NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 9

Selected answers and solutions

1. Find a formula for all analytic isomorphisms of
   (i) the first quadrant of $\mathbb{C}$ to itself;
   (ii) the right half plane to itself;
   (iii) the open ball $|z| < 2$ onto the unit ball $|z| < 1$.
   Remark: The expressions are not unique.

   [Hint: You may use the following facts proved in Tutorial 8 and the lecture notes:
   (a) The set of analytic automorphisms of the unit ball $|z| < 1$ consists of mappings of the form

   $$f(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z},$$

   where $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, and $\theta \in \mathbb{R}$.
   (b) The set of analytic automorphisms of the UHP Im $z > 0$ consists of mappings of the form

   $$f(z) = \frac{az+b}{cz+d}, \quad \text{with } a,b,c,d \in \mathbb{R},\ ad-bc > 0.$$

   Solution: (i) Consider the square map $g(z) = z^2$. It is an analytic isomorphism from the first quadrant of $\mathbb{C}$ to the UHP. Similarly its inverse mapping $g^{-1}(z) = z^{\frac{1}{2}}$ (principal branch) is an analytic isomorphism from the UHP to the first quadrant of $\mathbb{C}$. Then f is an analytic isomorphism from the first quadrant of $\mathbb{C}$ to itself if and only if $\phi = g \circ f \circ g^{-1}$ is an analytic isomorphism of the UHP, i.e., f is of the form $g^{-1} \circ \phi \circ g$. Thus the set of such analytic isomorphisms consists of mappings of the form

   $$f(z) = \Big(\frac{az^2+b}{cz^2+d}\Big)^{\frac{1}{2}}, \quad \text{with } a,b,c,d \in \mathbb{R},\ ad-bc > 0.$$

   (ii) The map $g(z) = e^{i\pi/2}z = iz$ is an analytic isomorphism from RHP to UHP. Its inverse map $g^{-1}(z) = -iz$ is an analytic isomorphism from UHP to RHP. Then f is an analytic automorphism of the RHP iff $\phi = g \circ f \circ g^{-1}$

is an analytic automorphism of the UHP, i.e., f is of the form $g^{-1}\circ\phi\circ g$. Thus the set of such analytic automorphism consists of mappings of the form

$$f(z) = -i\cdot\frac{aiz+b}{ciz+d}, \quad \text{with } a,b,c,d\in\mathbb{R},\ ad-bc>0.$$

(iii) The map $g(z)=\frac{z}{2}$ is an analytic isomorphism from the open ball $|z|<2$ to the unit ball $|z|<1$. Its inverse $g^{-1}(z)=2z$ is an analytic isomorphism from the unit ball $|z|<1$ to the open ball $|z|<2$. Then f is an analytic isomorphism from the open ball $|z|<2$ onto the unit ball $|z|<2$ iff $\phi=f\circ g^{-1}$ is an analytic automorphism of the unit ball $|z|<1$, i.e., f is of the form $\phi\circ g$. Therefore, the set of such analytic isomorphisms are mappings of the form

$$f(z)=e^{i\theta}\frac{\frac{z}{2}-\alpha}{1-\bar{\alpha}\frac{z}{2}},$$

where $\alpha\in\mathbb{C}$ with $|\alpha|<1$, and $\theta\in\mathbb{R}$.

Remark: The expressions are not unique. So you may get an answer which looks different.

2. (a) Let C denote the circle passing through the three points $1, i, 1+i$. Find the point z if z and $1-i$ are symmetric with respect to C.

   (b)Find a conformal isomorphism mapping the upper half plane onto $B(0,1)$ and sending i to 0 and ∞ to -1.
   [Hint: Use the Symmetry Principle.]

