NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II

Lecture Notes Part I

Chapter 1. Preliminary results/Review of Complex Analysis I

These are more detailed notes for the results studied in MA3111 Complex Analysis I. As many of these results are required for this module, you are expected to know them, especially the statement of the theorems and also the various formulae like the Cauchy integral formula, etc. However, you will not be required in this module to reproduce the proofs of these results (unless they occur in the proofs of theorems covered in this module). Note that some of the notation may differ from that used in the later parts of these notes.

1.1. Complex Numbers

(1.1.1) Notation and Basic Results

We denote the set of complex numbers by $\mathbb{C}$, that is

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

Remark. (i) For $z = x + iy$, x is called the **real part** of z, and y is called the **imaginary part** of z.
We usually write

$$x = \text{Re } z, \quad y = \text{Im } z.$$

(ii) (**Equality of two complex numbers.**) Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.
In other words, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 = z_2 \quad \Longleftrightarrow \quad x_1 = x_2 \quad \& \quad y_1 = y_2.$$

(iii) The **complex conjugate** $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\boxed{\bar{z} = x - iy.}$$

For any $z = x + iy$, $z_1, z_2 \in \mathbb{C}$, we have

$$\text{Re } z = x = \frac{z + \bar{z}}{2}, \qquad \text{Im } z = y = \frac{z - \bar{z}}{2i}.$$

(iv) The **modulus** $|z|$ of a complex number $z = x + iy$ is the distance of z from the origin 0 in the complex plane.

By the Pythagoras' theorem, we have

$$\boxed{|z| = \sqrt{x^2 + y^2}.}$$

(v) Geometrically, $|z_1 - z_2|$ is the distance between z_1 and z_2 in the complex plane.

(vi) For any $z \in \mathbb{C}$, we have

$$\boxed{z\bar{z} = |z|^2.}$$

(vii) For any $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1 z_2| = |z_1| \cdot |z_2|,$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

(viii) (**Triangle Inequality**) For any $z_1, z_2 \in \mathbb{C}$, we have

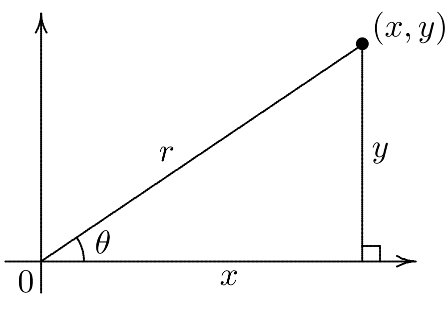
$$\boxed{\big||z_1| - |z_2|\big| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|.}$$

(ix) For more than two complex numbers, we have

$$|z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|.$$

Exercise: Prove the triangle inequality (viii) above.

(1.1.2) Polar form, Exponential form.



The **polar form** of a complex number $z = x + iy$ is given by

$$\boxed{z = r(\cos\theta + i\sin\theta),}$$

where $r = |z|$, and θ is an **argument** of z. Write

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

Then we have the **exponential form** of z given by

$$\boxed{z = re^{i\theta}.}$$

Sometimes, we also write

$$\exp(i\theta) := e^{i\theta},$$

so that the exponential form may also be written as

$$z = r\exp(i\theta).$$

Remark. The argument of z is defined only up to a multiple of 2π. The collection of all possible values of the argument of z is denoted by $\arg z$. Thus we may write

$$\boxed{\arg z = \theta + 2n\pi, \quad n \in \mathbb{Z},}$$

where θ is any one of the values of $\arg z$. Note: This ambiguity in the definition of the argument accounts for much of the interesting phenomena which occurs in the study of complex analysis.

• The **general polar form** of z is given by

$$\boxed{z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)], \quad n = 0, \pm 1, \pm 2, \cdots,}$$

where θ is any argument of z. Similarly, the **general exponential form** of z is given by

$$\boxed{z = r\exp[i(\theta + 2n\pi)], \quad n = 0, \pm 1, \pm 2, \cdots.}$$

• The **principal argument** of a non-zero z, denoted by $\operatorname{Arg} z$, is the unique value of the argument of z satisfying

$$-\pi < \operatorname{Arg} z \leq \pi.$$

• One has the following **de Moivre's formula**:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,}$$

which is valid for any rational number n.

• **Application:** To find all the n-th roots of a complex number c. Write $c = r_o\exp(i\theta_o)$. The problem is equivalent to solving the equation

$$\begin{aligned} z^n &= c \\ &= r_o\exp[i(\theta_o + 2k\pi)], \quad k = 0, \pm 1, \pm 2, \cdots \\ \Longrightarrow\ z_k &= \{r_o\exp[i(\theta_o + 2k\pi)]\}^{\frac{1}{n}}, \quad k = 0, \pm 1, \pm 2, \cdots \\ \Longrightarrow\ z_k &= r_o^{\frac{1}{n}}\exp\left[i\left(\frac{\theta_o + 2k\pi}{n}\right)\right], \quad k = 0, 1, 2, \cdots, n-1. \end{aligned}$$

Exercise: (i) Why is it necessary to consider only $k = 0, 1, \cdots, n-1$ in the solution above? (ii) Show that the four 4-th roots of -16 are $\sqrt{2} + \sqrt{2}i$, $-\sqrt{2} + \sqrt{2}i$, $-\sqrt{2} - \sqrt{2}i$, $-\sqrt{2} - \sqrt{2}i$.

1.2. Analytic functions

(1.2.1) Complex-valued function of a complex variable, limits, continuity

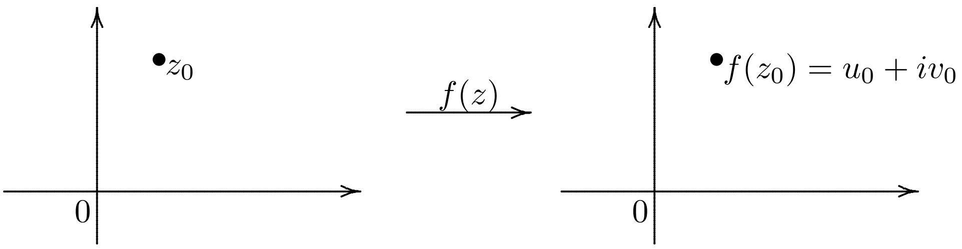
Let $f(z)$ be a complex-valued function of a complex variable. Write $z = x + iy$. Then we can write

$$f(z) = u(x, y) + iv(x, y),$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions in two real-variables x, y.
The function $u(x, y)$ is called the **real-part** of $f(z)$, while $v(x, y)$ is called the **imaginary part** of $f(z)$. And we simply write

$$u(x, y) = \text{Re } (f), \qquad v(x, y) = \text{Im } (f).$$

Geometrically,



For a function $f(z)$, we write $\lim_{z \to z_0} f(z) = w_o$ if the following condition is satisfied: For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

