 0$ on $L_1$. In particular, $z_1$ is a zero of $f$, but not an isolated zero of $f$. Then since $z_1$ is also the initial point of $L_2$, one can repeat the above argument to show that $f \equiv 0$ on $L_2$. By repeating the argument again and again, one can show that $f \equiv 0$ on the entire $L$, and thus $f(w) = 0$. Since $w$ is an arbitrary point of $D$, it follows that $f \equiv 0$ on $D$. This finishes the proof of the theorem.

Exercise: The usual definition for $D$ to be connected is that if $D = V_1 \cup V_2$ where $V_1 \cap V_2 = \emptyset$ and both $V_1$ and $V_2$ are open, then either $V_1$ or $V_2$ is the empty set (you cannot partition a connected set into two disjoint non-empty open sets). Use this definition to prove the theorem.

**Definition 2.1.3.** Consider a set $T \subset \mathbb{C}$. A point $z_0$ (not necessarily in $T$) is said to be an **accumulation point** of $T$ if $T \cap B(z_0, r) \setminus \{z_0\} \neq \emptyset$ for any $r > 0$, i.e., any open ball centered at $z_0$ contains a point in $T$ other than $z_0$.

**Remark 2.1.4.** By considering a sequence of open balls $B(z_0, r_n)$ with $r_n$ decreasing to 0, one can find a sequence of distinct points $\{z_n\}$ in $T$ such that $\lim_{n\to\infty} z_n = z_0$ (Exercise).

**Example.** Let $T = B(0,1) \setminus \{0\}$. Then 0 is an accumulation point of $T$ (note that $0 \notin T$).

**Theorem 2.1.5. (Zeros do not have accumulation point in $D$ if $f$ is not identically zero)** Let $f$ be an analytic function in a domain $D$, and let $T \subset D$ be such that $T$ has an accumulation point $z_0$ in $D$. If $f \equiv 0$ on $T$, then $f \equiv 0$ on $D$.

*Proof.* Since $z_0 \in D$ is an accumulation point of $T$, it follows from Remark 2.1.4 that one can find a sequence of distinct points $\{z_n\} \subset T$ such that $\lim_{n\to\infty} z_n = z_0$. Then by continuity of $f$ and since each $z_n \in T$, we have

$$f(z_0) = f(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} 0 = 0.$$

Thus $z_0$ is a zero of $f$. But $z_0$ is not an isolated zero of $f$ (Exercise). Thus by Proposition 2.1.1, there exists $\delta > 0$ such that $f(z) = 0$ for all $z \in B(z_0, \delta)$. Shrinking $\delta$ if necessary, we may assume that $B(z_0, \delta) \subset D$. Thus by Theorem 2.1.2, $f \equiv 0$ on $D$.

**Remark.**
(i) The result is not necessarily true if $z_0 \notin D$.
(ii) Examples of a set $T$ in a domain $D$ with an accumulation point in $D$ are given below:

(1) $T$ is a non-empty open subset of $D$;

(2) $T$ is a curve or line segment contained in $D$;

(3) $T$ consists of a sequence of distinct points $\{z_n\} \subset D$ which converges to a point $z_0 \in D$. For example, $D = B(0,2)$ and $T = \{1/n : n \in \mathbb{N}\}$ has the accumulation point 0 in $D$.

**Theorem 2.1.6 (Identity theorem for analytic functions)**
Let $f$ and $g$ be analytic in a domain $D$. If $f \equiv g$ on a subset $T \subset D$ which has an accumulation point in $D$ (such as an non-empty open subset of $D$ or a line segment in $D$), then $f \equiv g$ on $D$.

*Proof:* Consider $f - g$ on $D$. Since $f - g \equiv 0$ on $T \subset D$, and $T$ has an accumulation point $D$, it follows from Theorem 2.1.5 that $f - g \equiv 0$ on $D$, hence $f \equiv g$ on D.

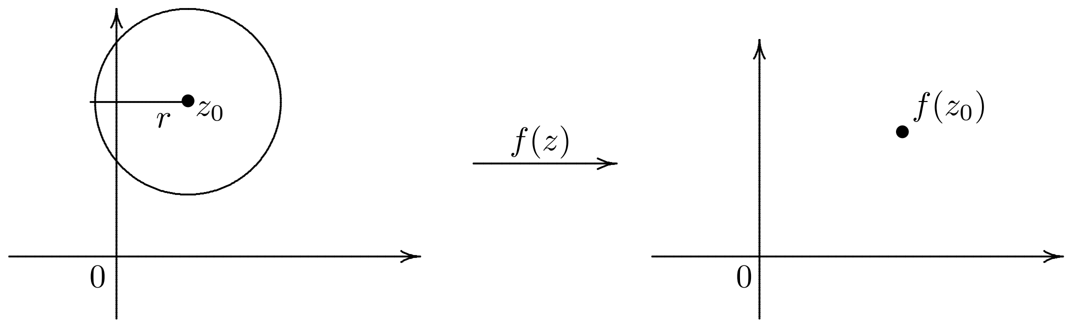
**Example 2.1.7.** Consider the functions $f(z) = \sin(2z)$ and $g(z) = 2\sin z \cos z$.

