NATIONAL UNIVERSITY OF SINGAPORE ■

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 7

Selected answers and solutions

1. Consider the linear fractional transformations

$$f(z) = \frac{2z+1}{3z+2} \quad \text{and} \quad g(z) = \frac{iz+2}{z+3}.$$

Find $f \circ g$, $g \circ f$ and also g^{-1} in the form of an LFT.

Answer: The matrix corresponding to f and g are

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} i & 2 \\ 1 & 3 \end{pmatrix}$$

respectively. Thus the matrix corresponding to $f \circ g$ is

$$AB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} i & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2i+1 & 7 \\ 3i+2 & 12 \end{pmatrix}.$$

Thus,

$$f \circ g(z) = \frac{(1+2i)z+7}{(2+3i)z+12}.$$

The matrix corresponding to $g \circ f$ is

$$BA = \begin{pmatrix} i & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 6+2i & 4+i \\ 11 & 7 \end{pmatrix}.$$

Thus,

$$g \circ f(z) = \frac{(6+2i)z+4+i}{11z+7}.$$

To find g^{-1}, we consider

$$\begin{aligned} w = g(z) = \frac{iz+2}{z+3} &\implies w(z+3) = iz+2 \\ &\implies wz + 3w = iz + 2 \\ &\implies wz - iz = -3w + 2 \\ &\implies z(w-i) = -3w+2 \\ &\implies z = \frac{-3w+2}{w-i} \\ &\implies g^{-1}(w) = \frac{-3w+2}{w-i} \\ &\implies g^{-1}(z) = \frac{-3z+2}{z-i} \end{aligned}$$

upon renaming the variable w as z. Alternatively, we may also use the matrix representation to obtain the inverse transformation.

2. Write the LFT $f(z) = \dfrac{3z - 4 - i}{iz - 1}$ as a composition of basic transformations (i.e., the inversion, rotations, dilations, and translations).

Solutions:

$$\begin{aligned} w = \frac{3}{i} + \frac{3z - 4 - i - \frac{3}{i}(iz - 1)}{iz - 1} &= -3i + \frac{3z - 4 - i - 3z - 3i}{iz - 1} \\ &= -3i + \frac{-4 - 4i}{iz - 1} \\ &= -3i + \frac{-4 - 4i}{i} \frac{1}{z + i} \\ &= -3i + \frac{-4 + 4i}{z + i} \\ &= -3i + \frac{4\sqrt{2}e^{i3\pi/4}}{z + i}. \end{aligned}$$

Thus we may decompose the LFT as

$$z \to z + i \to \frac{1}{z + i} \to \frac{e^{i3\pi/4}}{z + i} \to \frac{4\sqrt{2}e^{i3\pi/4}}{z + i} \to \frac{4\sqrt{2}e^{i3\pi/4}}{z + i} + 3i.$$

Letting $z_1 = f_1(z) = z + i$ (translation by i),
$z_2 = f_2(z_1) = \dfrac{1}{z_1}$ (inversion),
$z_3 = f_z(z_2) = \dfrac{e^{i3\pi/4}}{z_2}$ (rotation by $\frac{3\pi}{4}$),
$z_4 = f_z(z_3) = 4\sqrt{2}z_3$ (dilation by $4\sqrt{2}$),
$w = f_5(z_4) = z_3 + 3i$ (translation by $3i$).
Then

$$f(z) = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1(z)$$

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3. Find a linear map (i.e. a map of the form $f(z) = az + b$) that maps the circle $|z| = 1$ onto the circle $|w - 5| = 3$ and taking the point $z = i$ to $w = 2$. [Hint: What are the basic transformations needed?]

   Answer: Rotate by $\pi/2$ followed by dilation by 3 followed by translation by 5, this gives $w = 3iz + 5$.

4. Write the linear fractional transformation which sends the points
   (i) $2, 3i, 4$ to $\infty, 0, 1$ respectively;
   (ii) $0, i, \infty$ to $\infty, 0, 1$ respectively.

   Answer: (i) $Tz = \dfrac{(z - 3i)(4 - 2)}{(z - 2)(4 - 3i)}$.
   (ii) $Tz = \dfrac{z - i}{z - 0} = \dfrac{z - i}{z}$.

5. Find the Möbius transformation which sends the points

   (i) $-2, 2, i$ to $-1, 1, i$ respectively;

   (ii) $i, -i, 0$ to $1, -1, i$ respectively;

   (iii) $\infty, i, 0$ to $0, i, \infty$ respectively.

