

NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II

Homework 2 (due 21 Sep)

1. Fix any complex constant  $\alpha$  such that  $|\alpha| < 1$ . Consider the function

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha}z}.$$

- (i) Show that  $\phi_\alpha$  maps the unit circle  $C: |z| = 1$  to itself.  
(ii) Show also that  $\phi_\alpha$  is an analytic function from the open ball  $B(0, 1)$  into itself.

[Hint: Use (i).]

2. Let  $f$  be a non-constant entire function such that  $|f(z)| = 1$  for all complex numbers  $z$  satisfying  $|z| = 1$ .  
(i) Show that  $f(z) \in B(0, 1)$  for all  $z \in B(0, 1)$ .  
(ii) Show that for any complex number  $\alpha \in B(0, 1)$ , there exists a complex number  $z \in B(0, 1)$  such that  $f(z) = \alpha$ . (Hint: Use part (i), Question 1 above and Question 8 of Tutorial 2.)
3. Find the number of zeros of the polynomial  $p(z) = z^4 + 3z^3 + 2z + 3$ , counting multiplicity, in the annulus  $2 < |z| < 4$ . Justify your answer.
4. Find the number of roots of the equation  $e^{-z} + z^2 - 9 = 0$  in the right half plane  $\operatorname{Re} z > 0$ . Justify your answer carefully.