## §2.2. Maximum modulus principle and applications

**Theorem 2.2.1.(Gauss Mean Value Theorem)**
Suppose that $f$ is analytic everywhere within and on the circle $C: \ |z - z_0| = r$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

**Remark.** The formula says that the value of $f$ at the centre of the circle is the arithmetic mean of the values of $f$ on the circle.




*Proof.* By the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

Take the parametrization of $C$ given by $\gamma(\theta) = z_0 + re^{i\theta}$, $0 \leq \theta \leq 2\pi$, so that $\gamma'(\theta) = ire^{i\theta}$. Hence

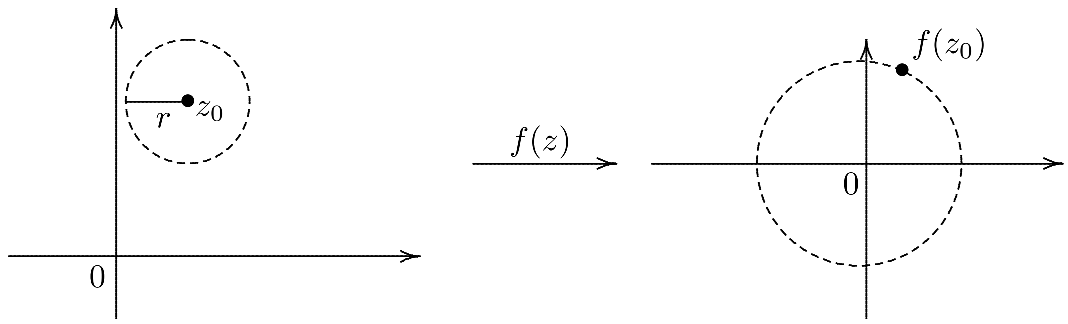
$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - z_0} \cdot ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta. \end{aligned}$$

**Example 2.2.2.** Let $f(z) = 2z^2 + z + 1$. Then

$$f(0) = 1 = \frac{1}{2\pi} \int_0^{2\pi} 2e^{2i\theta} + e^{i\theta} + 1 \, d\theta$$

**Proposition 2.2.3. (Local version of maximum modulus principle)**
Suppose that $f$ is analytic on an open ball $B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ centered at $z_0$ and of radius $r > 0$. If $|f(z)| \leq |f(z_0)|$ for each point $z \in B(z_0, r)$, then $f(z) \equiv f(z_0)$ on $B(z_0, r)$.

**Remark.** The above can be rephrased as "If $f$ is analytic on $B(z_0, r)$ and $|f(z)|$ attains its maximum in $B(z_0, r)$ at $z_0$, then $f$ is constant on $B(z_0, r)$".




*Proof.* Suppose $f$ satisfies the conditions of the Proposition, and let $0 < \rho < r$. Consider the circle $|z_1 - z_0| = \rho$. By Gauss' MVT,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta \qquad (1) \\ \Longrightarrow |f(z_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|d\theta. \qquad (2) \end{aligned}$$

On the other hand, by the assumption,

$$|f(z_0 + \rho e^{i\theta})| \leq |f(z_0)|, \quad 0 \leq \theta \leq 2\pi \qquad (3)$$

$$\Longrightarrow \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|d\theta \leq \int_0^{2\pi} |f(z_0)|d\theta = 2\pi|f(z_0)| \qquad (4)$$

$$\Longrightarrow |f(z_0)| \geq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|d\theta. \qquad (5)$$

By (2) and (5),

$$\begin{aligned} |f(z_0)| &= \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \\ \Longrightarrow |f(z_0)| \cdot \frac{1}{2\pi}\int_0^{2\pi} 1\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \\ \Longrightarrow \frac{1}{2\pi}\int_0^{2\pi} (|f(z_0)| - |f(z_0 + \rho e^{i\theta})|)\, d\theta &= 0. \qquad (6) \end{aligned}$$

By assumption, the integrand

$$(|f(z_0)| - |f(z_0 + \rho e^{i\theta})|) \geq 0 \quad \text{for all } 0 \leq \theta \leq 2\pi. \tag{7}$$