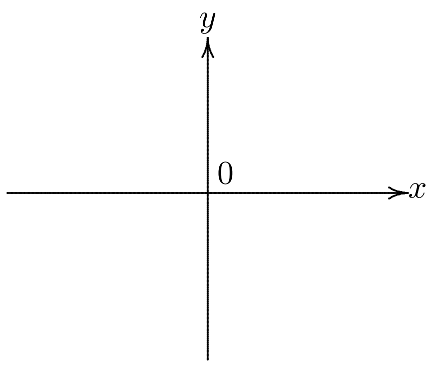
$$|f(z) - w_o| < \epsilon \quad \text{whenever } 0 < |z - z_0| < \delta.$$

A complex-valued function $f(z)$ is **continuous at** z_0 iff $\lim_{z \to z_0} f(z) = f(z_0)$.

Remark. Write $f(z) = u(x, y) + iv(x, y)$. Then $f(z)$ is a continuous function $\Longleftrightarrow$ $u(x, y)$ and $v(x, y)$ are continuous functions.

(1.2.2) Infinity

For complex numbers, there is only ONE infinity, ∞. It is often convenient to enlarge the complex plane $\mathbb{C}$ by adding the element ∞, and the enlarged set $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called the **extended complex plane**.



Remark. (i) $z \to \infty \iff |z| \to \infty$ (second limit is in the usual sense for real numbers);
(ii) $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$;
(iii) $\lim_{z \to \infty} f(z) = w_o \iff \lim_{s \to 0} f\big(\frac{1}{s}\big) = w_o$.

(1.2.3) Differentiations, Cauchy-Riemann equations

Definition: The **derivative** of f at z_0 is defined as

$$\frac{d}{dz} f(z)\Big|_{z=z_0} = f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided that the limit exists.
If $f'(z_0)$ exists,we say that f **is differentiable at** z_0.

Remark. Alternatively, we may define

$$f'(z_0) := \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}.$$

Remark. (i) If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.
(ii) The usual Power Rule, Addition Rule, Product Rule, Quotient Rule and Chain Rule hold.

Theorem. (Necessary conditions for differentiability)
If $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_o + iy_o$, then u and v satisfy the **Cauchy-Riemann equations** at (x_o, y_o), i.e.,

$$\begin{cases} \dfrac{\partial u}{\partial x}(x_o, y_o) = \dfrac{\partial v}{\partial y}(x_o, y_o), \\[2ex] \dfrac{\partial u}{\partial y}(x_o, y_o) = -\dfrac{\partial v}{\partial x}(x_o, y_o), \end{cases}$$

and

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

[In short, $f(z)$ differentiable $\implies$ $u_x = v_y,\ u_y = -v_x$ and $f'(z) = u_x + iv_x = v_y - iu_x$.]

Theorem. (A Sufficient Condition for Differentiability)
Let $f(z) = u(x, y) + iv(x, y)$ be defined near the point $z_0 = x_o + iy_o$. Suppose that
(i) u, v satisfy the Cauchy-Riemann equations at (x_o, y_o) i.e.

$$u_x = v_y,\ u_y = -v_x \quad \text{at } (x_o, y_o); \text{ and}$$

(ii) u_x, u_y, v_x, v_y are continuous at (x_o, y_o).
Then f is differentiable at z_0.

[In short, continuous partial derivatives + CR-equations $\Longrightarrow$ differentiable.]

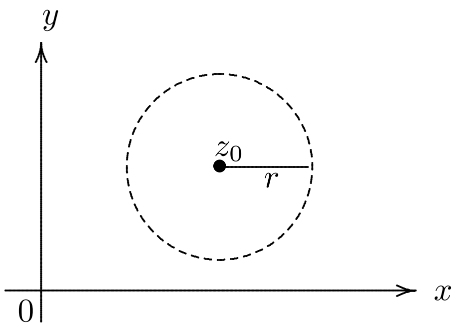
(1.2.4) Domains, Analytic functions

For $z_0 \in \mathbb{C}$ and $r > 0$, we define the ball centered at z_0 and of radius r by

$$\boxed{B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.}$$

We also use the following notation for the "deleted ball" (that is, with the centre removed):

$$\boxed{B'(z_0, r) := B(z_0, r) \setminus \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}.}$$

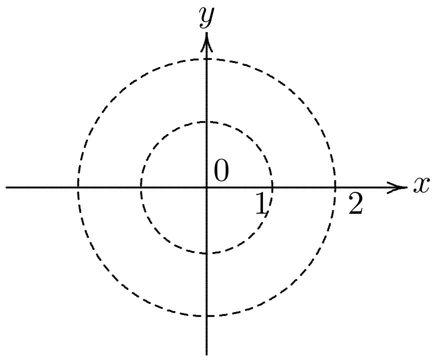


A subset U in $\mathbb{C}$ is said to be **open** if for each $z \in U$, there exists some $r > 0$ such that $z \in B(z, r) \subset U$.
(Here the radius r may depend on the point z)
[Roughly, an open set U is a set which contains no boundary points.]

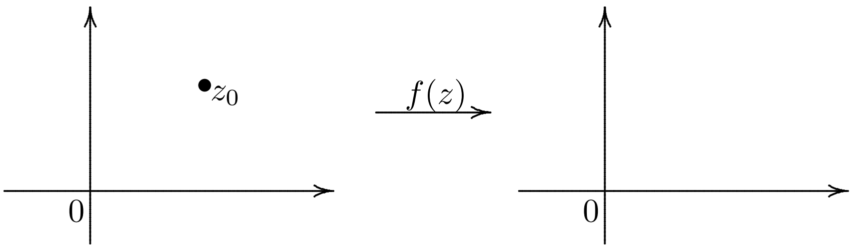
Definition. An open subset S of $\mathbb{C}$ is said to be **connected** if each pair of points z_1, z_2 in S can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, which lie entirely in S. (Remark: This is not the usual definition for connectedness. More generally, we have the notion of connectedness, path-connectedness, and polygonally path-connectedness. For open subsets of $\mathbb{C}$, these are all equivalent.)

