NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2009/2010 Semester I MA4247 Complex Analysis II Tutorial 10

Suggested answers and solutions:

1. Find a conformal isomorphism mapping the semi-infinite strip $x > 1,\ -1 < y < 1$ to the unit ball $|w| < 1$.

   [Hint: Recall that the function $f(z) = \sin z$ is a conformal isomorphism from the upper semi-infinite strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y > 0$ to the upper half plane Im $z > 0$.]

   Solution: Using $z_1 = f_1(z) = z - 1$, we first translate the semi-infinite strip to the left by 1 unit, so that the strip is mapped to the strip S_1 : $x > 0,\ -1 < y < 1$. Next using $z_2 = f_2(z_1) = e^{i\pi/2}z_1 = iz_1$, then S_1 becomes another strip $S_2 : -1 < x < 1, y > 0$. Then using the dilation $z_3 = f_3(z_2) = \frac{\pi}{2}z_2$, S_2 becomes the strip $S_3 : -\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0$. Now using $z_4 = f_3(z_3) = \sin z_3$, S_3 is mapped to the upper half plane Im $z_4 > 0$. Finally using the LFT $w = f_4(z_4) = \dfrac{z_4 - i}{z_4 + i}$, the upper half plane is mapped to $B(0, 1)$. Thus, the required conformal isomorphism is given by

$$w = f_4 \circ f_3 \circ f_2 \circ f_1(z) = \frac{\sin \frac{\pi}{2}i(z-1) - i}{\sin \frac{\pi}{2}i(z-1) + i}.$$

   Remark: The transformation is not unique. So you may get a different answer. An alternative method is to first map the semi-infinite strip to the semi-infinite strip $x < 0$, $0 < y < \pi$, map that to the upper half ball by the map e^z and then follow the steps of Q3, tutorial 8 and so on.

2. Show that the function $u(x, y) = 2xy + e^x \cos y$ is a harmonic function on $\mathbb{R}^2$ and find a harmonic conjugate to $u(x, y)$.

   [Answer: $v(x, y) = y^2 + e^x \sin y - x^2 + C$.]

3. Suppose that $f(z) = u + iv$ is analytic on a domain D. Show that $u + v$, $u^2 - v^2$ and uv are harmonic on D. What about $u^2 + v^2$?

   [You may use freely the fact that the real and imaginary parts of an analytic function are harmonic.]

Answer: $f - if = (u+v) + i(v-u)$, $f^2 = (u^2 - v^2) + i(2uv)$ are analytic, hence $u+v$, $u^2 - v^2$ and uv are harmonic.
u^2+v^2 is not harmonic, for example, take $u = x$, $v = y$. Then $\phi = u^2+v^2 = x^2 + y^2$, but $\phi_{xx} + \phi_{yy} = 1 + 1 = 2 \neq 0$.

4. Show that ϕ_x and ϕ_y are harmonic on a domain D if ϕ is harmonic on D.

   [Hint: Locally, write ϕ as the real part of an analytic function.]

   Answer: At each point $z \in D$ which is open, there exists an open ball $B(z_o, r) \subset D$ with $r > 0$. Since $B(z_o, r)$ is a simply connected domain, the harmonic function $\phi = \text{Re }(f)$ for some analytic function f on $B(z_o, r)$. Write $f = \phi + iv$, where v is the imaginary part of f. Then $f' = \phi_x + iv_x$. By CR equations, $v_x = -\phi_y$. Thus, $f' = \phi_x - i\phi_y$ is analytic (recall that derivatives of analytic functions are analytic). Thus, its real part ϕ_x and its imaginary part $-\phi_y$ are harmonic. Note that $-\phi_y$ is harmonic implies readily that ϕ_y is also harmonic in $B(z_o, r)$. By varying z_o in D, it follows that ϕ_x and ϕ_y are harmonic in D.
   Alternative solution: Let $g = \phi_x - i\phi_y$ on D. Show that g is analytic on D by noting that ϕ_x and $-\phi_y$ have continuous first order partial derivatives, and satisfy the CR-equations. Then ϕ_x and $-\phi_y$ are harmonic, since they are real and imaginary parts of an analytic function.

5. Consider the function

$$u(x,y) = \frac{1}{2}\ln(x^2 + y^2), \quad z = x + iy \in \mathbb{C} \setminus \{0\}.$$

   (i) Show that u is harmonic on $\mathbb{C} \setminus \{0\}$.
   (ii) Show that u has no harmonic conjugate on $\mathbb{C} \setminus \{0\}$.
   [Hint: First show that if v is a harmonic conjugate of u on $\mathbb{C} \setminus \{0\}$, then $u + iv = \text{Log}\, z + iC$ on $\mathbb{C} \setminus (-\infty, 0]$ for some real constant C.]

   Answer: (i) Direct calculation.
   (ii) We prove (ii) by contradiction. Suppose u has a harmonic conjugate v on $\mathbb{C} \setminus \{0\}$, so that $f = u + iv$ is analytic on $\mathbb{C} \setminus \{0\}$. Consider the function $\text{Log}\, z$ on the domain $D = \mathbb{C} \setminus (-\infty, 0]$. Note that $\text{Re }(\text{Log}\, z) = \ln|z| = \frac{1}{2}\ln(x^2 + y^2) = (x, y)$ on D. Thus, the real parts of f and $\text{Log}\, z$ are the same on D. Thus, the real part of the analytic function $f(z) - \text{Log}\, z$ is zero on the domain D, and thus $f(z) - \text{Log}\, z$ is a constant function (and purely imaginary) on D. Write $f(z) - \text{Log}\, z \equiv iC$ on D for some real constant C. Consider the two circular arcs $\gamma_1 : e^{i\theta},\ 0 \le \theta \le \pi$, and $\gamma_2 : e^{i\theta},\ -\pi < \theta \le 0$

in the upper and lower half plane respectively, which intersect at $z = -1$. Then since f is continuous at $z = -1$,

$$\begin{aligned}
&\lim_{\substack{z \to -1 \\ z \in \gamma_1}} f(z) = \lim_{\substack{z \to -1 \\ z \in \gamma_2}} f(z) = f(-1) \\
\implies &\lim_{\substack{z \to -1 \\ z \in \gamma_1}} \operatorname{Log} z + iC = \lim_{\substack{z \to -1 \\ z \in \gamma_2}} \operatorname{Log} z + iC \\
\implies &\lim_{\substack{z \to -1 \\ z \in \gamma_1}} \ln |z| + i\mathrm{Arg}(z) + iC = \lim_{\substack{z \to -1 \\ z \in \gamma_2}} \ln |z| + i\mathrm{Arg}(z) + iC \\
\implies &\lim_{\theta \to \pi -} \ln |e^{i\theta}| + i\mathrm{Arg}(e^{i\theta}) + iC = \lim_{\theta \to -\pi +} \ln |e^{i\theta}| + i\mathrm{Arg}(e^{i\theta}) + iC \\
\implies &0 + i\pi + iC = 0 - i\pi + iC,
\end{aligned}$$

which is a contradiction. Thus, f does not have a conjugate on $\mathbb{C} \setminus \{0\}$.