NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 8

Selected answers and solutions

1. Using the cross-ratio notation, write an equation defining a Möbius transformation that maps the half plane below the line $y = 2x - 3$ onto the interior of the circle $|w - 4| = 2$. Repeat for the exterior of the circle.
   [Hint: Recall the Orientation Principle.]
   Remark: The transformation is not unique.

   Solution: (i) Consider the extended line $L : y = 2x-3$. The points $-3i, -1-5i, \infty$ lie on L, and they determine the orientation of L so that the region below L is on the left of L. For the circle $C : |w - 4| = 2$. The points $6, 4 + 2i, 2$ lie on C, and they determine the positive orientation of C, so that the interior of C is on the left of C. By the Orientation Principle, a required LFT is given by

   $$(6, 4 + 2i; 2, w) = (-3i, -1 - 5i; \infty, z).$$

   (ii) The points $6, 4 - 2i, 2$ lie on C, and they determine the negative orientation of C, so that the exterior of C is on the left of C. By the Orientation Principle, a required LFT is given by

   $$(6, 4 - 2i; 2, w) = (-3i, -1 - 5i; \infty, z).$$

   Remark: The transformation is not unique. So you may get a different answer.

2. Find a Möbius transformation that maps
   (i) the region $\{z : |z - 1| > 1\}$ onto the open ball $|z - 1| < 1$.
   (ii) the region $\{z : |z - 1| > |z - i|\}$ onto the open ball $|z - 1| < 1$.

   Solution: (i) The points $2, 1 + i, 0$ determines the positive orientation of the circle $C : |w - 1| = 1$, and the interior $|w - 1| < 1$ is the domain of $\mathbb{C} \setminus C$ on the left side of C. Similarly, the points $2, 1 - i, 0$ lie on the circle $C : |z - 1| = 1$ and determines the negative orientation of C. The region $\{z : |z-1| > 1\}$ is one of the domain of $\mathbb{C}\setminus C$ on the left side of C. Therefore, by the orientation, an required LFT is given by the one mapping $2, 1-i, 0$ to $2, 1+i, 0$ respectively, i.e.,

$$(2, 1+i; 0, w) = (2, 1-i; 0, z).$$

(ii) Consider the line $L : \{z : |z-1| = |z-i|\}$, i.e, the line $y = x$, where $z = x + iy$. The points $\infty, 0, 1+i$ on the extended line L determines the orientation of L such that the region $\{z : |z-1| > |z-i|\}$ is the domain of $\mathbb{C} \setminus L$ on the left side of L. Thus by the Orientation Principle, an required LFT can be given by the one mapping the points $\infty, 0, 1+i$ to $2, 1+i, 0$ respectively, i.e.,

$$(2, 1+i; 0, w) = (\infty, 0; 1+i, z).$$

Remark: The transformation is not unique. So you may get a different answer.

3. Using the Orientation Principle, show that the map $f(z) = \dfrac{1+z}{1-z}$ is an analytic isomorphism of the upper half unit ball $\{z \in \mathbb{C} : |z| < 1, \ \mathrm{Im}\ z > 0\}$ onto the first quadrant of $\mathbb{C}$.
[Hint: What are the images of $-1, 0, 1$ under f? What are the images of $-1, 1, i$ under f? Explain carefully why f is both injective and surjective with respect to the given domains.]

Answer: Note that f is an LFT. Moreover, by the Orientation Principle, since f maps $-1, 0, 1$ to $0, 1, \infty$ respectively, it follows that f maps the region to the left of the real axis to the left of the real axis (with the orientation determined by these triples of points), i.e., f maps the upper half plane analytically isomorphically to the upper half plane. Similarly, since f maps the points $-1, 1, i$ to $0, \infty, i$ respectively, by using the orientation principle, one can conclude that f is an analytic isomorphism from the unit ball $B(0, 1)$ to the right half plane. Note that the intersection of the upper half plane with the unit ball is the upper half unit ball. The intersection of the upper half plane with the right half plane is the first quadrant. Therefore, f is an analytic function maping the upper half unit ball to the first quadrant (this follows from the general fact that $f(D_1 \cap D_2) \subset f(D_1) \cap f(D_2)$ with D_1 being the upper half plane and D_2 being the unit ball). Since the LFT f is one-to-one on $\hat{\mathbb{C}}$, it follows that f is also one-to-one on the upper half unit ball. Finally we need to check that the map from the upper half unit ball to the first quadrant is surjective. Take any point w in the first quadrant, we know that there exists a unique $z \in \hat{\mathbb{C}}$ such that $f(z) = w$, since f is an LFT and is bijective automorphism of $\hat{\mathbb{C}}$. Since f is an analytic isomorphism from the upper half plane to the upper half plane, we know that z must lie on the upper half plane. Similarly, we know that z must lie on the unit ball. Thus, z must lie on the upper half unit ball. Therefore f maps the upper half unit ball surjectively onto the first quadrant. Therefore, f is an analytic isomorphism from the upper half unit ball onto the first quadrant of $\mathbb{C}$.