Definition. A **domain** in $\mathbb{C}$ is a non-empty, open and connected subset of $\mathbb{C}$.

e.g. The annulus $U = \{z \in \mathbb{C} : 1 < |z| < 2\}$ is open.



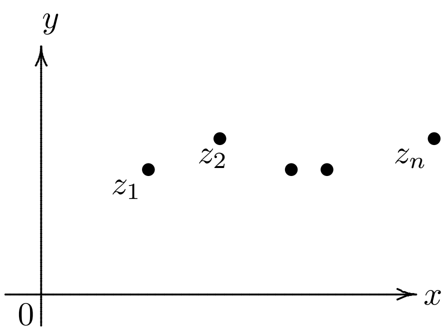
Definition. A function f is **analytic at a point** z_0 if f is differentiable everywhere in some open set U containing z_0.



Roughly speaking, 'analytic at z_0' means 'differentiable everywhere near z_0'.

Definition. A function f is **analytic in an open set** U if f is differentiable everywhere in U.

Remark. If a function f is differentiable only at a finite number of (isolated) points, say z_1, $z_2, \cdots, z_n$, then f is nowhere analytic.



Theorem. Let $f(z)$ be analytic in a domain D. If $f'(z) \equiv 0$ on D, then $f(z)$ is constant in D.

Definition. An **entire** function is a function which is analytic in the whole complex plane $\mathbb{C}$ (i.e. differentiable everywhere in $\mathbb{C}$).

1.3. Elementary Functions

(1.3.1) The exponential function

The (complex) exponential function given by

$$\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty).}$$

It can be shown that

$$\boxed{e^z = e^x(\cos y + i \sin y)} \quad \forall z = x + iy \in \mathbb{C}.$$

Some properties:
(i) When z is real, e^z coincides with the real exponential function e^x, i.e., when $z = x + i0$, $e^z = e^x$.

(ii) e^z is entire and $\dfrac{d}{dz}(e^z) = e^z$.

(iii)

$$|e^z| = e^x,$$
$$\arg(e^z) = y + 2n\pi, \quad n = 0, \pm 1, \pm 2, \cdots .$$

In particular, we have $e^z \neq 0$ for any $z \in \mathbb{C}$.

(1.3.2) The logarithmic function

Definition. The (complex) **logarithmic function** $\log z$ is defined as the inverse 'function' of the exponential function e^z, i.e.,

$$w = \log z \Longleftrightarrow z = e^w.$$

One easily sees that

$$\boxed{\log z = \ln |z| + i \arg z.}$$

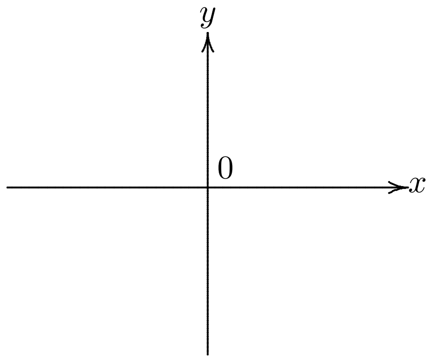
Remark. (i) $\log z$ is a multi-valued function. So it is strictly speaking, not a function under the usual definition.
(ii) $\log z$ is defined for $z \in \mathbb{C} \setminus \{0\}$.
Question: What can we do to try to obtain a well-defined log function?

Definition. The **principal logarithmic function** (or the principal branch of $\log z$) is given by

$$\boxed{\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z}.$$

Remark. (i) $\operatorname{Log} z$ is a single-valued function defined on $\mathbb{C} \setminus \{0\}$.

(ii) $\operatorname{Log} z$ is continuous on the **cut complex plane** $\mathbb{C} \setminus (-\infty, 0]$.



(iii) In fact, $\operatorname{Log} z$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$, and

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z} \quad \forall z \in \mathbb{C} \setminus (-\infty, 0].$$

Question: Can one define other branches of the log function? What must the domain satisfy for these branches?

(1.3.3) The trigonometric functions

The (complex) cosine and sine functions are given by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{for any } z \in \mathbb{C}.$$

Remark. (i) $\cos z$ and $\sin z$ are entire functions. (Why?)
(ii) When z is real, i.e. $z = x + i0$, we have

$$\cos z = \cos x, \quad \sin z = \sin x.$$

(iii) We have

$$\frac{d}{dz}(\cos z) = -\sin z, \quad \frac{d}{dz}(\sin z) = \cos z,$$
$$\cos^2 z + \sin^2 = 1.$$

(iv) (Exercise) $\sin z = 0 \Longleftrightarrow z = n\pi, \quad n \in \mathbb{Z}.$

We may define the other complex trigonometrical functions by:

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{1}{\tan z},$$

$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

Again, we have

$$\frac{d}{dz}(\tan z) = \sec^2 z, \qquad \frac{d}{dz}(\cot z) = -\csc^2 z,$$

$$\frac{d}{dz}(\sec z) = \sec z \tan z, \qquad \frac{d}{dz}(\csc z) = -\csc z \cot z.$$

Exercise: Determine the domains for each of the other trigonometric functions.

(1.3.4) The hyperbolic functions

The **(complex) hyperbolic cosine and sine functions** are given by

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

1. $\cosh z$, $\sinh z$ are entire functions.

2. When z is real, i.e. $z = x + 0i$,

$$\cosh z = \cosh x, \qquad \sinh z = \sinh x.$$

The other hyperbolic functions are given by

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{1}{\tanh z},$$

$$\operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.$$

Some Properties: One has

$$\cosh^2 z - \sinh^2 z = 1,$$

$$\frac{d}{dz}(\cosh z) = \sinh z, \qquad \frac{d}{dz}(\sinh z) = \cosh z.$$

Exercise: Find the relation between the functions sin and sinh, and cos and cosh.

(1.3.5) Complex Exponents z^c with $z, c \in \mathbb{C}$

For $z, c \in \mathbb{C}$ (where we regard z as the variable and c is a fixed complex constant), with $z \neq 0$, one defines

$$z^c := e^{c \log z}.$$

Remark. For fixed c, z^c is a multi-valued function in z. By fixing a branch of the log function (equivalently a branch of the argument), we may obtain a well-defined function on suitable subsets of $\mathbb{C}$. Note that when c is an integer, then z^c is well-defined.