If there is a value of $\theta$ for which the inequality in (7) is strict, then by continuity, we have strict inequality for an open interval of $\theta$ and

$$\int_0^{2\pi} (|f(z_0)| - |f(z_0 + \rho e^{i\theta})|) d\theta > 0,$$

which is a contradiction to (6). Hence we must have

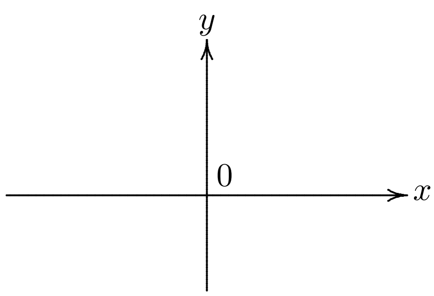
$$(|f(z_0) - |f(z_0 + \rho e^{i\theta})|) = 0, \qquad \text{for all } 0 \leq \theta \leq 2\pi,$$

i.e., $|f(z)| = |f(z_0)|$ for all $z$ on the circle $|z - z_0| = \rho$. By letting $\rho$ vary from 0 to $r$, it follows that $|f(z)| = |f(z_0)|$ for all $z \in B(z_0, r)$. Since $f$ is an analytic function such that $|f(z)|$ is constant on $B(z_0, r)$, it follows that that $f(z)$ itself is constant on $B(z_0, r)$ (see e.g. [Churchill, 7th ed., p. 74, Question 7]). □

**Theorem 2.2.4. (Maximum modulus principle)**
If $f$ is analytic and not constant on the domain $D$, then $|f(z)|$ has no maximum value in $D$.

*Proof:* We prove the theorem by contradiction. Suppose that $|f(z)|$ attains its maximum at some point $z_0 \in D$. Since $D$ is open, there exists $r > 0$ such that $z_0 \in B(z_0, r) \subset D$. Then $|f(z)| \leq |f(z_0)|$ for all $z \in B(z_0, r)$. Thus by the local version of the MMP (Proposition 2.2.3), $f$ is constant on $B(z_0, r)$. Together with the identity theorem for analytic functions (Theorem 2.1.6), it follows that $f(z)$ is constant on $D$, contradicting the assumption that $f$ is non-constant on $D$. Thus, $|f(z)|$ has no maximum value in $D$. □

**Remark.** The maximum modulus does not hold in the real case! Consider the function $f(x) = 1 - x^2, \quad -1 < x < 1$. Clearly, the function is non-constant and differentiable on $(-1, 1)$. But the maximum value of $|f(x)|$ is attained at the interior point $x = 0$ when $f(0) = 1$.



**Corollary 2.2.5.** Let $R \subset \mathbb{C}$ be a closed bounded set whose interior is a domain. Suppose $f$ is continuous on $R$ and analytic and not constant in the interior of $R$. Then the maximum value of $|f(z)|$ in $R$ occurs on the boundary of $R$ and never in the interior.

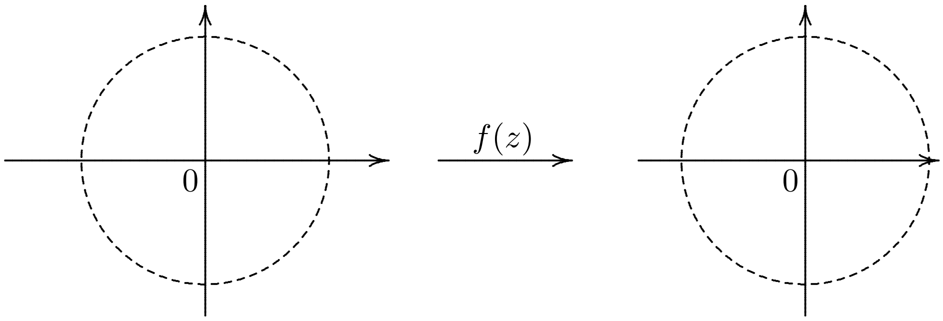
*Proof.* Note that $|f(z)|$ is a real-valued continuous function on the closed and bounded region $R$. Then it follows from the Extreme Value Theorem that $|f(z)|$ attains its maximum value at some point $z_0$ of $R$. By the maximum modulus principle, such $z_0$ cannot lie in the interior of $R$. Therefore, $z_0$ must lie in the boundary of $R$.

**Example 2.2.6. (minimum modulus principle)** Let $R \subset \mathbb{C}$ be a closed bounded set whose interior is a domain. Suppose $f$ is continuous on $R$ and analytic and not constant in the interior of $R$. If $f(z) \neq 0$ for any $z \in R$, then $|f(z)|$ attains its minimum value at the boundary of $R$ but not in the interior of $R$. (To be done in tutorial).