   Answer: (a) To find z, we solve the equation

$$\begin{aligned}
&(1,i;1+i,z)=\overline{(1,i;1+i,1-i)}\\
\Longrightarrow\ &\frac{(z-i)(1+i-1)}{(z-1)(1+i-i)}=\overline{\frac{(1-i-i)(1+i-1)}{(1-i-1)(1+i-i)}}\\
\Longrightarrow\ &\frac{(z-i)i}{(z-1)}=-1-2i\\
\Longrightarrow\ &z=\frac{3+i}{5}.
\end{aligned}$$

   (b) By symmetry the isomorphism maps $-i$ to ∞. We want a map taking $i,\infty,-i$ to $0,-1,\infty$ respectively. Thus the map is

$$\begin{aligned}
(0,-1;\infty,w)=(i,\infty;-i,w) &\Longrightarrow \frac{w+1}{w}=\frac{-i-i}{z-i}\\
&\Longrightarrow w=\frac{-z+i}{z+i}
\end{aligned}$$

3. (a) Suppose that C_1 and C_2 are two distinct concentric circles with centre a. Show that the only pair of points z and z^* in $\hat{\mathbb{C}}$ which are symmetric with respect to both C_1 and C_2 are a and ∞ (you may use the geometric interpretation of symmetry).

(b) Find two points z_1 and z_2 which are symmetric with respect to both the imaginary axis as well as the circle $|z+\frac{5}{2}|=2$. Hence or otherwise, find a linear fractional transformation which maps the imaginary axis and the circle $|z+\frac{5}{2}|=2$ to concentric circles centred at the origin.
[Hint: For part (b), you may need to use the result in part (a).]

Solution: (a) Clearly, a and ∞ are symmetric with respect to any circle centered at a, since by the invariance of cross ratio under LFT, for any 3 successive points z_1, z_2, z_3 on the circle of radius R, we have

$$\begin{aligned}
\overline{(z_1, z_2; z_3; a)} &= \overline{(z_1 - a, z_2 - a; z_3 - a; 0)} \\
&= (\overline{z_1 - a}, \overline{z_2 - a}; \overline{z_3 - a}, 0) \\
&= (\frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}; \frac{R^2}{z_3 - a}, 0) \\
&= (z_1 - a, z_2 - a; z_3 - a, \infty) \\
&= (z_1, z_2; z_3, \infty)
\end{aligned}$$

The above calculation also implies that if one point of the symmetric pair is ∞, then the other point is a. Now suppose z and z^* are two finite points which are symmetric with respect to the concentric circles C_1 and C_2 centered at z and with radius R_1 and R_2 respectively. Then by the geometric interpretation of the symmetry, one has

$$|z-a||z-z^*| = R_1^2 \quad \text{and} \quad |z-a||z-z^*| = R_2^2.$$

Hence $R_1 = R_2$, which implies $C_1 = C_2$, contradicting the assumption that the two circles are distinct. Hence the only pair of points symmetric wrt both C_1 and C_2 are a and ∞.

(b) Suppose z_1 and z_2 are symmetric with respect to both the imaginary axis and the circle $|z+5/2| = 2$. From the geometric interpretation of symmetry, one knows that the imaginary axis is the perpendicular bisector to the line segment joining two points, so the two points lie on a horizontal line. Also, the two points lie on a ray emanating from the center $-5/2$ of the circle. As all non-horizontal rays intersect a horizontal line at only one point, the two points must lie on a horizontal ray emanating from $-5/2$. Hence, the two points are two real numbers of the form x and $-x$ respectively with both x and $-x$ bigger than $-5/2$. Also, from the geometric interpretation of symmetry, we have

$$\begin{aligned} |x-(-\frac{5}{2})||-x-(-\frac{5}{2})| &= 2^2 \\ \implies (x-(-\frac{5}{2}))(-x-(-\frac{5}{2})) &= 4 \\ \implies -x^2+\frac{25}{4} &= 4 \\ \implies x &= \pm 3/2. \end{aligned}$$

Thus, the two points are $\frac{3}{2}$ and $-\frac{3}{2}$.

Next we want to find an LFT T mapping the imaginary axis L and the circle $C: |z+5/2| = 2$ to concentric circles C_1 and C_2 centred at the origin. Without loss of generality, we may assume that $T(L) = C_1$ and $T(C) = C_2$. Since $\frac{3}{2}$ and $-\frac{3}{2}$ are symmetric with respect to L and C, it follows from the Symmetry Principle that $T(\frac{3}{2})$ and $T(-\frac{3}{2})$ must be symmetric with respect to both concentric circles C_1 and C_2. By (i), it follows that $T(\frac{3}{2})$ and $T(-\frac{3}{2})$ must be the pair of points 0 (the common center of C_1 and C_2) and ∞. Recall that an LFT is determined completely by three distinct points z_1, z_2, z_3 and their images $T(z_1), T(z_2), T(z_3)$. We claim that any LFT T such that T sends $\frac{3}{2}$, $-\frac{3}{2}$ to $0, \infty$ (or ∞, 0) respectively will meet the requirements.