(1.3.6) Inverse Trigonometric/Hyperbolic Functions

One can define the the inverse trigonometric/hyperbolic functions (e.g. $\cos^{-1} z$, $\tanh^{-1} z, \cdots$, etc) in terms of $\log z$ by solving the appropriate equations.
Usually these are multi-valued functions.

Example. $\cos^{-1} z = -i \log \left[z + i(1 - z^2)^{1/2} \right]$.

1.4. Contour Integrals

(1.4.1) Contours

A **curve** in the complex plane is (parametrized by) a continuous function

$$\gamma : [a, b] \to \mathbb{C}.$$

That is, if we write $\gamma(t) = x(t) + iy(t)$, then $x(t)$ and $y(t)$ are continuous on $[a, b]$.

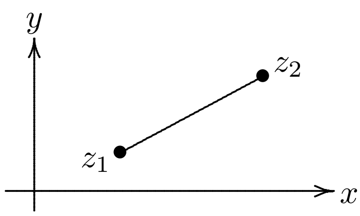
Remark. (i) γ is **simple** if $t_1 \neq t_2 \Longrightarrow \gamma(t_1) \neq \gamma(t_2)$, i.e. γ does not cross itself.

(ii) γ is **closed** if $\gamma(a) = \gamma(b)$.

(iii) γ is a **simple closed curve** if it is closed and $a < t_1 < t_2 < b \Longrightarrow \gamma(t_1) \neq \gamma(t_2)$, i.e. γ does not cross itself except at the end points.

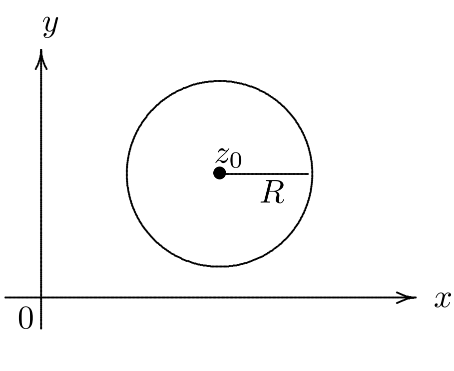
Example.
(i) **The line segment** $[z_1, z_2]$**.**
One can parametrize the straight line segment joining z_1 to z_2 by

$$\gamma(t) = z_1 + t(z_2 - z_1), \quad 0 \leq t \leq 1.$$



(ii) One can parametrize the circle centered at z_0 and of radius R by

$$\gamma(t) = z_0 + Re^{it}, \quad 0 \leq t \leq 2\pi.$$



A curve $\gamma : [a,b] \to \mathbb{C}$ is said to be **smooth** if
(i) $\gamma'(t) = x'(t) + iy'(t)$ exists and is continuous on $[a,b]$; and
(ii) $\gamma'(t) \neq 0$ for all $t \in [a,b]$.

Definition. The integral of f along a smooth curve γ is defined by

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

Definition. A **contour** γ is a finite sequence $\{\gamma_1, \gamma_2, ..., \gamma_n\}$ of smooth curves such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for $1 \leq k \leq n-1$. In this case, we write

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

The **contour integral** of f along a contour γ is defined to be

$$\int_\gamma f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \int_{\gamma_2} f(z)\ dz + \cdots + \int_{\gamma_n} f(z)\ dz.$$

Some basic properties:

(i) $\displaystyle\int_\gamma [f(z) + g(z)]\ dz = \int_\gamma f(z)\ dz + \int_\gamma g(z)\ dz.$

(ii) $\displaystyle\int_\gamma z_0 f(z)\ dz = z_0 \int_\gamma f(z)\ dz.$

(iii) $\displaystyle\int_{-\gamma} f(z)\ dz = -\int_\gamma f(z)\ dz.$

(iv) the value of $\displaystyle\int_\gamma f(z)\,dz$ remains unchanged under reparametrizaton of γ.

(v) **(ML-inequality)** Suppose that f is continuous along a contour γ, and

$$|f(z)| \leq M \qquad \forall z \in \gamma.$$

Then

$$\left|\int_\gamma f(z)\ dz\right| \leq ML,$$

where $L =$length of γ.

Note. The length of a curve $\gamma = \gamma(t) : [a,b] \to \mathbb{C}$ may be calculated by the formula

$$\text{Length of } \gamma = \int_a^b |\gamma'(t)|\,dt.$$

(1.4.2) Anti-derivatives, Cauchy-Goursat Theorem

Let f be a continuous function on a domain D. A function F such that

$$F'(z) = f(z) \qquad \forall z \in D$$

is called an **antiderivative** of f in D.

Theorem. Let f be continuous on a domain D. The following are equivalent:

(a) f has an **antiderivative** in D;
(b) for any **closed** contour γ in D, $\int_\gamma f(z)\ dz = 0$;
(c) the contour integrals of f are **independent of paths** in D, that is, if $z_1, z_2 \in D$ and γ_1, γ_2 are contours in D joining z_1 to z_2, then $\int_{\gamma_1} f(z)\ dz = \int_{\gamma_2} f(z)\ dz$.

Theorem. (Cauchy-Goursat Theorem) If a function f is analytic at all points inside and on a simple closed contour γ, then $\int_\gamma f(z)\ dz = 0$.

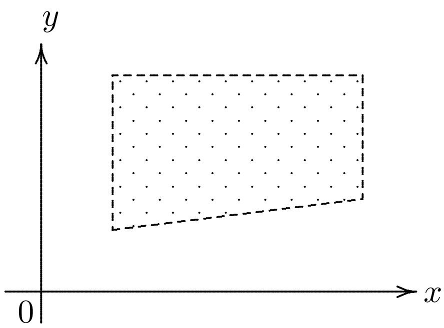
Definition. A domain D is **simply connected** if every simple closed contour in D encloses only points in D (i.e. D has no holes).

Remark. The interior of a simple closed contour is simply connected.

Theorem. (Cauchy-Goursat Theorem for simply-connected domains)
If f is analytic in a simply connected domain D, then

$$\int_\gamma f(z)\ dz = 0$$

for every closed contour γ in D.



Corollary. If f is analytic in a simply connected domain D, then it has an antiderivative in D.
Exercise: Find an analytic function in a non-simply connected domain D which does not have an anti-derivative in D.

