NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 6

Selected answers and solutions

1. Does there exist a non-constant entire function $f(z)$ such that its image $f(\mathbb{C}) \subset \mathbb{C} \setminus \{z = x + iy \in \mathbb{C} : x \geq 0 \ \& \ y \geq 0\}$?
   [Hint: Riemann Mapping Theorem.]

   Answer: No. The domain $D = \mathbb{C} \setminus \{z = x + iy : x \geq 0 \ \& \ y \geq 0\}$ is simply connected. By the Riemann Mapping Theorem, there exists an analytic automorphism ϕ from D to the ball $B(0, 1)$. Then the composite function $g = \phi \circ f$ is an entire function since composition of two analytic functions is analytic. Moreover, $g(\mathbb{C}) = \phi \circ f(\mathbb{C}) = g(f(\mathbb{C})) \subset g(D) \subset B(0, 1)$. In particular, $|g(z)| < 1$ for all $z \in \mathbb{C}$. Hence g is an entire and bounded. By Liouville's Theorem, g must be a constant function. Thus there exists a complex constant c such that $g(z) = \phi \circ f(z) = c$ for all $z \in \mathbb{C}$. By composing both sides by ϕ^{-1}, we get $\phi^{-1} \circ \phi \circ f(z) = \phi^{-1}(c) \implies \phi^{-1}(\phi(f(z))) = f(z) = \phi^{-1}(c)$ for all $z \in \mathbb{C}$. Hence f must be a constant function on $\mathbb{C}$.

2. Give an example of an entire function which is conformal on the entire complex plane $\mathbb{C}$ but not one-to-one on $\mathbb{C}$.

   Answer: An example is the exponential function $f(z) = e^z$. Clearly, $f'(z) = e^z \neq 0$ everywhere on $\mathbb{C}$, and thus f is conformal on $\mathbb{C}$. But $f(0) = f(2\pi i) = 1$, so f is not one-to-one on $\mathbb{C}$.

3. Show that if $w = f(z)$ is analytic at a point z_o and $f'(z)$ has a zero of order n at z_o, then f maps two oriented smooth curves in the z-plane intersecting at angle θ at z_o to two curves in the w-plane intersecting at the angle $(n+1)\theta$.
   [Hint: Let γ_1 and γ_2 be the two oriented smooth curves intersecting at z_o. You may assume that θ is given by

$$\theta = \lim_{\substack{z \to z_o \\ z \in \gamma_1}} \arg(z - z_o) - \lim_{\substack{z \to z_o \\ z \in \gamma_2}} \arg(z - z_o)$$

   with the understanding that one-sided limits are taken, i.e., z always lie on one of the two branches of γ_1 (or γ_2) separated by z_o. Similarly, you may assume that $f(\gamma_1)$ and $f(\gamma_2)$ intersect at $f(z_o)$ at the angle given by

$$\lim_{\substack{z \to z_o \\ z \in \gamma_1}} \arg(f(z) - f(z_o)) - \lim_{\substack{z \to z_o \\ z \in \gamma_2}} \arg(f(z) - f(z_o)).$$

Answer: Remark: there is a typo in the hint, γ_1 and γ_2 should be interchanged under the limits, similarly in the answer below:

Since $f'(z)$ has a zero of order n at z_o, it follows that

$$f'(z_o) = \cdots = f^{(n)}(z_o) = 0, \text{ and } f^{(n+1)}(z_o) \neq 0.$$

From Taylor's theorem and the above conditions on the derivatives of f at z_o, we may write

$$f(z) = f(z_o) + a_{n+1}(z - z_o)^{n+1} + \cdots$$

on the ball $|z - z_o| < r$ for some $r > 0$, where the Taylor coefficient $a_n \neq 0$. In turn, this means that we may write

$$f(z) = f(z_o) + (z - z_o)^{n+1}\phi(z)$$

on the ball $|z - z_o| < r$ for some analytic function $\phi(z)$ such that $\phi(z_o) \neq 0$ (the argument is similar to Tutorial 1 Question 4). Then

$$\begin{aligned}
&\lim_{\substack{z\to z_o\\ z\in\gamma_1}} \arg(f(z) - f(z_o)) - \lim_{\substack{z\to z_o\\ z\in\gamma_2}} \arg(f(z) - f(z_o)) \\
&= \lim_{\substack{z\to z_o\\ z\in\gamma_1}} \arg((z - z_o)^{n+1}\phi(z)) - \lim_{\substack{z\to z_o\\ z\in\gamma_2}} \arg((z - z_o)^{n+1}\phi(z)) \\
&= \lim_{\substack{z\to z_o\\ z\in\gamma_1}} (n+1)\arg(z - z_o) + \lim_{\substack{z\to z_o\\ z\in\gamma_1}} \arg(\phi(z)) - \lim_{\substack{z\to z_o\\ z\in\gamma_2}} (n+1)\arg(z - z_o) - \lim_{\substack{z\to z_o\\ z\in\gamma_2}} \arg(\phi(z)) \\
&= \lim_{\substack{z\to z_o\\ z\in\gamma_1}} (n+1)\arg(z - z_o) + \arg(\phi(z_o)) - \lim_{\substack{z\to z_o\\ z\in\gamma_2}} (n+1)\arg(z - z_o) - \arg(\phi(z_o)) \\
&= (n+1)\theta.
\end{aligned}$$

