NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 3

Selected answers and solutions

1. Prove the following variations of Schwarz's lemma:

   (i) Suppose that f is analytic in the open ball $|z| < R$ for some $R > 0$ and that $|f(z)| \leq 1$ for $|z| < R$, and $f(0) = 0$. Show that $|f(z)| \leq \dfrac{|z|}{R}$ for $|z| < R$.

   (ii) Suppose that f is analytic in the open ball $|z| < 1$, and that $|f(z)| \leq M$ for $|z| < 1$, where $M > 0$, and $f(0) = 0$. Show that $|f(z)| \leq M|z|$ for $|z| < 1$.

   [Hint: Use the Schwarz's lemma.]

   Solution: (i) Let $g(z) = f(Rz)$. Then g is analytic in $|z| < 1$, $|g(z)| \leq 1$ for $|z| < 1$ and $g(0) = 0$. By Schwarz lemma, $|g(z)| = |f(Rz)| \leq |z|$ for $|z| < 1$, or, letting $w = Rz$, we get $|f(w)| \leq |w|/R$ for $|w| < R$.

   (ii) Let $g(z) = f(z)/M$ and apply Schwarz lemma to $g(z)$.

2. Find all the analytic functions $f(z)$ on the open ball $|z| < 1$ such that $|f(z)| < 1$ for all $|z| < 1$, $f(0) = 0$ and $f(\frac{1}{2}) = -\frac{i}{2}$.

   Solution: By Schwarz's lemma, since $|f(z)| < 1$ for all $|z| < 1$, $f(0) = 0$, and $\left|f(\frac{1}{2})\right| = \left|-\frac{i}{2}\right| = \frac{1}{2}$, we must have $f(z) \equiv Cz$ on $|z| < 1$ for some constant C satisfying $|C = 1$. Since $f(\frac{1}{2}) = -\frac{i}{2}$, we must have $-\frac{i}{2} = C \cdot \frac{1}{2}$, and thus $C = -i$, and $f(z) = -iz$ on $|z| < 1$. Conversely, $f(z) = -iz$ clearly satisfies all the prescribed conditions. Thus, $f(z) = -iz$ is the only function satisfying the given conditions.

3. Which of the following functions are meromorphic in the whole plane?

   (a) $iz + z^5$ (b) $\text{Log } z$ (c) $\dfrac{\cos z}{z^3 + 1}$ (d) $e^{1/z}$

   (e) $\tan z$ (f) $\dfrac{2i}{(z-2)^3} + e^z$.

   Give brief explanation to your answer.

   Solution: (a), (c), (e) and (f). (b) is not analytic on the negative real axis (so singular on the negative real axis), (d) has an essential singularity at $z = 0$ (look at the Laurent series).

4. Consider the polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Explain why for each sufficiently large value of R,

$$\int_{|z|=R} \frac{P'(z)}{P(z)} dz = 2n\pi i.$$

Here the circle $|z| = R$ is positively oriented.
[Hint: Recall the Fundamental Theorem of Algebra.]
Solution: $P(z)$ is entire and by Fundamental Theorem of Algebra, $P(z)$ has n zeroes (counted with multiplicity). For R sufficiently large (greater than the maximum of the absolute values of the roots of $P(z)$), all the zeroes lie inside the circle $|z| = R$ and applying the argument principle, we get the result.

5.
$$\frac{1}{2\pi i} \int_{|z|=3} \frac{f'(z)}{f(z)} dz,$$

where $f(z) = \dfrac{z^2(z-i)^3 e^z}{3(z+2)^4(3z-18)^5}$.
Solution: Zeros and poles of $f(z)$ inside the circle $|z| = 3$ counted with multiplicity are as follows:
Zeroes: 0 (mult. 2), i, (mult. 3); Poles: -2, (mult. 4) Hence

$$\frac{1}{2\pi i} \int_{|z|=3} \frac{f'(z)}{f(z)} dz = Z - P = 5 - 4 = 1.$$

6. Let $f(z)$ be analytic on the closed disk $|z| \le \rho$, and suppose that $f(z) \neq w_0$ for all z on the circle $|z| = \rho$. Explain why the value of the integral

$$\frac{1}{2\pi i} \int_{|z|=\rho} \frac{f'(z)}{f(z) - w_0} dz$$

is bigger than or equal to the number of distinct solutions of the equation $f(z) = w_0$ inside the disk.
Solution: Let $g(z) = f(z) - w_0$. Then $g(z)$ is analytic on the closed disk $|z| \le \rho$, $g(z) \neq 0$ on the circle $|z| = \rho$. Note that $g'(z) = f'(z)$ and that the zeros of g counting multiplicity are exactly the solutions of $f(z) = w_0$. Applying the argument principle, one sees that the given contour integral is equal to the no. of zeros of g (i.e. the number of solutions of $f(z) = w_o$) counting multiplicity. Since the multiplicity of each (genuine) zero of g is ≥ 1, the result follows.

7. Suppose that a function f is analytic inside and on a positively oriented simple closed contour γ and that $f(z) \neq 0$ for all $z \in \gamma$. Show that if f has n zeroes z_k, $k = 1, \cdots, n$, inside γ, and each z_k is a zero of f of order m_k, then
$$\int_\gamma \frac{zf'(z)}{f(z)}\,dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

Solution: The integrand $\dfrac{zf'(z)}{f(z)}$ has singular points only at z_k, $k = 1, \cdots, z_k$. Since f has a zero of order m_k at z_k, we may write $f(z) = (z - z_k)^{m_k}\phi(z)$ near z_k for some function $\phi(z)$ analytic at z_k with $\phi(z_k) \neq 0$. Using Method I from the lecture notes, one may check that the residue of $\dfrac{zf'(z)}{f(z)}$ at z_k is $m_k z_k$. Then one may use the Cauchy residue theorem to get the given equality.