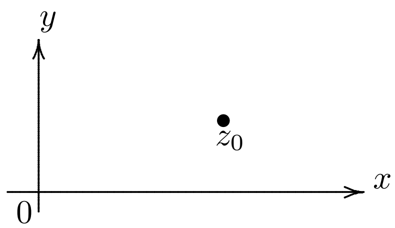
(1.4.3) Cauchy integral formula, Liouville's Theorem

A simple closed contour γ is **positively oriented** if the interior domain lies to the left of an observer tracing out the points in order.

Remark: A circle is positively oriented if it is traversed in the anti-clockwise direction.

Theorem. (Cauchy Integral Formula) Let γ be a positively oriented simple closed contour and let f be analytic everywhere within and on γ. Then for any z_0 inside γ,

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz.}$$



Theorem. (Cauchy integral formula for derivatives)
Let $f(z)$ be analytic everywhere inside and on a positively oriented simple closed contour C. Then for any z_0 inside C and any integer $n \geq 1$,

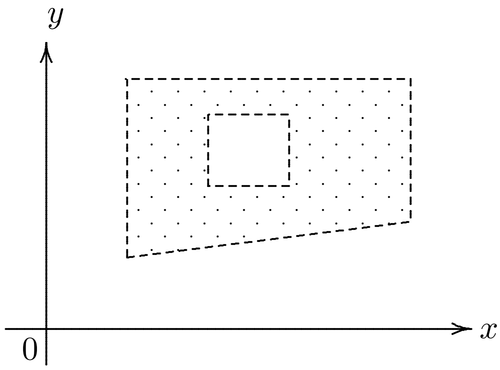
$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad (n = 1, 2, 3, ...).}$$

Corollary. If f is analytic in a domain D, then all its derivatives f', f'', f''',... exist and are analytic in D.

Corollary. (Morera's Theorem) If f is continuous on a domain D, and

$$\int_\gamma f(z)\, dz = 0$$

for every closed contour γ in D, then f is analytic in D.



Corollary. (Cauchy's inequality) If f is analytic within and on the circle γ_R centered at z_0 and of radius R, then for $n \in \mathbb{Z}^+$,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n}, \quad \text{where } M_R = \max_{z \in \gamma_R} |f(z)|.$$

Let $S \subset \mathbb{C}$. A function $f : S \to \mathbb{C}$ is **bounded** if there exists some $K > 0$ such that $|f(z)| \leq K \quad$ for all $z \in S$.

Theorem. (Liouville's Theorem) If an *entire* function f is *bounded*, then it must be a *constant* function.

[In short, entire + bounded $\implies$ constant.]

Theorem. (The Fundamental Theorem of Algebra)

Any polynomial $p(z)$ of degree ≥ 1 has a zero in $\mathbb{C}$ (i.e., $p(z) = 0$ has at least one solution.)

Idea of proof. Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/p(z)$ can be shown to be an entire and bounded function. Thus $1/p(z)$ and hence $p(z)$ are constant functions. This contradicts the assumption that $p(z)$ is of degree ≥ 1.

1.5. Power series, Taylor series, Laurent series

Definition. We say that a sequence of functions $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** to f on D if for any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{Z}^+$ such that

$$|f_n(z) - f(z)| < \epsilon \quad \text{for all } n > N \text{ and all } z \in D.$$

Note: Here N depends on ϵ but not on z.

Theorem. Let γ be a contour and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on γ. If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on γ, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

Theorem. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic functions on a domain D. If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on D, then f is also analytic on D. Moreover,

$$\lim_{n\to\infty} f_n'(z) = f'(z), \quad \text{i.e.,} \quad \lim_{n\to\infty} \frac{d}{dz} f_n(z) = \frac{d}{dz}\Big(\lim_{n\to\infty} f_n(z)\Big) \quad \forall z \in D.$$

Similar results hold for a series of functions.

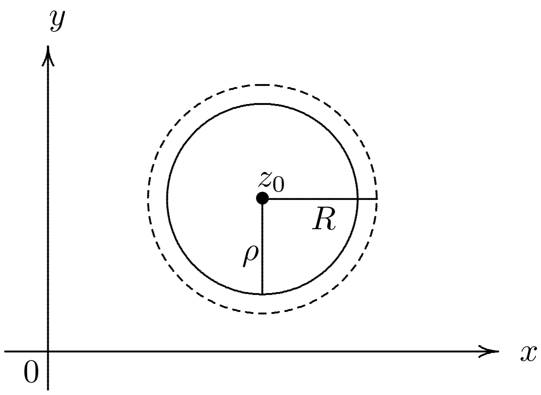
Theorem. Given any power series $\displaystyle\sum_{k=0}^{\infty} a_k (z-z_0)^k$, there is an associated number R, $0 \le R \le \infty$, called the radius of convergence, such that:

(i) $\displaystyle\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges absolutely at each point z satisfying $|z - z_0| < R$,

(ii) $\displaystyle\sum_{k=0}^{\infty} a_k (z-z_0)^k$ diverges at each z satisfying $|z-z_0| > R$, and

(iii) $\displaystyle\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges uniformly on the closed ball $\overline{B(z_0,\rho)}$ for any ρ satisfying $0 < \rho < R$. (Here, $\overline{B(z_0,\rho)} := \{z \in \mathbb{C} : |z - z_0| \le \rho\}$.)

Moreover, $R = \dfrac{1}{\limsup_{k\to\infty} |a_k|^{\frac{1}{k}}}$, and also $R = \dfrac{1}{\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}}$, if the limit exits.



Theorem. (Convergent power series are analytic functions) Let R be the radius of convergence of $\sum_{k=0}^{\infty} a_k(z-z_0)^k$. Then

(i) $S(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ is an analytic function on $B(z_0, R)$.

