NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA4247 Complex Analysis II Tutorial 5

Selected answers and solutions

1. Let $f_n(z)$, $n = 1, 2, \cdots$, be a sequence of functions analytic in the open ball $D: \ |z| < R$ with $R > 0$. Suppose the sequence of functions $\{f_n(z)\}$ converges uniformly to an analytic function $f(z)$ on D. Prove that if $f(z) \neq 0$ for all z on the circle $|z| = \delta$, where $0 < \delta < R$, then there exists a positive integer N such that for all $n > N$, $f_n(z)$ and $f(z)$ have the same number of zeroes (counting multiplicity) in the ball $|z| < \delta$.
   [Hint: Rouché's Theorem]

   Answer: We apply Rouché's theorem to the function $f(z)$ with perturbation $g_n(z) = f_n(z) - f(z)$ on the interior of the circle $|z| = \delta$. Sine the circle $|z| = \delta$ is a closed and bounded set, by the Extreme Value Theorem, the continuous function $|f(z)|$ attains a minimum value on $|z| = \delta$, which is necessarily positive, since $f(z) \neq 0$ everywhere on $|z| = \delta$. Let $\min\limits_{|z|=\delta} |f(z)| = \epsilon > 0$. Since f_n converges uniformly to f on D, there exists N such that $|f_n(z) - f(z)| < \frac{\epsilon}{2}$ for all $n > N$ and all $z \in D$. So

   $$|g_n(z)| = |f_n(z) - f(z)| < \frac{\epsilon}{2} < \epsilon \leq |f(z)| \quad \forall\, |z| = \delta.$$

   Then by Rouche's theorem, $f(z)$ and $f(z) + g_n(z) = f_n(z)$ have the same number of zeros in $|z| < \delta$ for all $n > N$.

2. Use the open mapping theorem to give a quick proof of the following familiar facts: If f is analytic in a domain D, then f is identically constant in D if any of the following conditions holds:
   (a) Re $f(z)$ is constant in D.
   (b) Im $f(z)$ is constant in D.
   (c) $|f(z)|$ is constant in D.

   Answer: In (a), $f(D) \subset \{iy : y \in \mathbb{R}\}$, the imaginary axis. In (b), $f(D) \subset \mathbb{R}$, the real axis. In (c), $f(D) \subset C$ for some circle $C : |z| = r$, where $r \geq 0$. In all cases, the image $f(D)$ cannot be open in $\mathbb{C}$, Hence, by the open mapping theorem, f must be constant.

3. Fix any complex constant α such that $|\alpha| < 1$. Consider the function

$$\phi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}.$$

(i) Show that ϕ_α maps the unit circle C: $|z| = 1$ to itself.

(ii) Show also that ϕ_α is an analytic function from the open ball $B(0,1)$ into itself.

[Hint: Use (i).]

(iii) Show that ϕ_α is an analytic automorphism of $B(0,1)$.

(iv) Show that the inverse of ϕ_α on $B(0,1)$ is $\phi_{-\alpha}$, i.e., show that

$$\phi_\alpha \circ \phi_{-\alpha}(z) = z = \phi_{-\alpha} \circ \phi_\alpha(z), \quad \text{for all } z \in B(0,1).$$

(v) Show that $\phi'_\alpha(0) = 1 - |\alpha|^2$ and $\phi'_\alpha(\alpha) = (1 - |\alpha|^2)^{-1}$.

[Remark: The $\phi'_\alpha s$ are very useful analytic automorphisms on $B(0,1)$, which we will see in the next question and in a number of occasions later. An important property of ϕ_α is that $\phi_\alpha(\alpha) = 0$ (check it).]

Answer:

(i) If $|z| = 1$, then

$$\begin{aligned}
&|1-\bar{\alpha}z|^2 - |z-\alpha|^2 \\
&= (1-\bar{\alpha}z)\overline{(1-\bar{\alpha}z)} - (z-\alpha)\overline{(z-\alpha)} \\
&= (1-\bar{\alpha}z)(1-\alpha\bar{z}) - (z-\alpha)(\bar{z}-\bar{\alpha}) \\
&= 1 - \bar{\alpha}z - \alpha\bar{z} + z\bar{z}\alpha\bar{\alpha} - (z\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha}) \\
&= 1 + |z|^2|\alpha|^2 - |z|^2 - |\alpha|^2 \\
&= (1-|z|^2)(1-|\alpha|^2) \\
&= 0.
\end{aligned}$$

Thus, if $|z| = 1$, then $|\phi_\alpha(z)| = \dfrac{|z-\alpha|}{|1-\bar{\alpha}z|} = 1$.

Alternatively, note that if z lies on the unit circle $|z| = 1$, $z\bar{z} = 1$ so that

$$|z-\alpha| = |z||1-\alpha\bar{z}| = |1-\bar{\alpha}z|,$$

so that $|\phi_\alpha(z)| = 1$.

