NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 2

Selected answers and solutions

1. Using the identity theorem for analytic functions, show that there exists at most one entire function f such that $f(x) = x + \sin x$ for all $x \in \mathbb{R}$.
Solution: Suppose there exist two entire functions f and g such that $f(x) = g(x) = x + \sin x$ for all $x \in \mathbb{R}$. Since $\mathbb{R}$ has an accumulation point in the domain $\mathbb{C}$, it follows from the identity theorem for analytic functions that $f(z) = g(z)$ for all $z \in \mathbb{C}$.

2. Use the identity $\sin 2x = 2 \sin x \cos x$ for all $x \in \mathbb{R}$ to deduce that
$$\sin 2z = 2 \sin z \cos z \quad \text{for all } z \in \mathbb{C}.$$
Solution: Consider the two entire functions $\sin 2z$ and $2 \sin z \cos z$. From the given identity $\sin 2x = 2 \sin x \cos x$ for all $x \in \mathbb{R}$, it follows that the two entire functions agree on $\mathbb{R}$, which has an accumulation point in the domain $\mathbb{C}$, it follows from the Identity Theorem for Analytic Functions that the two entire functions agree on the domain $\mathbb{C}$, i.e., $\sin 2z = 2 \sin z \cos z$ for all $z \in \mathbb{C}$.

3. Find all functions f analytic in $B(0, R) := \{z \in \mathbb{C} \,\big|\, |z| < R\}$ that satisfy $f(0) = i$ and $|f(z)| \leq 1$ for all z in $B(0, R)$. (Hint: where does the maximum modulus occur?)
Solution: $|f(0)| = 1$, so the maximum modulus occurs at the interior point 0, by the MMP, $f(z) \equiv i$.

4. If f is analytic in the annulus $1 \leq |z| \leq 2$ and $|f(z)| \leq 5$ for $|z| = 1$, and $|f(z)| \leq 20$ for $|z| = 2$, prove that $|f(z)| \leq 5|z|^2$ for $1 \leq |z| \leq 2$. (Hint: Consider $f(z)/5z^2$.)
Solution: $g(z) = f(z)/5z^2$ is analytic on the annulus and
$|g(z)| = |f(z)/5z^2| \leq 1$ on $|z| = 1$ and $|z| = 2$ (check!), so by the MMP, $g(z) \leq 1$, i.e., $|f(z)| \leq 5|z|^2$ for $1 \leq |z| \leq 2$.

5. Suppose that f is analytic inside and on the simple closed curve C and that $|f(z) - 1| < 1$ for all z on C. Prove that f has no zeroes inside C.

(Hint: Suppose $f(z_0) = 0$ for some z_0 inside C and consider the function $g(z) := f(z) - 1$.)

Solution: Suppose $f(z_0) = 0$ for some z_0 inside C. The function $g(z) := f(z) - 1$ is analytic on and inside C and $|g(z)| < 1$ for z on C while $|g(z_0)| = 1$. Hence the maximum value of $|g(z)|$ does not occur on the boundary, contradicting the MMP. Hence $f(z_0) \neq 0$ for all z_0 inside C.

6. Let f and g be analytic in the bounded domain D and continuous up to and including its boundary B. Suppose that g never vanishes. Prove that if the inequality $|f(z)| \leq |g(z)|$ holds for all z on B, then it must hold for all z in D.

   Solution: $f(z)/g(z)$ is analytic on D and continuous up to and including the boundary B since g never vanishes. We have $|f(z)/g(z)| \leq 1$ for all z on B, by the MMP, it also holds for all z in D.

7. Prove the **minimum modulus principle**: Let $R \subset \mathbb{C}$ be a closed bounded set whose interior is a domain. Suppose f is continuous on R and analytic and not constant in the interior of R. If $f(z) \neq 0$ for any $z \in R$, then $|f(z)|$ attains its minimum value at the boundary of R but not in the interior of R. (Hint: Consider the function $1/f(z)$.)
   Give an example to show why the non-zero condition is necessary.

   Solution: Consider the function $g(z) := 1/f(z)$ which is analytic and non-constant in the interior of R since f is non-zero in R. Clearly $g(z)$ is also continuous on R. By the MMP, $|g(z)| = |1/f(z)|$ attains its maximum value only on the boundary, i.e., there exists a point z_0 on the boundary such that $|1/f(z)| \leq |1/f(z_0)|$ for all z in R and $|1/f(z)| < |1/f(z_0)|$ for all z in the interior of R, that is, $|f(z)| \geq |f(z_0)|$ for all z in R, and $|f(z)| > |f(z_0)|$ for all z in the interior of R. Hence $|f(z)|$ attains its minimum only on the boundary.
   For the example, take $R = \{z : |z| \leq 1\}$ and $f(z) = z$.

8. Let the non-constant function f be analytic in the bounded domain D and continuous up to and including its boundary B. Prove that if $|f(z)|$ is constant on B, then f must have at least one zero in D.
   Solution: First note that $|f(z)| \equiv c > 0$ on the boundary B, otherwise, if $c = 0$, then by the MMP, $f(z) \equiv 0$ on D. Now suppose on the contrary that f has no zeros in D. By the minimum modulus principle, $|f(z)|$ also attains its minimum on B, so $|f| = c$ for all z in D and its boundary B. By MMP

again, f is constant on D, contradicting the fact that f is a non-constant function. Hence, f has at least one zero in D.