(ii) (**Term-by term integration**) If γ is a contour in $B(z_0, R)$ and $g(z)$ is continuous on γ, then

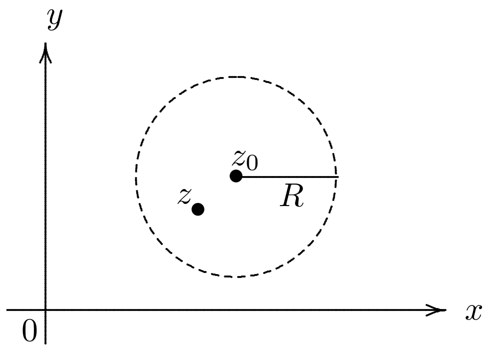
$$\int_\gamma g(z) \sum_{k=0}^{\infty} a_k(z-z_0)^k \, dz = \sum_{k=0}^{\infty} a_k \int_\gamma g(z)(z-z_0)^k \, dz.$$

(c) (**Term-by-term differentiation**)

$$\frac{d}{dz} \sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1} \quad \text{on } B(z_0, R).$$

Theorem. (Taylor's Theorem) Suppose $f(z)$ is analytic in $B(z_0, R)$. Then

$$\boxed{f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \qquad (|z-z_0| < R).}$$



This power series is called the **Taylor series** of $f(z)$ at z_0. The coefficients $\frac{f^{(k)}(z_0)}{k!}$ are called the **Taylor coefficents**.

Remark: (i) When $z_0 = 0$, the Taylor series at $z_0 = 0$ is also called the **Maclaurin series** of $f(z)$.

(ii) Roughly speaking, Taylor's Theorem says that **analytic functions are equal to their Taylor series**. Together with its preceding theorem, it follows that **power series** and **analytic functions** are more or less the same objects.

Theorem. (Uniqueness of Taylor series) Let $f(z)$ be an analytic function. If $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ for all $z \in B(z_0, R)$ for some $R > 0$, then $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is THE Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$ of $f(z)$ at z_0, i.e., $a_k = \frac{f^{(k)}(z_0)}{k!}$ for all $k = 0, 1, 2, \cdots$.

Consequences: We can use those standard power series to write $f(z)$ in the form $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, which will automatically be the Taylor series of $f(z)$ at z_0.

Some standard power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty).$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \qquad (|z| < \infty).$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \qquad (|z| < \infty).$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots \qquad (|z| < 1).$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + \cdots \qquad (|z| < 1).$$

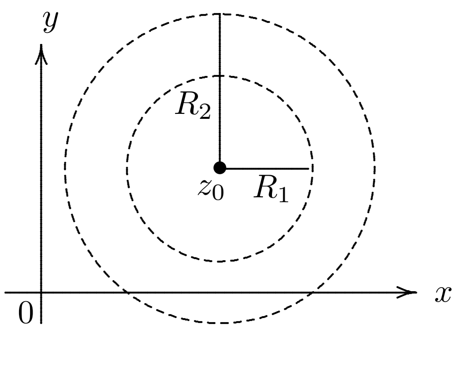
Theorem. (Laurent's Theorem) Suppose $f(z)$ is analytic in the annulus
$\mathcal{A} := \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$. Then

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad (R_1 < |z - z_0| < R_2),} \qquad (*)$$

where

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z_0)^{n+1}}\, ds \quad \text{and} \quad b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s-z_0)^{-n+1}}\, ds,$$

and γ is any positively oriented simple closed contour around z_0 and lying in $\mathcal{A}$.



Remark. The expression in (*) is called the **Laurent series** of f for the annulus $R_1 < |z - z_0| < R_2$. The coefficients $a_n,\ b_n$ are called the **Laurent coefficients**.

Theorem. (Uniqueness of Laurent series representation)
If an analytic function f in the annulus $R_1 < |z - z_0| < R_2$ satisfies

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{for } R_1 < |z - z_0| < R_2, \qquad (1)$$

then the expression in (1) is THE Laurent series of $f(z)$ for
$R_1 < |z - z_0| < R_2$.

Consequence: One can use the standard power series to find Laurent series of certain functions.

Example. Find the Laurent series of $\dfrac{5z + 14}{(z+2)(z+3)}$ for the annulus $2 < |z| < 3$.

Answer: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} z^n.$

1.6. Residues and poles

(1.6.1) Singular points and residues

Definition. (1) A point z_0 is said to be a **singular point** of a function f if

(i) f is not analytic at z_0; but

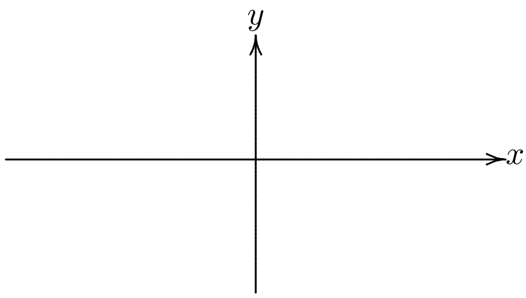
(ii) f is analytic at some point in $B(z_0, \epsilon)$ for all $\epsilon > 0$.

(2) A singular point z_0 of f is **isolated** if there exists $R > 0$ such that f is analytic in $B(z_0, R) \setminus \{z_0\}$.

Example.

(i) $f(z) = \dfrac{1}{\sin z}$.

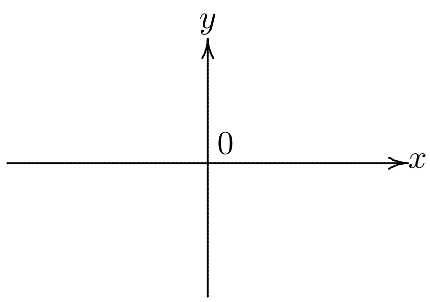
$f(z)$ has singular points at $\quad \sin z = 0 \Longleftrightarrow z = n\pi, \quad n \in \mathbb{Z}$.



(ii) $f(z) = \operatorname{Log} z$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$.

Thus, each point in $(-\infty, 0]$ is a singular point of $f(z)$.

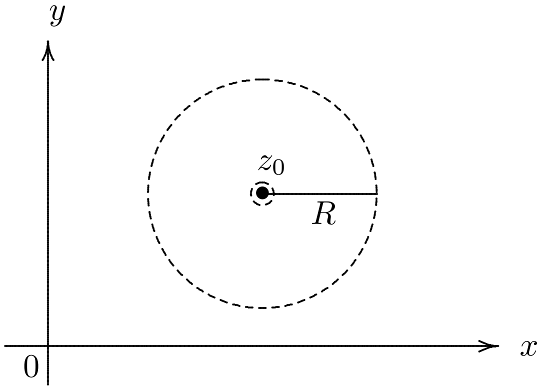
None of them is an isolated singular point of $f(z)$.