   (iv) $0, 1, \infty$ to $-1, -i, 1$ respectively.

   Express your answers in (ii), (iii) and (iv) in the form $w = f(z) = \dfrac{az+b}{cz+d}$.

   Solution:

   (i) $(-1, 1; i, w) = (-2, 2; i, z) \iff \dfrac{(w-1)(i+1)}{(w+1)(i-1)} = \dfrac{(z-2)(i+2)}{(z+2)(i-2)}$.

   Upon simplifying, one gets

$$w = \frac{3z+2i}{iz+6}.$$

   Note that the above expression is only unique up to a multiple of the coefficients.

   (ii) $(1, -1; i, w) = (i, -i; 0, z) \iff \dfrac{(w+1)(i-1)}{(w-1)(i+1)} = \dfrac{(z+i)(0-i)}{(z-i)(0+i)}$. Upon simplifying, one gets

$$w = \frac{-iz+i}{z+1}.$$

   (iii) $(0, i; \infty, w) = (\infty, i; 0, z) \iff \dfrac{(w-i)}{(w-0)} = \dfrac{(z-i)}{(0-i)}$. On simplifying, we get

$$w = -\frac{1}{z}.$$

   (iv) $(-1, -i; 1, w) = (0, 1; \infty, z) \iff \dfrac{(w+i)(1+1)}{(w+1)(1+i)} = \dfrac{(z-1)}{(z-0)}$. On simplifying, we get

$$w = \frac{z-i}{z+i}.$$

6. Find the fixed points in $\hat{\mathbb{C}}$ of the mappings:

   (a) $w = \dfrac{z-1}{z+1}$, (b) $w = \dfrac{z}{z+1}$, (c) $w = z+1$.

   [Recall that fixed points of f are points z such that $f(z) = z$.]

   Answer: (a) To find fixed points, we solve the equation $f(z) = z$ for finite z, and check whether $f(\infty) = \infty$ or not. Now,

$$f(\infty) = \lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{z-1}{z+1} = 1 \neq \infty.$$

   Thus, ∞ is not a fixed point of f. For finite z, we solve the equation

$$\begin{aligned} f(z) = z \implies& \frac{z-1}{z+1} = z \\ \implies& z-1 = z(z+1) \\ \implies& z^2 = -1 \\ \implies& z = i, -i. \end{aligned}$$

Therefore, the fixed points of f are $i, -i$.

b) 0; (c) ∞.

7. (a) Find the linear fractional transformation which has 0 and ∞ as fixed points and which maps $1+i$ onto $2+3i$.
(b) Suppose the transformation $w = \dfrac{az+b}{cz+d}$, where $ad-bc \neq 0$, is the same as its inverse. Show that either (i) $d=-a$; or (ii) $b=c=0$ and $d=a\neq 0$.

Solution: (a) Let $f(z) = \dfrac{az+b}{cz+d}$ be the required LFT. Since f fixes ∞, we must have $c=0$ (otherwise, $f(-d/c)=\infty$), and thus $d\neq 0$ (otherwise, ad-bc=0). Then dividing the numerator and denominator by d, we may assume that $d=1$, and thus we may write $f(z)=az+b$. Furthermore, since f fixes 0, we have $0=f(0)=b$, i.e., $b=0$ so that $f(z)=az$. Substituting the values in, we have $a=\dfrac{2+3i}{1+i}=\dfrac{5+i}{2}$, i.e., $f(z)=\dfrac{5+i}{2}z$.
(b) For the second part, we have $f^{-1}(z)=\frac{dz-b}{-cz+a}$ from which we get

$$\frac{az+b}{cz+d} = \frac{dz-b}{-cz+a}, \quad \forall z$$

$(az+b)(-cz+a) = (dz-b)(cz+d)$

$$-acz^2 + (a^2-bc)z + ab = cdz^2 + (d^2-bc)z - bd, \quad \forall z$$

Comparing coefficients, we get

$$-ac = cd, \quad a^2 = b^2, \quad ab = -bd.$$

From the second equation, we get $a=d$ or $a=-d$. Case 1: $a=-d$, then the other two equations are automatically satisfied, i.e., b and c are arbitrary with $ad-bc\neq 0$;

Case 2. $a=d\neq 0$ (note that the case $a=d=0$ can be regarded as a special case of Case 1 above), then from the other two equations, we easily see that $b=c=0$.

Thus we $a=d\neq 0$, and $b=c=0$.