NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 4

Selected answers and solutions, note that details for questions 3,4,5,6,7,8 and 9 should be filled in yourself

1. Let $h(z) = \frac{1}{2}(z + \frac{1}{z})$. Prove that if w is any complex number **not** in the closed interval $[-1, 1]$, then there is exactly one z in the unit open ball $B(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ such that $h(z) = w$.
   Hint: Fix a w not in $[-1, 1]$, and let
   $$f(z) = h(z) - w.$$
   Show that as z traces around the unit circle $C : |z| = 1$ once in the positive direction $f(C)$ does not pass through the origin in the w plane and has winding number 0 about the origin. Then apply the argument principle.

   Solution: Fix a w not in $[-1, 1]$, and let
   $$f(z) = h(z) - w.$$
   We will show that as z traces around the unit circle $C : |z| = 1$ once in the positive direction $f(C)$ does not pass through the origin in the w plane and has winding number 0 about the origin. Then as $f(z)$ has only one simple pole at $z = 0$ inside C, by the argument principle, this implies that it also has a simple zero inside C which gives us the required conclusion.
   Parametrize C by $z = \cos\theta + i\sin\theta$, $0 \le \theta \le 2\pi$. Then $f(z) = h(z) - w = \cos\theta - w$, where $0 \le \theta \le 2\pi$ and since $w \notin [-1, 1]$, $f(z) \ne 0$ for $z \in C$. Now as θ varies from 0 to 2π, $f(z)$ varies along the horizontal line segment from $-w+1$ to $-w-1$ and back to $-w+1$ again. This clearly has winding number 0.

2. Use Rouché's theorem to show that the polynomial $z^5 + 3z^2 + 1$ has exactly two zeroes in the disk $|z| < 1$ counting multiplicity.
   Solution: Use $f(z) = 4z^2$ and $g(z) = z^6 - 1$. On $|z| = 1$, $|g(z)| = |z^6 - 1| \le |z^6| + |1| = 2 < 4 = |f(z)|$. By Rouché's theorem, number of zeros of $z^5 + 3z^2 + 1$ is equal to number of zeroes of $4z^2$ inside $|z| < 1$ which is two.

3. Prove that the equation $z^3 + 9z + 27 = 0$ has no roots in the disk $|z| < 2$.
   Solution: Use $f(z) = 27$ and $g(z) = z^3 + 9z$.

4. Find the number of roots of the equation $6z^4 + z^3 - 2z^2 + z - 1 = 0$ in the disk $|z| < 1$.
   Solution: Use $f(z) = 6z^4$ and $g(z) = z^3 - 2z^2 + z - 1$.

5. Give an example to show that the conclusion of Rouché's theorem may be false if the strict inequality $|g(z)| < |f(z)|$ is replaced by $|g(z)| \leq |f(z)|$ on C.
   Solution: Use $f(z) = z$ and $g(z) = -z$ with $\gamma : |z| = 1$ for example. Another example: Use $f(z) = z$ and $g(z) = 1$ with $\gamma : |z| = 1$.

6. Prove that all the roots of the equation $z^6 - 5z^2 + 10 = 0$ lie in the annulus $1 < |z| < 2$.
   Solution: For $|z| = 1$, use $f(z) = 10$ and $g(z) = z^6 - 5z^2$; for $|z| = 2$, use $f(z) = z^6$ and $g(z) = -5z^2 + 10$.

7. Let $a, b \in \mathbb{C}$, and $n \in \mathbb{Z}^+$. Show that $az^n + be^z$ has n zeroes counting multiplicity in the interior of the unit circle $|z| = 1$ if $|a| > |b|e$.
   Solution: Use $f(z) = az^n$ and $g(z) = be^z$. Note that if $z = x + iy$, then $|e^z| = e^x$, so that for z on the unit circle $|z| = 1$, $x \leq 1$ so $|e^z| \leq e^1 = e$.

8. Prove that the equation $z = 2 - e^{-z}$ has exactly one root in the right half plane. Why must this root be real?
   Solution: Use the right half semi-circular contour C_R of radius R (with diameter given by the line segment from $-Ri$ to Ri) and let R approach ∞. Use $f(z) = z - 2$ and $g(z) = -e^{-z}$ for this contour. Note that if $R \geq 4$, then for $z \in C_R$, $|z - 2|$ is the distance of z from the point 2, which is ≥ 2. To obtain an upper bound for $|g(z)|$, adapt the argument from the previous question. Finally, use intermediate value theorem (restricting the function $h(z) = z - 2 + e^{-z}$ to the positive real axis) to deduce there is at least one positive real root.

9. Determine the number of zeroes of the polynomial $3z^7 + 5z - 1$ counting multiplicity which lie in the annulus $1 < |z| < 2$.
   Solution: Use $f(z) = 5z$ and $g(z) = 3z^7 - 1$ to show that there is one zero of $3z^7 + 5z - 1$ in the disk $|z| < 1$ (and none on $|z| = 1$). Then use $f(z) = 3z^7$ and $g(z) = 5z - 1$ to show that there are 7 zeroes of $3z^7 + 5z - 1$ in $|z| < 1$. Therefore, there are 7-1=6 zeroes of $3z^7 + 5z - 1$ in $1 < |z| < 2$.