NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Homework 3 (due 26 Oct)

1. Suppose f is an analytic function on the unit ball $B(0,1)$ such that $f(0) = \dfrac{1}{2}$ and $|f(z)| < 1$ for all $|z| < 1$.
(a) Show that

$$\left| \frac{f(z) - \frac{1}{2}}{1 - \frac{f(z)}{2}} \right| \leq |z| \quad \text{for all } z \in B(0,1).$$

(b) Deduce that $f(z) \neq 0$ for all z satisfying $|z| < \dfrac{1}{2}$.
[For part (a), you may use freely the following fact: For any α with $|\alpha| < 1$, the function $\phi_\alpha(z) = \dfrac{z - \alpha}{1 - \overline{\alpha} z}$ is an analytic automorphism of $B(0,1)$ such that $\phi_\alpha(\alpha) = 0$.]

2. (a) Let z_1, z_2, z_3, z_4 be four distinct complex numbers. Suppose $(z_1, z_2; z_3, z_4) = 1 + i$. Find the value of $(\bar{z}_1 + i,\ \bar{z}_2 + i;\ \bar{z}_3 + i,\ \bar{z}_4 + i)$. Justify your answer. (Note that $\bar{z}$ is just the complex conjugate of z).

(b) Consider the domain $D = \{z = x + iy \in \mathbb{C} : y < x^2 \text{ and } z \neq -i\}$. Does there exist a non-constant entire function F such that $F(\mathbb{C}) \subset D$? Justify your answer.

(c) Suppose T is a Möbius transformation such that $T(0) = \infty$ and $T(\infty) = 0$. Is it true that T is of the form $T(z) = \dfrac{a}{z}$ for some non-zero complex constant a? Justify your answer.