**Example 2.2.7.** Consider the function $f(z) = e^z$ on the closed disk $\overline{B(1,2)} = \{z \in \mathbb{C} \ : \ |z-1| \leq 2\}$. Then the maximum and minimum values of $|f(z)|$ in $\overline{B(1,2)}$ occurs on the boundary, and not on the interior. (Exercise: Find the points where the maximum and the minimum values for $|f(z)|$ occurs).

An important application of the MMP is the following

**Theorem 2.2.8. (Schwarz's Lemma)**
Consider the unit ball $B(0,1) := \{z \in \mathbb{C} : \ |z| < 1\}$. Suppose that $f$ is analytic on $B(0,1)$ such that
(i) $|f(z)| \leq 1 \quad$ for each $z \in B(0,1)$, and
(ii) $f(0) = 0$.
Then
(a) $|f(z)| \leq |z| \quad$ for each $z \in B(0,1)$, and
(b) $|f'(0)| \leq 1$.
Moreover, if the equality in (b) holds or the equality in (a) holds for at least one non-zero point in $B(0,1)$, then $f(z) \equiv az$ on $B(0,1)$ for some complex constant $a$ with $|a| = 1$ (that is, $f(z) = e^{i\alpha} z$ is a rotation by some angle $\alpha$ about the origin).




*Proof.* (Key step:) Let $g(z) := \dfrac{f(z)}{z}$. Clearly, $g(z)$ analytic on $B(0,1) \setminus \{0\}$, and it has an isolated singular point at $z = 0$. From (ii), we have $f(0) = 0$. By Taylor's theorem,

$$f(z) = 0 + f'(0)z + \frac{f''(0)}{2!} z^2 + \cdots, \quad \text{for } |z| < 1.$$

Thus the Laurent series of $g(z)$ at $z = 0$ is given by

$$\begin{aligned} g(z) &= \frac{1}{z}\Big[f'(0)z + \frac{f''(0)}{2} z^2 + \cdots\Big] \\ &= f'(0) + \frac{f''(0)}{2} z + \cdots, \quad 0 < |z| < 1. \end{aligned}$$

In particular, $z = 0$ is a removable singular point of $g(z)$, and we can define $g(0) = f'(0)$ so that the extended function $g(z)$ becomes analytic on $B(0,1)$ (see (1.6.3)). Now, since $|f(z)| \leq 1$ for $z \in B(0,1)$, we see that

$$|g(z)| \leq \frac{1}{|z|}, \quad \text{for } 0 < |z| < 1. \tag{1}$$

Now for any $z_1 \in B(0,1)$, we choose an $r$ such that $|z_1| < r < 1$. The function $g(z)$ is analytic on the closed disk $\overline{B(0,r)} := \{z \in \mathbb{C} : \ |z| \leq r\}$

and $|g(z)| \leq \frac{1}{r}$ on the circle $|z| = r$, which is the boundary of $\overline{B(0,r)}$. Hence, by the MMP,

$$|g(z_1)| \leq \frac{1}{r}. \tag{2}$$

Then by taking the limit as $r \to 1^-$, one gets from (2) that

$$|g(z_1)| = \lim_{r \to 1^-} |g(z_1)| \leq \lim_{r \to 1^-} \frac{1}{r} = 1. \tag{3}$$

Hence, upon renaming $z_1$ as $z$ (since $z_1$ was an arbitrary point in $B(0,1)$), we have

$$|g(z)| \leq 1 \quad \text{for all } |z| < 1. \tag{4}$$

Therefore, $|f(z)| \leq |z|$ for $0 < |z| < 1$. The inequality clearly holds when $z = 0$, since $f(0) = 0$. Thus, $|f(z)| \leq |z|$ for all $|z| < 1$, which gives (a). Recall that $g(0) = f'(0)$. Thus by (4), we also have $|f'(0)| = |g(0)| \leq 1$, which gives (b). Finally, if the equality in (b) holds (i.e., $|f'(0)| = |g(0)| = 1$) or the equality in (a) holds at some non-zero point in $B(0,1)$ (which implies $|g(z)| = 1$ for some non-zero $z \in B(0,1)$), then together with (3), it follows $|g(z)|$ attains its maximum value 1 at an interior point $z_0 \in B(0,1)$. Thus by MMP, $g(z)$ is a constant function on $B(0,1)$, i.e., $g(z) \equiv a = e^{i\alpha}$ on $B(0,1)$. Note also that $|a| = |g(z_0)| = 1$. Thus, $f(z) \equiv az$ on $B(0,1) \setminus \{0\}$ with $|a| = 1$. Since $f(0) = 0$, the equality also holds at $z = 0$. This finishes the proof of Schwarz's Lemma. $\square$