To see this, we let T be any such LFT, say $T(\frac{3}{2}) = 0$ and $T(-\frac{3}{2}) = \infty$. Then as an LFT, T maps L and C to circles or extended straight lines. However, since either $T(\frac{3}{2})$ and $T(-\frac{3}{2})$ is ∞, and the two points $\frac{3}{2}$, $-\frac{3}{2}$ are not on L or C, it follows that $T(L)$ and $T(C)$ cannot contain ∞ (since an LFT is bijective on $\hat{\mathbb{C}}$). Thus, $T(L)$ and $T(C)$ cannot be extended straight lines, and thus $T(L)$ and $T(C)$ are circles, say C_1 and C_2 respectively. Since $\frac{3}{2}$ and $-\frac{3}{2}$ are symmetric with respect to L and C, it follows from the Symmetry Principle that $T(\frac{3}{2}) = 0$ and $T(-\frac{3}{2}) = \infty$ are symmetric with respect to both C_1 and C_2. Then by (i), 0 must then be the centre of both C_1 and C_2, i.e., C_1 and C_2 are concentric circles with center at the origin 0.

Thus we may choose any third point $a \neq \pm\frac{3}{2}$, and choose any point $b \neq 0, \infty$. Then the LFT $w = T(z)$ mapping $\frac{3}{2}, -\frac{3}{2}, a$ to $0, \infty, b$ respectively will meet the requirement, i.e., $w = T(z)$ can be given by $(\frac{3}{2}, -\frac{3}{2}; a, z) = (0, \infty; b, w)$.

For example, we may take $a = 0$, $b = 1$, so that one such T can be given by $(\frac{3}{2}, -\frac{3}{2}; 0, z) = (0, \infty; 1, w)$.

Remark. The LFT is not unique.

4. Find a conformal isomorphism mapping the infinite vertical strip $0 < \text{Re } z < 2$ to the unit ball $|z| < 1$.

   Solution: Recall the exponential map maps an infinite horizontal strip to the unit ball. So we first use the rotation $z_1 = f_1(z) = e^{i\pi/e} = iz$ to map the vertical strip $0 < \text{Re } z < 2$ to the horizontal strip $0 < \text{Im } z_1 < 2$. Next we use the dilation map $z_2 = f_2(z_1) = \dfrac{\pi}{2}z_1$ to map the horizontal strip $0 < \text{Im } z_1 < 2$ to another horizontal strip $0 < \text{Im } z_2 < \pi$. Then the exponential map $z_3 = f_3(z_2) = e^{z_2}$ will map the strip $0 < \text{Im } z_2 < \pi$ to the upper half plane $\text{Im } z_2 > 0$. Finally the LFT $w = f_4(z_3) = \dfrac{z_3 - i}{z_3 + i}$ maps the upper half plane to the unit ball $|w| < 1$. Thus an required conformal isomorphism is

$$w = f_4 \circ f_3 \circ f_2 \circ f_1(z) = \frac{e^{\frac{\pi}{2}iz} - i}{e^{\frac{\pi}{2}iz} + i}.$$

5. Find an analytic isomorphism from the region $0 < \arg z < \dfrac{\pi}{3}$ to the open ball $|z - 1| < 2$.

   Solution. First open the region $0 < \arg z < \dfrac{\pi}{3}$ to a half plane, using

$$z_1 = f_1(z) = z^3$$

   which maps the first quadrant isomorphically to the UHP. Next, map the UHP to the unit ball $|z_2| < 1$ with the LFT

$$z_2 = f_2(z_1) = \frac{z_1 - i}{z_1 + i}.$$

   Then we use the dilation $z_3 = f_3(z_2) = 2z_2$ to map the unit ball $|z_2| < 1$ to another open ball $|z_3| < 2$. Finally we use the translation $w = f_4(z_3) = z_3 + 1$ to map the open ball $|z_3| < 2$ to the open ball $|w - 1| < 2$. So an required transformation can be chosen to be

$$w = f_4 \circ f_3 \circ f_2 \circ f_1(z) = 2\frac{z^3 - i}{z^3 + i} + 1.$$

   Remark: The transformation is not unique. So you may get a different answer.