NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Extra problems

These questions are optional, you may work in your groups, or with other classmates to come up with the solutions and submit them to me or the forum page of the IVLE.

1. Suppose that f is a non-constant entire function such that $|f(z)| = 1$ on $|z| = 1$. Prove that $f(z) = Cz^n$ for some constant C with $|C| = 1$. (Hint: Use the maximum and minimum modulus theorems to show that

$$f(z) = C \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z}$$

for some complex constants α_j.

2. Let $f : B(0,1) \longrightarrow B(0,1)$ be analytic. Prove that if there exist two distinct points α, β in the unit disk which are fixed points (i.e. $f(\alpha) = \alpha$, $f(\beta) = \beta$), then $f(z) = z$ for all z in $B(0,1)$. (Hint: If one of α or β is 0, use Schwarz's lemma, if $\alpha, \beta \neq 0$, find two fixed points of $\phi_\alpha \circ f \circ \phi_{-\alpha}$), where

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}, \qquad |\alpha| < 1.$$

3. Show that a function f which is analytic on and inside a simple closed contour C and which only takes real values on C reduces to a constant function (i.e. is identically constant).