Suppose that $f(z)$ has an isolated singular point at z_0. Then there exists $R > 0$ such that f is analytic in the punctured ball

$$B(z_0, R) \setminus \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}, \qquad (*)$$

which may be regarded an annulus.



By Laurent's theorem, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (0 < |z - z_0| < R),$$

where

$$b_n = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{(s - z_0)^{-n+1}}\, ds, \quad \text{and thus} \quad b_1 = \frac{1}{2\pi i} \int_\gamma f(s)\, ds.$$

Definition. The **residue** of $f(z)$ at z_0 is given by:

$$\begin{aligned} \operatorname*{Res}_{z=z_0} f(z) &= b_1 \\ &= \text{coeff. of } \frac{1}{z - z_0} \text{ in the Laurent series of } f \text{ at } z_0. \end{aligned}$$

Theorem. (Cauchy's Residue Theorem) If γ is a positively oriented simple closed contour and $f(z)$ is analytic everywhere inside and on γ except for a finite number of isolated singular points z_k ($k = 1, 2, \cdots, n$) inside γ, then

$$\boxed{\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).} \tag{*}$$

(1.6.3) Classification of isolated singularities

Definition. Suppose $f(z)$ has an isolated singular point at z_0. Consider the Laurent series of $f(z)$ at z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \qquad (0 < |z - z_0| < R).$$

$\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ is called the **principal part** of the Laurent series of $f(z)$ at z_0.

(i) Removable singular point:
If $b_n = 0$ for all $n = 1, 2, \cdots$ (i.e. the principal part vanishes), we say that z_0 is a **removable singular point** of $f(z)$. In this case, the Laurent series of $f(z)$ reduces to a power series in $z - z_0$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (0 < |z - z_0| < R), \text{ and}$$
$$\operatorname*{Res}_{z=z_0} f(z) = 0.$$

(ii) Essential singular point:
If $b_n \neq 0$ for infinitely many n (i.e. the principal part has infinitely many non-zero terms), then we say that z_0 is an **essential singular point** of $f(z)$.

Note that in this case, some of the b_n's may still be zero.

(iii) Pole
If there exists $m \in \mathbb{Z}^+$ such that $b_m \neq 0$ but $b_n = 0$ for all $n > m$ (i.e. the principal part has finitely many non-zero terms) so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m},$$

then we say that z_0 is a **pole of order** m for $f(z)$.

We also say that z_0 is a **simple pole** of $f(z)$ if $m = 1$, and
we also say that z_0 is a **double pole** of $f(z)$ if $m = 2$.

Exercise: Give examples of functions with removable and essential singularities and poles.

Behavior of a function near a singular point

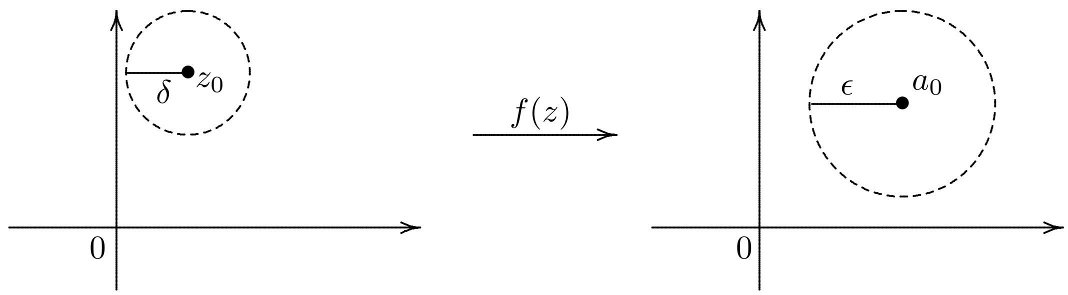
(a) Behavior of a function near a removable singular point

Suppose a function $f(z)$ has a removable singular point at a point z_0. By Laurent's theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad (0 < |z-z_0| < R) \qquad (*)$$
$$= a_0 + a_1(z-z_0) + \cdots$$

for some $R > 0$ (recalling that all $b_n = 0$). Note that the RHS of (*) is a convergent power series, and thus it is an analytic (and thus continuous) function on $|z-z_0| < R$. It follows that one has

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} [a_0 + a_1(z-z_0) + \cdots] = a_0 + a_1 \cdot 0 + \cdots = a_0.$$



Observe that if one extend $f(z)$ across the point z_0 by letting $f(z_0) = a_0$, then (*) will hold everywhere on $|z-z_0| < R$. The (extended) function $f(z)$ is equal to the analytic function $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ everywhere on $|z-z_0| < R$.

Example. The function $\dfrac{\sin z}{z}$ has an isolated singular point at $z = 0$. For $0 < |z| < \infty$,

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots. \qquad (*)$$

Thus $\dfrac{\sin z}{z}$ has a removable singular point at $z = 0$. Now we define

$$f(z) = \begin{cases} \dfrac{\sin z}{z} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases} \qquad (**)$$

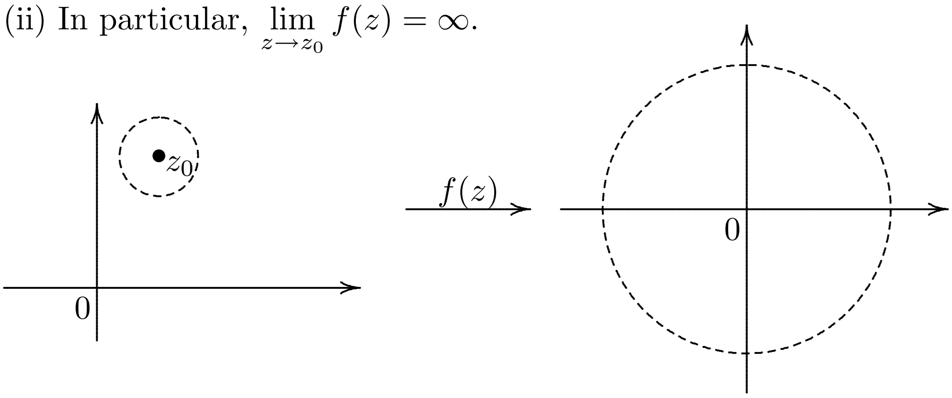
Then $f(z)$ is equal to the convergent power series $1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} + \cdots$ at all $z \in \mathbb{C}$. (By (*) and (**), both $f(z)$ and the power series are equal to $\frac{\sin z}{z}$ for $z \neq 0$; at $z = 0$, both are equal to 1.) The power series is necessarily analytic at $z = 0$. Hence f is also analytic at $z = 0$.