(ii) ϕ_α is not analytic only at $z = 1/\overline{\alpha}$, which is outside $B(0,1)$, since

$$|\frac{1}{\overline{\alpha}}| = \frac{1}{|\alpha|} > 1.$$

Thus ϕ_α is analytic and clearly non-constant on the closed unit ball $|z| \le 1$ (say, $\phi(\alpha) = 0$, but $\phi_\alpha(z) \ne 0$ if $z \ne \alpha$). Since $|\phi_\alpha(z)| = 1$ on the boundary circle $|z| = 1$, so by the MMP, $|\phi_\alpha(z)| < 1$ for $|z| < 1$.
(iii) By (ii), ϕ_α is analytic from $B(0,1)$ to itself. To show that ϕ_α is an analytic automorphism on $B(0,1)$, it remains to show that ϕ_α is one-to-one and onto.
One-to-one: Let $z_1, z_2 \in B(0,1)$. Then

$$\begin{aligned} \phi_\alpha(z_1) &= \phi_\alpha(z_2) \\ \implies \frac{z_1 - \alpha}{1 - \overline{\alpha} z_1} &= \frac{z_2 - \alpha}{1 - \overline{\alpha} z_2} \\ \implies (z_1 - \alpha)(1 - \overline{\alpha} z_2) &= (z_2 - \alpha)(1 - \overline{\alpha} z_1) \\ \implies z_1 - \alpha\overline{\alpha} z_2 &= z_2 - \alpha\overline{\alpha} z_2 \\ \implies (z_1 - z_2)(1 - |\alpha|^2) &= 0 \\ \implies z_1 &= z_2, \end{aligned}$$

since $|\alpha| < 1$. Hence ϕ_α is one-to-one on $B(0,1)$.
Onto: For $|w| < 1$, we solve for the equation

$$\begin{aligned} f(z) = w \Leftrightarrow \frac{z - \alpha}{1 - \overline{\alpha} z} &= w \\ \Leftrightarrow z &= \frac{w + \alpha}{1 + \bar{\alpha} w}. \end{aligned}$$

Note that if $|w| < 1$, then $\left| \dfrac{w + \alpha}{1 + \bar{\alpha} w} \right| < 1$ (by a calculation similar to (i), try it). Hence if $w \in B(0,1)$, then $z = \dfrac{w + \alpha}{1 + \bar{\alpha} w} \in B(0,1)$ and $f(\dfrac{w + \alpha}{1 + \bar{\alpha} w}) = w$. Hence ϕ_α is onto.

Alternatively, (iii) can be deduced from (iv) since if the composition of two functions $f \circ g$ is one-to-one and onto, then f is one-to-one and g is onto, we can use this to deduce that $\rho_\alpha(z)$ is one-to-one and onto on $B(0,1)$.

(iv) Note that $\phi_{-\alpha} = \frac{z + \alpha}{1 + \bar{\alpha} z}$. Then check directly that

$$\phi_\alpha \circ \phi_{-\alpha}(z) = z = \phi_{-\alpha} \circ \phi_\alpha(z), \qquad \text{for all } z \in B(0,1).$$

(v) By direct calculation,

$$\phi'_\alpha(z) = \frac{1 - |\alpha|^2}{(1 - \overline{\alpha} z)^2}$$

and the result follows.

4. (Another generalization of Schwarz's lemma).
Let $f(z)$ be an analytic function on the open ball $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ such that $|f(z)| < 1$ for all $|z| < 1$. Show that

$$|f'(0)| \leq 1 - |f(0)|^2.$$

[Hint: Let $\alpha = f(0)$. Consider the composite function $g = \phi_\alpha \circ f$, where ϕ_α is as in Question 3. Then apply Schwarz's lemma to the function. You may also need to use the results in Question 3.]

Answer: Let $\alpha = f(0)$, so that $|\alpha| < 1$. Then consider the function $g(z) = \phi_\alpha \circ f(z) = \phi_\alpha(f(z))$ for $|z| < 1$. By Question 3, we know that ϕ_α is an analytic automorphism on $B(0,1)$. Note that f is an analytic function mapping $B(0,1)$ to $B(0,1)$. Thus, $g(z) = \phi_\alpha \circ f(z)$ is a well-defined analytic function mapping $B(0,1)$ to $B(0,1)$. In particular, $|g(z)| < 1$ for all $|z| < 1$. Now

$$g(0) = \phi_\alpha(f(0)) = \phi_\alpha(\alpha) = \frac{\alpha - \alpha}{1 - \bar{\alpha}\alpha} = 0.$$

Thus by Schwarz's lemma, we have $|g'(0)| \leq 1$. By the Chain Rule, we have $g'(0) = \phi'_\alpha(f(0)) \cdot f'(0)$. Therefore, we have

$$\begin{aligned}
|g'(0)| &= |\phi_\alpha(f(0))| \cdot |f'(0)| \leq 1 \\
\Longrightarrow & |\phi'_\alpha(f(0))| \cdot |f'(0)| \leq 1 \\
\Longrightarrow & |\phi'_\alpha(\alpha)| \cdot |f'(0)| \leq 1 \\
\Longrightarrow & |(1 - |\alpha|^2)^{-1}| \cdot |f'(0)| \leq 1 \\
\Longrightarrow & |f'(0)| \leq |1 - |\alpha|^2| = 1 - |\alpha|^2 = 1 - |f(0)|^2,
\end{aligned}$$

which gives the result.