$\blacksquare$

Here we used the fact that $\phi(z)$ is continuous at z_o and $\phi(z_o) \neq 0$ implies that we can make a consistent choice (branch) of $arg(\phi(z))$ for z near z_o so that $arg(\phi(z))$ is a continuous function in z, and thus

$$\lim_{\substack{z\to z_o\\ z\in\gamma_1}} \arg(\phi(z)) = \lim_{\substack{z\to z_o\\ z\in\gamma_2}} \arg(\phi(z)) = \arg(\phi(z_o)).$$

4 Consider the principal logarithmic function $w = \text{Log } z$ defined on $\mathbb{C} \setminus \{0\}$. Write $w = u + iv$. Describe and sketch the level curves of u and v associated to the function.

Answer: Recall that $u + iv = w = \text{Log } z = ln|z| + i\text{Arg } z$. Thus we have $u = ln|z|$ and $v = \text{Arg } z$. For the vertical line $u = u_o$ with fixed u_o, the corresponding level curve is given by $ln|z| = u_o \implies |z| = e^{u_o}$, which is a circle centered at 0 and of radius e^{u_o}. For the horizontal line $v = v_o$ with fixed v_o satisfying $-\pi < v_o \leq \pi$, the horizonta line $v = v_o$ corresponds to Arg $z = v_o$, which is a half line (ray) making an angle v_o with the positive real axis.

5. Describe the image of each of the following domains under the mapping $w = e^z$:
(i) the strip $0 < \text{Im} \quad z < \pi/2$;
(ii) the half strip $\text{Re} \quad z < 0$, $0 < \text{Im} \quad z < \pi$;
(iii) the half planes $\text{Re} \quad z > 0$ and $\text{Re} \quad z < 0$.

Answer: Write $z = x + iy$. Note that $w = e^z = e^{x+iy} = e^x \cdot e^{iy}$. Thus $|w| = e^x$ and $\arg w = y$.
(i) For the strip $0 < \text{Im} \quad z = y < \pi/2$, the corresponding region in the w-plane is thus $0 < \arg w < \pi/2$, which is the first quadrant.
(ii) For the strip $x < 0$, $0 < y < \pi$, the corresponding region is $ln|w| < 0$, $0 < \arg w < \pi$ or equivalently, the upper half unit disk centred at the origin given by $|w| < 1$,$0 < \arg w < \pi$.
(iii) For the half plane $x > 0$, the corresponding region is $ln|w| > 0$ or equivalently $|w| > 1$, i.e., the exterior of the unit circle centred at the origin. For the half plane $x < 0$, the corresponding region is $ln|w| < 0$ or equivalently $0 < |w| < 1$, i.e., the punctured unit ball $B(0,1) \setminus \{0\}$.

6. Find the image of the circle $|z| = 1$ under the maps
(i) $w = \dfrac{1}{z-1}$;
(ii) $w = \dfrac{1}{z-2}$.

Answer:
(i) $w = 1/(z-1) \Longrightarrow z = (1+w)/w$. $|z| = 1 \Longrightarrow |(1+w)/w| = 1 \Longrightarrow |1+w| = |w|$. Write $w = u + iv$. Then

$$\begin{aligned}
|1+w| = |w| &\Longleftrightarrow |1+w|^2 = |w|^2 \\
&\Longleftrightarrow |(u+1)+iv|^2 = |u+iv|^2 \\
&\Longleftrightarrow (u+1)^2 + v^2 = u^2 + v^2 \\
&\Longleftrightarrow u = -\frac{1}{2}.
\end{aligned}$$

which a vertical straight line through $u = -\dfrac{1}{2}$.

(ii) $w = 1/(z-2) \implies z = (2+w)/w$. $|z| = 1 \implies |(1+2w)/w| = 1 \implies |1+2w| = |w|$. Write $w = u + iv$. Then

$$\begin{aligned}
|1+2w| = |w| &\iff |1+2w|^2 = |w|^2 \\
&\iff |1+2(u+iv)|^2 = |u+iv|^2 \\
&\iff |(2u+1)+iv|^2 = |u+iv|^2 \\
&\iff (2u+1)^2 + v^2 = u^2 + v^2 \\
&\iff 3u^2 + 3v^2 + 4u + 1 = 0 \\
&\iff (u + \frac{2}{3})^2 + v^2 = \frac{1}{9},
\end{aligned}$$

which is a circle centred at $(-2/3, 0)$ and of radius $1/3$.