4. Let $w = f(z)$ be the Möbius transformation that maps the points $0, \lambda, \infty$ to $-i, 1, i$ respectively, where λ is real. For what values of λ is the upper half plane mapped onto the unit ball $B(0, 1)$? Justify your answer.

   Solution: For real λ, 0, λ, ∞ determines an orientation of the real axis denoted by L, and the UHP is one of the two domains of $\mathbb{C} \setminus L$. Note that $-i, 1, i$ determines the positively oriented unit circle $C : |w| = 1$, and the unit ball $|w| < 1$ is the domain of $\mathbb{C} \setminus C$ on the left side of C. By the Orientatin Principle, f maps the UHP onto the unit ball $B(0, 1)$ iff the UHP lies to the left of L iff the point i lies to the left of L iff

$$\text{Im } (0, \lambda; \infty, i) > 0 \Longleftrightarrow \text{Im } \left(\frac{i - \lambda}{i - 0} \right) = \lambda > 0.$$

   The required range of λ is: $\lambda > 0$.

5. Prove that the set of analytic automorphisms of the unit ball $B(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ consists of mappings of the form

$$f(z) = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z} \qquad (*)$$

   where α and θ are constants such that $|\alpha| < 1$ and $\theta \in \mathbb{R}$.
   Hint: First show that if f maps 0 to 0, then $f(z) = e^{i\theta} z$. For the general case, compose f by a suitable ϕ_a in Tutorial 5 Question 3. (You may use the results in Tutorial 5 Question 3 that each

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z},$$

   with $|\alpha| < 1$, is an automorphism of $B(0, 1)$ such that $\phi(\alpha) = 0$. Also, recall that the inverse of ϕ_α is $\phi_{-\alpha}$.

   Answer: First we show that any mapping f of the form in (*) is an automorphism of $B(0, 1)$. To see this, we just observe that we may write

$f = R_\theta \circ \phi_\alpha$, where $R_\theta(z) = e^{i\theta} z$ denotes the rotation through the angle θ about the origin. Clearly, R_θ is an analytic automorphism of $B(0,1)$ (do the exercise, if you have any doubt). We have also seen in Tutorial 5 that each ϕ_α is an analytic automorphism of $B(0,1)$. Since composition of analytic automorphisms remains an analytic automorphism, it follows that f is an analytic automorphism.

Next we show that each analytic automorphism of $B(0,1)$ can be written in the form of (*) for some suitable θ and α.

To show that any analytic automorphism of $B(0,1)$ is of the form in (*), we first consider the special case that $f(0) = 0$. Then by Schwarz's lemma, $|f(z)| \leq |z|$ for $|z| < 1$. Similarly, since f^{-1} is an analytic automorphism of $B(0,1)$, $|f^{-1}(w)| \leq |w| \implies |z| \leq |f(z)|$ for $|w| < 1$ where $f(z) = w$, or equivalently, $f^{-1}(w) = z$. Hence $|f(z)| = |z|$ for all $z \in B(0,1)$, and by the second part of Schwarz's lemma, $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. (Remark: An alternative is to apply Schwarz's lemma to $f'(0)$ and $(f^{-1})'(0)$)

Now we consider the general case, and let f be any analytic automorphism of $B(0,1)$. Since f is surjective, there exists an $\alpha \in B(0,1)$ such that $f(\alpha) = 0$. Note that ϕ_α is also an analytic automorphism of $B(0,1)$ such that $\phi_\alpha(\alpha) = 0$. To compare the two maps, we observe that $f \circ \phi_\alpha^{-1}$ is an analytic automorphism of $B(0,1)$, with $f \circ \phi_\alpha^{-1}(0) = f(\alpha) = 0$. Thus, from the special case we consider earlier, we must have $f \circ \phi_\alpha^{-1}(z) = e^{i\theta} z$ for some real constant θ and all $z \in B(0,1)$. Now since z is arbitrary, and ϕ_α is an automorphism, we may replace z by $\phi_\alpha(z)$ and get $f \circ \phi_\alpha^{-1}(\phi_\alpha(z)) = e^{i\theta}\phi_\alpha(z)$, i.e.,

$$f(z) = e^{i\theta}\phi_\alpha(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}.$$