(b) Behavior of a function near a pole

Proposition. Suppose f has a pole of order m at z_0 $(m \geq 1)$. Then
(i) $\exists R > 0$ such that

$$\boxed{f(z) = \frac{\phi(z)}{(z-z_0)^m} \quad \text{for } 0 < |z-z_0| < R,} \tag{1}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.
(ii) In particular, $\lim_{z \to z_0} f(z) = \infty$.



Proof of Proposition. (i) Suppose that f has a pole of order m at z_0. Then there exists $R > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m}, \quad (0 < |z-z_0| < R),$$

where $b_m \neq 0$. Then for $0 < |z - z_0| < R$,

$$\begin{aligned}&(z-z_0)^m f(z)\\ &= \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + \cdots + b_{m-1}(z-z_0) + b_m.\end{aligned}$$

Consider the power series

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + \cdots + b_{m-1}(z-z_0) + b_m$$

for $z \in B(z_0, R)$. Then $\phi(z)$ converges everywhere on $B(z_0, R) \backslash \{z_0\}$ (being convergent to $(z-z_0)^m f(z)$). Clearly $\phi(z)$ converges at z_0 (Why?). Thus the radius of convergence of $\phi(z)$ is at least R. Therefore, the power series $\phi(z)$ is convergent on $B(z_0, R)$, and thus it is an analytic function on $B(z_0, R)$. So we have

$$(z-z_0)^m f(z) = \phi(z) \quad \text{with } \phi(z_0) = b_m \neq 0,$$

and $\phi(z)$ is analytic on $B(z_0, R)$. This proves (i).
To prove (ii), we let $\phi(z)$ be as in (i). Then

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z-z_0)^m}{\phi(z)} = \frac{(z_0 - z_0)^m}{\phi(z_0)} = 0.$$

Thus, we have $\lim_{z \to z_0} f(z) = \infty$.

(c) Behavior of a function near an essential singular point

It turns out that a function has complicated behaviors near an essential singular point, which was beyond the scope of MA3111. Let me just state (in a rough manner) without proof one result along this direction:

Theorem. (Casorati-Weierstrass Theorem)
Suppose $f(z)$ has an essential singular point at $z = z_0$ and let w_0 be any complex number. Then, for any positive number ε, the inequality

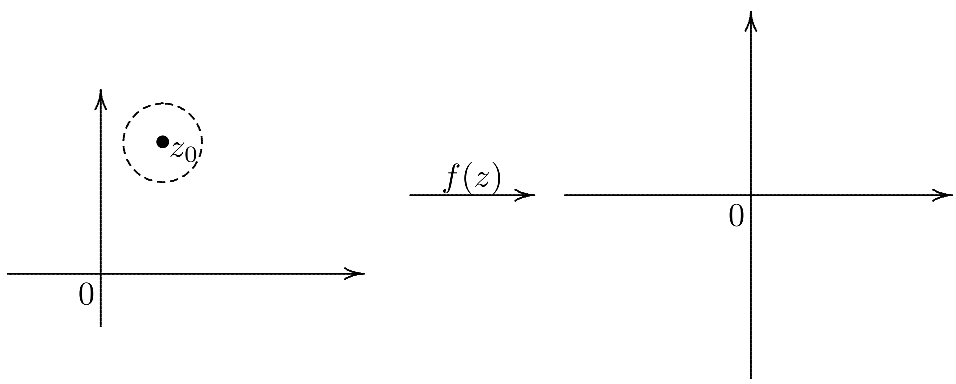
$$|f(z) - w_0| < \varepsilon$$

is satisfied at some point z in each deleted neighborhood $0 < |z - z_0| < \delta$ of z_0.

Roughly speaking, this says that for each (small) $r > 0$, the image $f(B(z_0, r))$ is "dense" in $\mathbb{C}$. Note that this cannot hold if z_0 is a removable singularity or pole (why?)

For more details, see [Churchill, p. 249(7th ed) or p. 202-203(6th ed)].

Exercise: Write up a proof for the Casorati-Weierstrass theorem.



(1.6.4) Methods for computing residues

Proposition. (Method I) Suppose $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{z - z_0} \quad \text{near } z_0$$

for some function $\phi(z)$ analytic at z_0. Then

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0).}$$

Remark. Such f has either a simple pole or a removable singular point at z_0.

Proposition. (Method II) Suppose $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \quad \text{near } z_0$$

for some function $\phi(z)$ analytic at z_0 and $m \geq 1$. Then

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.}$$

Remark. Such f has a pole of order $\leq m$ or a removable singular point at z_0.

Proposition. (Method III)
If $p(z)$ and $q(z)$ are analytic at z_0, and $q(z)$ has a simple zero at z_0, (i.e. $q(z_0) = 0$ and $q'(z_0) \neq 0$), then

$$\boxed{\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.}$$

Remark. Such $\frac{p(z)}{q(z)}$ has either a simple pole at z_0 (if $p(z_0) \neq 0$) or a removable singular point at z_0 (if $p(z_0) = 0$).

Method IV. (None of the above) When the above methods all fail, we can still directly compute the residue b_1, by first finding the Laurent series of $f(z)$ at z_0 using standard power series, and then reading off the appropriate Laurent coefficient b_1.

Summary of methods of finding residues

	$f(z)$	$\operatorname*{Res}_{z=z_0} f(z)$
$I.$	$\frac{\phi(z)}{z - z_0}$	$\phi(z_0)$
$II.$	$\frac{\phi(z)}{(z - z_0)^m}$	$\frac{\phi^{(m-1)}(z_0)}{(m-1)!}$
$III.$	$\frac{p(z)}{q(z)}$	$\frac{p(z_0)}{q'(z_0)}$
$IV.$	None of above.	Find Laurent series and read b_1.

1.7. Applications of Residues

Some of the typical applications of the Cauchy's residue theorem (and the methods of evaluating residues) are in the evaluation of certain improper integrals of certain functions from $(-\infty, \infty)$ and in the evaluation of certain trigonometric integrals over $[0, 2\pi]$. See [Churchill, Chapter 7] for details. We will see later that it also plays an important role in the proof of the argument principle and Rouché's theorem.